# Lectures on Supersymmetry
## Lecture 1

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**Abstract**

These are the notes for lecture one of a course on Supersymmetry to be given by Mark Goodsell, Karim Benakli and Pietro Slavich at Jussieu in 2024.

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# Contents



# 1 Introduction

These notes are intended to be as self-contained as possible. However, supersymmetry (SUSY) is a subject that has been studied for more than forty years now, so there is a lot of material that cannot be included. It is a good idea to consult at least two other sources.

There are by now a number of excellent references on SUSY. Some books are:

- Wess and Bagger, *Supersymmetry and Supergravity.* The "classic" reference book, but more of a collection of formulae. Most people use a slightly adapted version of their notation.
- Weinberg, *The quantum theory of fields*, vol. 3. Deep conceptually (as are the first two volumes) but rather inconvenient use of four-spinor notation.
- Drees, Godbole and Roy, *Sparticles.* Practical from a phenomenological point of view.
- Bailin and Love, *Supersymmetric gauge field theory and string theory.* Divides into two parts, the first of which is about SUSY. Is very clear, concise and cheap, although contains some typos.

Some lecture notes are:

- Martin, *A Supersymmetry primer*, arXiv:hep-ph/9709356. A famous set of notes from a phenomenologist's point of view.

- Quevedo (written up by Krippendorff and Schlotterer), *Cambridge Lectures on Supersymmetry and Extra Dimensions*, arXiv:1011.1491.
- Bertolini, *Lectures on Supersymmetry*, `http://people.sissa.it/~bertmat/teaching.htm`. Somewhat more formal notes with discussion about SUSY breaking and mediation.

Be warned that the notation is often similar but not identical between references! This can lead to different signs and numerical factors between formulae. My notation will be very close to that of Martin.

# 2 Why Supersymmetry?

The first question you might be asking is, "why study supersymmetry?" This is difficult to answer properly before we have first answered "what is SUSY?" but I will outline some of the arguments before returning to some of the most important ones in detail later in the lectures. You will see that there is a justifiably large amount of research that has been and continues to be dedicated to it!

## 2.1 Deficiencies of the Standard Model

An important part of SUSY research is motivated by the study of physics beyond the Standard Model. We know that the Standard Model has many deficiencies that it cannot explain:

- Quantum gravity.
- Inflation.
- The strong CP problem.
- Baryogenesis.
- Dark matter.
- Dark energy.

In addition, there are several problems that may either be aesthetic or fundamental:

- The hierarchy problem.
- The origin of neutrino masses (formally there is no source of neutrino mass in the Standard Model).
- The pattern of quark and lepton Yukawa couplings.

Finally, there is the longstanding $3\sigma$ discrepancy between the Standard Model calculation and experimental measurement of the muon anomalous magnetic moment; not to mention several other potential anomalies which have so far had a shorter lifespan.

Supersymmetry fits into possible explanations for several of these – more details below. But it also provides a robust framework for studying all of the others in a consistent manner, rather than e.g. adding *ad hoc* fields and couplings to the Standard Model to solve one particular issue.

### 2.2 Simpler field theories

The other branch of SUSY research consists of formal explorations of its properties and consequences. Particularly, it allows us to construct *simpler* field theories (even though they may have more fields) because the symmetries restrict many of the properties/couplings. For example, the powerful supersymmetric non-renormalisation theorems show that certain quantities are not renormalised in perturbation theory, and indeed some SUSY theories are finite. Some people have even conjectured that a particular supersymmetric theory of gravity is finite. By simplifying calculations we can investigate quantities out of reach otherwise, and access non-perturbative information or all-orders calculations, e.g. maximally supersymmetric Yang-Mills theory is dual to string theory in AdS space; its amplitudes can be described to all orders by an "amplituhedron" and possess many other remarkable properties. It may also be possible to relate SUSY theories to condensed matter systems. So this part of SUSY theory exists independently of Beyond-the-Standard-Model phenomenology.

### 2.3 Hierarchy problem

One of the widely cited motivations for supersymmetry is the "hierarchy problem," broadly speaking the problem that the Planck scale of $M_P \equiv \sqrt{\frac{\hbar c}{G}} \simeq 1.22 \times 10^{19}$ GeV$/c^2$ is so much larger than the electroweak scale of $\mathcal{O}(100)$ GeV. To be more precise, consider a theory with fundamental scalars (such as the Higgs boson)

$$\mathcal{L} \supset -\mu^2|H|^2 - \frac{\lambda}{4}|H|^4. \tag{2.1}$$

If we regard this as an *effective theory* with some cutoff $\Lambda$, above which new physics comes in, then there is nothing that protects $\mu^2$ from being much smaller than $\Lambda^2$. One way to see it is to imagine that we fix $\mu^2$ in the high energy theory, and then compute the 1PI action at low energies, e.g. at zero external momentum

$$\langle H\overline{H}\rangle \propto \int_0^\Lambda \frac{d^4k}{(2\pi)^4}\frac{\lambda}{k^2-\mu^2_{\text{bare}}} \propto \lambda\Lambda^2. \tag{2.2}$$

This means that we have to tune the value of $\mu^2_{\text{bare}}$ to allow $\mu^2 \ll \lambda\Lambda^2$. However, this argument is perhaps misleading if we consider dimensional regularisation, where there are no quadratic divergences; it can instead by imagining that in the high energy theory there are other heavy particles that couple to the Higgs, e.g. through

$$\mathcal{L}_{\text{high}} \supset -\lambda_S|S|^2|H|^2 - m_S^2|S|^2, \tag{2.3}$$

where $m_S \sim \Lambda$, which, when we integrate them out, we find

$$\Delta\mu^2 = \frac{\lambda_S}{16\pi^2}\left[\Lambda^2 - 2m_S^2\log\frac{\Lambda}{m_S}\right] + \ldots \tag{2.4}$$

This is a genuine physical problem of fine tuning which cannot be transformed away. We also find the same quadratic divergences from fermions coupling to the Higgs. The conclusion is that either

1. Nature is fine tuned. Perhaps this can be explained by an anthropic principle.
2. Or, there is no new physics at high scales. This would exclude "grand unified theories" which unite the gauge groups. But suppose we throw them out; in any case we expect that quantum gravity should be associated with new states around the Planck scale, so we either have to put our head in the sand or find another mechanism to protect the Higgs sector specifically from gravity (see e.g. "asymptotic safety").
3. Or, there is something that protects the Higgs mass from large quantum corrections from heavy physics *in general*.

One possibility to realise option 3 above is supersymmetry.[1] To get a flavour of this, let us first look at why the problem is unique to fundamental scalars:

- Gauge bosons will remain massless unless their gauge symmetry is spontaneously broken. Their masses are protected by the gauge symmetry.
- If we have a theory with a massless fermion, it cannot obtain a mass through quantum effects, integrating out heavy fields etc, due to the chiral symmetry:

$$\begin{aligned}
\overline{\Psi} i \not{D} \Psi \underbrace{\longrightarrow}_{\Psi \to e^{i\alpha\gamma_5}\Psi} \overline{\Psi} e^{i\alpha\gamma_5} i \not{D} e^{i\alpha\gamma_5} \Psi = \overline{\Psi} i \not{D} \Psi \\
m\overline{\Psi}\Psi \underbrace{\longrightarrow}_{\Psi \to e^{i\alpha\gamma_5}\Psi} m\overline{\Psi} e^{2i\alpha\gamma_5}\Psi \neq m\overline{\Psi}\Psi
\end{aligned}$$

- Is there an equivalent for scalars?

To understand this a bit better, consider:

- If a theory is invariant under some symmetry, then quantum corrections cannot generate terms in the effective action which violate that symmetry.
- NB even if a global symmetry is anomalous it will only be violated *nonperturbatively* (anomalies in local symmetries are fundamentally inconsistent, however).
- If we add a term $\delta\mathcal{L}$ to the Lagrangian that violates this symmetry, then it will induce other terms in the effective action that violate the symmetry ...
- But they must all be proportional to $\delta\mathcal{L}$, since when we set it to zero the symmetry is restored.
- Put another way, if we *spontaneously* break the symmetry then the vacuum is not invariant, even if the theory as a whole remains invariant[2] $\rightarrow$ we can construct new terms in the effective action that break the symmetry that are proportional to the breaking parameter.

[1] I shall not discuss other options, but be aware that they exist: e.g. composite Higgs models, large extra dimensions, the relaxion. The key difference between SUSY and all other approaches is that it has *several* motivations, as opposed to being uniquely introduced to solve the hierarchy problem.

[2] I.e. the symmetry becomes *non-linearly realised*, and the symmetry transformations now change the expectation values.

- E.g. for fermion masses, the breaking parameter is the Higgs vacuum expectation value, and this generates a term $-m_\Psi \overline{\Psi}\Psi$ in the Lagrangian where $m_\Psi = \langle H \rangle Y_\Psi$ for Yukawa coupling $Y_\Psi$. But then the renormalisation of the fermion mass must be proportional to $m_\Psi$, e.g. for the electron mass
$$m_e = m_e^{\text{bare}} \left[ 1 + \frac{3\alpha}{4\pi} \log \frac{m_e}{\Lambda} + ... \right]$$

So what we are looking for is a symmetry that prohibits scalar mass terms, but is then spontaneously broken. Supersymmetry achieves this by changing scalars into fermions and vice versa – i.e. it co-opts the fermions' chiral symmetry to work for scalars too!

### 2.4 Dark matter and Dark energy

Cosmology has given us many indications that there must be physics beyond the Standard Model coupled to Einstein gravity. These include

- Galaxy rotation curves: by measuring the quantity of visible matter in galaxies, we find that they do not rotate as we expect. They behave as if there is a additional dark matter extending out to large distances where there cannot possibly be visible matter. More specifically, the rotation curves are flat ($v^2$ =const), implying
$$M(r) = v^2 r/G.$$
If the dark matter is distributed in a spherical halo, this implies that
$$\int dr 4\pi r^2 \rho(r) \propto r \rightarrow \rho(r) \propto \frac{1}{r^2}$$
out to some scale where the density must drop off. Simulations of galaxy formation allowing cold dark matter particles lead to proposed halo models of e.g. the Navarro-Frenk-White form
$$\rho_{DM} \propto \frac{1}{r} \frac{1}{(1+r/R_s)^2} \rightarrow M_{DM}(r) \propto \left[ \log \left( 1 + \frac{r}{R_s} \right) - \frac{1}{1+R_s/r} \right]$$
where for e.g. the Milky Way $R_{\max}/R_s \simeq 10$ is the cutoff. This works best at larger radii and other models can fit the behaviour at smaller radii better (there remains a "cusp-core problem"). Theories of modified gravity also work well to explain this behaviour.

- Galaxy clusters: clusters of galaxies probe physics at much larger scales than galaxies, and also mutually rotate much faster than could be possible from baryonic matter alone. In particular their rotation is consistent with dark matter, less so with modified gravity.

- Gravitational lensing: by observing double copies of galaxies and clusters we can see the effect of light being bent as it passes around heavy objects. By quantifying the amount of lensing we can infer the mass, and this again implies extra matter.

- Dwarf galaxies: galaxies contain 100 to 1000 times fewer stars than normal galaxies, and most of their visible matter is in the form of hydrogen. From their rotation they can be seen to contain much more dark matter than normal galaxies; in particular they disfavour modifications of gravity.
- The bullet cluster: held as the "smoking gun" of dark matter, in this cluster, formed by the collision of two smaller clusters, by observing the gravitational lensing the dark matter and visible matter can be seen to be located in two separate locations.
- Expansion of the universe: the universe was originally observed to be expanding at an increasing rate by looking at distant galaxies and comparing the spectra of standard candles.
- Precision measurements of the cosmic microwave background: the universe has a background of photons with a black-body spectrum of around 2.7K. This is the imprint of the cooling of the universe (the photons interacted with the hot plasma until decoupling). After first COBE and now the Planck satellite their spectrum and its variation accross the sky has been measured to incredible precision, and now give us precise values for the parameters of the ΛCDM model of cosmological constant plus dark matter. In particular, no other model can (so far) fit the peaks of the spectrum.

In conclusion, most physicists are convinced that these observations imply dark matter, implying a new (meta)stable particle that interacts little with visible matter. That the particle should be long-lived on the age of the universe implies that there is some new symmetry. While it is not essential for supersymmetric theories to contain such a candidate particle with an appropriate symmetry, they do accommodate both almost trivially. So far, however, the direct search for dark matter has produced only null results, and so we are still waiting for an indication as to its nature.

### 2.5 Gauge coupling unification

The idea of "grand unification" of all three gauge groups came shortly after the unification of the electromagnetism and weak forces. It was noticed, for example, that (almost) all of the matter fields of the Standard Model could be fit into representations of $SU(5)$, where under breaking to $SU(3) \times SU(2) \times U(1)_Y$ we find

$$\begin{aligned} \overline{\mathbf{5}} \to& (\overline{\mathbf{3}}, 1)_{1/3} \oplus (1, \mathbf{2})_{-1/2} \to d^c \oplus l \\ \mathbf{10} \to& (\mathbf{3}, 2)_{1/6} \oplus (\overline{\mathbf{3}}, 1)_{-2/3} \oplus (1, 1)_1 \to q \oplus u^c \oplus e^c \end{aligned} \tag{2.5}$$

The exception is the Higgs field, which fits into

$$\mathbf{5} \to (\mathbf{3}, 1)_{-1/3} \oplus (1, \mathbf{2})_{1/2} \to ?? \oplus H \tag{2.6}$$

where we have a missing Higgs triplet which, if it were light, would be very dangerous and mediate proton decay, but thankfully since it is a scalar we can easily give it a mass around the unification

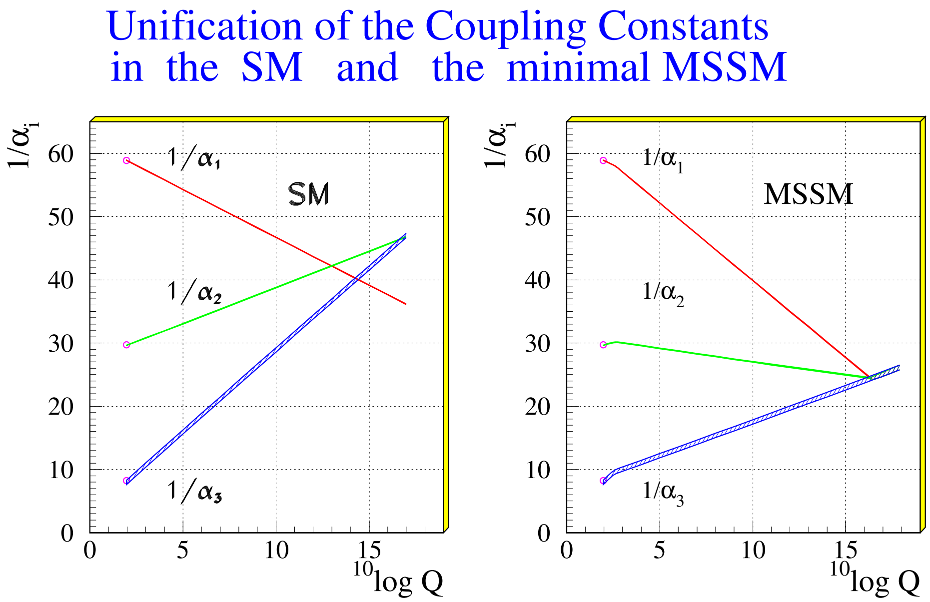


Figure 1: Left: running of Standard model gauge couplings. Right: running of gauge couplings in the Minimal Supersymmetric Standard Model (MSSM). Taken from [1].

scale. The Yukawa couplings can then be written

$$\mathcal{L}_{\text{Yukawa}} \supset -y^{ij}_{u,e}\mathbf{10}_i\mathbf{10}_j\mathbf{5} - y^{ij}_d\mathbf{10}_i\overline{\mathbf{5}}_j(\mathbf{5})^*. \tag{2.7}$$

In particular, this predicts unification of the lepton and up-type Yukawa couplings; an alternative, called "flipped $SU(5)$" adds a $U(1)$ gauge group and, roughly speaking, "flips" the role of $u$ and $d$. In these models right-handed neutrinos must be added as a gauge singlet. If we want to include right-handed neutrinos as representations of our grand unified group, then $SO(10)$ includes them.

These theories therefore make various predictions for the Yukawa couplings at the GUT ("grand unified theory") scale, but they also predict that the gauge couplings should be unified, although the normalisation of the $U(1)$ may change; for example in $SU(5)$ the generators are all normalised to $\text{tr}((T^a)^2) = 1/2$ so the unbroken generator of the $U(1)$ is

$$T^1 = \sqrt{3/5}\ \text{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}).$$

But we want the hypercharges $Y$ to be fractions, so we identify

$$g_1 T^1 = g_Y Y \to g_Y = \sqrt{3/5} g_1.$$

We can then investigate these predictions by running the couplings up in energy and seeing if they unify. Unfortunately for the Standard Model this does not work very well; see the left plot in figure 1.

On the other hand, the simplest SUSY extension of the Standard Model, the Minimal Supersymmetric Standard Model (MSSM), adds partners to the Standard Model fields near the electroweak scale, and if we recompute the running we find in the right plot of figure 1 – amazingly and surprisingly – that they seem to unify. It has since been realised that this also works in some other variants of SUSY models such as "Split SUSY" (which we shall discuss later in these lectures). Supersymmetry was not invented to do this, and it seems an amazing coincidence; if SUSY is not realised in nature at low energies then it would seem a cruel joke!

### 2.6 Vacuum stability

By running *all* of the couplings of the Standard Model up to high energies, we can also investigate whether its vacuum is *stable*: if we find that at some scale the quartic coupling in the Higgs potential $\lambda$ becomes less than zero then the potential is unbounded from below! In this case, we can calculate the time it would take for the vacuum to decay, and check whether it is less than the age of the universe (in which case it is unstable) or greater (metastable). In figure 2 we can see that the current values are nearly $3\sigma$ away from the region of stability of the potential, and most likely $\lambda < 0$ between $10^9$ and $10^{12}$ GeV.

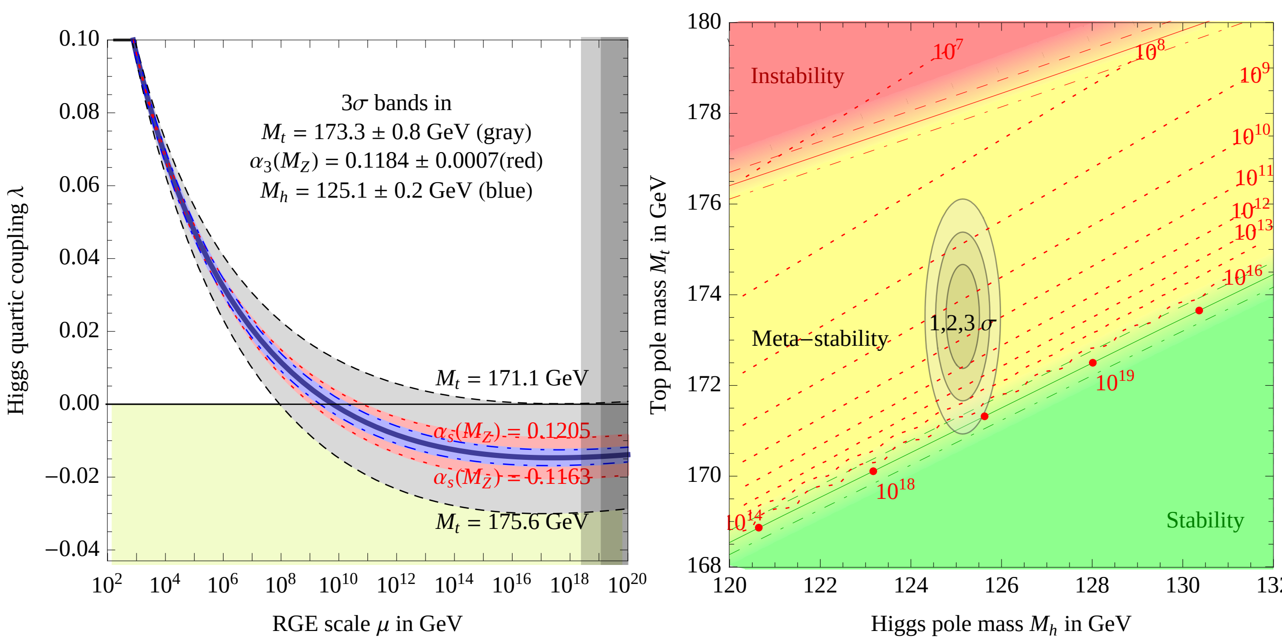


Figure 2: (Meta)stability of the Standard Model, taken from [2]. Left: running of the quartic coupling to high energies. Right: metastable region in the Higgs mass – top mass plane; the dotted contours give the instability scale in GeV.

Some people believe that there must be a deep (potentially anthropic) reason (other than supersymmetry) that we live in the *tiny* metastability region. However, another conclusion is that there is new physics that enters below the instability scale. Indeed, we expect that a sector associated with an axion may be associated with such scales, but supersymmetry would have a much stronger direct effect on the Higgs sector and render the potential stable.

### 2.7 The strong CP problem

In the lagrangian of the strong force we can write a term which violates CP:

$$\mathcal{L} \supset \frac{\theta}{64\pi^2} G^a_{\mu\nu} G^a_{\rho\kappa} \epsilon^{\mu\nu\rho\kappa}. \tag{2.8}$$

Such a term would (through pion loops) induce an electric dipole moment for the neutron, of the order

$$d_n \sim |\theta| e \frac{m_\pi^2}{m_N^2} \simeq 10^{-16}\ e\,\mathrm{cm}. \tag{2.9}$$

However, the current limit is

$$d_n \leq 3.0 \times 10^{-26}\ e\,\mathrm{cm}, \longrightarrow |\theta| < 10^{-10}\ ! \tag{2.10}$$

Even if we suppose that the strong force preserves $CP$ and set $\theta = 0$, CP is violated in the Standard Model and, when we diagonalise the quark masses and remove their complex phases, we find

$$\delta\theta = \log\det \mathcal{M}_f \tag{2.11}$$

where $\mathcal{M}_f$ is the mass matrix of all the quarks. While we do not know what the phases in the matrix are – naively we would expect that they contain $\mathcal{O}(1)$ phases, giving an $\mathcal{O}(1)$ value for $\theta$ – we do know the CKM angles, and, for example, $\delta_{13} \simeq 1.2$ radians. Therefore there seems to be some extreme fine-tuning of this parameter in the lagrangian.

To solve this problem, the widely accepted solution is an "axion," which couples to the strong field strength:

$$\mathcal{L} \supset \frac{a}{64\pi^2 f_a} G^a_{\mu\nu} G^a_{\rho\kappa} \epsilon^{\mu\nu\rho\kappa}, \tag{2.12}$$

where $a$ is the axion and $f_a$ is the "axion decay constant." The idea is that the strong force generates a potential for the axion which minimises $\langle \theta + \frac{a}{f_a} \rangle$.

While not necessarily a sign of supersymmetry, the axion is expected to be some new physics that should enter at some high scale – indeed, constraints from searches give $f_a \gtrsim 10^9$ GeV – and also we find that axions appear quite naturally when we formulate supersymmetric field theories.

### 2.8 String theory

In the search for a quantum theory of gravity, string theory is the most developed – and most succesful. Originally conceived as a theory of the strong force, it was later realised that it automatically contains spin 2 particles in the spectrum – gravitons. However, in order to formulate the theory without tachyons, it was necessary to add fermions, and then in order to cancel anomalies it was found that the set of consistent string theories is very restricted, and that supersymmetry plays a crucial role. It has since passed all consistency tests as a theory of both particle physics and quantum gravity, and is incredibly rich, even if it remains a framework (on a similar footing to quantum field theory) rather than a unique theory. Indeed, it has the property of removing ultra-violet divergences from field theory calculations while automatically preserving Lorentz and gauge symmetries (in contrast to other approaches) – so implements the Wilsonian approach to renormalisation automatically, and can thus be used as an ultra-violet completion of field theory and gravity computations. Thus, even if supersymmetry is not found at low energies, it is very likely that nature requires it at some scale.

# 3 What is Supersymmetry?

## 3.1 Poincaré algebra

I will follow the conventions of Martin's primer with the choice of metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The Poincaré group consists of Lorentz transformations and translations:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}. \tag{3.1}$$

We can derive the algebra by considering infinitesimal translations on spacetime points; clearly translations can be written in terms of the energy-momentum operator $P^{\lambda}$ where $(P^{\lambda})_{\mu} = i\delta^{\lambda}_{\mu}$ as

$$\delta x^{\mu} = -ia_{\lambda}(P^{\lambda})_{\mu} \tag{3.2}$$

i.e. there are four independent translations hence four parameters $a_{\lambda}$. For the rotations and Lorentz transformations there are $3+3$ parameters which can be written in terms of an antisymmetric matrix $\omega_{\mu\nu}$ as

$$\begin{aligned} \delta x^{\mu} =& -\frac{1}{2} i\omega_{\mu\nu}(M^{\mu\nu})^{\rho}_{\sigma} x^{\sigma} \\ (M^{\mu\nu})_{\rho\sigma} =& i(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}) \leftrightarrow (M^{\mu\nu})^{\rho}_{\sigma} = i(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\eta^{\nu\rho}) \end{aligned} \tag{3.3}$$

For example a boost $\beta$ in the 3 direction corresponds to

$$\begin{aligned} \delta x^0 =& -\beta x^3, \qquad \delta x^3 = -\beta x^0 \\ \leftrightarrow \delta x^{\rho} =& \omega_{03}\left[\eta^{0\rho}x^3 - \eta^{3\rho}x^0\right] \end{aligned} \tag{3.4}$$

so $\omega_{03} = -\beta$ is the boost etc. Similarly a rotation $\theta$ in three dimensions about the 3 axis is

$$\begin{aligned} \delta x^1 =& -\theta x^2, \qquad \delta x^2 = \theta x^1 \\ \leftrightarrow \delta x^{\rho} =& \omega_{12}\left[\eta^{1\rho}x^2 - \eta^{2\rho}x^1\right] \end{aligned} \tag{3.5}$$

so now $\omega_{12} = \theta$; this implies $J_k = \frac{1}{2}\epsilon_{ijk}M^{ij} \leftrightarrow M_{ij} = \epsilon_{ijk}J_k$. From the above representation we can derive the algebra

$$[P^\lambda, P^\mu] = 0 \tag{3.6}$$

$$[M^{\mu\nu}, P^\lambda] = i(\eta^{\nu\lambda}P^\mu - \eta^{\mu\lambda}P^\nu) \tag{3.7}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}). \tag{3.8}$$

### 3.2 Fermions and representations of Lorentz symmetry

You'll be familiar with the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \tag{3.9}$$

which comes from the lagrangian density

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{3.10}$$

where $\overline{\Psi} \equiv \psi^\dagger\gamma^0$. In four dimensions massive spinors have four components (in $d$ dimensions it is $2^{d/2}$ for $d$ even) and massless ones have two. Usually we write the solutions for these equations in terms of spinors $u, v$ which each have two components. Equally, these can be regarded as a left- and right-handed Weyl spinor. As we shall discuss below, these are the natural building blocks for spinors in four dimensions, in particular for supersymmetry. In fact, in principle we can abandon four-spinors completely and write everything just in terms of two component spinors. To understand how this comes about, we first ask the question: how does a spinor transform under the Lorentz algebra?

If we look at the generators of Lorentz symmetry and write $K_i = M^{0i}$ as the generator of boosts then we see we have two sets of generators, $J_i$ and $K_i$. In four dimensions both have three components, and so we can write

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{3.11}$$

If we construct two combinations

$$J_i^\pm \equiv \frac{1}{2}(J_i \pm iK_i) \tag{3.12}$$

then they satisfy two independent $SU(2)$ algebras

$$[J_i^\pm, J_j^\pm] = i\epsilon_{ijk}J_k^\pm, \qquad [J_i^\pm, J_j^\mp] = 0, \tag{3.13}$$

which shows

$$SO(1,3) \simeq SU(2) \oplus SU(2). \tag{3.14}$$

Note that, from the above definitions for e.g. our explicit representation and the example in equations (3.4) and (3.5)

$$\begin{aligned}\delta x^1 =& - \theta x^2 = -\frac{1}{2} i\theta (M^{12})^{\rho}_{\sigma} x^{\rho} = -\frac{1}{2} i\theta (J^3)^{\rho}_{\sigma} x^{\rho} \\ \delta x^0 =& - \beta x^3 = \frac{1}{2} i\beta (M^{03})^{\rho}_{\sigma} x^{\rho} = \frac{1}{2} i\beta (K^3)^{\rho}_{\sigma} x^{\rho}\end{aligned} \tag{3.15}$$

and so, since $\theta, \beta$ are real, we must have $J^i, K^i$ pure imaginary. Therefore $(J_i^{\pm})^* = -(J_i^{\pm})$ so complex conjugation exchanges the two $SU(2)$s. This means that all representations of the Lorentz group can be decomposed into representations of two $SU(2)$s, e.g. $(0, \frac{1}{2})$, and that complex conjugation identifies $(0, \frac{1}{2})^* = (\frac{1}{2}, 0)$; in general $(a, b)^* = (-b, -a)$ and this represents the action of the charge conjugation operator. The scalar representation is obviously $(0, 0)$, but we see that fermions are represented by $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Each of these has, however, only two components, which would correspond to massless fermions, so we will identify them with left- and right-handed fermions respectively.

The above also relates to the homomorphism $SO(1, 3) \to SL(2, \mathbf{C})$, the group of $2 \times 2$ complex matrices having unit determinant ($SL(2, \mathbf{C})$ is a double cover of $SO(1, 3)$). This is to say that we can represent our Lorentz algebra in terms of $2 \times 2$ matrices, as we require to act on our fermionic representations.

We shall write left handed spinors with undotted indices such as $\psi_\alpha$ where $\alpha = 1, 2$ and right-handed spinors with dotted indices such as $\bar{\chi}_{\dot{\alpha}}$; we write the second with a bar because the second $SU(2)$ can be exchanged with the first by complex conjugation, as we showed above. So then the Lorentz transformation sends

$$\psi_\alpha \to M_\alpha^\beta \psi_\beta, \qquad \bar{\psi}_{\dot{\alpha}} \to (M^*)_{\dot{\alpha}}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}}. \tag{3.16}$$

These two representations *are not equivalent* because they involve the two different $SU(2)$s $J^{\pm}$. However, for a single $SU(2)$ the fundamental and antifundamental representations are equivalent, and we change from one to the other using the antisymmetric tensor $\epsilon^{\alpha\beta}$. This means that we can write the fermions with raised or lowered indices:

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \qquad \overline{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \overline{\psi}_{\dot{\beta}} \tag{3.17}$$

$$\psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \qquad \overline{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \overline{\psi}^{\dot{\beta}}. \tag{3.18}$$

We shall use the conventions

$$\begin{aligned}\epsilon^{\alpha\beta} &= \epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma^2, \qquad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\alpha\beta} = -i\sigma^2 \\ \epsilon^{12} &= 1, \qquad \epsilon_{12} = -1\end{aligned} \tag{3.19}$$

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.20}$$

This gives

$$\begin{aligned}\psi^2 &= -\psi_1 \\ \psi^1 &= \psi_2.\end{aligned} \tag{3.21}$$

To see the equivalence, note that the matrices for the fundamental and antifundamental of $SU(2)$ are always $\exp(ia_i\sigma^i), \exp(-ia_i(\sigma^i)^T)$, and using the identity

$$\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^T \tag{3.22}$$

we have

$$\begin{aligned}\epsilon^{\alpha\beta}\psi'_\beta =& i\sigma^2 \exp(ia_i\sigma^i)\psi \\ =& \exp(ia_i\sigma^2\sigma^i\sigma^2)(i\sigma^2\psi) \\ =& \exp(-ia_i(\sigma^i)^T)(i\sigma^2\psi).\end{aligned} \tag{3.23}$$

Now, to form gauge invariant quantities we must always contract raised and lowered indices. However, since the fermions are anticommuting quantities we can form the scalar products:

$$\begin{aligned}\psi\chi \equiv \psi^\alpha\chi_\alpha =& -\chi_\alpha\psi^\alpha = \epsilon_{\alpha\beta}\psi^\alpha\chi^\beta = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha \\ \bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} =& -\bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha}.\end{aligned} \tag{3.24}$$

Note that because of these definitions the order which we write the product is not important:

$$\begin{aligned}\psi\chi =& \chi\psi \\ \bar\psi\bar\chi =& \bar\chi\bar\psi.\end{aligned} \tag{3.25}$$

We also see that the action of charge conjugation must be

$$(\psi_\alpha)^\dagger = \overline{\psi}_{\dot\alpha}, \qquad (\psi\chi)^\dagger = (\chi_\alpha)^\dagger(\psi^\alpha)^\dagger = \bar\chi\bar\psi. \tag{3.26}$$

To write the representations of the Lorentz transformations, we require an antisymmetric tensor. To construct these we introduce the notation

$$\begin{aligned}\sigma^\mu_{\alpha\dot\alpha} =& (1_2, \sigma) \\ (\overline{\sigma}^\mu)^{\dot\alpha\alpha} =& (1_2, -\sigma) = \sigma_\mu.\end{aligned} \tag{3.27}$$

The basic identities related to the Grassman commutator are

$$\begin{aligned}\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu =& 2\eta^{\mu\nu} \\ \overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu =& 2\eta^{\mu\nu}.\end{aligned} \tag{3.28}$$

We also define

$$
\begin{aligned}
(\sigma^{\mu\nu})_\alpha^{\ \beta} \equiv& \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)_\alpha^{\ \beta} \\
(\overline{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \equiv& \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)^{\dot\alpha}_{\ \dot\beta}.
\end{aligned} \tag{3.29}
$$

These are the crucial quantities that we will need. In particular, we have

$$
\begin{aligned}
\sigma^{0i} =& -\frac{i}{2}\sigma^i, \qquad \overline{\sigma}^{0i} \equiv \frac{i}{2}\sigma^i \\
\sigma^{ij} =& \frac{1}{2}\epsilon^{ijk}\sigma^k = \overline{\sigma}^{ij}.
\end{aligned} \tag{3.30}
$$

To construct the Lorentz transformations, consider that the representations have

$$
\begin{aligned}
J_3^+|\frac{1}{2},0\rangle =& \pm\frac{1}{2}|\frac{1}{2},0\rangle, \qquad J_3^-|\frac{1}{2},0\rangle = 0 \\
\rightarrow J_3|\frac{1}{2},0\rangle =& \pm\frac{1}{2}|\frac{1}{2},0\rangle, \qquad K_3|\frac{1}{2},0\rangle = \mp\frac{i}{2}|\frac{1}{2},0\rangle
\end{aligned} \tag{3.31}
$$

or more generally (not distinguishing between raised and lowered indices for three dimensional spins)

$$
\begin{aligned}
J_i|\frac{1}{2},0\rangle =& \frac{1}{2}\sigma^i|\frac{1}{2},0\rangle, \qquad K_i|\frac{1}{2},0\rangle = -\frac{i}{2}\sigma^i|\frac{1}{2},0\rangle \\
J_i|0,\frac{1}{2}\rangle =& \frac{1}{2}\sigma^i|0,\frac{1}{2}\rangle, \qquad K_i|0,\frac{1}{2}\rangle = +\frac{i}{2}\sigma^i|0,\frac{1}{2}\rangle.
\end{aligned} \tag{3.32}
$$

So then we see that

$$
\begin{array}{rcl|rcl}
-\frac{i}{2}\omega_{ij}M^{ij}|\frac{1}{2},0\rangle &=& -\frac{i}{2}\omega_{ij}\epsilon_{ijk}J_k|\frac{1}{2},0\rangle & -\frac{i}{2}\omega_{ij}M^{ij}|0,\frac{1}{2}\rangle &=& -\frac{i}{2}\omega_{ij}\epsilon_{ijk}J_k|0,\frac{1}{2}\rangle \\
&=& -\frac{i}{4}\omega_{ij}\epsilon_{ijk}\sigma^k|\frac{1}{2},0\rangle & &=& -\frac{i}{4}\omega_{ij}\epsilon_{ijk}\sigma^k|0,\frac{1}{2}\rangle \\
&=& -\frac{i}{2}\omega_{ij}\sigma^{ij}|\frac{1}{2},0\rangle & &=& -\frac{i}{2}\omega_{ij}\overline{\sigma}^{ij}|0,\frac{1}{2}\rangle \\
\hline
-\frac{i}{2}\omega_{0i}M^{0i}|\frac{1}{2},0\rangle &=& -\frac{i}{2}\omega_{0i}K_i|\frac{1}{2},0\rangle & -\frac{i}{2}\omega_{0i}M^{0i}|0,\frac{1}{2}\rangle &=& -\frac{i}{2}\omega_{0i}K_i|0,\frac{1}{2}\rangle \\
&=& -\frac{1}{4}\omega_{0i}\sigma^i|\frac{1}{2},0\rangle & &=& \frac{1}{4}\omega_{0i}\sigma^i|0,\frac{1}{2}\rangle \\
&=& -\frac{i}{2}\omega_{0i}\sigma^{0i}|\frac{1}{2},0\rangle & &=& -\frac{i}{2}\omega_{0i}\bar{\sigma}^{0i}|0,\frac{1}{2}\rangle
\end{array}
$$

from which we conclude that

$$
[M^{\mu\nu},\psi_\alpha] = (\sigma^{\mu\nu})_\alpha^{\ \beta}\psi_\beta, \qquad [M^{\mu\nu},\bar\psi^{\dot\alpha}] = (\overline{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\bar\psi^{\dot\beta}, \tag{3.33}
$$

or equivalently the action of a Lorentz transformation on the two-component spinors is

$$
\psi_\alpha \rightarrow \exp\big(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\big)_\alpha^{\ \beta}\psi_\beta, \qquad \bar\psi^{\dot\alpha} \rightarrow \exp\big(-\frac{i}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}\big)^{\dot\alpha}_{\ \dot\beta}\bar\psi^{\dot\beta}. \tag{3.34}
$$

We might be at first a bit confused about how we passed from two separate $SU(2)$ transformations (each with only three generators) to what appears to be six generators acting on each spinor. This is resolved when we realise that not all of the components of $\sigma^{\mu\nu}$ are independent: we have from the above that

$$0 = 2i\sigma^{0i} - \epsilon^{ijk}\sigma^{jk} = -2i\bar{\sigma}^{0i} - \epsilon^{ijk}\bar{\sigma}^{jk} \tag{3.35}$$

or equivalently, in covariant form

$$\sigma^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}\sigma^{\rho\sigma}, \qquad \overline{\sigma}^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}\overline{\sigma}^{\rho\sigma}. \tag{3.36}$$

To continue our development of two-component spinor formalism, we need to be able to write the Dirac equation for them. To do this, we write the gamma matrices as

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix}. \tag{3.37}$$

So then we can write Dirac spinors in terms of a left-handed and right-handed spinor, written as

$$\Psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{3.38}$$

Incidentally, the Lorentz transformations are generated by

$$\Sigma^{\mu\nu} \equiv \begin{pmatrix} 2\sigma^{\mu\nu} & 0 \\ 0 & 2\overline{\sigma}^{\mu\nu} \end{pmatrix} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}] \to \delta\Psi = -\frac{i}{4}\omega_{\mu\nu}\Sigma^{\mu\nu}\Psi. \tag{3.39}$$

This is true in any number of dimensions.

The Dirac equation itself then reads

$$\begin{aligned} i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\bar{\chi}^{\dot{\alpha}} - m\psi_{\alpha} &= 0 \\ i(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\partial_{\mu}\psi_{\alpha} - m\bar{\chi}^{\dot{\alpha}} &= 0. \end{aligned} \tag{3.40}$$

These are two coupled equations, so if we want to construct perturbation theory entirely in terms of two component spinors we will in general have matrix propagators, but this does not pose a problem:

$$\begin{aligned} \langle 0|T(\psi_{\alpha}(x), \overline{\psi}_{\dot{\alpha}}(y))|0\rangle &= \int \frac{d^d p}{(2\pi)^d}\frac{i}{p^2 - m^2 + i\epsilon}p_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\ e^{-ip\cdot(x-y)} \\ \langle 0|T(\chi_{\alpha}(x), \overline{\chi}_{\dot{\alpha}}(y))|0\rangle &= \int \frac{d^d p}{(2\pi)^d}\frac{i}{p^2 - m^2 + i\epsilon}p_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\ e^{-ip\cdot(x-y)} \\ \langle 0|T(\psi_{\alpha}(x), \chi_{\beta}(y))|0\rangle &= \int \frac{d^d p}{(2\pi)^d}\frac{-im\epsilon_{\alpha\beta}}{p^2 - m^2 + i\epsilon}\ e^{-ip\cdot(x-y)}. \end{aligned} \tag{3.41}$$

Before continuing, we will now simply note several important identities for the two-spinor algebra that we will need later on. First we note that from a Dirac spinor, we can contruct three other spinors; $\bar{\Psi}$, as well as its charge conjugate $\Psi^c$ and $\bar{\Psi}^c$:

$$\begin{aligned} \bar{\Psi} &= (\chi^{\alpha}, \bar{\psi}_{\dot{\alpha}}) \\ \Psi^c &= \begin{pmatrix} \chi_{\alpha} \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi}^c = (\psi^{\alpha}, \bar{\chi}_{\dot{\alpha}}). \end{aligned} \tag{3.42}$$

Now $\Psi^c = \bar{\Psi}^T$ transforms differently to $\Psi$:

$$\begin{aligned}
0 =& [i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\Psi \\
=& [-i(\partial_\mu + ieA_\mu)(\gamma^\mu)^* - m]\Psi^* \\
=& [-i(\partial_\mu + ieA_\mu)\Big((\gamma^\mu)^\dagger\Big)^T - m](\Psi^\dagger)^T \\
=& \gamma^0[-i(\partial_\mu + ieA_\mu)\Big(\gamma^0(\gamma^\mu)\gamma^0\Big)^T - m](\overline{\Psi}\gamma^0)^T \\
=& [-i(\partial_\mu + ieA_\mu)(\gamma^\mu)^T - m]\overline{\Psi}^T
\end{aligned} \tag{3.43}$$

Hence $\overline{\Psi}^T$ is a Dirac spinor with opposite charge and matrices $-(\gamma^\mu)^T$: we have exchanged the SU(2)s. *It is the charge conjugate spinor.* We can write a charge-conjugation matrix $C$ which transforms the gammas as $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$; then we can define $\Psi^c \equiv C\overline{\Psi}^T$. Insisting that performing the operation twice we return to $\Psi$ we get $C = -i\gamma^0\gamma^2$, so

$$C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \tag{3.44}$$

We therefore see that $\Psi^c = -i\gamma^0\gamma^2\bar{\Psi}^T$.

Also note that we can write a Majorana spinor that is equal to its charge conjugate $\Psi = \Psi^c$ as

$$\Psi_M \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{3.45}$$

Finally we shall require some properties of the scalar and vector objects that can be constructed:

$$\begin{aligned}
(\sigma^\mu)^* =& (\sigma^\mu)^T \\
(\sigma^\mu)^T =& \sigma^2\overline{\sigma}^\mu\sigma^2 \\
(\chi\sigma^\mu\overline{\psi})^\dagger =& \psi\sigma^\mu\overline{\chi} \\
(\chi\sigma^\mu\overline{\sigma}^\nu\psi)^\dagger =& \bar{\psi}\overline{\sigma}^\nu\sigma^\mu\bar{\chi} \to (\chi\sigma^{\mu\nu}\psi)^\dagger = \bar{\psi}\overline{\sigma}^{\mu\nu}\bar{\chi} \\
\overline{\chi}_{\dot{\alpha}}(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha}\psi_\alpha =& -\psi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} \\
\chi\sigma^{\mu\nu}\psi =& \psi\sigma^{\mu\nu}\chi.
\end{aligned} \tag{3.46}$$

### 3.2.1 Lagrangians with two-component spinors

We already saw the Dirac equation; to write Lagrangians we have

$$\begin{aligned}
\mathcal{L} \supset& i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi = i\overline{\psi}_{\dot{\alpha}}(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha}D_\mu\psi_\alpha = -iD_\mu\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} \\
\to& i\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}D_\mu\overline{\psi}^{\dot{\alpha}}
\end{aligned} \tag{3.47}$$

where on the second line we need to integrate by parts, and use the fact that the gauge representation of $\overline{\psi}$ is conjugate to $\psi$.

The above contains the gauge interactions

$$
\begin{aligned}
i\overline{\psi}\overline{\sigma}^{\mu}D_{\mu}\psi =& i\overline{\psi}_i\overline{\sigma}^{\mu}(\partial_{\mu} + iA^a_{\mu}T^a_{ij})\psi_j \\
\supset & -\overline{\psi}_i\overline{\sigma}^{\mu}A^a_{\mu}T^a_{ij}\psi_j.
\end{aligned}
\tag{3.48}
$$

So we now just need Yukawa interactions:

$$
\mathcal{L}_{\text{Yukawa}} = -y\phi\psi\chi - y^*\phi^*\overline{\psi}\overline{\chi} \tag{3.49}
$$

for general $\phi, \psi, \chi$. Compare this to four-spinor notation:

$$
\mathcal{L}_{\text{Yukawa}} = -y_L\phi\overline{\Psi}_1 P_L\Psi_2 - y_R\phi\overline{\Psi}_1 P_R\Psi_2 - y_L^*\phi^*\overline{\Psi}_2 P_R\Psi_1 - y_R^*\phi^*\overline{\Psi}_2 P_L\Psi_1. \tag{3.50}
$$

### 3.3 The Coleman Mandula theorem

One very fruitful approach to studying a theory is to look first to see how much symmetries can constrain their properties, before trying to calculate any dynamics. (A recent example of this is the bootstrap program). So it is very reasonable to ask if there could be any more symmetries of spacetime that *extend* that of Poincaré (by extend I do not mean here "internal" symmetries – such as gauge symmetries – which commute with the Poincaré; we could in principle have any number). However, Coleman and Mandula [3] found a theorem that tells us that there are none in any dimension greater than $1 + 1$, roughly provided:

- Particles have positive energy, and their spectrum is discrete (i.e. we have a finite number of particles of mass less than any given $M$). Also at least one of the particles has a mass (for massless theories we can also have conformal symmetries).
- Scattering amplitudes are analytic functions of the kinetimatic variables (except at normal thresholds). I.e. the theory has local interactions.
- The theory has some interactions.

Witten gave a nice argument explaining why this is true: broadly speaking, in any two body elastic collision, Lorentz symmetry leaves only an unknown scattering angle because momentum $P_{\mu}$ and angular momentum/centre of mass momentum $M_{\mu\nu}$ are conserved. Any additional "exotic" symmetries would then fix the angle, which would either prevent analyticity of the functions or non-trivial scattering.

To be more concrete (but not rigorous), assume that there is some symmetry generator $Q_{\mu\nu}$ such that $[Q_{\mu\nu}, P_{\rho}] \neq 0$. It should be symmetric and traceless so that it is a non-trivial irreducible Lorentz tensor (if it had a trace it would be reducible). So then

$$
\langle p|Q_{\mu\nu}|p\rangle \propto p_{\mu}p_{\nu} - \frac{1}{d}\eta_{\mu\nu}p^2. \tag{3.51}
$$

If we consider then the scattering of two-particle states this is just the tensor product so

$$
\langle p_1, p_2|Q_{\mu\nu}|p_1, p_2\rangle = \langle p_1|Q_{\mu\nu}|p_1\rangle + \langle p_2|Q_{\mu\nu}|p_2\rangle \tag{3.52}
$$

and therefore for scattering to $q_1, q_2$ we have

$$\begin{aligned}\langle p_1, p_2|Q_{\mu\nu}|p_1, p_2\rangle =&\langle q_1, q_2|Q_{\mu\nu}|q_1, q_2\rangle \\ \to p_{1,\mu}p_{1,\nu} + p_{2,\mu}p_{2,\nu} =&q_{1,\mu}q_{1,\nu} + q_{2,\mu}q_{2,\nu}\end{aligned} \tag{3.53}$$

since the masses are the same on both sides. Since we must also apply momentum conservation, consider the rest frame and put the initial momenta along direction 3. Then for $i, j \neq 0, 3$ we have $q_{1,i} = -q_{2,i}$ and

$$\begin{aligned}0 =&q_{1,i}q_{1,j} + q_{2,i}q_{2,j} \\ =&2q_{1,i}q_{1,j} \\ \to 0 =&q_{1,1} = q_{1,2} = q_{2,1} = q_{2,2}.\end{aligned} \tag{3.54}$$

Hence in the centre of mass frame the scattering angle is zero.

### 3.4 SUSY algebra

On the other hand, "supergauge transformations" were studied in two-dimensional theories in the context of "dual models" (which would go on to become string theory) by Neveu-Schwarz [4] and P. Ramond [5]. Adding fermions was required to make the models more "realistic" models of pions, and also turned out to be relevant for avoiding ghosts. Wess and Zumino then found that they could extend the algebra to four dimensions and the study of supersymmetric theories was born; Haag, Łopuszański and Sohnius later extended the Coleman-Mandula theorem to show that enlargement of the Poincaré algebra is possible when we allow *anticommuting* brackets. Here we shall show how considering the simplest possible theory of fermions and bosons and allowing as symmetry that exchanges them will lead to the supersymmetry algebra.

The simplest example is the non-interacting Wess-Zumino model. We begin with just one complex scalar and one Weyl fermion:

$$S_{\rm WZ} = \int d^4x\ \partial^\mu\phi^*\partial_\mu\phi + i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi. \tag{3.55}$$

The first thing to note is that the theory has two global $U(1)$ symmetries where we rotate either the scalar or the fermion. Now, our supersymmetry transformation must turn the scalar into a fermion, and so it should identify these symmetries without breaking them; in fact, the $U(1)$ symmetry will become chiral symmetry (because it will apply the same for all chiral fermions, when we add more to our theory), so the simplest tranformation we can write down is

$$\delta_\epsilon\phi = \sqrt{2}\epsilon\psi, \qquad \delta_\epsilon\phi^* = \sqrt{2}\bar{\epsilon}\bar{\psi}, \tag{3.56}$$

where $\epsilon$ is an *anticommuting* parameter, and the factor of $\sqrt{2}$ is included for historical reasons. *We shall take $\epsilon, \bar{\epsilon}$ to be constant, i.e. global supersymmetry transformations, but the algebra we shall find* *must of course be valid in the local case too.* Transforming the action above we have

$$\delta_\epsilon \mathcal{L}_{\text{scalar}} = \sqrt{2}\epsilon\partial^\mu\psi\partial_\mu\phi^* + \sqrt{2}\bar{\epsilon}\partial^\mu\overline{\psi}\partial_\mu\phi. \tag{3.57}$$

This must be matched by an appropriate shift of the fermion. From the above we see that the mass dimensions of $\phi, \psi, \epsilon$ are $1, 3/2, -1/2$, so we will need a derivative; again to preserve the $U(1)$ symmetry we have, *up to a multiplicative factor*

$$\delta_\epsilon\psi_\alpha = -\sqrt{2}i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu\phi, \qquad \delta\overline{\psi}_{\dot{\alpha}} = \sqrt{2}i(\epsilon\sigma^\mu)_{\dot{\alpha}}\partial_\mu\phi^* \tag{3.58}$$

Applying this above we find

$$\begin{aligned}
\frac{1}{\sqrt{2}}\delta_\epsilon\mathcal{L}_{\text{fermion}} =& -\epsilon\sigma^\mu\overline{\sigma}^\nu\partial_\nu\psi\partial_\mu\phi^* + \overline{\psi}\overline{\sigma}^\nu\sigma^\mu\bar{\epsilon}\partial_\mu\partial_\nu\phi \\
=& -\epsilon\partial^\mu\psi\partial_\mu\phi^* + 2i\epsilon\sigma^{\mu\nu}\partial_\nu\psi\partial_\mu\phi^* + \bar{\psi}\bar{\epsilon}\partial^\mu\partial_\mu\phi - 2i\overline{\psi}\overline{\sigma}^{\mu\nu}\partial_\mu\partial_\nu\phi \\
=& -\epsilon\partial^\mu\psi\partial_\mu\phi^* - (\partial^\mu\bar{\psi}\bar{\epsilon})\partial_\mu\phi + \partial_\mu\left(\bar{\psi}\bar{\epsilon}\partial_\mu\phi - \phi^*\epsilon\sigma^{\mu\nu}\partial_\nu\psi\right) \\
=& -(\epsilon\partial^\mu\psi)\partial_\mu\phi^* - (\bar{\epsilon}\partial^\mu\bar{\psi})\partial_\mu\phi + \text{total derivative}.
\end{aligned} \tag{3.59}$$

Here we used the fact that $\epsilon$ is constant. We arrive at

$$\delta S = \int d^4x \ \ (\delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{fermion}}) = 0, \tag{3.60}$$

justifying our guess of the numerical multiplicative factor made in eq. (3.67).

We are not quite finished in showing that the theory above is supersymmetric. We must also show that the supersymmetry algebra closes; in other words, that the commutator of two supersymmetry transformations parameterized by two different spinors $\epsilon_1$ and $\epsilon_2$ is another symmetry of the theory:

$$\begin{aligned}
(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi \equiv& \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) \\
=& \delta_{\epsilon_2}(\epsilon_1\psi) - \delta_{\epsilon_1}(\epsilon_2\psi) \\
=& 2(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)\, i\partial_\mu\phi = 2(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)(-P_\mu\phi)
\end{aligned} \tag{3.61}$$

where we noted $-i\partial_\mu = P_\mu$. We can also check this for the fermion:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha \ = \ -2i(\sigma^\mu\bar{\epsilon}_1)_\alpha\,\epsilon_2\partial_\mu\psi + 2i(\sigma^\mu\bar{\epsilon}_2)_\alpha\,\epsilon_1\partial_\mu\psi. \tag{3.62}$$

To reorganise this, we use the fact that spinors are anticommuting and have two components to write for any three spinors $\psi_\alpha, \chi_\beta, \eta_\gamma$

$$\psi_\alpha(\chi\eta) = -\eta_\alpha(\psi\chi) - \chi_\alpha(\psi\eta). \tag{3.63}$$

This can be easily proved (recalling $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$) using the identities

$$\begin{aligned}
\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} =& -\delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\delta\delta_\beta^\gamma \\
\to 0 =& \epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}.
\end{aligned} \tag{3.64}$$

Applying this to the above we have

$$\begin{aligned}-i(\sigma^\mu\bar{\epsilon}_1)_\alpha\epsilon_2\partial_\mu\psi =& i\epsilon_{2,\alpha}(\partial_\mu\psi\sigma^\mu\bar{\epsilon}_1) + i\partial_\mu\psi_\alpha(\epsilon_2\sigma^\mu\bar{\epsilon}_1)\\ \to (\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha =& (-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)\, 2i\partial_\mu\psi_\alpha + 2i\epsilon_{1\alpha}\,\bar{\epsilon}_2\overline{\sigma}^\mu\partial_\mu\psi - 2i\epsilon_{2\alpha}\,\bar{\epsilon}_1\overline{\sigma}^\mu\partial_\mu\psi.\end{aligned} \tag{3.65}$$

The last two terms vanish by the equation of motion $\overline{\sigma}^\mu\partial_\mu\psi = 0$, and the first are *exactly what we had before*, showing that the algebra closes. However, because we had to use the equations of motion, we say that it closes "on-shell," which is perhaps worrying because in quantum theories particles are not always on-shell. However, this can be easily fixed when we realise that, on-shell there are 2 bosonic degrees of freedom and 2 fermionic ones, but off-shell there are an extra 2 fermionic degrees of freedom. So we should add some extra bosonic ones – a complex "auxiliary" scalar $F$. This should vanish on-shell, so does not propagate – so cannot have any time derivative. The only Lorentz-invariant lagrangian we can then write is

$$\mathcal{L}_{\text{auxiliary}} = F^*F\,. \tag{3.66}$$

Its dimensions are $[\text{mass}]^2$. It can be easily integrated out (since it does not propagate) via its equations of motion, which for this theory are trivial. Now we want its SUSY transformation to correct the fermionic one (not the scalar) so we put

$$\delta\psi_\alpha = -\sqrt{2}i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu\phi + \sqrt{2}\epsilon_\alpha F. \tag{3.67}$$

Now we require

$$\begin{aligned}\delta_{\epsilon_2}(\epsilon_{1,\alpha}F) - \delta_{\epsilon_1}(\epsilon_{2,\alpha}F) =& -2i\epsilon_{1\alpha}\,\bar{\epsilon}_2\overline{\sigma}^\mu\partial_\mu\psi + 2i\epsilon_{2\alpha}\,\bar{\epsilon}_1\overline{\sigma}^\mu\partial_\mu\psi\\ \to \delta F =& -\sqrt{2}i\bar{\epsilon}\overline{\sigma}^\mu\partial_\mu\psi.\end{aligned} \tag{3.68}$$

With these modifications we see that the SUSY algebra closes for the fermions! Now we should check it for the auxiliary field too:

$$\begin{aligned}(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})F =& -2\bar{\epsilon}_1\overline{\sigma}^\mu\sigma^\nu\bar{\epsilon}_2\partial_\mu\partial_\nu\phi - 2i\bar{\epsilon}_1\overline{\sigma}^\mu\epsilon_2\partial_\mu F + 2\bar{\epsilon}_2\overline{\sigma}^\mu\sigma^\nu\bar{\epsilon}_1\partial_\mu\partial_\nu\phi + 2i\bar{\epsilon}_2\overline{\sigma}^\mu\epsilon_1\partial_\mu F\\ =& -(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)2P_\mu F.\end{aligned} \tag{3.69}$$

We can also check that the action is still unchanged: now we have

$$\delta\mathcal{L}_{\text{auxiliary}} = -\sqrt{2}i\bar{\epsilon}\overline{\sigma}^\mu\partial_\mu\psi\, F^* + \sqrt{2}i\partial_\mu\bar{\psi}\overline{\sigma}^\mu\epsilon\, F \tag{3.70}$$

which clearly compensates the change in the fermionic action.

In summary, we find that

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}) = -(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)2P_\mu. \tag{3.71}$$

From this we can deduce the (some of) the supersymmetry algebra. A SUSY transformation should be generated by "supercharges" that must be fermionic, so we can write

$$\delta_\epsilon X \propto [\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}, X]. \tag{3.72}$$

Since we have freedom to normalise the charges as we want, the historical choice is

$$\delta_\epsilon X = - i[\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}, X]. \tag{3.73}$$

$$\begin{aligned}
-(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X =& [\epsilon_2^\alpha Q_\alpha + \bar{\epsilon}_{2,\dot{\alpha}}\bar{Q}^{\dot{\alpha}}, [\epsilon_1^\beta Q_\beta + \bar{\epsilon}_{1,\dot{\beta}}\bar{Q}^{\dot{\beta}}, X]] - [\epsilon_1^\beta Q_\beta + \bar{\epsilon}_{1,\dot{\beta}}\bar{Q}^{\dot{\beta}}, [\epsilon_2^\alpha Q_\alpha + \bar{\epsilon}_{2,\dot{\alpha}}\bar{Q}^{\dot{\alpha}}, X]] \\
=& [\epsilon_2^\alpha Q_\alpha + \bar{\epsilon}_{2,\dot{\alpha}}\bar{Q}^{\dot{\alpha}}, \epsilon_1^\beta Q_\beta + \bar{\epsilon}_{1,\dot{\beta}}\bar{Q}^{\dot{\beta}}]X \\
=& [\epsilon_2 Q, \epsilon_1 Q] + [\bar{\epsilon}_2 \bar{Q}, \epsilon_1 Q] + [\epsilon_2 Q, \bar{\epsilon}_1 \bar{Q}] + [\bar{\epsilon}_2 \bar{Q}, \bar{\epsilon}_1 \bar{Q}] \\
=& 2(-\epsilon_1 \sigma^\mu \bar{\epsilon}_2 + \epsilon_2 \sigma^\mu \bar{\epsilon}_1) P_\mu.
\end{aligned} \tag{3.74}$$

Since $\epsilon_1, \epsilon_2$ are arbitrary complex numbers, we conclude that

$$\begin{aligned}
-2\epsilon_1^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\epsilon}_2^{\dot{\alpha}} P_\mu =& [\bar{Q}_{\dot{\alpha}} \bar{\epsilon}_2^{\dot{\alpha}}, \epsilon_1^\alpha Q_\alpha] \\
=& - \epsilon_1^\alpha \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \epsilon_2^{\dot{\alpha}} \\
\longrightarrow \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} =& 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \qquad \{Q_\alpha, Q_\beta\} = 0, \qquad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0.
\end{aligned} \tag{3.75}$$

Now, you might complain that the reason that the second two anticommutators vanish is because we imposed that the $U(1)$ symmetries should be identified rather than broken. Could we have a theory where this is not true? To investigate this, we need first consider the commutator of the supercharges with the momentum operator, $P_\mu$: since the supercharge is a spinor, it must transform as such under the Poincaré group, so we must find

$$[P_\mu, Q_\alpha] = [P_\mu, Q_{\dot{\alpha}}] = 0 \tag{3.76}$$

and

$$[M^{\mu\nu}, Q_\alpha] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta, \qquad [M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}}. \tag{3.77}$$

So then if we use the Jacobi identity

$$[A, \{B, C\}] - \{C, [A, B]\} + \{B, [C, A]\} = 0 \tag{3.78}$$

we can write

$$\begin{aligned}
0 =& [P_\mu, \{Q_\alpha, Q_\beta\}] - \{Q_\beta, \underbrace{[P_\mu, Q_\alpha]}_{=0}\} - \{Q_\alpha, \underbrace{[P_\mu, Q_\beta]}_{=0}\} \\
\longrightarrow 0 =& [P_\mu, \{Q_\alpha, Q_\beta\}].
\end{aligned} \tag{3.79}$$

Now the only operator that is already in the algebra that we could construct with the correct dimensions is

$$\{Q_\alpha, Q^\beta\} \stackrel{?}{=} k(\sigma^{\mu\nu})_\alpha^\beta M_{\mu\nu} \tag{3.80}$$

but we know that this doees not commute with $P_\mu$, so $k = 0$. For $N = 1$ SUSY theories in four dimensions this then means that (3.75) is complete. However, if we were to allow more supercharges labelled $A, B$ taking values between 1 and $N$ – in our toy model above we would need to add more fermions and bosons – then we could imagine a constant term:

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB} \tag{3.81}$$

with $Z^{AB}$ antisymmetric (this is why it will not work with $N = 1$). These are called "central charges" because they must commute with all of the operators of the algebra (exercise: prove this for $M^{\mu\nu}$ and $Q_\alpha$).

So now we can write the full SUSY algebra for arbitrary supercharges:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} =& 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{AB} \\ \{Q_\alpha^A, Q_\beta^B\} =& \epsilon_{\alpha\beta} Z^{AB}, \qquad \{\bar{Q}_{\dot{\alpha}}^A, \bar{Q}_{\dot{\beta}}^B\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{AB})^* \\ [P_\mu, Q_\alpha^A] =& [P_\mu, Q_{\dot{\alpha}}^A] = 0. \end{aligned} \tag{3.82}$$

We note now one caveat which appears in string theory: when we allow *extended* objects (which therefore *spontaneously* violate Lorentz invariance) to appear in the theory we could have the charges associated with them appearing in the algebra – because we would have new Lorentz structures, e.g.

$$\{Q_\alpha, Q^\beta\} \supset (\sigma^{\mu\nu})_\alpha^\beta \tilde{Q}_{\mu\nu}. \tag{3.83}$$

These appear in string theory when studying D-branes.

# References


[1] D. I. Kazakov, "Beyond the standard model: In search of supersymmetry," in *2000 European School of high-energy physics, Caramulo, Portugal, 20 Aug-2 Sep 2000: Proceedings*, pp. 125–199. 2000. arXiv:hep-ph/0012288 [hep-ph].

[2] D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio, and A. Strumia, *Investigating the near-criticality of the Higgs boson.* JHEP **12** (2013) 089, arXiv:1307.3536 [hep-ph].

[3] S. R. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix.* Phys. Rev. **159** (1967) 1251–1256.

[4] A. Neveu and J. H. Schwarz, *Factorizable dual model of pions.* Nucl. Phys. **B31** (1971) 86–112.

[5] P. Ramond, *Dual Theory for Free Fermions.* Phys. Rev. **D3** (1971) 2415–2418.