Lectures on Supersymmetry
Lecture 3

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Abstract

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3.7 R symmetry, local and global symmetries

We have given the whole super-poincaré algebra, and for massive theories there are no other extensions of the poincaré algebra. However, we should not forget gauge transformations and other global symmetries: their generators must commute with all of the super-poincaré algebra, but as a result all symmetry transformations will be promoted to supertransformations; consider some generator T inducing a shift $\delta_\alpha X = [\alpha T, X]$, then

$$\delta_\epsilon \delta_\alpha X = \delta_\alpha \delta_\epsilon X \tag{3.1}$$

so we can define some combined supersymmetry-gauge shift that the theory must be invariant under.

However, you may have noticed that there is a redundancy in our description of our supertranformations: because the parameter ϵ is complex, we can rotate the supercharges by a unitary tranformation without changing anything! Indeed, we see that it must be unitary from the requirement that

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{AB}$$

is unchanged. We can promote this to an internal symmetry with generators R_A such that

$$[R^A, Q^B_\alpha] = -(r_A)^{BC}\ Q^C_\alpha, \qquad [R^A, \overline{Q}_{B,\dot{\alpha}}] = (r^A)^{CB}\ \overline{Q}^C_{\dot{\alpha}}, \tag{3.2}$$

where the differing signs are because

$$\begin{aligned} Q \to \exp(-i(r^A)\ )Q \longrightarrow & \overline{Q}^T \to \overline{Q}^T \Big[\exp(-i(r^A))\Big]^\dagger \\ & = \overline{Q}^T \exp(i(r^A)) \longrightarrow \overline{Q} \to \exp(i(r^A)^T)\overline{Q}. \end{aligned} \tag{3.3}$$

Hence for N supersymmetry generators we have an $U(N)$ "R-symmetry" group; for $N = 1$ this is just a $U(1)$ rotation.

Clearly R must commute with all of the other generators of the algebra; however we do see that the central charges must transform as

$$\begin{aligned} \{Q^A_\alpha, Q^B_\beta\} \to & \{\exp(iR^C)Q^A_\alpha, \exp(iR^C)Q^B_\beta\} = \epsilon_{\alpha\beta}\big[\exp(iR^C)Z\exp(-iR^C)\big]^{AB} = \epsilon_{\alpha\beta}(Z^{AB} + i[R^C, Z]^{AB} + ...) \\ & = \epsilon_{\alpha\beta}Z^{AB} + i\{[R^C, Q^A_\alpha], Q^B_\beta\} + i\{Q^A_\alpha, [R^C, Q^B_\beta]\} + ... \\ \to [R^C, Z^{AB}] = & -(r^C)^{AD}Z^{DB} - Z^{AD}(r^C)^{BD}. \end{aligned} \tag{3.4}$$

Also, by applying the Jacobi identity we conclude that now the central charges must have non-trivial commutation relations among themselves! However, we will not require them in these lectures, where we are concerned mostly with $N = 1$ SUSY, and $Z^{AB} = 0$ for the cases where $N > 1$.

Before discussing representations of the SUSY algebra, note that in the simple model we considered above, we only allowed global SUSY transformations. However, the algebra itself is completely general, and now that we have found it we could promote the transformations to local ones, i.e. allow $\epsilon, \bar{\epsilon}$ to vary in space. What we know from gauge theories is that this requires adding a gauge boson; for kinetic terms to be covariant so that under $\phi \to \exp(i\alpha(x)T)\phi$ we want

$$\nabla_\mu \phi \to \exp(i\alpha(x)T)\nabla_\mu\phi \to \nabla_\mu = \partial_\mu + iA_\mu,$$

where under gauge transformations

$$A_\mu \to A_\mu - \nabla_\mu(\alpha T) = A_\mu - \partial_\mu(\alpha T) - i[A_\mu, \alpha T]. \tag{3.5}$$

Now suppose we do this for the Poincaré algebra and make our theory invariant under transformations such as $a^\mu(x)P_\mu$; what is the gauge boson? It is a graviton! So if we make our supersymmetry transformations $\epsilon, \bar{\epsilon}$ depend on position, then because $\{Q, \overline{Q}\} = 2\sigma^\mu P_\mu$ we need to make our Poincaré transformations position invariant and we have a theory of supergravity! Indeed, now we should have a graviton and a gravitino ...

3.8 Representations of the SUSY algebra

We are now ready to discuss what possible representations the SUSY algebra can have – or in other words what sets of particles we will find.

3.8.1 First observations

Firstly, we see that $P^2 = P_\mu P^\mu$ is a Casimir of the super-poincaré algebra, because it commutes with all generators. Now consider an irreducible representation of the SUSY algebra. By this, we mean that starting from any state in the representation, and acting with $Q_\alpha, \overline{Q}_{\dot\alpha}, P_\mu$ we generate all elements of the representation. Since P^2 is a casimir, all elements of this irrep – which we shall refer to as a "multiplet" – have the same mass.

Next, take the irrep to have some P_μ and consider summing $\mathrm{tr}[(-1)^{2S} P_\mu]$ for all states in the multiplet, where and the operator S is the spin operator (so half-integer for fermions, and integer for bosons):

$$\begin{aligned}
2\sigma^\mu_{\alpha\dot\alpha} p_\mu \mathrm{tr}(-1)^{2S} =& 2\sigma^\mu_{\alpha\dot\alpha}\sum_i \langle i|(-1)^{2S}P_\mu|i\rangle \\
=&\sum_i \langle i|(-1)^{2S}(Q_\alpha\overline{Q}_{\dot\alpha} + \overline{Q}_{\dot\alpha}Q_\alpha)|i\rangle \\
=&\sum_i \langle i|(-1)^{2S}Q_\alpha\overline{Q}_{\dot\alpha}|i\rangle + \sum_{i,j}\langle i|(-1)^{2S}\overline{Q}_{\dot\alpha}|j\rangle\langle j|Q_\alpha|i\rangle \\
=&\sum_i \langle i|(-1)^{2S}Q_\alpha\overline{Q}_{\dot\alpha}|i\rangle + \sum_j \langle j|Q_\alpha(-1)^{2S}\overline{Q}_{\dot\alpha}|j\rangle \\
=&0.
\end{aligned}$$

This shows that in a given multiplet there are *an equal number of fermionic and bosonic degrees of freedom, all having the same mass.*

A simple corollary of the above is to consider the vacuum; if it is invariant under SUSY then $Q_\alpha|0\rangle = \bar{Q}_{\dot\alpha}|0\rangle = 0$. Now the energy of the vacuum is the Hamiltonian; therefore

$$\langle 0|H|0\rangle = \langle 0|P_0|0\rangle = \frac{1}{2}(\overline{\sigma}^0)^{\dot\alpha\alpha}\langle 0|(Q_\alpha\overline{Q}_{\dot\alpha} + \overline{Q}_{\dot\alpha}Q_\alpha)|0\rangle = 0. \tag{3.6}$$

On the other hand, for any state

$$\begin{aligned}
\langle i|P_0|i\rangle =& \frac{1}{2}(\overline{\sigma}^0)^{\dot\alpha\alpha}\langle i|(Q_\alpha\overline{Q}_{\dot\alpha} + \overline{Q}_{\dot\alpha}Q_\alpha)|i\rangle \\
=&\frac{1}{2}\sum_\alpha \langle i|(Q_\alpha(Q_\alpha)^\dagger + (Q_\alpha)^\dagger Q_\alpha)|i\rangle \\
=&\frac{1}{2}\sum_\alpha \left|(Q_\alpha)^\dagger|i\rangle\right|^2 + \left|Q_\alpha|i\rangle\right|^2 \\
\geq& 0.
\end{aligned} \tag{3.7}$$

In particular, this also holds for the vacuum in a theory with spontaneously broken supersymmetry: if $Q_\alpha|0\rangle \neq 0$ and/or $\overline{Q}_{\dot\alpha}|0\rangle \neq 0$ then we conclude that

$$\langle 0|P_0|0\rangle > 0. \tag{3.8}$$

This means that the potential in such a theory is non-zero. In addition, the same argument can be used to show that it is not possible to spontaneously break $N = 2$ SUSY to $N = 1$ SUSY – but in that case there is an important loophole, see e.g. the work of Antoniadis, Partouche and Taylor [1].

3.8.2 Massless multiplets

Consider a massless particle in a frame where $p_\mu = (E,\ 0,\ 0,\ E)$. Consider the algebra

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} \;=\; 2\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu \;=\; 2\,E\left(\sigma^0 \,+\, \sigma^3\right)_{\alpha\dot\beta} \;=\; 4\,E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta}\,, \tag{3.9}$$

which implies that Q_2 is zero in the representation:

$$\{Q_2, \bar{Q}_{\dot2}\} \;=\; 0 \;\Longrightarrow\; \langle p^\mu, \lambda|\,\bar{Q}_{\dot2}\,Q_2\,|\tilde{p}^\mu, \tilde\lambda\rangle \;=\; 0 \;\Longrightarrow\; Q_2 \;=\; 0 \tag{3.10}$$

where λ is the spin of the particle. The Q_1 satisfy $\{Q_1, \bar{Q}_{\dot1}\} = 4E$, so defining creation- and annihilation operators a and $a^\dagger$ via

$$a^\dagger = \frac{Q_1}{2\sqrt{E}}\,, \qquad a = \frac{\bar{Q}_{\dot1}}{2\sqrt{E}}, \tag{3.11}$$

we have the anticommutation relations

$$\{a, a^\dagger\} \;=\; 1\,, \qquad \{a, a\} \;=\; \{a^\dagger, a^\dagger\} \;=\; 0. \tag{3.12}$$

Also, since

$$\begin{aligned} [J_3, Q_1] =& \frac{1}{2}Q_1, \qquad [J_3, \overline{Q}^{\dot2}] = -\frac{1}{2}\overline{Q}^{\dot2} \to [J_3, \overline{Q}_{\dot1}] = -\frac{1}{2}\overline{Q}_{\dot1} \\ \to [J_3, a^\dagger] =& \frac{1}{2}a^\dagger, \qquad [J_3, a] = -\frac{1}{2}a \end{aligned} \tag{3.13}$$

we have

$$J_3\left(a\,|p^\mu, \lambda\rangle\right) = \left([J_3, a] + aJ^3\right)\,|p^\mu, \lambda\rangle = \left(a\,J_3 \;-\; \frac{a}{2}\right)|p^\mu, \lambda\rangle = \left(\lambda - \frac{1}{2}\right)a\,|p^\mu, \lambda\rangle\;. \tag{3.14}$$

$a|p^\mu, \lambda\rangle$ has spin $\lambda - \frac{1}{2}$, and by similar reasoning, find that the spin of $a^\dagger|p^\mu, \lambda\rangle$ is $\lambda + \frac{1}{2}$. To build the representation, start with a vacuum state of minimum spin λ, let's call it $|\Omega\rangle$. Obviously $a|\Omega\rangle = 0$ (otherwise $|\Omega\rangle$ would not have lowest spin) and $a^\dagger a^\dagger|\Omega\rangle = 0|\Omega\rangle = 0$, so the whole multiplet consists of

$$|\Omega\rangle \;=\; |p^\mu, \lambda\rangle\;, \qquad a^\dagger\,|\Omega\rangle \;=\; |p^\mu, \lambda + \tfrac{1}{2}\rangle\;. \tag{3.15}$$

For any given $n, n+\frac{1}{2}$, because we have only massless fields and therefore only a single chirality of fermion, we must also add the CPT conjugate states; the conjugate fields have quantum numbers

$$|p^\mu, -\lambda - \frac{1}{2}\rangle, |p^\mu, -\lambda\rangle \tag{3.16}$$

just by complex conjugation. Since we are discussing massless particles, for the spin 1/2 fermions the spin and helicity must be the same so we can identify chiral or antichiral fermions easily. For the case of spin zero particles the two states exchanged by CPT are a single complex scalar and its complex conjugate.

So we can build up the set of allowed multiplets by starting with $\lambda = 0$. We have:

Name	λ	Lowest state	$\lambda + \frac{1}{2}$ state
Chiral multiplet	0	Complex scalar	Chiral fermion
Vector multiplet	$\frac{1}{2}$	Gaugino	Gauge boson
Gravity multiplet	$\frac{3}{2}$	Gravitino	Graviton

In field theory, we can only quantise states with spin up to 1, and so the gravity multiplet is aptly named. In these lectures we shall therefore be concerned only with chiral and vector multiplets. In addition, even in gravitational theories we cannot have spins greater than 2 while retaining non-trivial interactions *and* a finite number of fields. Finally, it does not seem that the multiplet with $n = 1$ is of interest phenomenologically.

Note also that we can define R-charges (under R-symmetry) for our states, and since $[R, Q_1] = -Q_1, [R, \overline{Q}]_{\dot{1}} = -\overline{Q}_{\dot{1}}$ we can add an R-charge to the set of quantum numbers; for any multiplet we then have $R|p^\mu, \lambda\rangle = r|p^\mu, \lambda\rangle$ and

$$Ra^\dagger|p^\mu, \lambda\rangle = (r-1)a^\dagger|p^\mu, \lambda\rangle \tag{3.17}$$

so we can label our multiplets plus CPT conjugates as

$$|p^\mu, \lambda, r\rangle, \quad |p^\mu, \lambda + \frac{1}{2}, r-1\rangle, \quad |p^\mu, -\lambda - \frac{1}{2}, 1-r, \rangle, |p^\mu, -\lambda, -r\rangle. \tag{3.18}$$

For a chiral multiplet, the R-symmetry is actually a chiral symmetry that rotates all of the fermions, although we have the freedom to choose their charges; and note that the bosons will have charges increased by one. However, for a vector multiplet, we require that the spin 1 state must have R-charge 0, because it must be a real field! Hence the gaugino in a vector multiplet must have R-charge 1.

3.8.3 Massive multiplets

In case of $m \neq 0$, there are P^μ - eigenvalues $p^\mu = (m,\ 0,\ 0,\ 0)$. Again, the anticommutation - relation for Q and $\bar{Q}$ is the key to get the states:

$$\left\{Q_\alpha\ ,\ \bar{Q}_{\dot{\beta}}\right\} \;=\; 2\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu \;=\; 2\,m\,(\sigma^0)_{\alpha\dot{\beta}} \;=\; 2\,m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}}$$

Since both Q's have nonzero anticommutators with their $\bar{Q}$ - partner, define two sets of ladder operators

$$a^\dagger_{1,2} = \frac{Q_{1,2}}{\sqrt{2m}} , \qquad a_{1,2} = \frac{\bar{Q}_{\dot{1},\dot{2}}}{\sqrt{2m}} , \tag{3.19}$$

with anticommutation relations

$$\{a_p, a^\dagger_q\} = \delta_{pq} , \qquad \{a_p, a_q\} = \{a^\dagger_p, a^\dagger_q\} \quad = \quad 0. \tag{3.20}$$

Assume that we have some state with spin j_3; then we can obtain the rest of the multiplet using

$$\begin{aligned} a_1 \left|j_3\right\rangle &= \left|j_3 - \tfrac{1}{2}\right\rangle , \qquad & a^\dagger_1 \left|j_3\right\rangle &= \left|j_3 + \tfrac{1}{2}\right\rangle \\ a_2 \left|j_3\right\rangle &= \left|j_3 + \tfrac{1}{2}\right\rangle , \qquad & a^\dagger_2 \left|j_3\right\rangle &= \left|j_3 - \tfrac{1}{2}\right\rangle . \end{aligned}$$

We still define a_1, a_2 to annihilate the vacuum, but now $a^\dagger_1$ raises the spin and $a^\dagger_2$ lowers it, so any given multiplet will contain more than one fermion; but of course because we are considering massive multiplets spin 1/2 fermions must now have both spins. For the simplest case we can start with a state with $j_3 = 0$ annihilated by $a_{1,2}$, and add states with $j_3 = -1/2, 0, 1/2$. We can then impose that the whole multiplet is invariant under CPT, in which case we have a massive complex scalar and a massive Majorana fermion, which is therefore non-chiral.

More generally, let us call some given pair of fermion states in a massive multiplet ψ, χ. Since these fermions must be in the same representation of any gauge group, we can identify the antifield of χ with an opposite chirality fermion in the conjugate representation, which we can denote χ^c. We can then form a non-chiral pair ψ, χ^c. Of course, this is *necessary* to give the fermions a Dirac mass; however, it also has the consequence that the representation is not chiral – so cannot be used to construct the Standard Model, which is a chiral theory.

3.8.4 $N \geq 2$

Following the logic of the previous two subsections, suppose we have more SUSY generators, and again just consider the massless representations. Now we have two or more raising operators on our vacuum state: $a^\dagger_I$ for $I = 2$ for $N = 2$ SUSY or 4 for $N = 4$. The entire logic follows as before, so now we have for $N = 2$

Name	λ	Lowest state	$\lambda + \frac{1}{2}$ state	$\lambda + 1$ state
Hypermultiplet	$-\frac{1}{2}$	Chiral fermion	2 Complex scalars	Chiral fermion
$N = 2$ Vector multiplet	0	Complex scalar	2 gauginos	Gauge boson
Gravity multiplet	1	Graviphoton	2 Gravitinos	Graviton

In particular, the logic is similar for massive multiplets of $N = 1$: the fermions must always come in non-chiral pairs. For example, for the hypermultiplet we start with an antichiral state and find a chiral fermion in the same representation of the gauge group; when we add in the CPT conjugates of these states we can choose instead to write the multiplet as two $N = 1$ multiplets in opposite representations of the gauge group, which is necessarily non-chiral. Likewise the vector multiplet can be written as the sum of an $N = 1$ vector and $N = 1$ chiral multiplet.

For $N = 4$, in field theory there is no longer a separation between vector and matter multiplets: there is only an $N = 4$ multiplet which contains 1 vector, four fermions of spin 1/2 and three complex scalars. These can be written as an $N = 1$ vector multiplet and three chiral multiplets. Since it contains a vector, they must transform under the adjoint representation of the gauge group.

3.9 *Constructing SUSY theories from the supersymmetry algebra

In the previous section we defined the SUSY algebra and showed that only massless representations of $N = 1$ supersymmetry are chiral and therefore interesting for phenomenology (with the exception of $N = 2$ vector multiplets). From now on we shall restrict ourselves to this case. Here we will learn how to construct supersymmetric field theories, and in particular we shall stick to the *global* SUSY case for now.

3.9.1 Massive Wess-Zumino Model

Let us recall our simple example example model, and let us try to add a mass term. We showed already that the fermion and boson must have the same mass to sit in the same representation, so let us add the mass terms

$$\mathcal{L}_{\text{naive mass}} \overset{?}{=} -|M|^2\phi^*\phi - \frac{1}{2}\Big[M\psi\psi + h.c.\Big]. \tag{3.21}$$

Now look at the SUSY transformation:

$$\delta\mathcal{L}_{\text{naive mass}} = -\sqrt{2}|M|^2\Big[\phi^*\epsilon\psi + \phi\bar{\epsilon}\bar{\psi}\Big] + \sqrt{2}\Big[iM\psi\sigma^\mu\bar{\epsilon}\partial_\mu\phi - \epsilon\psi MF - iM^*\epsilon\sigma^\mu\bar{\psi}\partial_\mu\phi^* - \bar{\epsilon}\bar{\psi}M^*F^*\Big].$$

But we have a problem: the SUSY transformations of both ψ and F involve derivatives of the scalars, so we cannot cancel the first term off! However, a clue as to what we should do is found if we go on-shell and remove the auxiliary fields again; there we would find (after integrating by parts and dropping a surface term)

$$\delta\mathcal{L}_{\text{naive mass}} \overset{F\to 0}{\to} \sqrt{2}M\phi\bar{\epsilon}_{\dot{\alpha}}\Big[i(\overline{\sigma}^\mu\partial_\mu\psi)^{\dot{\alpha}} - M\overline{\psi}^{\dot{\alpha}}\Big] + h.c. \tag{3.22}$$

which again vanishes on-shell! So maybe we are on the right track but the auxiliary field plays some role. Now when we go off-shell, let us drop the naive scalar mass, and look at what other renormalisable terms we could put. We have only $\phi F, \phi^2 F$, their complex conjugates and $|F|^2$. Now we already have $|F|^2$, but modifying it would give the same problem as for $|\phi|^2$: the transformation will lead to a term with F and derivatives of ψ that cannot be cancelled by anything else. Also $\phi^2 F$ is dimensionless so is independent of M. So let us look at

$$\mathcal{L}_{\text{aux mass}} = M\phi F + h.c. \tag{3.23}$$

Then

$$\delta\mathcal{L}_{\text{aux mass}} = \sqrt{2}\epsilon\psi M F - \sqrt{2} i M\phi\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi + h.c.$$
$$= \sqrt{2}\epsilon\psi M F - i\sqrt{2} M\psi\sigma^\mu\bar{\epsilon}\partial_\mu\phi + h.c. \tag{3.24}$$

This exactly cancels the transformation of the fermion term! But what about the scalar mass? Shouldn't they be degenerate? To see how this comes about, we now write the lagragnian

$$\mathcal{L}_{\text{massive WZ}} = \partial^\mu\phi^*\partial_\mu\phi + i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + F^*F$$
$$+ \left[M\phi F - \frac{1}{2}M\psi\psi + h.c.\right]. \tag{3.25}$$

Now look at the equation of motion of the auxiliary:

$$F^* + M\phi = 0. \tag{3.26}$$

So if we now integrate out the auxiliary field we find

$$\mathcal{L}_{\text{massive WZ}} \to \partial^\mu\phi^*\partial_\mu\phi + i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi - \left(\frac{1}{2}M\psi\psi + h.c.\right) - |M|^2|\phi|^2, \tag{3.27}$$

which contains exactly the mass term we were looking for!

3.9.2 *General renormalisable field theory

We can continue this procedure to find all the allowed renormalisable interactions for a theory with fermions and scalars. To generalise a little, let us consider n chiral multiplets $\Phi_i = (\phi_i, \psi_i, F_i)$. We can then write the flavour indices for the complex conjugates as raised so $\Phi_i^* = \Phi^i = (\bar{\phi}^i, \bar{\psi}^i, \bar{F}^i)$. The SUSY transformations are

$$\begin{aligned} \delta\phi_i &= \sqrt{2}\epsilon\psi_i, & \delta\bar{\phi}^i &= \sqrt{2}\bar{\epsilon}\bar{\psi}^i, \\ \delta\psi_{i\,\alpha} &= -\sqrt{2}i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu\phi_i + \sqrt{2}\epsilon_\alpha F_i, & \delta\bar{\psi}^i_{\dot{\alpha}} &= -\sqrt{2}i(\epsilon\sigma^\mu)_{\dot{\alpha}}\partial_\mu\bar{\phi}^i + \sqrt{2}\bar{\epsilon}_{\dot{\alpha}}\bar{F}^i, \\ \delta F_i &= -\sqrt{2}i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi_i, & \delta\bar{F}^i &= \sqrt{2}i\partial_\mu\bar{\psi}^i\bar{\sigma}^\mu\epsilon. \end{aligned} \tag{3.28}$$

Note that the SUSY transformations do not mix the ϕ_i with the $\bar{\psi}_i$ or F_i. This is a consequence of the need for $\{Q, Q\}$ to vanish, but it has far-reaching consequences: we will not be able to cancel the transformations if we try to write down "mixed" terms such as ϕ^*F or $\bar{\phi}\psi\psi$ in the lagrangian, so the interactions are "holomorphic" – we will come back to this later.

For now, having ruled out adding purely scalar terms and non-holomorphic terms to the lagrangian, let us start by adding fermion masses and Yukawa terms:

$$\mathcal{L}_{\text{mass, Yukawa}} = -\frac{1}{2}M^{ij}\psi_i\psi_j - \frac{1}{2}W^{ijk}\phi_i\psi_j\psi_k + h.c. \tag{3.29}$$

We saw previously that the transformations of $\psi\psi$ were cancelled by a ϕF term; let us now write

$$\mathcal{L}_{\text{aux}} = L^i F_i + c^{ij}\phi_i F_j + \frac{1}{2}c^{ijk}F_i\phi_j\phi_k + h.c. \tag{3.30}$$

The term $L^i F_i$ has a variation that is just a surface term, so we can retain it. To simplify the check of the transformations, let us also define

$$W^{ij} \equiv M^{ij} + W^{ijk}\phi_k, \qquad \tilde{L}^i \equiv L^i + c^{ij}\phi_j + \frac{1}{2}c^{ijk}\phi_j\phi_k \tag{3.31}$$

Now looking at the transformations and using the symmetry of W^{ij} we have

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left(\delta\mathcal{L}_{\text{mass, Yukawa}} + \delta\mathcal{L}_{\text{aux}}\right) \\
=&W^{ij}\left[i\psi_i\sigma^\mu\bar{\epsilon}\partial_\mu\phi_j - \epsilon\psi_i F_j\right] - \frac{1}{2}W^{ijk}(\epsilon\psi_i)\psi_j\psi_k - iL^i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi_i + F_i(\epsilon\psi_j)\frac{\partial\tilde{L}^i}{\partial\phi_j} + h.c. \\
=&W^{ij}\left[i\psi_i\sigma^\mu\bar{\epsilon}\partial_\mu\phi_j - \epsilon\psi_j F_i\right] + \frac{\partial\tilde{L}^i}{\partial\phi_j}\left[-i\psi_i\sigma^\mu\bar{\epsilon}\partial_\mu\phi_j + F_i(\epsilon\psi_j)\right] - \frac{1}{2}W^{ijk}(\epsilon\psi_i)\psi_j\psi_k + h.c.
\end{aligned} \tag{3.32}$$

from which we conclude that SUSY is preserved if

$$W^{ij} = \frac{\partial\tilde{L}^i}{\partial\phi_j} \tag{3.33}$$

and that W^{ijk} is symmetric on exhange of $i \leftrightarrow j \leftrightarrow k$; clearly for terms with only one fermion flavour $(\epsilon\psi_i)\psi_i\psi_i = 0$, but if we have the extra symmetry then we can use the identities (**??**) to show that $W^{ijk}(\epsilon\psi_i)\psi_j\psi_k$ vanishes. This means that we can define

$$W \equiv W^0 + W^i\phi_i + \frac{1}{2}W^{ij}\phi_i\phi_j + \frac{1}{6}W^{ijk}\phi_i\phi_j\phi_k \tag{3.34}$$

which we shall call the "superpotential," (the constant term W^0 plays no role in global supersymmetry so we can set it to zero) and then write

$$\mathcal{L}_{\text{aux}} + \mathcal{L}_{\text{mass, Yukawa}} = \frac{\partial W}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j. \tag{3.35}$$

If we then examine what we find for the scalar potential after integrating out the auxiliary fields, we find

$$V_{\text{scalar}} = \sum_i \left|\frac{\partial W}{\partial\phi_i}\right|^2. \tag{3.36}$$

3.10 Gauge interactions

The interactions of a vector superfield cannot be described by a superpotential: they are real fields consisting of a vector A^a_μ and a Weyl spinor λ^a. The λ is known as the *gaugino* and its interactions must be determined by the gauge symmetry! We shall see how to derive them next time, but for now we can write down the lagrangian:

$$S_V = \int d^4x\, \left[\frac{1}{2g^2}D^a D^a + \frac{i}{g^2}\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda - \frac{1}{4g^2}F^{a,\mu\nu}F^a_{\mu\nu} + \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}F^a_{\rho\kappa}\right]. \tag{3.37}$$

Again we need an auxiliary. Note that the gaugino is in the adjoint representation so

$$D_\mu \lambda = \partial_\mu \lambda + i[A_\mu, \lambda], \qquad \lambda \equiv \lambda^a t^a. \tag{3.38}$$

The SUSY transformations become

$$\begin{aligned} \delta A_\mu &= -\left(\bar{\epsilon}\bar{\sigma}_\mu \lambda^a + \bar{\lambda}\bar{\sigma}_\mu \epsilon\right) \\ \delta \lambda_\alpha &= -\tfrac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \epsilon)_\alpha \, F_{\mu\nu} + \epsilon_\alpha \, D \\ \delta D &= i\left(-\bar{\epsilon}\bar{\sigma}^\mu \nabla_\mu \lambda^a + \nabla_\mu \bar{\lambda}\bar{\sigma}^\mu \epsilon\right). \end{aligned} \tag{3.39}$$

We can also add supersymmetric interactions between gauge fields and chiral superfields. Supersymmetry gives us new Yukawa couplings!

$$\begin{aligned} &= (D_\mu \phi)^\dagger (D^\mu \phi) + i\psi \sigma^\mu D_\mu^\dagger \bar{\psi} + F^\dagger F \\ &\quad - \sqrt{2} g(\phi^\dagger t^a \lambda^a \psi + \bar{\psi} t^a \bar{\lambda}^a \phi) + g\phi^\dagger t^a D^a \phi. \end{aligned} \tag{3.40}$$

4 Superspace

4.1 Superspace basics

At the end of part 1 we showed how to construct supersymmetric field theories using only the SUSY transformations, that we derived from considering a free theory. In this section we will show how this can be embedded in the concept of "superspace," which hugely simplifies constructing and working with $N = 1$ SUSY theories (for $N \geq 2$ SUSY we can still use $N = 1$ superspace; but a true $N = 2$ superspace becomes vastly more complicated).

The first part of this – the derivation of superspace, the construction of superfields in superspace, and the beginnings of writing down lagrangians in superspace – was covered by Karim in lecture 2. I give my notes here to have a unified presentation, but the material for lecture 3 will pick up where he left off.

We start by supposing that we take some function and want to act with a group element of the SUSY algebra on it. If we just consider the subgroup of translations, then we have the action on a field as

$$\begin{aligned} \phi(x^\mu + y^\mu) &= \phi(x^\mu) + y^\nu \partial_\nu \phi(x^\mu) + \ldots \\ &= \exp(y^\mu \partial_\mu)\phi(x) \equiv \exp(iy^\mu P_\mu)\phi(x) \\ \to P_\mu &\equiv -i\partial_\mu. \end{aligned} \tag{4.1}$$

This is the action of the operator on a state in the Hilbert space. If we extend it to the group operation, then a transformation on operators acts as

$$\exp(iy^\mu P_\mu)\phi(x^\mu)\exp(-iy^\mu P_\mu) = \phi(x^\mu) + iy^\mu [P_\mu, \phi(x^\mu)] + \ldots \tag{4.2}$$

This rule is universal: the operator $\phi(x^\mu)$ acts at x^μ, so to find the action at $x^\mu + a^\mu$ we translate first to x^μ, act with ϕ, and then translate back to $x^\mu + a^\mu$. Equating the two procedures gives us

$$P_\mu \phi(x^\mu) = [P_\mu, \phi(x^\mu)] = -i\partial_\mu \phi(x^\mu) \tag{4.3}$$

which gives us our classic representation of P_μ as $-i\partial_\mu$.

Now to extend this to a supersymmetric transformation, recall that we defined

$$\delta_\epsilon X \equiv -i[\epsilon Q + \bar{\epsilon}\bar{Q}, X]. \tag{4.4}$$

This means that the group element should be

$$g(y, \theta, \bar{\theta}, \omega^{\mu\nu}) = \exp i\big[y^\mu P_\mu - \theta Q - \bar{\theta}\bar{Q} + \omega^{\mu\nu} M_{\mu\nu}\big]. \tag{4.5}$$

Now imagine that we apply a supersymmetry transformation to $\phi(x^\mu)$. We can use this to define ϕ as a function of spinor coordinates:

$$\phi(x^\mu, \theta, \bar{\theta}) \equiv g(0, \theta, \bar{\theta}, 0)\phi(x^\mu) g(0, -\theta, -\bar{\theta}, 0). \tag{4.6}$$

So now imagine combining two transformations; we expect to recover another member of the group:

$$\begin{aligned} g(0, \epsilon, \bar{\epsilon}, 0) g(0, \theta, \bar{\theta}, 0) &= \exp -i\big[\epsilon Q + \bar{\epsilon}\bar{Q}\big] \exp -i\big[\theta Q + \bar{\theta}\bar{Q}\big] \\ &= \exp\Big[-i(\epsilon + \theta)Q - i(\bar{\epsilon} + \bar{\theta})\overline{Q} - \frac{1}{2}\big([\epsilon Q, \overline{\theta Q}] + [\bar{\epsilon}\overline{Q}, \theta Q]\big)\Big] \\ &= \exp\Big[-i(\epsilon + \theta)Q - i(\bar{\epsilon} + \bar{\theta})\overline{Q} - \big(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon}\big)P_\mu\Big] \\ &= \exp i\Big[-(\epsilon + \theta)Q - (\bar{\epsilon} + \bar{\theta})\overline{Q} + P_\mu i\big(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon}\big)\Big] \end{aligned} \tag{4.7}$$

For this we used the Baker-Campbell-Hausdorff formula

$$\begin{aligned} \exp A \exp B &= \exp\left[A + B + \sum_{n=2}^{\infty} \frac{1}{n!} C_n\right] \\ C_2 &= [A, B], \qquad C_3 = \frac{1}{2}[A, [A, B]] + \frac{1}{2}[B, [B, A]] + ... \end{aligned} \tag{4.8}$$

and the fact that in our case $[A, B]$ commutes with A and B. Then we see that

$$g(0, \epsilon, \bar{\epsilon}, 0) g(0, \theta, \bar{\theta}, 0) = g(i\big(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon}\big), \theta + \epsilon, \bar{\theta} + \bar{\epsilon}, 0). \tag{4.9}$$

If we now consider the action on ϕ then

$$\begin{aligned} -i[\epsilon Q + \bar{\epsilon}\bar{Q}, \phi(0, \theta, \bar{\theta})] &= \phi(i\big(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon}\big), \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) - \phi(0, \theta, \bar{\theta}) \\ &= \left[i\big(\epsilon\sigma^\mu\bar{\theta} + \bar{\epsilon}\overline{\sigma}^\mu\theta)\partial_\mu + \epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} + \bar{\epsilon}_{\dot{\alpha}} \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\right]\phi(0, \theta, \bar{\theta}) \equiv -i(\epsilon Q + \bar{\epsilon}\bar{Q})\phi \end{aligned} \tag{4.10}$$

and therefore when we promote ϕ to a superfield we can write teh generators as

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu\bar{\theta})_\alpha\partial_\mu \tag{4.11}$$

$$\bar{Q}^{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - (\overline{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu \to \bar{Q}_{\dot{\alpha}} = -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu. \tag{4.12}$$

Now we can define Minkowski space as the coset of the Poincaré group divided by the Lorentz group: the translations then become all of spacetime by the map

$$x^\mu \to \exp(-ix^\mu P_\mu). \tag{4.13}$$

Similarly we can imagine taking the coset of all of the supertransformations divided by the Lorentz group to get a “superspace”. This would have space and spinor coordinates as we have considered above. We can then define differentiation and integration over these Grassmann coordinates as we shall describe below.

4.1.1 Grassmann coordinates

Imagine that we now treat the θ^α like anticommuting coordinates of spacetime. They obey

$$\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} = 0 \tag{4.14}$$

and so, since each has two components, any product of three or more vanishes:

$$\theta_\alpha\theta_\beta\theta_\gamma = 0. \tag{4.15}$$

This means that any function can be expanded *exactly*:

$$S(x,\theta,\bar{\theta}) = a + \theta\xi + \bar{\theta}\overline{\chi} + \theta\theta b + \bar{\theta}\bar{\theta}c + \theta\sigma^\mu\bar{\theta}v_\mu + \bar{\theta}\bar{\theta}\theta\eta + \theta\theta\bar{\theta}\overline{\zeta} + \theta\theta\bar{\theta}\bar{\theta}d, \tag{4.16}$$

where all of the coefficients are functions of x^μ. Other consequences are that we can write (recalling $\epsilon^{12} = 1 = -\epsilon_{12}$):

$$\begin{array}{ll|ll} \theta\theta & = \theta^\alpha\theta_\alpha & \bar{\theta}\bar{\theta} & = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \\ & = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta & & = \epsilon_{\dot{\alpha}\dot{\beta}}\overline{\theta}^{\dot{\beta}}\overline{\theta}^{\dot{\alpha}} \\ & = 2\epsilon_{12}\theta^1\theta^2 & & = 2\epsilon_{\dot{2}\dot{1}}\overline{\theta}^1\overline{\theta}^2 \\ & = -2\theta^1\theta^2 & & = 2\overline{\theta}^{\dot{1}}\overline{\theta}^{\dot{2}} \\ & = -2\theta_1\theta_2 & & = 2\overline{\theta}_{\dot{1}}\overline{\theta}_{\dot{2}} \end{array} \tag{4.17}$$

and

$$\begin{array}{rl|rl} \theta^\alpha\theta^\beta = & \epsilon^{\alpha\beta}\theta^1\theta^2 & \overline{\theta}^{\dot{\alpha}}\overline{\theta}^{\dot{\beta}} = & \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\theta}^{\dot{1}}\overline{\theta}^{\dot{2}} \\ = & -\frac{1}{2}\epsilon^{\alpha\beta}(\theta\theta) & = & \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}(\overline{\theta\theta}) \\ \theta_\alpha\theta_\beta = & -\epsilon_{\alpha\beta}\theta_1\theta_2 & \overline{\theta}_{\dot{\alpha}}\overline{\theta}_{\dot{\beta}} = & -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}_1\bar{\theta}_2 \\ = & \frac{1}{2}\epsilon_{\alpha\beta}(\theta\theta) & = & -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\theta}\bar{\theta}). \end{array}$$

We can define differentiation easily enough:

$$\begin{aligned}\frac{\partial}{\partial\theta^\beta}(\theta^\alpha) = \delta^\alpha_\beta, \qquad & \frac{\partial}{\partial\bar\theta^{\dot\beta}}(\bar\theta^{\dot\alpha}) = \delta^{\dot\alpha}_{\dot\beta} \\ \frac{\partial}{\partial\theta_\beta}(\theta_\alpha) = \delta^\beta_\alpha, \qquad & \frac{\partial}{\partial\bar\theta_{\dot\beta}}(\bar\theta_{\dot\alpha}) = \delta^{\dot\beta}_{\dot\alpha}\end{aligned} \tag{4.18}$$

which leads to, using $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta^\gamma_\alpha$:

$$\begin{aligned}\delta^\alpha_\beta =& \frac{\partial}{\partial\theta^\beta}(\epsilon^{\alpha\gamma}\theta_\gamma) \to \epsilon_{\delta\beta} = \frac{\partial}{\partial\theta^\beta}(\theta_\delta) \to \delta^\alpha_\delta = -\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}(\theta_\delta) \\ \to \frac{\partial}{\partial\theta_\alpha} =& -\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}.\end{aligned} \tag{4.19}$$

and we require

$$\begin{aligned}\{\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta}\} =& 0 \\ \frac{\partial}{\partial\theta^\alpha}(AB) =& \frac{\partial A}{\partial\theta^\alpha}B + (-1)^{F_A}A\frac{\partial B}{\partial\theta^\alpha}\end{aligned} \tag{4.20}$$

where F_A is 1 if A is fermionic and 0 if A is a bosonic operator. Then for example

$$\begin{aligned}\frac{\partial}{\partial\bar\theta_{\dot\beta}}(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}) =& 2\bar\theta^{\dot\beta} \\ \frac{\partial}{\partial\bar\theta^{\dot\beta}}(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}) =& -\frac{\partial}{\partial\bar\theta^{\dot\beta}}(\bar\theta^{\dot\alpha}\bar\theta_{\dot\alpha}) = -2\bar\theta_{\dot\beta} = -2\epsilon_{\dot\beta\dot\alpha}\bar\theta^{\dot\alpha} \\ \to \frac{\partial}{\partial\bar\theta^{\dot\beta}} =& -\epsilon_{\dot\beta\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}.\end{aligned} \tag{4.21}$$

Similarly

$$\begin{aligned}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}(\bar\theta\bar\theta) =& 4 \to \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \equiv 4\frac{\partial}{\partial\bar\theta^2} \\ \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}(\theta\theta) =& 4 \to \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha} \equiv 4\frac{\partial}{\partial\theta^2}.\end{aligned} \tag{4.22}$$

Now we can define integration on superspace. The rule that we want to preserve is that surface elements should integrate to zero; then consider just the integral over one variable

$$\int d\theta \frac{\partial f}{\partial\theta} = 0. \tag{4.23}$$

But then a function of this variable is $f(\theta) = f_0 + f_1\theta$ and $\frac{\partial f}{\partial\theta} = f_1$, so we must have

$$\int d\theta = 0. \tag{4.24}$$

So then we should define the integral of θ; we do not want the integral to be trivial, and since $d\theta\theta$ is bosonic it must be a number, so we take

$$\int d\theta\ \theta = 1. \tag{4.25}$$

Now we can define the "volume element" for two variables by insisting

$$
\begin{aligned}
1 =& \int d^2\theta\ \theta\theta = \frac{1}{2}\int d\theta_1 d\theta_2\ (2\theta_2\theta_1) \\
\to d^2\theta =& \frac{1}{2} d\theta_1 d\theta_2 = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \\
1 =& \int d^2\bar\theta\ \bar\theta\bar\theta = -\frac{1}{2}\int d\bar\theta_{\dot 1} d\bar\theta_{\dot 2} (2\bar\theta_{\dot 2}\bar\theta_{\dot 1}) \\
\to d^2\bar\theta =& -\frac{1}{4} d\bar\theta_{\dot\alpha} d\bar\theta_{\dot\beta} \epsilon^{\dot\alpha\dot\beta}.
\end{aligned} \tag{4.26}
$$

4.1.2 Superderivatives

We already gave the definitions of the SUSY generators in terms of Grassmann coordinates. We can use them to derive the transformations of a general superfield by identifying

$$
-i[\epsilon Q + \bar\epsilon \bar Q, S(x^\mu, \theta, \bar\theta)] = \delta S. \tag{4.27}
$$

It is an instructive if mechanical exercise to do this.

But since we have expanded spacetime to superspace, and the algebra mixes SUSY generators with the momentum operators, we might expect that SUSY derivatives are not covariant:

$$
g(x^\mu, \xi, \bar\xi, 0)\frac{\partial}{\partial\theta^\alpha} g(x^\mu, -\xi, -\bar\xi, 0) \neq \frac{\partial}{\partial(\theta^\alpha + \xi^\alpha)} = \frac{\partial}{\partial\theta^\alpha}. \tag{4.28}
$$

Among other consequences, this means that $\partial_\alpha S$ is *not* a superfield, i.e. we cannot expand it as in equation (4.16). What we need is a covariant derivative

$$
\{D_\alpha, Q_\beta\} = \{\overline{D}_{\dot\alpha}, Q_\beta\} = 0. \tag{4.29}
$$

We can check that these are given by

$$
\begin{aligned}
D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\overline\theta^{\dot\alpha}\partial_\mu, \qquad & D^\alpha &= -\frac{\partial}{\partial\theta_\alpha} + i\bar\theta^{\dot\alpha}\overline\sigma^{\mu}_{\dot\alpha\alpha}\partial_\mu, \\
\overline{D}_{\dot\alpha} &= -\frac{\partial}{\partial\overline\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad & \overline{D}^{\dot\alpha} &= \frac{\partial}{\partial\overline\theta_{\dot\alpha}} - i\overline\sigma^\mu_{\dot\alpha\alpha}\theta^\alpha\partial_\mu.
\end{aligned} \tag{4.30}
$$

This gives

$$
\begin{aligned}
\{D_\alpha, D_\beta\} =& 0 \\
\{D_\alpha, \bar D_{\dot\alpha}\} =& 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu.
\end{aligned} \tag{4.31}
$$

We can then use these to construct new superfields $D_\alpha S, \overline{D}_{\dot\alpha} S$ ... and acting with more derivatives. We shall do this below, but first we note some important identities:

$$
D_\alpha D_\beta D_\gamma = 0, \qquad \overline{D}_{\dot\alpha}\overline{D}_{\dot\beta}\overline{D}_{\dot\gamma} = 0 \tag{4.32}
$$

follow from the anticommutation and that they have two indices. Then

$$\int d^2\theta D_\alpha(S) = -i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}} \int d^2\theta \partial_\mu(S) \tag{4.33}$$

for any superfield S, and similarly for $\int d^2\overline{\theta}\overline{D}_{\dot{\alpha}}(S)$. Then we have a total derivative which vanishes once we integrate over d^4x. Consider also

$$\begin{aligned}
\int d^4x d^2\theta \overline{D}^2(S) =& \int d^4x d^2\theta \epsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}(S) \\
=& \int d^4x d^2\theta 2\frac{\partial}{\partial\bar{\theta}^1}\frac{\partial}{\partial\bar{\theta}^2}(S) \\
=& -4\int d^4x d^2\theta \frac{\partial}{\partial\bar{\theta}^2}(S) \\
=& -4\int d^4x d^2\theta d^2\bar{\theta} S.
\end{aligned} \tag{4.34}$$

4.1.3 Chiral superfields

In the previous subsections we gave the expression for a general superfield. However, if we count the components there are too many for it to represent either a chiral or vector multiplet:

$$S(x,\theta,\bar{\theta}) = a + \theta\xi + \bar{\theta}\overline{\chi} + \theta\theta b + \bar{\theta}\bar{\theta}c + \theta\sigma^\mu\bar{\theta}v_\mu + \bar{\theta}\bar{\theta}\theta\eta + \theta\theta\bar{\theta}\bar{\zeta} + \theta\theta\bar{\theta}\bar{\theta}d,$$

It is therefore a *reducible representation of SUSY.* So in order to construct a field that represents theories that we are interested in we should impose some constraints upon it. To find matter fields, we want a chiral multiplet: i.e. we want to remove three of the four fermions and v_μ. The simplest thing that we can try is

$$\begin{aligned}
\overline{D}_{\dot{\alpha}}S =& 0 \\
=& \cancel{\overline{\chi}_{\dot{\alpha}}} + \cancel{2\bar{\theta}_{\dot{\alpha}}c} + (\theta\sigma^\mu)_{\dot{\alpha}}v_\mu + \cancel{2\bar{\theta}_{\dot{\alpha}}(\theta\eta)} + \bar{\zeta}_{\dot{\alpha}}\theta^2 + 2\theta^2\bar{\theta}_{\dot{\alpha}}d \\
&+ \underbrace{i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\left[a + \theta\xi + \cancel{\bar{\theta}\overline{\chi}} + \theta\sigma^\nu\bar{\theta}v_\nu + \cancel{\bar{\theta}\bar{\theta}c} + \cancel{\bar{\theta}\bar{\theta}\theta\eta}\right]}_{=i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu a - \frac{i}{2}\theta\theta(\xi\sigma^\mu)_{\dot{\alpha}} + \frac{i}{2}\theta^2\epsilon_{\dot{\alpha}\dot{\gamma}}(\overline{\sigma}^\mu\sigma^\nu)^{\dot{\gamma}}_{\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu v_\nu}.
\end{aligned} \tag{4.35}$$

Simply by looking at unmatched terms we immediately set

$$\overline{\chi}_{\dot{\alpha}} = 0, \qquad c = 0, \qquad \eta = 0. \tag{4.36}$$

We can then also write the set of equations

$$\begin{aligned}
0 =& v_\mu + i\partial_\mu a \\
=& \bar{\zeta}_{\dot{\alpha}} - \frac{i}{2}(\partial_\mu\xi\sigma^\mu)_{\dot{\alpha}} \\
=& 2\bar{\theta}_{\dot{\alpha}}d + \frac{i}{2}\epsilon_{\dot{\alpha}\dot{\gamma}}(\overline{\sigma}^\mu\sigma^\nu)^{\dot{\gamma}}_{\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu v_\nu.
\end{aligned} \tag{4.37}$$

Which therefore leads to

$$
\begin{aligned}
0 =& 2\bar{\theta}_{\dot{\alpha}} d + \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\gamma}}(\bar{\sigma}^\mu \sigma^\nu)^{\dot{\gamma}}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \partial^\mu a \\
=& 2d + \frac{1}{2}\partial_\mu \partial^\mu a.
\end{aligned} \tag{4.38}
$$

We can then write Φ for the new superfield, and renaming $a \to \phi, \xi \to \sqrt{2}\psi, b \to F$ we have:

$$
\Phi = \phi + \sqrt{2}(\theta\psi) + \theta\theta F - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{i}{\sqrt{2}}\theta\theta(\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\phi. \tag{4.39}
$$

This has *exactly* the correct number of components for a chiral superfield including the auxiliary F! Then to find an antichiral superfield we need only $D_\alpha\overline{\Phi} = 0$, and find

$$
\overline{\Phi} = \overline{\phi} + \sqrt{2}(\bar{\theta}\overline{\psi}) + \bar{\theta}\bar{\theta}\overline{F} + i\theta\sigma^\mu\bar{\theta}\partial_\mu\overline{\phi} - \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}(\theta\sigma^\mu\partial_\mu\overline{\psi}) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\overline{\phi}. \tag{4.40}
$$

In fact, there is a simple way of finding the above expansion if we start from the constraint $\overline{D}_{\dot{\alpha}}\Phi = 0$ and observe that

$$
\begin{aligned}
y^\mu \equiv& x^\mu - i\theta\sigma^\mu\bar{\theta} \\
\overline{D}_{\dot{\alpha}} y^\mu =& 0.
\end{aligned} \tag{4.41}
$$

This implies that if we write a function in terms of $y^\mu, \theta, \bar{\theta}$ then

$$
\begin{aligned}
\left(\frac{\partial}{\partial\theta^\alpha} f(y^\mu,\theta,\bar{\theta})\right)_{x^\mu,\bar{\theta}} =& \frac{\partial}{\partial\theta^\alpha} f(x^\mu - i\theta\sigma^\mu\bar{\theta},\theta,\bar{\theta}) \\
=& \left(\frac{\partial}{\partial\theta^\alpha} f(y^\mu,\theta,\bar{\theta})\right)_{y^\mu,\bar{\theta}} - i(\sigma^\mu\bar{\theta})_\alpha \left(\frac{\partial}{\partial y^\mu} f(y^\mu,\theta,\bar{\theta})\right)_{\theta,\bar{\theta}}.
\end{aligned} \tag{4.42}
$$

Similarly,

$$
\begin{aligned}
\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} f(y^\mu,\theta,\bar{\theta})\right)_{x^\mu,\theta} =& \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} f(x^\mu - i\theta\sigma^\mu\bar{\theta},\theta,\bar{\theta}) \\
=& \left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} f(y^\mu,\theta,\bar{\theta})\right)_{y^\mu,\bar{\theta}} + i(\theta\sigma^\mu)_{\dot{\alpha}} \left(\frac{\partial}{\partial y^\mu} f(y^\mu,\theta,\bar{\theta})\right)_{\theta,\bar{\theta}}.
\end{aligned} \tag{4.43}
$$

This means that we can write

$$
\begin{aligned}
\overline{D}_{\dot{\alpha}} =& -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\frac{\partial}{\partial x^\mu} = \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right)_{y^\mu,\theta} \\
D_\alpha =& \frac{\partial}{\partial\theta^{\dot{\alpha}}} - i(\sigma^\mu\bar{\theta})_\alpha\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar{\theta})_\alpha\frac{\partial}{\partial y^\mu}.
\end{aligned} \tag{4.44}
$$

Hence the constraint $\overline{D}_\alpha\Phi = 0$ implies that $\Phi(y,\theta,\bar{\theta})$ has no $\bar{\theta}$ components, so

$$
\Phi(y,\theta,\bar{\theta}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{4.45}
$$

Expanding out in y gives exactly the expression we found above. For the antichiral superfield we can use $\overline{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and swap the roles of $\overline{D}_{\dot{\alpha}}, D_\alpha$.

4.1.4 Real superfields

Having constructed the superfield for an off-shell chiral multiplet, we can now look at doing the same for the vector superfield. We start again from the general superfield

$$S(x,\theta,\bar{\theta}) = a + \theta\xi + \bar{\theta}\bar{\chi} + \theta\theta b + \bar{\theta}\bar{\theta}c + \theta\sigma^\mu\bar{\theta}v_\mu + \bar{\theta}\bar{\theta}\theta\eta + \theta\theta\bar{\theta}\bar{\zeta} + \theta\theta\bar{\theta}\bar{\theta}d,$$

and notice that now we want to keep the vector; but in the above it is a complex vector. We also want to remove three of the four scalars and fermions (we need to keep one of the scalars which shall become another auxiliary). To get a real vector, we can impose that the whole superfield be real; we define it to be a "vector superfield" with $V = V^*$:

$$a = a^*, \qquad \bar{\chi} = \bar{\xi}, \qquad c = b^*, \qquad v_\mu = v_\mu^*, \qquad \bar{\zeta} = \bar{\eta}, \qquad d = d^*. \tag{4.46}$$

It is also convenient and traditional to define:

$$\eta_\alpha = \lambda_\alpha - \frac{i}{2}(\sigma^\mu\partial_\mu\bar{\xi})_\alpha, \qquad v_\mu = A_\mu, \qquad d = \frac{1}{2}D - \frac{1}{4}\partial_\mu\partial^\mu a. \tag{4.47}$$

The component expansion of the vector superfield is then

$$\begin{aligned} V(x,\theta,\theta^\dagger) &= a + \theta\xi + \bar{\theta}\bar{\xi} + \theta\theta b + \bar{\theta}\bar{\theta}b^* + \theta\sigma^\mu\bar{\theta}A_\mu + \bar{\theta}\bar{\theta}\theta(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\xi}) \\ &\quad + \theta\theta\bar{\theta}(\bar{\lambda} - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\xi) + \theta\theta\bar{\theta}\bar{\theta}(\frac{1}{2}D - \frac{1}{4}\partial_\mu\partial^\mu a). \end{aligned} \tag{4.48}$$

If we want A_μ to have canonical mass dimension 1, then since θ has dimension $-1/2$ we need V to be dimensionless. So then the dimensions of a, b, ξ, λ, D are $0, 1, 1/2, 3/2, 2$. This means that only λ can be the fermion of our vector multiplet, and D is an auxiliary field that can have a lagrangian

$$\mathcal{L} \supset \frac{1}{2}D^2.$$

Suppose we have a $U(1)$ gauge field; then we can write the free lagrangian

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2 \tag{4.49}$$

So what to do with the remaining terms? If we had a *massive* vector field, then we would need one real scalar and an extra fermion; we could potentially write a mass term

$$\int d^4\theta m^2 V^2 \supset m^2\left[\frac{1}{2}A_\mu A^\mu - (\lambda\xi) - (\bar{\lambda}\bar{\xi}) + 2|b|^2\right], \tag{4.50}$$

where we used the identity

$$(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}\eta^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}. \tag{4.51}$$

While a could clearly play the role of the real scalar, and ξ can then give a mass to the gaugino λ, we expect that b should be "supergauged" away; and indeed a and ξ should be removed for a massless gauge field. On the other hand, suppose that we broke the gauge symmetry without breaking supersymmetry, by giving an expectation value to some scalar. We should equivalently be able to write mass terms

$$\mathcal{L} \supset \frac{1}{2}(mA_\mu - \partial_\mu \omega)^2 - m(\lambda\psi) - m(\bar{\lambda}\bar{\psi}) \tag{4.52}$$

for some scalar ω and fermion ψ. These “matter” fields should sit inside a chiral multiplet, and since only the real scalar appears, we conclude that we can make the “supergauge transformation”:

$$V \quad \to \quad V + \frac{i}{2}(\Omega^* - \Omega), \tag{4.53}$$

where Ω is a chiral superfield gauge transformation parameter, $\Omega = \phi + \sqrt{2}\theta\psi + \theta\theta F + \ldots$. In components, this transformation is

$$a \quad \to \quad a + \frac{i}{2}(\phi^* - \phi), \tag{4.54}$$

$$\xi_\alpha \quad \to \quad \xi_\alpha - \frac{i}{\sqrt{2}}\psi_\alpha, \tag{4.55}$$

$$b \quad \to \quad b - \frac{i}{2}F, \tag{4.56}$$

$$A_\mu \quad \to \quad A_\mu - \partial_\mu \frac{1}{2}(\phi + \phi^*), \tag{4.57}$$

$$\lambda_\alpha \quad \to \quad \lambda_\alpha, \tag{4.58}$$

$$D \quad \to \quad D. \tag{4.59}$$

In particular, supergauge transformations can eliminate the auxiliary fields a, ξ_α, and b completely. A superspace Lagrangian for a vector superfield must be invariant under the supergauge transformation eq. (4.53) in the Abelian case, or a suitable generalization given below for the non-Abelian case. After making a supergauge transformation to eliminate a, ξ, and b, the vector superfield is said to be in Wess-Zumino gauge, and is simply given by

$$V_{\text{WZ gauge}} \;=\; \theta\sigma^\mu\bar{\theta}A_\mu + \bar{\theta}\bar{\theta}\theta\lambda + \theta\theta\bar{\theta}\bar{\lambda} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D. \tag{4.60}$$

The restriction of the vector superfield to Wess-Zumino gauge is not consistent with the linear superspace version of supersymmetry transformations. However, a supergauge transformation can always restore $\delta_\epsilon(V_{\text{WZ gauge}})$ to Wess-Zumino gauge. Adopting Wess-Zumino gauge is equivalent to partially fixing the supergauge, while still maintaining the full freedom to do ordinary gauge transformations.

Because of its simplicity, the Wess-Zumino gauge is extremely useful. In particular, we note that

$$V^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}A^\mu A_\mu, \qquad V^{n>2} = 0. \tag{4.61}$$

These will be very helpful when we consider non-abelian theories, but also gauge interactions even in the abelian case.

4.1.5 Other types of superfields

Before passing on to the construction of Lagrangians, we remark that we have not exhausted the possible sets of superfields – just those that are the most important ones for phenomenology, because they describe chiral and vector multiplets. However, there are other types of superfield that are of interest.

- Linear superfields $\mathbf{L}$; we impose $D^2\mathbf{L} = 0$ *and* $L = \overline{L}$ (or equivalently $D^2\mathbf{L} = \overline{D}^2\mathbf{L} = 0$). We can write the expansion as

$$\mathbf{L} = L + \sqrt{2}\theta l + \sqrt{2}\bar{\theta}\bar{l} - i\theta\sigma^\mu\bar{\theta}j_\mu - \frac{i}{\sqrt{2}}\theta\theta(\bar{\theta}\bar{\sigma}^\mu\partial_\mu l) - \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}(\theta\sigma^\mu\partial_\mu\bar{l}) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu L.$$

  It therefore contains a real component L, a current satisfying $\partial^\mu j_\mu = 0$ and a fermion l. The reality condition means that it cannot carry a complex representation of a gauge group, but the presence of a current means we can use it as a gauge current; it is also useful in Supergravity where we write $j_\mu = \partial_\mu a$ for an axion field a.

- We can also give multiplets Lorentz indices and have e.g. fermionic multiplets. The best example is the Ferrara-Zumino multiplet $\mathcal{J}_{\alpha\dot{\alpha}}$ which is a real multiplet satisfying in addition $\overline{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha X$ for X a chiral multiplet. This multiplet contains both the stress-energy tensor $T_{\mu\nu}$ and the supercurrent $S_{\mu\alpha}$.

4.2 Lagrangians for chiral superfields from superspace

We already derived the general lagrangian for chiral superfields, which we found was given by the kinetic terms and a superpotential. The interaction terms were all holomorphic, in that they consisted only of scalars ϕ_i, fermions ψ_i and auxiliaries F_i not mixing with their hermitian conjugates. Moreover, since we are interested in renormalisable terms, the interactions of the chiral superfields did not have any derivatives (we will treat gauge interactions in the next subsection).

We ought to recover this structure when we enlarge spacetime to superspace. Naively we expect that the lagrangian should be given as the integration of some function, say K, over all of superspace:

$$S_K = \int d^4x \int d^2\theta d^2\bar{\theta}\ K \tag{4.62}$$

Here K must be a real function of the chiral superfields. The first question we should ask is, is this really invariant under SUSY? We can show that it is, by considering K as a general real superfield and writing

$$\begin{aligned} K =& ... + \theta\theta\big[K\big]_{\theta\theta} + \bar{\theta}\bar{\theta}\big[K\big]_{\bar{\theta}\bar{\theta}} + ... + \theta\theta\bar{\theta}\bar{\theta}\big[K\big]_{\theta\theta\bar{\theta}\bar{\theta}} \\ \to S_K =& \int d^4x\big[K\big]_{\theta\theta\bar{\theta}\bar{\theta}}. \end{aligned} \tag{4.63}$$

Now, because $\big[K\big]_{\theta\theta\bar{\theta}\bar{\theta}}$ is the top component, when we act with the SUSY generators upon K we find:

$$\delta\big[K\big]_{\theta\theta\bar{\theta}\bar{\theta}} = -(\xi\sigma^\mu\bar{\theta})\partial_\mu\big[K\big]_{\theta\theta\bar{\theta}} - (\bar{\xi}\overline{\sigma}^\mu\theta)\partial_\mu\big[K\big]_{\bar{\theta}\bar{\theta}\theta} \tag{4.64}$$

which is a total derivative.

Next, we realise that since $\theta, \bar{\theta}$ must have mass dimension of $-1/2$ and $d\theta$ has dimension $1/2$, this means that $[K]$ has dimension 2, and so if we want to stick to only renormalisable interactions this means we can only have one or two chiral superfields.

However, notice that if Φ_i, Φ_j are both chiral superfields, then

$$\overline{D}_{\dot{\alpha}}(\Phi_i \Phi_j) = 0 \rightarrow \Phi_i \Phi_j \text{ is a chiral superfield.} \tag{4.65}$$

In particular, any product of chiral superfields must have the form

$$\prod_i \Phi_i = \prod_i \phi_i + \sqrt{2}\theta \big[f_2^j(\phi_i)\psi_j\big] + \theta\theta f_3(\phi_i, F_i) + ... - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu \partial^\mu \left(\prod_i \phi_i \right). \tag{4.66}$$

This means that if we were to construct K out of only chiral superfields, we would get

$$\int d^4x \int d^4\theta \prod \Phi_i = -\frac{1}{4} \int d^4x \partial_\mu \partial^\mu \left(\prod_i \phi_i \right) = 0. \tag{4.67}$$

Hence the function K should be built from both chiral and antichiral superfields – so this cannot give us our superpotential. This means that for a renormalisable lagrangian the only term we can write is

$$\Phi^{*i}\Phi_j = ... + \theta\theta\bar{\theta}\bar{\theta}\Big[F^{*i}F_j + \frac{1}{2}\partial^\mu \phi^{*i}\partial_\mu \phi_j - \frac{1}{4}\phi^{*i}\partial^\mu \partial_\mu \phi_j - \frac{1}{4}\phi_j \partial^\mu \partial_\mu \phi^{*i} + \frac{i}{2}\bar{\psi}^i \bar{\sigma}^\mu \partial_\mu \psi_j + \frac{i}{2}\psi_j \sigma^\mu \partial_\mu \bar{\psi}^i\Big]. \tag{4.68}$$

This is exactly what we need for our free lagrangian! We can therefore write

$$K = K_i^j \Phi^{*i}\Phi_j. \tag{4.69}$$

If we work in a diagonal basis then we write $K = \Phi^{*i}\Phi_i$ and identify terms up to integration by parts we find

$$S_K = \int d^4x \int d^4\theta \Phi^{*i}\Phi_i = F^{*i}F_i + \partial^\mu \phi^{*i}\partial_\mu \phi_i + i\bar{\psi}^i \bar{\sigma}^\mu \partial_\mu \psi_i. \tag{4.70}$$

We still need to find our interaction terms. Recall that we found that our interactions involved only holomorphic terms and antiholomorphic terms, that is there were no terms mixing terms from the chiral superfields with those from the complex conjugates. Note that our superpotential was a function of mass dimension 3. So let us now consider integrating over *half* of superspace, and including the hermitian conjugate:

$$S_W \equiv \int d^4x \left[\int d^2\theta \; W(\Phi_i) \right]_{\bar{\theta}=0} + h.c. = \int d^4x \big[W\big]_{\theta\theta} + h.c. \tag{4.71}$$

Note that since we need the result to be independent of the Grassman parameters we need to set $\bar{\theta} = 0$ in the first part. Often in references or textbooks this is not emphasised – and indeed in the following we shall be sloppy and often drop it; other references use a variant of $\big[W\big]_{\theta\theta}$, such as $\big[W\big]_F$.

Now if W is a function of *only* chiral superfields, we can apply the same reasoning as before to show that it is invariant under SUSY, because $\delta[W]_{\theta\theta}$ must be a total derivative:

$$\delta[W]_{\theta\theta} = - i\bar{\xi}_{\dot{\alpha}}[W]_{\theta\theta\bar{\theta}_{\dot{\alpha}}} + \text{total derivative} \tag{4.72}$$

but for a chiral superfield $[W]_{\theta\theta\bar{\theta}_{\dot{\alpha}}} = \frac{i}{2}\overline{\sigma}_{\dot{\alpha}\alpha}\partial_\mu[W]_{\theta_\alpha}$ which is also a total derivative.

Furthermore, if we write our superpotential as a function of superfields now then

$$W(\Phi_i) \equiv W^0 + W^i\Phi_i + \frac{1}{2}W^{ij}\Phi_i\Phi_j + \frac{1}{6}W^{ijk}\Phi_i\Phi_j\Phi_k \tag{4.73}$$

then, using

$$\begin{aligned} \Phi_i =& \phi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i \\ \theta\psi_i\theta\psi_j =& - \theta^\alpha\theta^\beta\psi_{i,\alpha}\psi_{j,\beta} = \frac{1}{2}\epsilon^{\alpha\beta}\theta\theta\psi_{i,\alpha}\psi_{j,\beta} = -\frac{1}{2}\theta\theta(\psi_i\psi_j), \end{aligned} \tag{4.74}$$

we find

$$\big[W\big]_{\theta\theta} = \frac{\partial W}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j, \tag{4.75}$$

which was exactly the set of interaction terms we wanted!

In summary, a renormalisable theory of chiral superfields is specified entirely by

$$S = S_K + S_W = \int d^4x \int d^4\theta K_i^j\overline{\Phi}^i\Phi_j + \Big[\int d^2\theta W(\Phi_i) + h.c.\Big]. \tag{4.76}$$

4.3 Supergauge interactions

Suppose that we now want to gauge some global symmetry of the Φ_i, Φ^{*i} fields, which can now be either abelian or non-abelian. For the scalars and fermions, we must have transformations $\phi \to e^{i\omega^a T^a(R)}\phi, \psi \to e^{i\omega^a T^a(R)}\psi$ where $T^a(R)$ are the generator matrices of the representation R. This must translate to a superfield transformation of $\Phi \to e^{i\Omega^a T^a(R)}\Phi$; let us define $\Omega \equiv \Omega^a T^a$ so then e.g. $\overline{\Phi} \to \overline{\Phi}e^{-i\overline{\Omega}}$ where we now require $\overline{\Omega} \equiv \Omega^\dagger$ in gauge space. But when we consider that our gauge field transforms (for a $U(1)$) as $V \to V + \frac{i}{2}(\overline{\Omega} - \Omega)$ we see that our supergauge rotation has become complex and thus our kinetic term $\overline{\Phi}\Phi$ is no longer supergauge invariant. We also must find a way to obtain the covariant derivative for matter fields of

$$[\nabla_\mu(\phi)]_i = \partial_\mu\phi_i + iA_\mu^a(T^a)_i{}^j\phi_j. \tag{4.77}$$

However, for a $U(1)$ transformation we see that, defining $\tilde{V} \equiv 2VQ$, we find that

$$S_{K,G} = \int d^4x \int d^4\theta\overline{\Phi}e^{\tilde{V}}\Phi \tag{4.78}$$

is supergauge invariant. Notice that we can write this because V is dimensionless. So when we write a non-abelian gauge transformation we define $\tilde{V} \equiv 2V^a T^a(R)$ and we require that the supergauge transformation should become

$$e^{\tilde{V}} \to e^{i\overline{\Omega}} e^{\tilde{V}} e^{-i\Omega} \tag{4.79}$$

(because $e^{-i\overline{\Omega}} e^{i\overline{\Omega}} = 1$ etc). Recalling that $V^{n>2} = 0$ in Wess-Zumino gauge, we can then solve for an infinitesimal transformation to be first order in Ω, so that using the Baker-Campell-Hausdorff formula we find

$$\begin{aligned}
e^{i\overline{\Omega}} e^{\tilde{V}} e^{-i\Omega} =& \exp(i\overline{\Omega}) \exp\left(\tilde{V} - i\Omega + \frac{1}{2}[\tilde{V}, -i\Omega] + \frac{1}{2}[\tilde{V}, [\tilde{V}, -i\Omega]]\right) + \mathcal{O}(\Omega^2) \\
=& \exp\left(\tilde{V} + i(\overline{\Omega} - \Omega) - \frac{1}{2}[\tilde{V}, i\Omega] + \frac{1}{12}[\tilde{V}, [\tilde{V}, -i\Omega]] + \frac{1}{2}[i\overline{\Omega}, \tilde{V}] + \frac{1}{12}[\tilde{V}, [\tilde{V}, i\overline{\Omega}]]\right) + \mathcal{O}(\Omega^2) \\
\delta\tilde{V} =& i(\overline{\Omega} - \Omega) - \frac{i}{2}[\tilde{V}, \Omega + \overline{\Omega}] - \frac{i}{12}[\tilde{V}, [\tilde{V}, \Omega - \overline{\Omega}]].
\end{aligned} \tag{4.80}$$

Most pertinently this gives exactly

$$\begin{aligned}
\delta A_\mu =& -\partial_\mu \frac{1}{2}(\omega + \overline{\omega}) - \frac{i}{2}[A_\mu, (\omega + \overline{\omega})], \qquad \delta\lambda = -\frac{i}{2}[\lambda, \omega + \overline{\omega}] \\
\delta(D - \frac{1}{2}\partial_\mu\partial^\mu a) =& -\frac{i}{2}[D, \omega + \overline{\omega}] - \frac{i}{6}\eta^{\mu\nu}[A_\mu, [A_\nu, \omega - \overline{\omega}]]
\end{aligned} \tag{4.81}$$

which is what we need when we define $A_\mu \equiv A_\mu^a T^a$ etc. Of particular interest is the final term for the D-term transformation, which shows that a transformation with an imaginary component of ω takes us away from the Wess-Zumino gauge.

Finally we can write down the gauge interactions by expanding out the terms and integrating over superspace:

$$\begin{aligned}
S_{K,G} =& \overline{F}^i F_i + \nabla_\mu \overline{\phi}^i \nabla^\mu \phi_i + i\overline{\psi}^i \overline{\sigma}^\mu \nabla_\mu \psi_i - \sqrt{2}(\overline{\phi} T^a \psi)\lambda^a - \sqrt{2}\overline{\lambda}(\overline{\psi} T^a \phi) \\
&+ (\overline{\phi} T^a \phi) D^a.
\end{aligned} \tag{4.82}$$

The final term is very important phenomenologically. When we combine it with the auxiliary field action that we need to make the gauge kinetic terms supersymmetrically invariant off-shell (that we shall also find in the following section):

$$\mathcal{L}_{\mathrm{aux}} \supset \frac{1}{2g^2}(D^a)^2 \tag{4.83}$$

and integrate out the auxiliary fields we find the "D-term potential"

$$V_D = \frac{g^2}{2}\left(\sum_i \overline{\phi}^i T^a \phi_i\right)^2. \tag{4.84}$$

Hence the scalar potential of a supersymmetric theory consists of the combination of the F-term and D-term potentials.

4.4 Lagrangians for vector superfields

4.4.1 $U(1)$

Since the vector superfield is dimensionless, we cannot follow the same argument as for chiral superfields. Moreover, the kinetic term for a gauge field is given by

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}. \tag{4.85}$$

However, the curvature is not present in the vector superfield – we need to add two derivatives. But we cannot write the kinetic term involving just the vector superfield and spacetime derivatives, because then we would never find the gaugino kinetic term. So instead we need to consider superderivatives. If we write the kinetic term as an integral over all of superspace, then it should involve two vector superfields and four superspace derivatives; a reasonable guess is then

$$\int d^4\theta \left(\overline{D}_{\dot{\alpha}} D^{\alpha} V\right) \left(\overline{D}^{\dot{\alpha}} D_{\alpha} V\right). \tag{4.86}$$

However, we showed that $\int d^4\theta S = -\frac{1}{4} \int d^2\theta \overline{D}^2 S$ and, because $\overline{D}^3 = 0$ this is equivalent to

$$\int d^2\theta \left(\overline{D}^2 D^{\alpha} V\right) \left(\overline{D}^2 D_{\alpha} V\right). \tag{4.87}$$

This is the square of a (fermionic) chiral superfield! It is conventional to include a factor of $-\frac{1}{4}$, so we define:

$$\mathcal{W}_\alpha \equiv -\frac{1}{4} \overline{D}\overline{D} D_\alpha V, \qquad \overline{\mathcal{W}}_{\dot{\alpha}} \equiv -\frac{1}{4} D D \overline{D}_{\dot{\alpha}} V. \tag{4.88}$$

Note that they have mass dimension 3/2. Interestingly they are *supergauge* invariant, which we can see by

$$\begin{aligned} \mathcal{W}_\alpha \to -\frac{1}{4}\overline{D}\overline{D} D_\alpha \big[V + \frac{i}{2}(\Omega^* - \Omega)\big] &= \mathcal{W}_\alpha + \frac{i}{8}\overline{D}\overline{D} D_\alpha \Omega \\ &= \mathcal{W}_\alpha - \frac{i}{8}\overline{D}^{\dot{\beta}}\{\overline{D}_{\dot{\beta}}, D_\alpha\}\Omega \\ &= \mathcal{W}_\alpha + \frac{1}{4}\sigma^{\mu}_{\alpha\dot{\beta}}\partial_\mu \overline{D}^{\dot{\beta}}\Omega \\ &= \mathcal{W}_\alpha. \end{aligned} \tag{4.89}$$

This means that we can work out its component expression in the Wess-Zumino gauge without loss of generality. It is also consistent with containing the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which is gauge invariant for a $U(1)$ field.

To see how it looks in components, we write V in Wess-Zumino gauge as a function of $y^\mu, \theta, \bar{\theta}$ so that

$$V(y^\mu, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta} A_\mu(y) + \bar{\theta}\bar{\theta}\theta\lambda(y) + \theta\theta\bar{\theta}\bar{\lambda}(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(y) + i\partial_\mu A^\mu(y)\right]. \tag{4.90}$$

Then we have

$$
\begin{aligned}
\mathcal{W}_\alpha =& \frac{1}{4}\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot\alpha}}\left(\frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial y^\mu}\right)V \\
=& \frac{\partial}{\partial\bar\theta^2}\left[\frac{\partial}{\partial\theta^\alpha}\left(\bar\theta\bar\theta\theta\lambda(y) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y) + i\partial_\mu A^\mu(y)\right]\right) - 2i(\sigma^\nu\bar\theta)_\alpha\frac{\partial}{\partial y^\nu}\left(\theta\sigma^\mu\bar\theta A_\mu(y) + \theta\theta\bar\theta\bar\lambda(y)\right)\right] \\
=& \lambda_\alpha + \theta_\alpha\left[D(y) + i\partial_\mu A^\mu(y)\right] + i\theta\theta(\sigma^\nu\partial_\nu\bar\lambda)_\alpha + i\theta^\beta\sigma^\mu_{\beta\dot\beta}\sigma^\nu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\partial_\nu A_\mu \\
=& \lambda_\alpha + \theta_\alpha D(y) + i\big[\eta^{\mu\nu}\delta^\beta_\alpha - \frac{1}{2}(\sigma^\nu\overline{\sigma}^\mu)_\alpha^{\ \beta}\big]\theta_\beta\partial_\nu A_\mu \\
=& \lambda_\alpha + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\overline{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + i\theta\theta(\sigma^\mu\partial_\mu\bar\lambda)_\alpha = \lambda_\alpha + \theta_\alpha D - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + i\theta\theta(\sigma^\mu\partial_\mu\bar\lambda)_\alpha,
\end{aligned}
\tag{4.91}
$$

where we used and similarly

$$
\overline{\mathcal{W}}_{\dot\alpha} = \bar\lambda^{\dot\alpha} + \bar\theta^{\dot\alpha}D + \frac{i}{2}(\overline{\sigma}^\mu\sigma^\nu\bar\theta)^{\dot\alpha}F_{\mu\nu} + i\bar\theta\bar\theta(\overline{\sigma}^\mu\partial_\mu\lambda)^{\dot\alpha}, \tag{4.92}
$$

where these are functions of y^μ and $y^{*\mu}$ respectively; if we are looking at the lagrangian term integrated over half of superspace then we can then simply replace $y^\mu \to x^\mu, y^{*\mu} \to x^\mu$ because the differences with $y^\mu, y^{*\mu}$ contain respectively $\bar\theta$ and θ which vanish.

Now we can easily write down the kinetic terms from the lagrangian, although we need:

$$
\begin{aligned}
\epsilon^{\alpha\gamma}(\sigma^{\mu\nu})_\gamma^{\ \beta}\theta_\beta(\sigma^{\rho\kappa})_\alpha^{\ \delta}\theta_\delta =& \frac{1}{2}\theta\theta\,\mathrm{tr}\big(\sigma^2\sigma^{\mu\nu}\sigma^2(\sigma^{\rho\kappa})^T\big) \\
=& -\frac{1}{2}\theta\theta\,\mathrm{tr}\big(\sigma^{\mu\nu}\sigma^{\rho\kappa}\big) \\
=& \frac{1}{4}\theta\theta\left(-(\eta^{\mu\rho}\eta^{\nu\kappa} - \eta^{\mu\kappa}\eta^{\nu\rho}) + i\epsilon^{\mu\nu\rho\kappa}\right)
\end{aligned}
\tag{4.93}
$$

where we define

$$
\epsilon^{0123} = 1, \qquad \epsilon_{0123} = -1, \tag{4.94}
$$

and we find

$$
[\mathcal{W}^\alpha\mathcal{W}_\alpha]_{\theta\theta} = D^2 + 2i\lambda\sigma^\mu\partial_\mu\bar\lambda - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{4.95}
$$

where now all fields on the right side are functions of x^μ. In order to obtain the correct normalisation of the gauge kinetic term we then write

$$
S_V \equiv \int d^4x\,\frac{1}{4}[\mathcal{W}^\alpha\mathcal{W}_\alpha]_{\theta\theta} + \text{c.c.} = \int d^4x\,\left[\frac{1}{2}D^2 + i\bar\lambda\overline{\sigma}^\mu\partial_\mu\lambda - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]. \tag{4.96}
$$

On the other hand, it is often useful to write the gauge coupling as a prefactor for the kinetic term, replacing the $\frac{1}{4} \to \frac{1}{4g^2}$. In a supersymmetric theory there is no reason that this parameter should not be a complex number; in fact we can define

$$
\tau \equiv \frac{1}{g^2} - i\frac{\theta}{8\pi^2} \tag{4.97}
$$

and write

$$S_V \to \int d^4x\, \frac{1}{4}[\tau \mathcal{W}^\alpha \mathcal{W}_\alpha]_{\theta\theta} + \text{c.c.} = \int d^4x \left[\frac{1}{2g^2}D^2 + \frac{i}{g^2}\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda - \frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\kappa}F_{\mu\nu}F_{\rho\kappa}\right]. \tag{4.98}$$

Thus the θ angle is automatically packaged with the gauge coupling!

Finally, before we move on, we note that there is an additional "Fayet-Iliopoulos" term that we can write in the action for a $U(1)$ gauge field only: we can put

$$\mathcal{L}_{\text{FI}} \equiv \int d^4\theta 2\xi V = \xi D. \tag{4.99}$$

This is supergauge invariant because the integral of a chiral or antichiral superfield over all of superspace vanishes; for a non-abelian gauge field (as we shall see) this is no longer true. However, it has important phenomenology for the $U(1)$ case, and modifies the D-term potential to

$$V_D = \frac{1}{2}g^2\left(\overline{\phi}^i Q\phi_i + \xi\right)^2 \tag{4.100}$$

where Q is now the charge operator for the $U(1)$.

4.4.2 Non-abelian case

For a non-abelian gauge group we will need the additional terms in the field strength and under gauge transformations ω, using $A_\mu^a = A_\mu^a T^a, \omega = \omega^a T^a(R)$:

$$\begin{aligned} F_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \\ \delta A_\mu =& -\partial_\mu \omega - i[A_\mu, \omega]. \end{aligned} \tag{4.101}$$

We shall treat A_μ, ω as gauge objects; to write the above out in components we can take generators of the gauge group in representation R to be $T^a(R)$ and then

$$[A_\mu, \omega]^a = i f^{abc} A_\mu^b \omega^c, \qquad [\nabla_\mu(X)]^a = \Big[\partial_\mu X + i[A, X]\Big]^a = \partial_\mu X^a - f^{abc}A^b X^c. \tag{4.102}$$

We shall also work in the basis where the kinetic term of the gauge field is given by

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\,\mu\nu} = -\frac{1}{2g^2}\text{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{4.103}$$

where on the right we assume the generators to be normalised to $1/2$. To change to the canonical basis without the g^2 factor in the denominator we simply replace $V \to gV$ everywhere. Now the gaugino should be in the adjoint representation of the gauge group, so it must have kinetic term

$$\mathcal{L}_{\text{gaugino}} = \frac{i}{g^2}\bar{\lambda}\bar{\sigma}^\mu \nabla_\mu \lambda, \qquad \nabla_\mu \lambda = \partial_\mu \lambda + i[A_\mu, \lambda]. \tag{4.104}$$

To obtain these from superspace, we need to find a quantity that is invariant under the supergauge transformation $e^{\tilde{V}} \to e^{i\overline{\Omega}} e^{\tilde{V}} e^{-i\Omega}$ that reduces to $-\frac{1}{4}\overline{D}^2 D_\alpha V$ for the abelian case. To do this, we can take advantage of $\overline{D}_{\dot{\alpha}}\Omega = 0$ and

$$e^{-\tilde{V}} \to e^{i\Omega} e^{-\tilde{V}} e^{-i\overline{\Omega}} \tag{4.105}$$

so $\overline{D}^2 e^{-\tilde{V}} \to e^{i\Omega}\overline{D}^2 e^{-\tilde{V}} e^{-i\overline{\Omega}}$; therefore

$$\mathcal{W}_\alpha \equiv -\frac{1}{8}\overline{D}^2\Big(e^{-\tilde{V}} D_\alpha e^{\tilde{V}}\Big), \qquad \overline{\mathcal{W}}_{\dot{\alpha}} \equiv \frac{1}{8}D^2\Big(e^{\tilde{V}}\overline{D}_{\dot{\alpha}} e^{-\tilde{V}}\Big) \tag{4.106}$$

and

$$\begin{aligned}\mathcal{W}_\alpha \to &-\frac{1}{8}e^{i\Omega}\overline{D}^2\Big(e^{-\tilde{V}} D_\alpha(e^{\tilde{V}} e^{-i\Omega})\Big)\\ =&-\frac{1}{4}e^{i\Omega}\overline{D}^2\Big(e^{-\tilde{V}} D_\alpha e^{\tilde{V}}\Big)e^{-i\Omega} - \frac{1}{4}e^{i\Omega}\overline{D}^2 D_\alpha e^{-i\Omega}\\ =&e^{i\Omega}\mathcal{W}_\alpha e^{-i\Omega}.\end{aligned} \tag{4.107}$$

Then we find

$$\mathcal{W}_\alpha = -\frac{1}{8}\overline{D}^2\Big(D_\alpha\tilde{V} - [\tilde{V}, D_\alpha\tilde{V}] + ...\Big) \tag{4.108}$$

which truncates to these terms in the Wess-Zumino gauge. Writing again in terms of the variable y and θ, we see that the $[A, A]$ term comes from

$$\begin{aligned}\mathcal{W}_\alpha \supset &-\frac{1}{2}\frac{\partial}{\partial\bar{\theta}^2}\Big([\theta\sigma^\mu\bar{\theta}2A_\mu, (\sigma^\nu\bar{\theta})_\alpha 2A_\nu]\Big)\\ =&-(\sigma^\nu\overline{\sigma}^\mu\theta)_\alpha[A_\mu, A_\nu]\end{aligned} \tag{4.109}$$

which, when we write $W_\alpha = 2W^a_\alpha T^a$ we have

$$W^a_\alpha \supset \frac{i}{2}(\sigma^\mu\overline{\sigma}^\nu\theta)_\alpha f^{abc}A^b_\mu A^c_\nu, \tag{4.110}$$

and thus we have

$$W^a_\alpha = \lambda^a_\alpha + \theta_\alpha D^a - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + i\theta\theta(\sigma^\mu\nabla_\mu\overline{\lambda}^a)_\alpha, \tag{4.111}$$

$$\begin{aligned}S_V =& \int d^4x \frac{1}{4}[\tau\mathcal{W}^{a,\alpha}\mathcal{W}_{a,\alpha}] + c.c.\\ =& \int d^4x \left[\frac{1}{2g^2}D^a D^a + \frac{i}{g^2}\bar{\lambda}\overline{\sigma}^\mu\partial_\mu\lambda - \frac{1}{4g^2}F^{a,\mu\nu}F^a_{\mu\nu} + \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}F^a_{\rho\kappa}\right].\end{aligned} \tag{4.112}$$

4.5 R-symmetry revisited

R-symmetry has a particularly simple interpretation in superspace. Recall that

$$[R, Q] = -Q, \quad [R, \overline{Q}] = \overline{Q} \qquad \longrightarrow Q_\alpha \overset{R}{\to} e^{-i\alpha} Q_\alpha, \quad \overline{Q}_{\dot\alpha} \overset{R}{\to} e^{i\alpha} \overline{Q}_{\dot\alpha}.$$

We can then apply this rotation to the supercoordinates $\theta_\alpha, \bar\theta_{\dot\alpha}$; since this is an internal rotation of the supercharges the spacetime coordinates must be unaffected and, using the definition of the supercharges in superspace we see

$$\theta_\alpha \overset{R}{\to} e^{i\alpha}\theta_\alpha, \quad \bar\theta_{\dot\alpha} \overset{R}{\to} e^{-i\alpha}\bar\theta_{\dot\alpha}. \tag{4.113}$$

Similarly we need that the differentials or derivatives transform oppositely and so

$$\begin{aligned} \frac{\partial}{\partial\theta_\alpha} &\overset{R}{\to} e^{-i\alpha}\frac{\partial}{\partial\theta_\alpha} \\ d^2\theta &\overset{R}{\to} e^{-2i\alpha} d^2\theta, \qquad d^2\bar\theta \overset{R}{\to} e^{2i\alpha} d^2\bar\theta. \end{aligned} \tag{4.114}$$

The second line is particularly important, because it means that *if R-symmetry is preserved, then the kinetic terms* $\int d^4\theta K_i^j \overline{\Phi}^i \Phi_j$ *are invariant, but the superpotential* W *must transform with R-charge 2*! In particular, this means that W_α has R-charge 1 and so does its lowest component, the gaugino λ_α. Note also that different fields in the supermultiplet have different R-charges; e.g. $\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F + ...$ implies that the R-charges of (ϕ, ψ, F) are $(r, r-1, r-2)$.

References

[1] I. Antoniadis, H. Partouche, and T. R. Taylor, *Spontaneous breaking of N=2 global supersymmetry*. Phys. Lett. **B372** (1996) 83–87, `arXiv:hep-th/9512006 [hep-th]`.