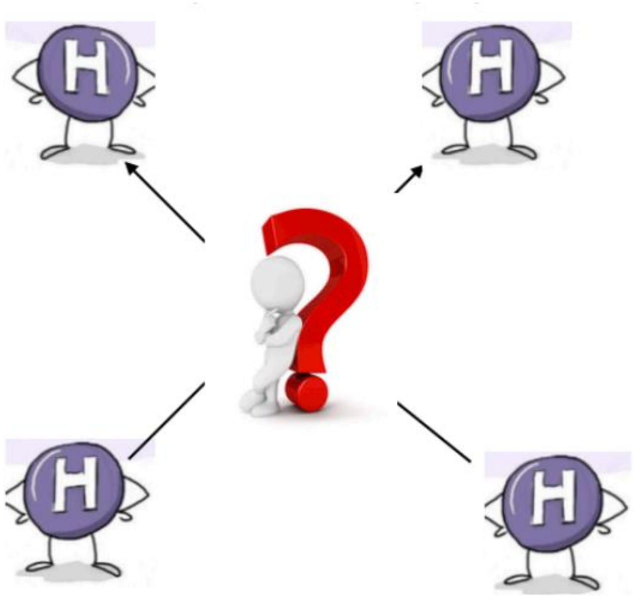
Extremal Higgs couplings



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CERN-Annecy-Geneva-EPFL (CAGE) Meeting

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Motivation

- **Analyticity ($\mathcal{A}$)**, **Unitarity ($\mathcal{U}$)**, **Crossing Symmetry ($\mathcal{CS}$)** of the $2 \to 2$ scattering amplitude

  are very powerful general principles to constrain **low energy physics**.

- Central object: $2 \to 2$ interacting scattering amplitude, no Lagrangians, no perturbation theory etc.

  Input: Particle spectrum of the theory. $\to$ *Output:* Bounds on low energy coefficients.

- I will first try to convince you how the machinery works on a toy model.

  Then we move to a more realistic model, a step towards Higgs sector in SM as an EFT.

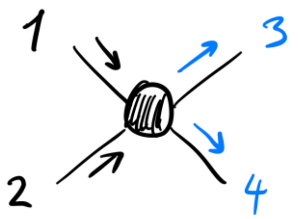
Motivation

Toy model: Massive real scalar ϕ

with Z_2 symmetry (no cubic vertex)

- Interacting part of the amplitude

$$\mathcal{S}_{2\to2} \equiv \langle\, P_3\, P_4, out \mid P_1\, P_2, in\,\rangle \qquad \mathcal{S}_{2\to2} = \mathrm{Id}_{2\to2} + i\,\mathcal{M}_{2\to2}$$



- Low energy expansion around $\bar{x} \equiv x - 4m^2/3$ to preserve $\mathcal{CS}$

$$M(\bar{s},\bar{t},\bar{u}) = c_0 + c_2\,(\bar{s}^2 + \bar{t}^2 + \bar{u}^2) + c_3\,\bar{s}\bar{t}\bar{u}\ ...$$

Notice that there is no dim-6 coefficient c_1 due to $(*)$.

Mandelstam variables:

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

$$s + t + u = 4m^2$$
$$\bar{s} + \bar{t} + \bar{u} = 0 \quad *$$

- **Question :** Can coefficients $\{c_0, c_2, ...\}$ attain any value?

Answer : Let us consider few examples …

Motivation - Example I

- **Unitarity:** $\mathcal{S}^\dagger\mathcal{S} = \mathrm{Id}$ implies

$$2\,\mathrm{Im}\, M_{2\to2}(s,t) = \sum_{n\geq2}\int d\mathrm{LIPS}_{2\to n}\, |M_{2\to n}|^2$$
$$\geq \int d\mathrm{LIPS}_2 |M_{2\to2}|^2$$
$$\geq 0$$

"optical theorem"

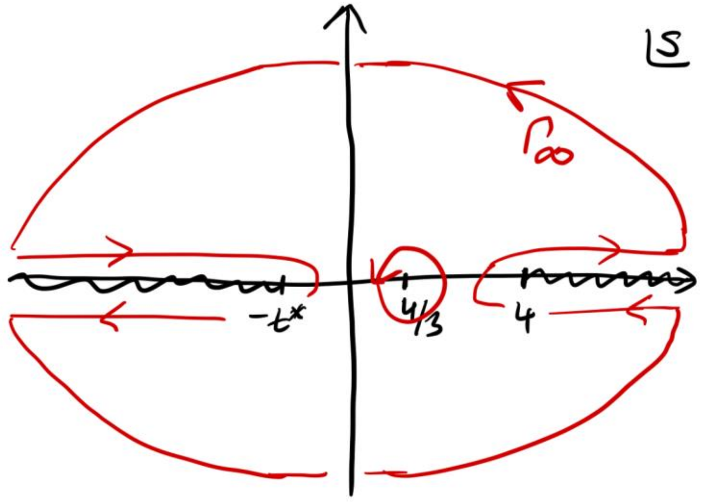
$d\mathrm{LIPS}_n$: Lorentz invariant n-particle phase space

- **Analyticity:** Cauchy's theorem $c_2 = \frac{1}{2\pi i}\oint dz \frac{M(z, 4m^2/3)}{(z-4m^2/3)^3}$ and blow up the contour

$$c_2 = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{dz}{(z-4m^2/3)^3} \underbrace{\mathrm{Im}\, M(z, 4m^2/3)}_{\text{positive}} \geq 0$$

$\Rightarrow$ Positivity bounds on some $\{c_i\}$ & ratios of $\{c_i\}$.

[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06]
….
[Bellazzini, Elias Miró, Rattazzi, Riembau, Riva '20]
[Caron-Huot, Van Duong '20]
[Tolley, Wang, Zhou '20]
…

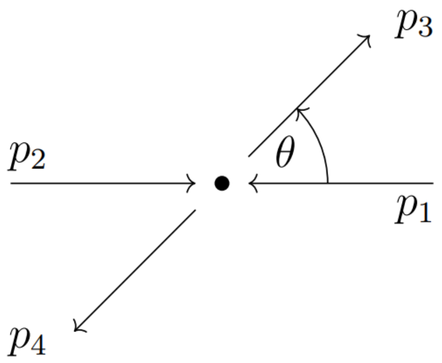


Motivation - Example II

- Partial wave expansion:

$$M(s,t) = \sum_{\ell=0}^{\infty} 16\pi\,(2\ell+1)\, f_\ell(s)\, P_\ell(\cos\theta)$$

$$\mathrm{Im}\, M(s,t) = \sum_{\ell=0}^{\infty} 16\pi\,(2\ell+1)\, \mathrm{Im} f_\ell(s)\, P_\ell(\cos\theta)$$



- Partial wave unitarity circle – for physical $s \geq 4m^2$:

$$2\,\mathrm{Im} f_\ell\,(s) \geq \rho^2(s) \cdot |f_\ell(s)|^2 \qquad \Rightarrow \qquad 2/\rho^2 \geq \mathrm{Im} f_\ell(s) \geq 0$$

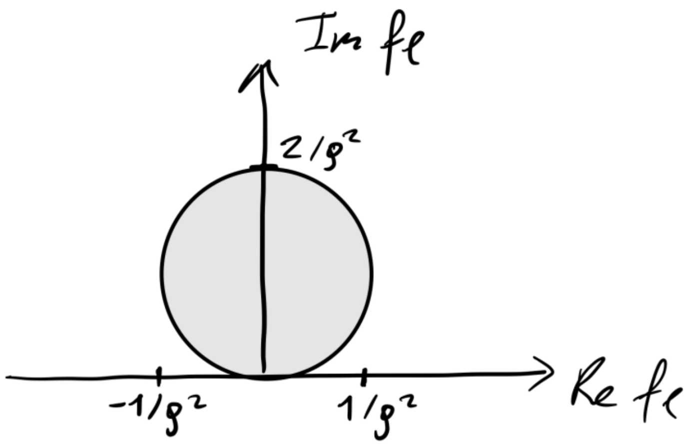
$\rho^2(s) \equiv \sqrt{s-4m^2}/\sqrt{s}$
two-body phase space factor

- Upper bound on $c_2 \propto \int dz\, \Sigma_\ell\, \frac{\mathrm{Im} f_\ell(z)}{(z-4m^2/3)^3} \leq \int dz\, \Sigma_\ell\, \frac{2/\rho^2(z)}{(z-4m^2/3)^3}$

+ crossing symmetry $M(s,t) = M(t,s) = M(s, 4m^2 - s - t)$,

one can show that $0 \leq c_2\, \Lambda^2 \leq O(1) \cdot (4\pi)^2$ in an EFT approximation.

[Caron-Huot, Van Duong ’20]



Motivation - Example III

- What about c_0? It is *non-dispersive*, i.e.

  Blowing up $M(s) = \frac{1}{2\pi i}\oint dz \frac{M(z)}{(z-s)}$ around $z = s$ gives $M(s) = c_\infty + \frac{1}{\pi}\int_{\text{cuts}} dz \frac{\text{Im}\, M(z)}{(z-s)}$

  $\Rightarrow$ $c_0 = M(s = 4m^2/3)$ gets contribution from both $\{\,\text{Im}M(s), c_\infty\}$

- Alternative way to write – choosing a different subtraction constant.

  Blowing up $M(s) = \frac{(s-s_0)^2}{2\pi i}\oint \frac{dz\, M(z)}{(z-s)(z-s_0)^2}$ gives $\frac{M(s) - M(s_0)}{(s-s_0)^2} = \frac{1}{\pi}\int_{\text{cuts}} dz \frac{\text{Im}\, M(z)}{(z-s)(z-s_0)^2}$

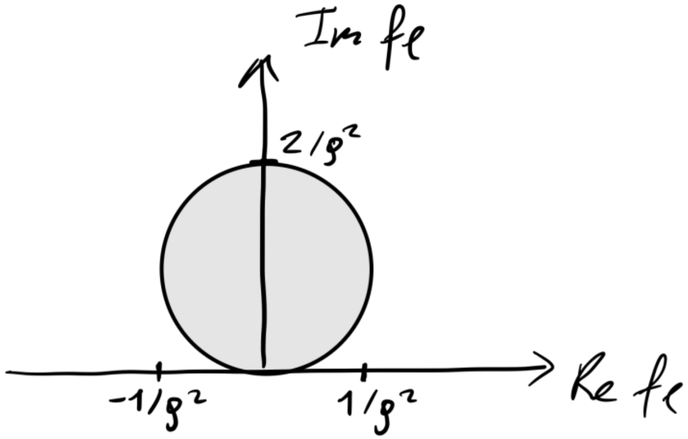
  $\Rightarrow$ We traded c_∞ for another unknown $M(s_0)$.

- Projecting onto spin-zero (Roy equation)

$$\frac{c_0}{16\pi} = \underbrace{\text{Re}\, f_0(s)} - \frac{1}{\pi}\, p.v. \int_{4m^2}^{\infty} dz\, \Sigma_{\ell=0}^{\infty} \ker_{0,\ell}(s,z)\, \underbrace{\text{Im}\, f_\ell(z)}$$

bounded by unitarity circle!

bounded by unitarity circle!



Motivation - Space of low energy coefficients

- We learnt that **the answer:** No, $\{c_i\}$ cannot take arbitrary values!

- What are the allowed values then?

We can use **the tool:** Numerical primal/dual S-matrix bootstrap

to study a single coefficient

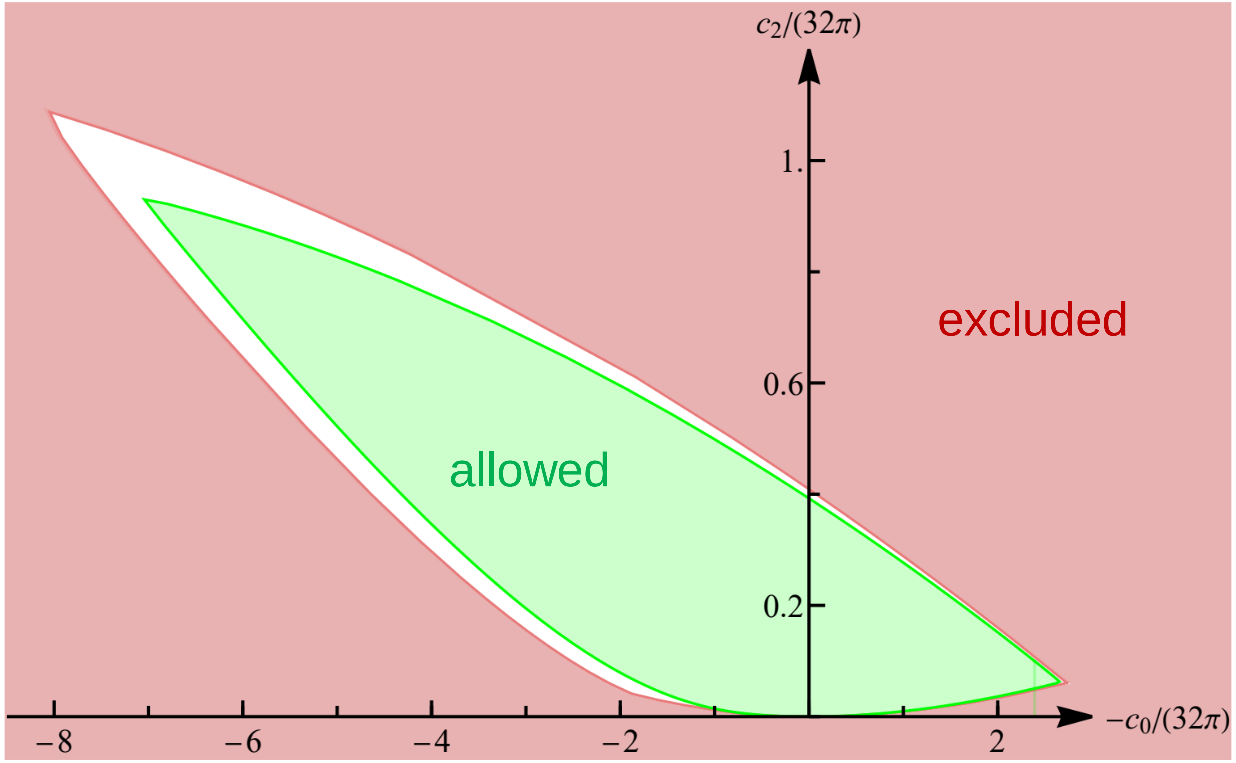
$$-8.02.. < c_0 \cdot (32\pi) < 2.6613..$$

or a multi-dimensional system

$$(c_0, c_2) \ = \ F[\mathrm{Re} f_0(s), \mathrm{Im} f_\ell(s)]$$

with F a linear functional.

[Elias Miró, Guerrieri, MAG '22]
[Chen, Fitzpatrick, Karateev '22]
[Tourkine, Zhiboedov '23]



Interlude: S-matrix Bootstrap

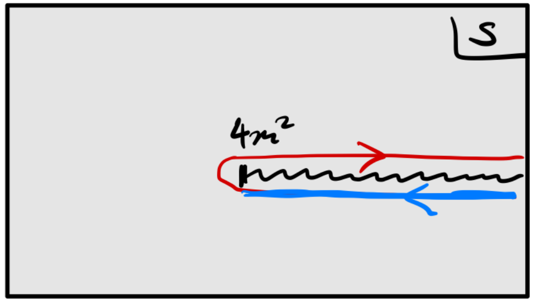
Interlude – Numerical S-matrix Bootstrap

Idea: Parametrize the amplitude in a basis of functions that makes manifest a subset of $\{\mathcal{A}, \mathcal{U}, \mathcal{CS}\}$.

- Partial wave expansion $\mathrm{M} = \sum_\ell f_\ell(s)$ diagonalizes $\mathcal{U}$, but $\{\mathcal{A}, \mathcal{CS}\}$ mixes them.
- ρ-expansion manifestly solves for $\{\mathcal{A}, \mathcal{CS}\}$, but $\mathcal{U}$ is non-trivial.

[Paulos, Penedones, Toledo, van Rees, Vieira, Homrich '16 – '19]

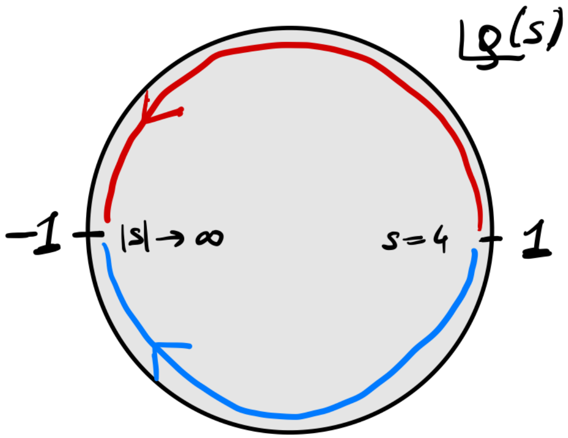
$$M = \sum_{a,b,c} \alpha_{(abc)} \rho(s)^a \rho(t)^b \rho(u)^c \quad \text{with } \rho(s) = \frac{2-\sqrt{4m^2-s}}{2+\sqrt{4m^2-s}}$$



Look for numerical solutions to the missing subset

In case of $\mathcal{U}$, by using semi-definite linear programming.

$$\begin{pmatrix} 1-(\rho^2/2)\,\mathrm{Im} f_\ell & \rho\,\mathrm{Re}\, f_\ell \\ \rho\,\mathrm{Re}\, f_\ell & 2\,\mathrm{Im}\, f_\ell \end{pmatrix} \succcurlyeq 0$$



Optimize for the desired objectives $\{c_i\}$, giving us

the bounds $c_i^{min/max}$,

the extremal solution $M^{min/max}(s,t,u)$.

Interlude – Numerical S-matrix Bootstrap

- An optimization problem P with the objective $\mathcal{O}$ over the set of admissible $M(s,t)$.

		analyticity	
$\mathcal{O} = \max_{\{M(s,t)\}} \pm c_i$	subject to	**unitarity:**	$2\,\mathrm{Im}\,M \geq \int |M|^2$
		crossing:	$M(s,t) = M(t,s) = M(s,u)$

- P admits two approaches.

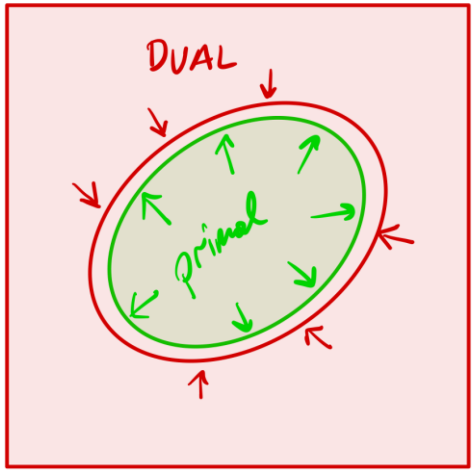
Primal approach:

Construct valid set of solutions and search for its max/min. ("filling inside").

Dual approach:

Construct a dual problem $\bar{P}$ with the dual objective $\bar{\mathcal{O}} > \mathcal{O}$.

Solving it "rules out" solutions.



Continuing

Motivation – Example IV

- S-matrix Bootstrap hinted at optimal constraints on low energy parameters.

  Can we make use of it for potential BSM applications?

- Consider the Higgs sector of the SM
  - in custodial symmetric limit $SO(4) \simeq SU(2)_L \times SU(2)_R$
  - assume $g_{SM} \ll g_{BSM}$

- A Lagrangian description of this limit

$$\mathcal{L} = \frac{1}{2}\left(\partial\vec{\phi}\right)^2 - \frac{m^2}{2}\vec{\phi}^2 - \frac{\lambda}{8}\vec{\phi}^4 - \frac{g_H}{4}\vec{\phi}^2\left(\partial\vec{\phi}\right)^2 + \cdots$$

  Leading dim-6 deviation from the SM Lagrangian:

$$\vec{\phi}^2\left(\partial\vec{\phi}\right)^2 \sim \partial_\mu\left(H^\dagger H\right)\partial^\mu\left(H^\dagger H\right) \equiv \mathcal{O}_H$$

Example IV

Next model: Massive real $O(n)$ scalars $\phi^{a\in\{1\ldots n\}}$

- Two-to-two amplitude

$$\mathbf{M}_{ab}^{cd} = M(s|t,u)\,\delta_{ab}\delta^{cd} + M(t|u,s)\delta_a^c\delta_b^d + M(u|s,t)\delta_a^d\delta_b^c$$

$M(s|t,u)$ is symmetric only in $t \leftrightarrow u$.

- Low energy expansion

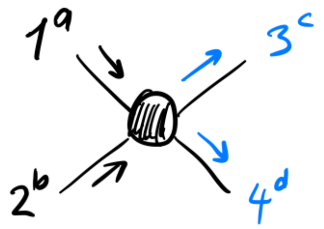
$$\frac{M(s|t,u)}{(4\pi)^2} = c_\lambda + c_H\bar{s} + c_2(\bar{t}^2+\bar{u}^2) + c_2'\,\bar{s}^2 + O(\bar{s}^4,\bar{t}^4,\bar{u}^4)$$

Notice that there is now a dim-6 coefficient c_H!

- We study the bounds on the non-dispersive coeff.s (cL,cH)

$$c_\lambda\, 2\pi = \mathrm{Re}\, f_0^{(sym)}(s) - \frac{1}{\pi}\int_{4m^2}^{\infty} dz\, \Sigma_{\ell,\mathrm{rep}}\, \mathrm{K}_{1,\ell}^{(sym,rep)}(s,z)\, \mathrm{Im}\, f_\ell^{(rep)}(z)$$

$$c_H\frac{\pi}{3}(s-4) = \mathrm{Re}\, f_1^{(anti)}(s) - \frac{1}{\pi}\int_{4m^2}^{\infty} dz\, \Sigma_{\ell,\mathrm{rep}}\, \mathrm{L}_{1,\ell}^{(anti,rep)}(s,z)\, \mathrm{Im}\, f_\ell^{(rep)}(z)$$



Mandelstam variables:

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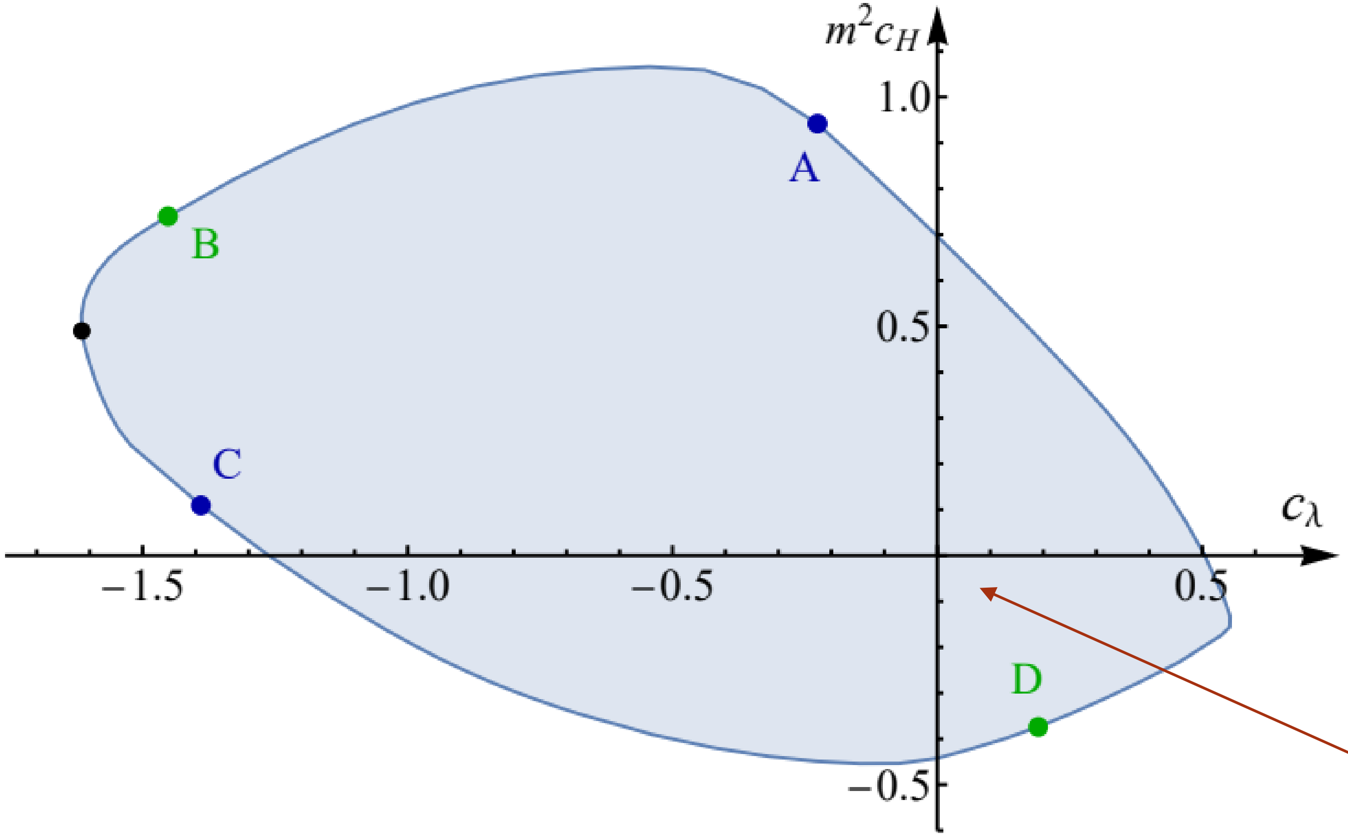
Example IV

Next model: Massive real $O(n)$ scalars $\phi^{a\in\{1\ldots n\}}$



- Nonperturbative two-sided (dual) bound:

$$-0.46 < c_H \cdot m^2 < 1.07$$

- The boundary is strongly coupled.
  - We detect various threshold singularities in each section of $A - B - C - D$.
  - We observe $c_\lambda \sim O(1)$ and $c_H \sim O(1)$

  but, typically in an EFT, we expect

  $c_\lambda \sim O(1)$ and $c_H \sim O(1) \cdot m^2/\Lambda^2$

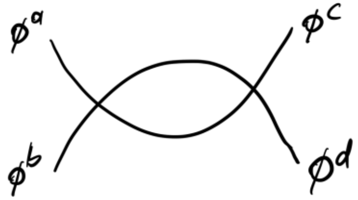
- Weakly coupled EFTs near the origin.

Example IV

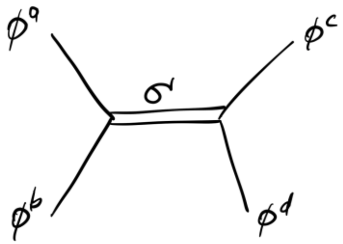
A simple UV completion confirms this insight:

$$\mathcal{L} = \frac{1}{2}\left(\partial\vec{\phi}\right)^2 - \frac{1}{2}m^2\vec{\phi}^2 - \frac{\lambda}{8}\left(\vec{\phi}\cdot\vec{\phi}\right)^2 + \frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}M^2\sigma^2 - g\,\sigma\,(\vec{\phi}\cdot\vec{\phi})$$

- If we compute two-to-two scattering for ϕ^a and expand for small energies



$$= -\lambda + \frac{\lambda^2}{16\pi^2}\left(\frac{3(n+2)}{16}\left(3\sqrt{2}\,\tan^{-1}\frac{1}{\sqrt{2}} - 2\right)\right)\bar{s} + \cdots$$

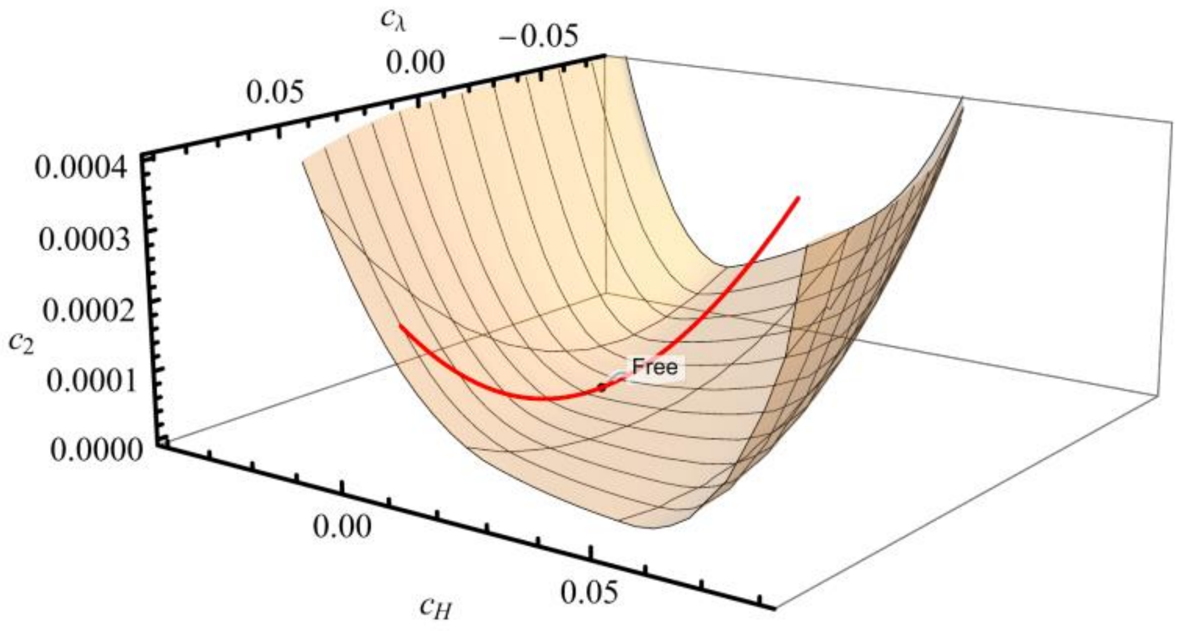


$$= \frac{g^2}{M^2-4m^2/3} + \frac{g^2}{(M^2-4m^2/3)^2}\,\bar{s} + \frac{g^2}{(M^2-4m^2/3)^3}\,\bar{s}^2 + \cdots$$

- c_H w.r.t. c_λ is either coupling (λ) supressed or scale (m^2/M^2) suppressed.
- Let us zoom further in the origin to locate EFT-like amplitudes.

Example IV – EFT approximation

Study the 3d-system $\{c_\lambda, c_H, c_2 \geq 0\}$



Notice the agreement of one-loop $\vec{\phi}^4$ amplitude (red) with the nonperturbative boundary.

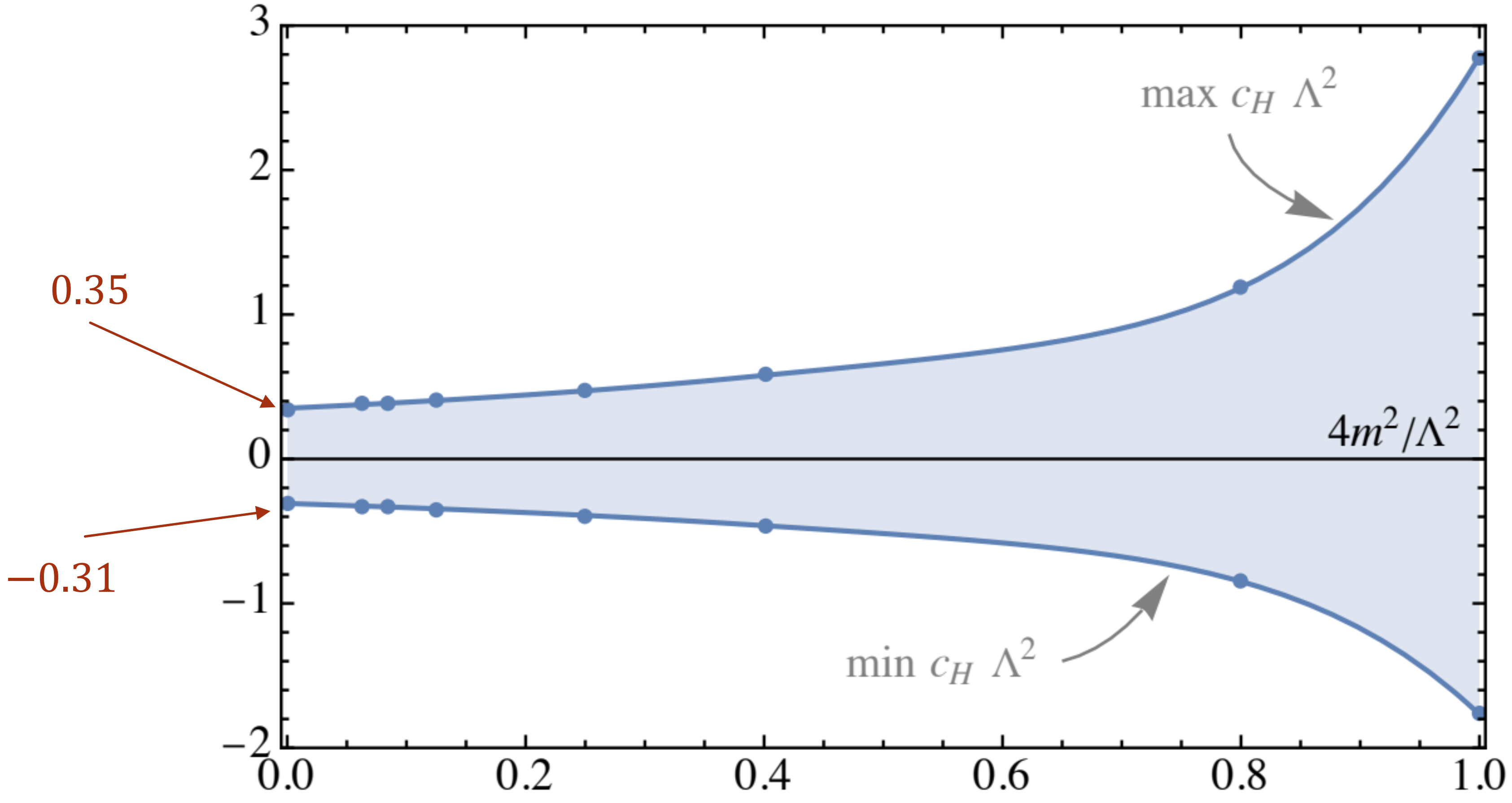
How can we isolate EFTs in the vast space of strongly coupled amplitudes?

- Introduce the two scales m^2 and Λ^2 and the scale separation $m^2/\Lambda^2 \ll 1$ through the discontinuity $Im\ f_\ell(s)$

- We call this limit “exact UV domination”
- It is possible to refine it by fixing $\mathrm{Im} f_\ell^{IR}(s)$ to some desired profile obtained from experiments.

Example IV – EFT approximation

- We dial up $\Lambda^2 \in [4m^2, 64m^2]$ and discover a scaling limit where $-0.31 < c_H \cdot \Lambda^2 < 0.35$
- Global fits from experiments report: $|\, c_H \cdot \Lambda^2 \,| < O(1) \cdot 1\, TeV$



Conclusions

- Numerical S-matrix bootstrap allows us to study the space of theories with high precision, given a set of very general assumptions, such as unitarity, analyticity, and crossing symmetry.

  Check out other 4d examples, such as fluxtubes [1906.08098], neutral Goldstones [2310.06027] etc.

- Theoretical constraints can be useful / complementary to experimental ones (see bounds on $c_H \cdot \Lambda^2$)

Some future outlook:

- Add other particles in the spectrum (transverse + longitudinal modes of heavy gauge bosons)
- Further modelling of the IR data – the more constraints in the input, the stricter the bounds we get.
- Adding inelasticity (particle production) would lead to tighter constraints.

Thank you!

Dual for O(n) amplitudes

- We dualize the problem by means of a Lagrangian:

$$\mathcal{L}(\mathcal{P},\mathcal{D}) = c_H[f_\ell] + w \cdot \text{Roy eq.}\,[f_\ell] + \lambda \cdot \text{unitarity}[f_\ell] + \nu \cdot \text{crossing}[f_\ell]$$

$\mathcal{P} \equiv \{f_\ell(s)\}$ are primal variables, and $\mathcal{D} \equiv \{w, \lambda, v\}$ are Lagrange multipliers.

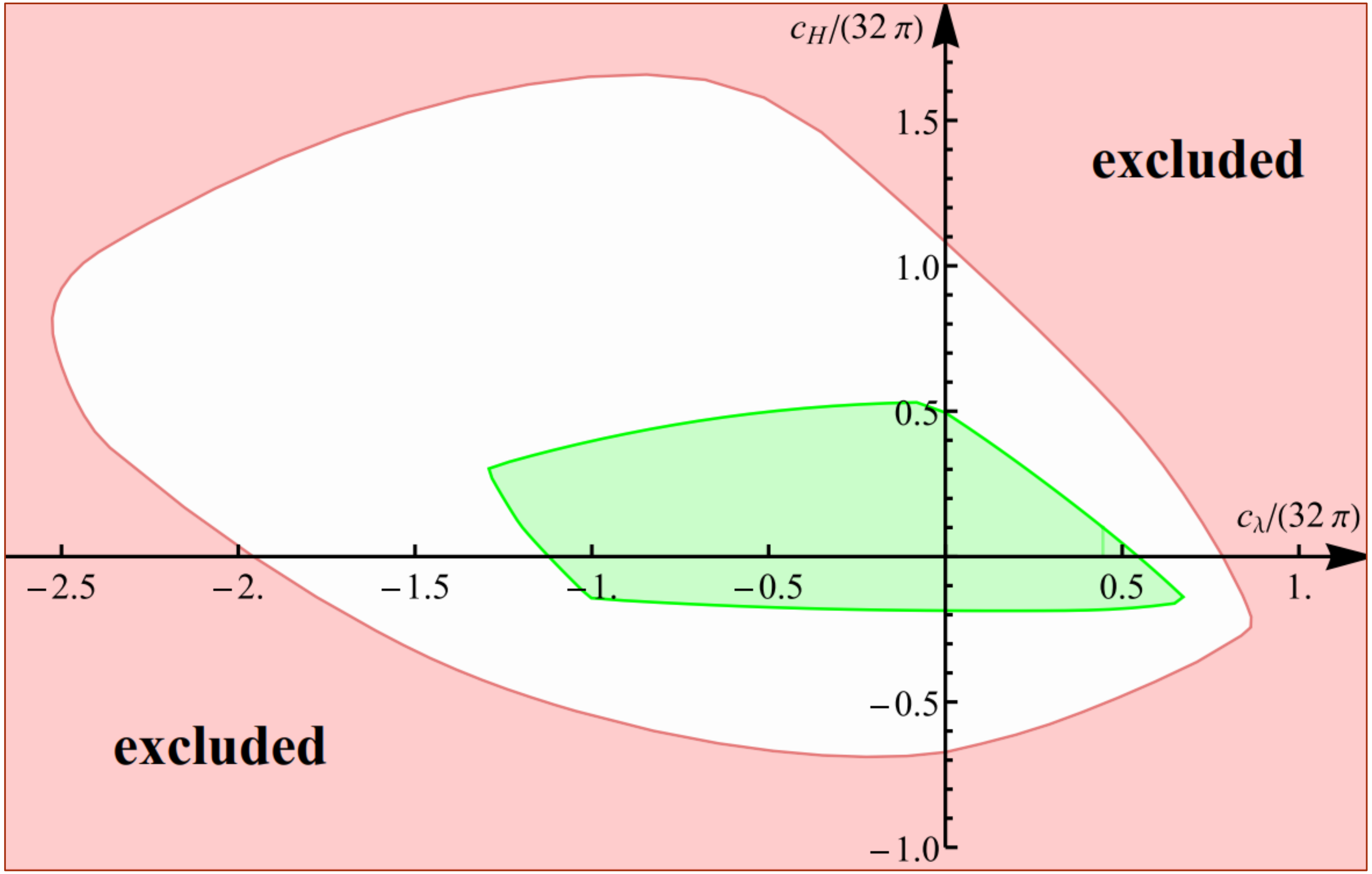
- Integrating out $\mathcal{P}$, gives the dual functional $\mathrm{d}(\mathcal{D}) \equiv \max_{\mathcal{P}} \mathcal{L}(\mathcal{P},\mathcal{D})$.

- For **all** choices of $\{w, \lambda, v\}$: $\mathrm{d}(\mathcal{D}) \geq c_i$

- To do the optimal choice:

  Nonlinear → Semidefinite linear conditions → SDPB.

Duality gap in O(n)

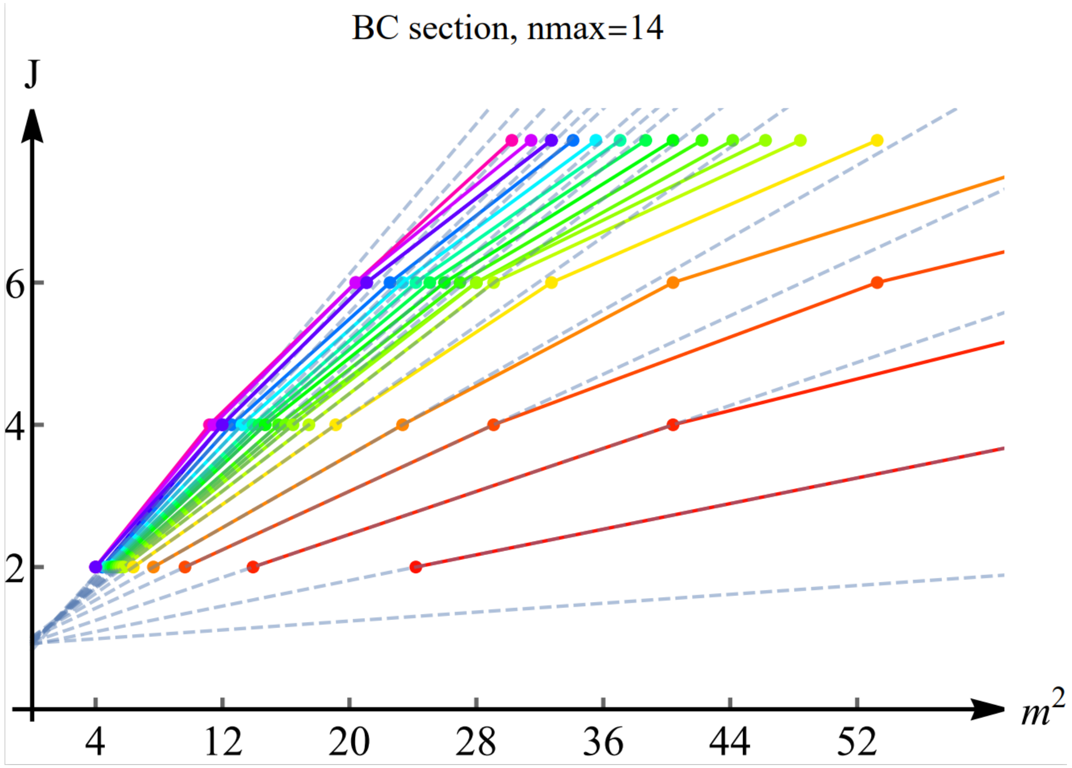


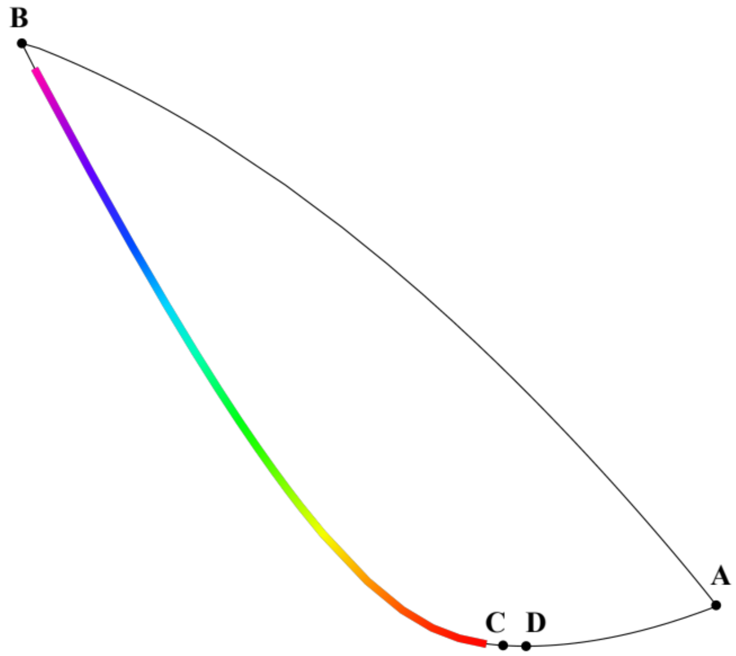
Regge Physics of O(1)

- **Higher spin resonances.**

  π phase rotations in each $S_\ell(s)$.

  They come in ~ linear Regge trajectories + align with asymptotic growth $M(s,t) \sim \beta(t) s^{\alpha(t)}$.





Analyticity

- Lehmann ellipses (small/large) for $\sum_\ell a_\ell P_\ell(z)$
  - For the amplitude, $z_{small} = 1 + 8m^2/(s-4m^2)$
  - For its discontinuity, $z_{large} = 2z_{small}^2 - 1 = 1 + 32m^2/(s-4m^2)^2$

- Ellipses shrink to 1 for large s.

- Martin's extension using unitarity + N-subtracted dispersion relations.
  - Analytic region for $|t| < 4m^2$ independent of s.

  + elastic unitarity:
  - Double subtracted dispersion relation holds for any real $-28m^2 < t < 4m^2$.

- Dual approach is limited by $\Lambda_c^2 = 12m^2$ for fixed-t dispersion relations.