Theory of Light Muonic Atoms

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EPIC 2024
22–27 September 2024
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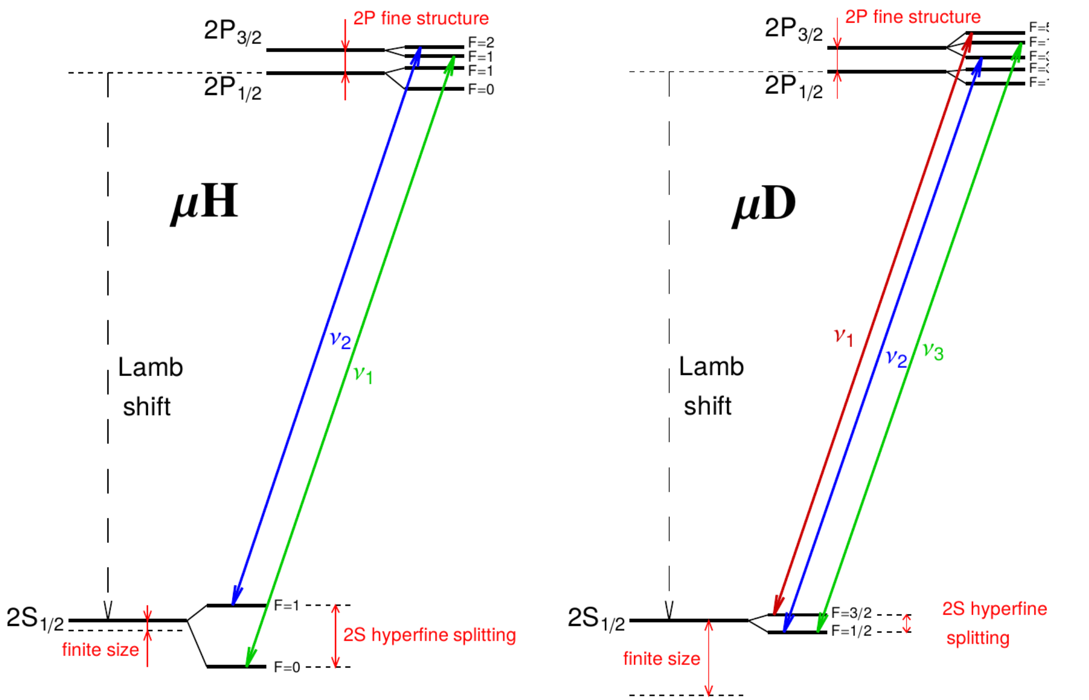
Light Muonic (H-like) Atoms and Ions

- Light Muonic Atoms: Z=1,2,…?
- This talk focuses on μH and μD (predominantly)
- Mostly Lamb shift is considered (a few words on μH HFS)



- Basic ideas about the theory (QED, FS, NS)
- Nuclear structure contribution: Two-photon exchange (TPE)
- Data-driven or effective field theory: Treatment of TPE
- Results for μH
- Remarks about heavier nuclei
- Results for μD
- Outlook

A Scary Table

TABLE I Contributions to the $2P_{1/2} - 2S_{1/2}$ energy difference E_L in meV, with the charge radii r_C given in fm. All corrections larger than 3% of the overall uncertainty are included. Theoretical predictions for E_L are $E_L(\text{theo}) = E_{\text{QED}} + \mathcal{C}\, r_C^2 + E_{\text{NS}}$. The last two rows show the values of r_C determined from a comparison of $E_L(\text{theo})$ to $E_L(\text{exp})$.

Sec.	Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
III.A	$\alpha\,(Z\alpha)^2$	eVP$^{(1)}$	205.007 38	227.634 70	1641.886 2	1665.773 1
III.A	$\alpha^2(Z\alpha)^2$	eVP$^{(2)}$	1.658 85	1.838 04	13.084 3	13.276 9
III.A	$\alpha^3(Z\alpha)^2$	eVP$^{(3)}$	0.007 52	0.008 42(7)	0.073 0(30)	0.074 0(30)
III.B	$(Z, Z^2, Z^3)\,\alpha^5$	light-by-light eVP	−0.000 89(2)	−0.000 96(2)	−0.013 4(6)	−0.013 6(6)
III.C	$(Z\alpha)^4$	recoil	0.057 47	0.067 22	0.126 5	0.295 2
III.D	$\alpha\,(Z\alpha)^4$	relativistic with eVP$^{(1)}$	0.018 76	0.021 78	0.509 3	0.521 1
III.E	$\alpha^2(Z\alpha)^4$	relativistic with eVP$^{(2)}$	0.000 17	0.000 20	0.005 6	0.005 7
III.F	$\alpha\,(Z\alpha)^4$	μSE$^{(1)}$ + μVP$^{(1)}$, LO	−0.663 45	−0.769 43	−10.652 5	−10.926 0
III.G	$\alpha\,(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$, NLO	−0.004 43	−0.005 18	−0.174 9	−0.179 7
III.H	$\alpha^2(Z\alpha)^4$	μVP$^{(1)}$ with eVP$^{(1)}$	0.000 13	0.000 15	0.003 8	0.003 9
III.I	$\alpha^2(Z\alpha)^4$	μSE$^{(1)}$ with eVP$^{(1)}$	−0.002 54	−0.003 06	−0.062 7	−0.064 6
III.J	$(Z\alpha)^5$	recoil	−0.044 97	−0.026 60	−0.558 1	−0.433 0
III.K	$\alpha\,(Z\alpha)^5$	recoil with eVP$^{(1)}$	0.000 14(14)	0.000 09(9)	0.004 9(49)	0.003 9(39)
III.L	$Z^2\alpha\,(Z\alpha)^4$	nSE$^{(1)}$	−0.009 92	−0.003 10	−0.084 0	−0.050 5
III.M	$\alpha^2(Z\alpha)^4$	$\mu F_1^{(2)}$, $\mu F_2^{(2)}$, μVP$^{(2)}$	−0.001 58	−0.001 84	−0.031 1	−0.031 9
III.N	$(Z\alpha)^6$	pure recoil	0.000 09	0.000 04	0.001 9	0.001 4
III.O	$\alpha\,(Z\alpha)^5$	radiative recoil	0.000 22	0.000 13	0.002 9	0.002 3
III.P	$\alpha\,(Z\alpha)^4$	hVP	0.011 36(27)	0.013 28(32)	0.224 1(53)	0.230 3(54)
III.Q	$\alpha^2(Z\alpha)^4$	hVP with eVP$^{(1)}$	0.000 09	0.000 10	0.002 6(1)	0.002 7(1)
IV.A	$(Z\alpha)^4$	r_C^2	$-5.197\,5\, r_p^2$	$-6.073\,2\, r_d^2$	$-102.523\, r_h^2$	$-105.322\, r_\alpha^2$
IV.B	$\alpha\,(Z\alpha)^4$	eVP$^{(1)}$ with r_C^2	$-0.028\,2\, r_p^2$	$-0.034\,0\, r_d^2$	$-0.851\, r_h^2$	$-0.878\, r_\alpha^2$
IV.C	$\alpha^2(Z\alpha)^4$	eVP$^{(2)}$ with r_C^2	$-0.000\,2\, r_p^2$	$-0.000\,2\, r_d^2$	$-0.009(1)\, r_h^2$	$-0.009(1)\, r_\alpha^2$
V.A	$(Z\alpha)^5$	TPE	0.029 2(25)	1.979(20)	16.38(31)	9.76(40)
V.B	$\alpha^2(Z\alpha)^4$	Coulomb distortion	0.0	−0.261	−1.010	−0.536
V.C	$(Z\alpha)^6$	3PE	−0.001 3(3)	0.002 2(9)	−0.214(214)	−0.165(165)
V.D	$\alpha\,(Z\alpha)^5$	eVP$^{(1)}$ with TPE	0.000 6(1)	0.027 5(4)	0.266(24)	0.158(12)
V.E	$\alpha\,(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$ with TPE	0.000 4	0.002 6(3)	0.077(8)	0.059(6)
III	E_{QED}	point nucleus	206.034 4(3)	228.774 0(3)	1644.348(8)	1668.491(7)
IV	$\mathcal{C}\, r_C^2$	finite size	$-5.225\,9\, r_p^2$	$-6.107\,4\, r_d^2$	$-103.383\, r_h^2$	$-106.209\, r_\alpha^2$
V	E_{NS}	nuclear structure	0.028 9(25)	1.750 3(200)	15.499(378)	9.276(433)
	$E_L(\text{exp})$	experiment[a]	202.370 6(23)	202.878 5(34)	1258.598(48)	1378.521(48)
	r_C	this review	0.840 60(39)	2.127 58(78)	1.970 07(94)	1.678 6(12)
	r_C	previous work[a]	0.840 87(39)	2.125 62(78)	1.970 07(94)	1.678 24(83)

Pachucki, VL, Hagelstein, Li Muli, Bacca, Pohl
– theory review (2022)
[a] experiment:
CREMA (2013-2023)

Contributions to LS of Light Muonic Atoms/Ions

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(Partial) expansion in powers of α, $Z\alpha$: light means that you can still expand

Recoil (expansion in powers of m_μ / M_A): more important than in ordinary atoms

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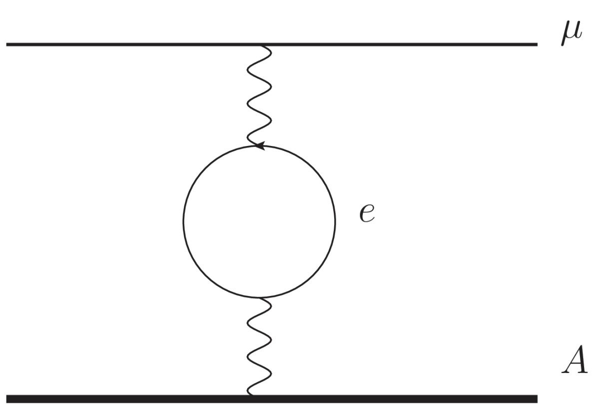

$$R_B \simeq (Z\alpha\, m_\mu)^{-1} \simeq m_e^{-1}$$

Electron loops are enhanced (matching scales)!

Recall that in normal hydrogen eVP is a small term ~0.5% on top of the electron vertex correction

Eides, Grotch, Shelyuto 2000 (review), 2007 (book)

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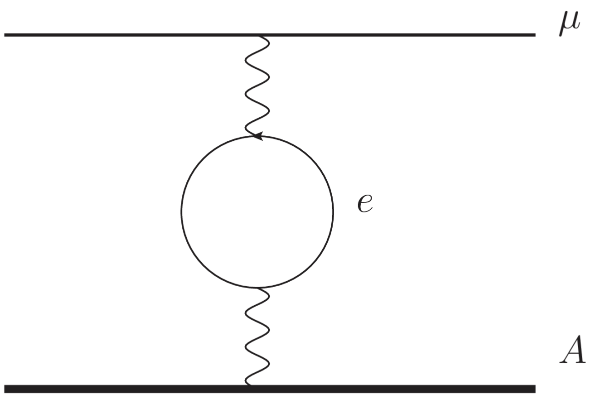

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E_{QED}

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Finite size correction is also enhanced (2nd most important term)

$\mathcal{C}\, r_C^2$

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E_{NS}

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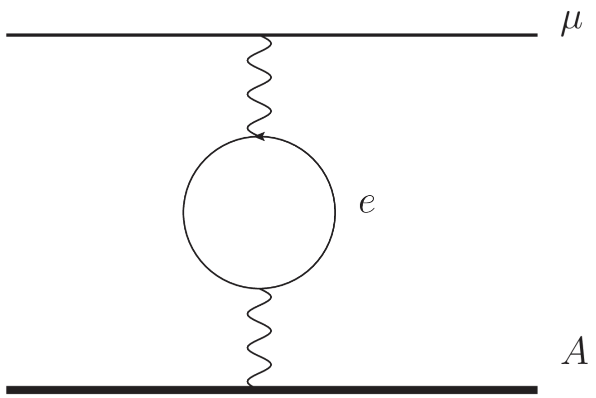

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$\}\; E_{\text{QED}}$

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Finite size correction is also enhanced (2nd most important term) $\}\; \mathcal{C}\, r_C^2$

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Nuclear structure corrections are enhanced, too, and, most importantly, they dominate the overall uncertainty! $\}\; E_{\text{NS}}$

Finite Size and Nuclear Structure

$$E_{LS}(\text{theo}) = E_{\text{QED}} + \mathcal{C}\, r_C^2 + E_{\text{NS}}$$

- Squeezed Table

	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
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E_{NS}	nuclear structure	$0.0289(25)$	$1.7503(200)$	$15.499(378)$	$9.276(433)$
$E_L(\text{exp})$	experiment[a]	$202.3706(23)$	$202.8785(34)$	$1258.612(86)$	$1378.521(48)$

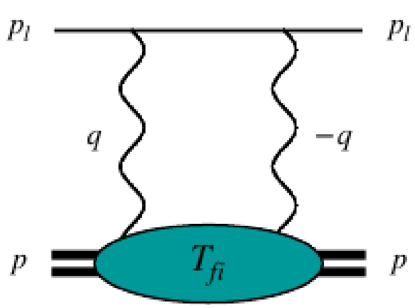
Bohr radius $a = (Z\alpha m_r)^{-1}$

r_C : charge radius

R_F: Friar radius

- Dominant nuclear structure effect:
  Two-Photon Exchange (TPE)

$\propto \alpha^5$

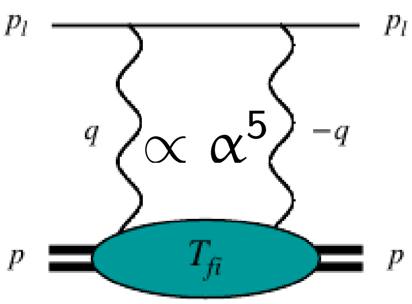


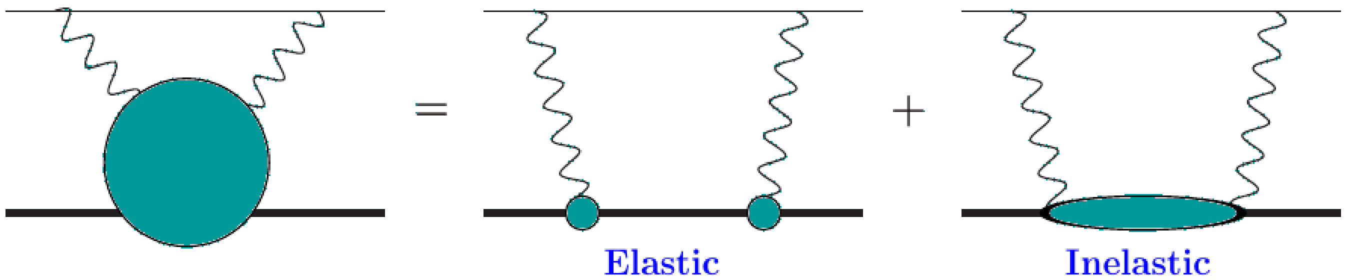
$$\Delta E_{nS} = \frac{2\pi Z\alpha}{3}\frac{1}{\pi(an)^3}\left[r_C^2 - \frac{Z\alpha m_r}{2} R_F^3\right] + \dots$$

- TPE also dominates the uncertainty (90-95%)
- Finite size enhanced (by a factor ~10^8) – great sensitivity!
- Also greater sensitivity to subleading nuclear structure

TPE and VVCS

- TPE is naturally described in terms of (doubly virtual fwd) Compton scattering (VVCS)
- Elastic ($\nu = \pm Q^2/2M_{\rm target}$, elastic e.m. form factors) and inelastic (~ nuclear generalised polarisabilities)





- Forward spin-1/2 VVCS amplitude

$$\alpha_{\rm em}\, M^{\mu\nu}(\nu, Q^2) = -\left\{\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu, Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2}\, q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}\, q^\nu\right) T_2(\nu, Q^2)\right.$$
$$\left. + \frac{i}{M}\,\epsilon^{\nu\mu\alpha\beta}\, q_\alpha s_\beta\, S_1(\nu, Q^2) + \frac{i}{M^3}\,\epsilon^{\nu\mu\alpha\beta}\, q_\alpha(p\cdot q\; s_\beta - s\cdot q\; p_\beta)\, S_2(\nu, Q^2)\right\}$$

~HFS

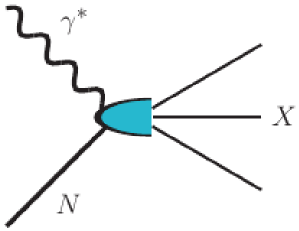
Lamb Shift: $$E^{2\gamma}_{nS} = -8i\pi\alpha m\,[\phi_n(0)]^2 \int_s \frac{d^4q}{(2\pi)^4}\,\frac{(Q^2 - 2\nu^2)\, T_1(\nu, Q^2) - (Q^2+\nu^2)\, T_2(\nu, Q^2)}{Q^4(Q^4 - 4m^2\nu^2)}$$

VVCS and Structure Functions

- Forward spin-1/2 VVCS amplitude

$$\alpha_{em} M^{\mu\nu}(\nu, Q^2) = -\Big\{ \Big(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \Big) T_1(\nu, Q^2) + \frac{1}{M^2} \Big(p^\mu - \frac{p \cdot q}{q^2} q^\mu \Big) \Big(p^\nu - \frac{p \cdot q}{q^2} q^\nu \Big) T_2(\nu, Q^2)$$
$$+ \frac{i}{M} \epsilon^{\nu\mu\alpha\beta} q_\alpha s_\beta S_1(\nu, Q^2) + \frac{i}{M^3} \epsilon^{\nu\mu\alpha\beta} q_\alpha (p \cdot q \, s_\beta - s \cdot q \, p_\beta) S_2(\nu, Q^2) \Big\}$$

- Unitarity and analyticity, data-driven: dispersive relations



Structure functions $F_1(x, Q^2)$, $F_2(x, Q^2)$, $g_1(x, Q^2)$, $g_2(x, Q^2)$

$$T_1(\nu, Q^2) = T_1(0, Q^2) + \frac{32\pi M \nu^2}{Q^4} \int_0^1 dx \, \frac{x F_1(x, Q^2)}{1 - x^2(\nu/\nu_{el})^2 - i0^+},$$
$$T_2(\nu, Q^2) = \frac{16\pi M}{Q^2} \int_0^1 dx \, \frac{F_2(x, Q^2)}{1 - x^2(\nu/\nu_{el})^2 - i0^+}$$

- The subtraction function is not directly accessible in experiment
- Data on structure functions is sometimes deficient (in practice, for any light nuclei heavier than proton)

EFTs for TPE (and *vice versa*)

Lamb Shift: $$E^{2\gamma}_{nS} = -8i\pi\alpha m\,[\phi_n(0)]^2 \int_s \frac{d^4q}{(2\pi)^4}\frac{(Q^2-2\nu^2)\,T_1(\nu,Q^2)-(Q^2+\nu^2)\,T_2(\nu,Q^2)}{Q^4(Q^4-4m^2\nu^2)}$$

- Typical energies in (muonic) atoms are small: natural to use EFTs
- Chiral EFT (covariant, HB, …) or (even) pionless EFT for nuclear effects
- Expansion in powers of a small parameter, order-by-order uncertainty
- TPE effect is needed to high precision to extract radii

	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
$E_{\rm QED}$	point nucleus	206.034 4(3)	228.774 0(3)	1644.348(8)	1668.491(7)
$\mathcal{C}\,r_C^2$	finite size	$-5.225\,9\,r_p^2$	$-6.107\,4\,r_d^2$	$-103.383\,r_h^2$	$-106.209\,r_\alpha^2$
$E_{\rm NS}$	nuclear structure	0.028 9(25)	1.750 3(200)	15.499(378)	9.276(433)
E_L(exp)	experiment[a]	202.370 6(23)	202.878 5(34)	1258.612(86)	1378.521(48)

  - a rather high order calculation of these effects is typically needed

- If TPE can be extracted (e.g. isotope shifts and/or known radii), this provides a benchmark for the theory
- Can calculate either VVCS or structure functions

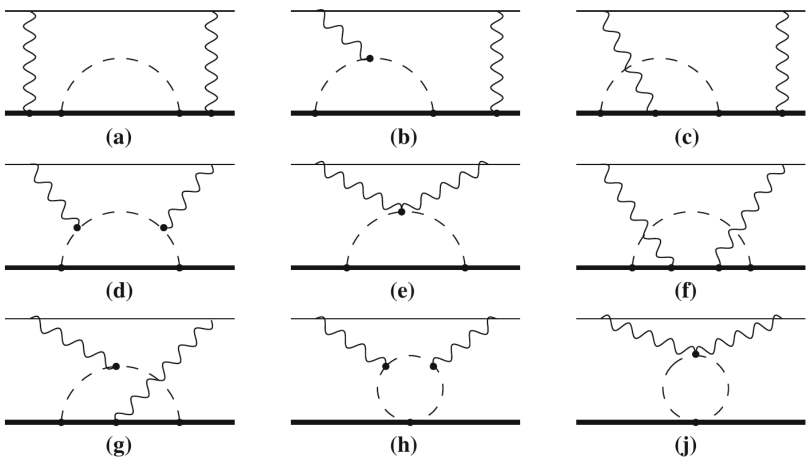
Lamb Shift of μH in Covariant BχPT

- Delta counting: $\Delta = M_\Delta - M \gg m_\pi$ — Pascalutsa, Phillips (2003)
- The contributions of the Delta isobar are suppressed by powers of m_π/Δ
- Expansion in powers of

$$p/\Delta \sim m_\pi/\Delta \sim 0.5$$

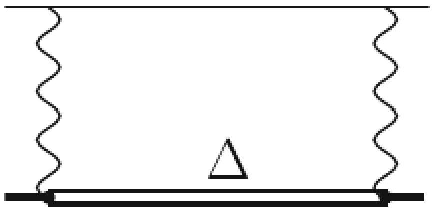
- LO BχPT: pion-nucleon loops

$$\Delta E_{2S}^{\text{LO, pol}} = -9.6^{+1.4}_{-2.9}\ \mu\text{eV}$$



Alarcon, VL, Pascalutsa (2014)

- Delta exchange:



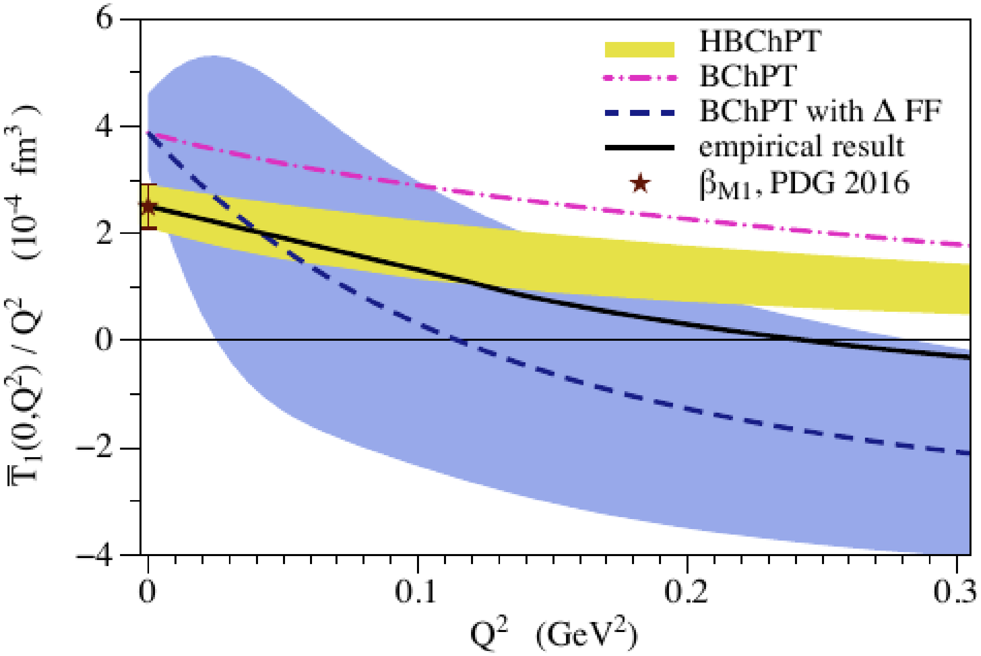
  - suppressed in $\Delta E_{2S}^{\text{pol}}$ but affects the subtraction
  - insert transition form factors (Jones-Scadron)

$$\Delta E_{2S}^{\Delta-\text{pole}} = 0.95 \pm 0.95\ \mu\text{eV}$$

VL, Hagelstein, Pascalutsa, Vanderhaeghen (2017)

Various Subtraction Functions

- The diversity of the results for the proton subtraction function $T_1(0, Q^2)$
  - HBChPT: dipole FF, matches β_{M1}[PDG] and the slope at 0
    modification of Birse, McGovern (2012)
  - BChPT: transition FFs change the subtraction function
  - Empirical: Regge asymptotic at high energy subtracted
    Tomalak, Vanderhaeghen (2015)



VL, Hagelstein, Pascalutsa, Vanderhaeghen (2017)

- Zero crossing at low Q^2 – emerges in BChPT with FFs; established in the empirical derivation, but the position not well known (0.1..0.4 GeV^2)
- Big cancellations between different mechanisms (πN and πΔ loops vs. Δ pole), also cancellations in the LS integral because of the sign change
- Empirical derivation has sizeable errors towards $Q^2 = 0$ (not shown)
  attributed to mismatch between structure function fit in the resonance region (Christy-Bosted) and at high energies (Donnachie-Landshoff) =>
  needs a better (combined) structure function parametrization

Lamb Shift of μH in Various Approaches

Table 1 Forward 2γ-exchange contributions to the $2S$-shift in μH, in units of μeV.

Reference	$E_{2S}^{(\mathrm{subt})}$	$E_{2S}^{(\mathrm{inel})}$	$E_{2S}^{(\mathrm{pol})}$	$E_{2S}^{(\mathrm{el})}$	$E_{2S}^{\langle 2\gamma\rangle}$
DATA-DRIVEN DISPERSIVE EVALUATION					
(75) Pachucki '99	1.9	−13.9	−12(2)	−23.2(1.0)	−35.2(2.2)
(76) Martynenko '06	2.3	−16.1	−13.8(2.9)		
(77) Carlson *et al.* '11	5.3(1.9)	−12.7(5)	−7.4(2.0)		
(78) Birse and McGovern '12	4.2(1.0)	−12.7(5)	−8.5(1.1)	−24.7(1.6)	−33(2)
(79) Gorchtein *et al.*'13 [a]	−2.3(4.6)	−13.0(6)	−15.3(4.6)	−24.5(1.2)	−39.8(4.8)
(80) Hill and Paz '16					−30(13)
(81) Tomalak'18	2.3(1.3)		−10.3(1.4)	−18.6(1.6)	−29.0(2.1)
LEADING-ORDER BχPT					
(82) Alarcòn *et al.* '14			$-9.6^{+1.4}_{-2.9}$		
(83) Lensky *et al.* '17 [b]	$3.5^{+0.5}_{-1.9}$	−12.1(1.8)	$-8.6^{+1.3}_{-5.2}$		
LATTICE QCD					
(84) Fu *et al.* '22					−37.4(4.9)

[a] Adjusted values due to a different decomposition into the elastic and polarizability contributions.
[b] Partially includes the Δ(1232)-isobar contribution.

Antognini, Hagelstein, Pascalutsa (2022)

- Agreement between different approaches, also on the size of the subtraction contribution separately – despite the variation in $T_1(0, Q^2)$
- Still, $T_1(0, Q^2)$ carries the biggest uncertainty, and needs to be further constrained [esp. in view of a more precise experiment]

see Randolf Pohl's talk

HFS of μH in Covariant BχPT

$$E_{\mathrm{hfs}}(nS) = \frac{8}{3}\frac{Z\alpha}{(an)^3}\frac{1+\kappa}{mM}\left(1+\Delta_{\mathrm{QED}}+\Delta_{\mathrm{weak}}+\Delta_{\mathrm{strong}}\right)$$

$$\Delta_{\mathrm{strong}} = \Delta_Z + \Delta_{\mathrm{recoil}} + \Delta_{\mathrm{pol}}$$

$$\Delta_{\mathrm{pol.}} = \Delta_1 + \Delta_2 = \frac{Zm}{2\pi(1+\kappa)M}(\delta_1+\delta_2),$$

$$\delta_1 = 2\int_0^\infty \frac{\mathrm{d}Q}{Q}\left\{\frac{5+4v_l}{(v_l+1)^2}\left[4I_1(Q^2)/Z^2+F_2^2(Q^2)\right]-\frac{32M^4}{Q^4}\int_0^{x_0}\mathrm{d}x\,x^2 g_1(x,Q^2)\right.$$
$$\left.\times\frac{1}{(v_l+v_x)(1+v_x)(1+v_l)}\left(4+\frac{1}{1+v_x}+\frac{1}{v_l+1}\right)\right\},$$

$$\delta_2 = 96M^2\int_0^\infty\frac{\mathrm{d}Q}{Q^3}\int_0^{x_0}\mathrm{d}x\,g_2(x,Q^2)\left(\frac{1}{v_l+v_x}-\frac{1}{v_l+1}\right)$$

$$I_1(Q^2) = \frac{2M^2Z^2}{Q^2}\int_0^{x_0}\mathrm{d}x\,g_1(x,Q^2)$$

The generalised GDH integral

$$v_l = \sqrt{1+1/\tau_l},\ \ v_x = \sqrt{1+x^2\tau^{-1}},\ \ \tau_l = Q^2/4m^2,\ \ \tau = Q^2/4M^2$$

Kinematic functions

HFS of μH in Covariant BχPT

$$E_{\text{hfs}}(nS) = \frac{8}{3}\frac{Z\alpha}{(an)^3}\frac{1+\kappa}{mM}\left(1+\Delta_{\text{QED}}+\Delta_{\text{weak}}+\Delta_{\text{strong}}\right)$$

$$\Delta_{\text{strong}} = \Delta_Z + \Delta_{\text{recoil}} + \Delta_{\text{pol}}$$

$$\Delta_{\text{pol.}} = \Delta_{LT} + \Delta_{TT} + \Delta_{F_2} = \frac{m}{2\pi(1+\kappa)M}\left(\delta_{LT} + \delta_{TT} + \delta_{F_2}\right),$$

$$\delta_{LT} = \frac{4M}{\alpha\pi^2}\int_0^\infty \mathrm{d}Q \int_0^{x_0} dx\, \frac{1}{v_l + v_x}\frac{1}{x^2+\tau}\left[1 - \frac{1}{(1+v_l)(1+v_x)}\right]\sigma_{LT}(x, Q^2),$$

$$\delta_{TT} = \frac{4M^2}{\alpha\pi^2}\int_0^\infty \frac{\mathrm{d}Q}{Q}\int_0^{x_0}\frac{\mathrm{d}x}{x}\frac{1}{1+v_l}\left[\frac{2\tau}{x^2+\tau} + \frac{1}{(v_l+v_x)(1+v_x)}\right]\sigma_{TT}(x, Q^2),$$

$$\delta_{F_2} = 2\int_0^\infty \frac{Q}{Q}\frac{5+4v_l}{(v_l+1)^2}F_2^2(Q^2)$$

$v_l = \sqrt{1 + 1/\tau_l}$, $v_x = \sqrt{1 + x^2\tau^{-1}}$, $\tau_l = Q^2/4m^2$, $\tau = Q^2/4M^2$ Kinematic functions

- Rewritten in terms of scattering cross sections

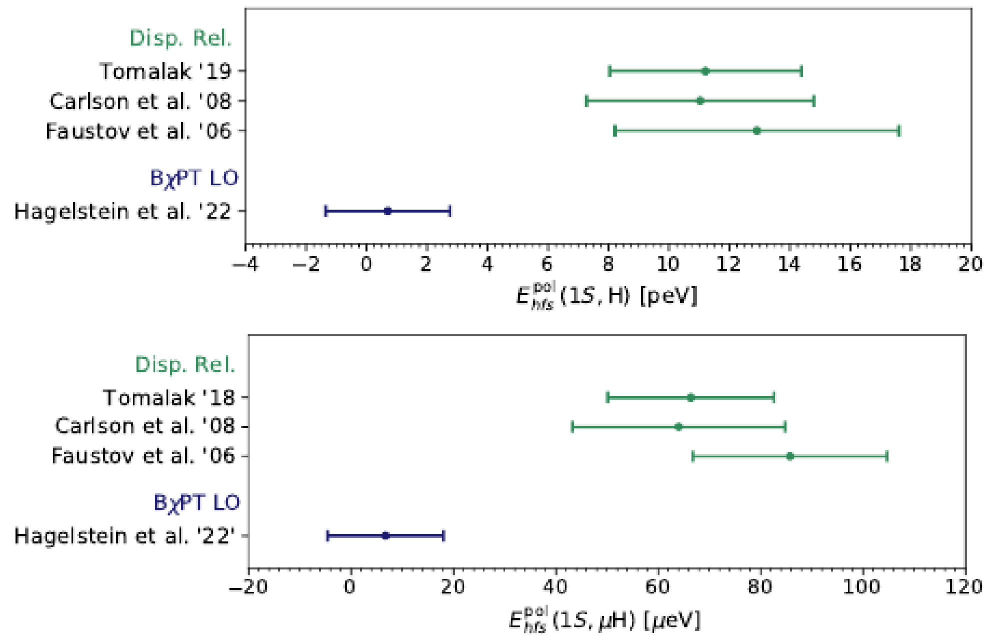
HFS of μH in Covariant BχPT: Cancellations

- LO BχPT result

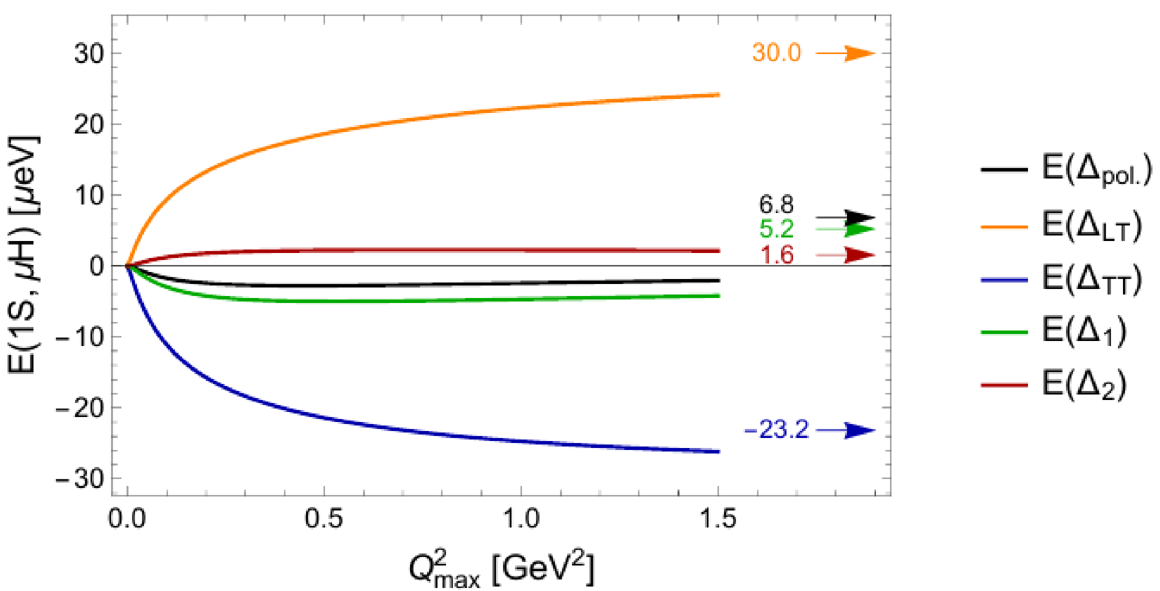
$$E_{\text{hfs}}^{\langle\text{LO}\rangle\,\text{pol.}}(1S,\text{H}) = 0.69(2.03)\,\text{peV}$$

$$E_{\text{hfs}}^{\langle\text{LO}\rangle\,\text{pol.}}(1S,\mu\text{H}) = 6.8(11.4)\,\mu\text{eV}$$

- Consistent with zero
- Cancellations!



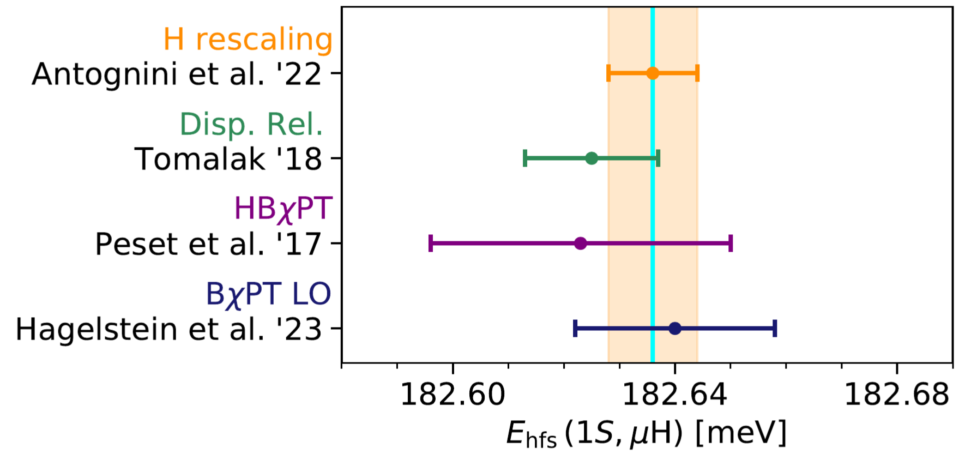
Hagelstein, VL, Pascalutsa (2023)



- The LT and TT contributions are large and almost cancel each other
- The LO BχPT result is nearly zero
- Sizeable uncertainty

HFS of μH

- Compare with expected experimental precision (cyan line)
- Theory needs to do better than that



- Rescaling non-recoil contributions from the HFS in H

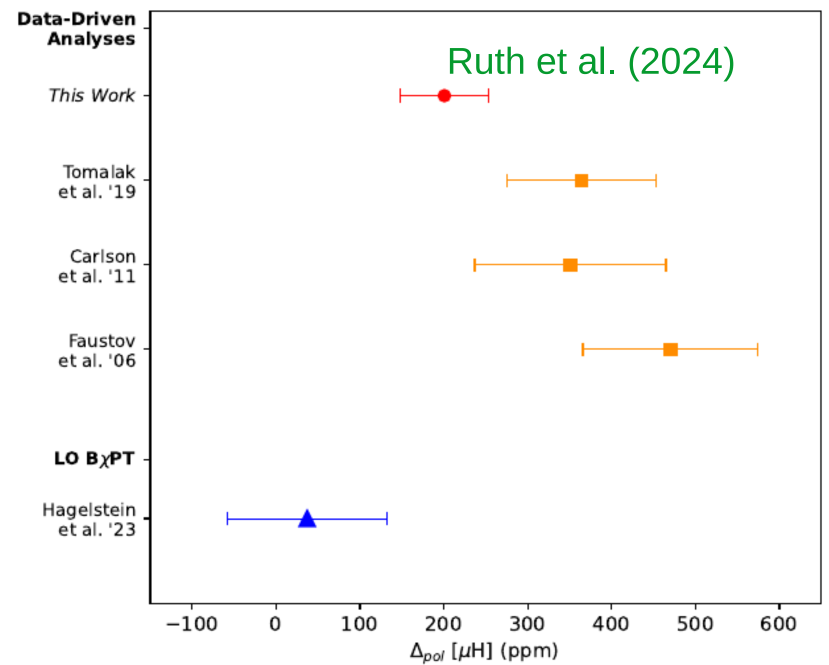
Antognini, Hagelstein, Pascalutsa (2022)

$$E^{\mathrm{Z+pol}}_{nS-\mathrm{hfs}}(\mu\mathrm{H}) = \frac{E_\mathrm{F}(\mu\mathrm{H})\, m_r(\mu\mathrm{H})\, b_{nS}(\mu\mathrm{H})}{n^3 E_\mathrm{F}(\mathrm{H})\, m_r(\mathrm{H})\, b_{1S}(\mathrm{H})} E^{\mathrm{Z+pol}}_{1S-\mathrm{hfs}}(\mathrm{H}) - \frac{E_\mathrm{F}(\mu\mathrm{H})}{n^3} \Delta_{\mathrm{pol}}(\mu\mathrm{H}) \underbrace{\left[c_{1S}(\mathrm{H}) \frac{b_{nS}(\mu\mathrm{H})}{b_{1S}(\mathrm{H})} - c_{nS}(\mu\mathrm{H})\right]}_{\simeq 10^{-5}}$$

- New results from g2p@JLab shrink discrepancy between data and BχPT

$$0.02\ \mathrm{GeV}^2 < Q^2 < 0.12\ \mathrm{GeV}^2$$

- Can one do better than that?
  One needs to limit the frequency scan region! see Randolf Pohl's talk



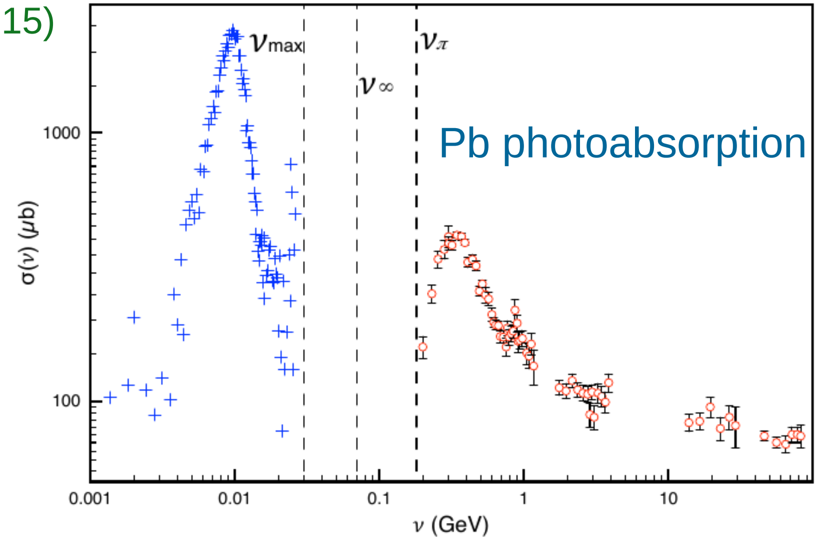
Nuclei Heavier than Proton

- Most of the TPE correction is nuclear (as with pointlike nucleons)
- Nuclear part of subtraction function converges (finite energy sum rule)
  - TPE with nuclear response functions calculated ab initio will converge
  - Most widely used method

$$E_{\rm pol} = -\frac{4\pi\alpha^2}{3}\,\phi^2(0)\int_{E_T} dE\,\sqrt{\frac{2\mu}{E}}\,|\langle \Phi_N|\tilde{d}|E\rangle|^2$$

See Nir Barnea's talk for more details on this method!

Gorchtein (2015)



- Single-nucleon contributions are treated separately
  - relatively more important in heavier nuclei
  - sizeable uncertainty!
  - neutron not so well constrained empirically (especially important in μ^3H)

Ji et al. (2018)

	$\delta^A_{\rm Zem}$	$\delta^A_{\rm pol}$	$\delta^N_{\rm Zem}$	$\delta^N_{\rm pol}$	$\delta_{\rm TPE}$
μ^2H	−0.423(04)	−1.245(13)	−0.030(02)	−0.020(10)	−1.718(17)
μ^3H	−0.227(06)	−0.480(11)	−0.033(02)	−0.031(17)	−0.771(22)
μ^3He$^+$	−10.49(23)	−4.23(18)	−0.52(03)	−0.25(13)	−15.49(33)
μ^4He$^+$	−6.14(31)	−2.35(13)	−0.54(03)	−0.34(20)	−9.37(44)

nuclear — individual nucleons

Deuteron VVCS in Pionless EFT

Kaplan, Savage, Wise (1998)
Chen, Rupak, Savage (1999)
Phillips, Rupak, Savage (1999)

- Nucleons are non-relativistic → $E \simeq p^2/M = O(p^2)$
- Loop integrals $dE\, d^3p = O(p^5)$
- Nucleon propagators $(E - p^2/2M)^{-1} = O(p^{-2})$
- Typical momenta $p \sim \gamma = \sqrt{ME_d} \simeq 45$ MeV
- Photon momenta $|\vec{q}| \sim p, \quad \nu \sim p^2$
- Expansion parameter $p/m_\pi \simeq \gamma/m_\pi \simeq 1/3$
- *NN* system has a low-lying bound/virtual state → enhance S-wave coupling constants, resum the LO *NN* S-wave scattering amplitude
- z-parametrization (reproducing deuteron S-wave asymptotics at NLO)

- Easy to solve (analytic results for *NN*)
- Explicit gauge invariance and renormalisability
- A field theory treatment!

Counting for VVCS and TPE: Predictive Powers

- Longitudinal and Transverse amplitudes

$$f_L(\nu, Q^2) = -T_1(\nu, Q^2) + \left(1 + \frac{\nu^2}{Q^2}\right) T_2(\nu, Q^2), \qquad f_T(\nu, Q^2) = T_1(\nu, Q^2)$$

Lamb Shift:

$$\Delta E_{nl} = -8i\pi m\, [\phi_{nl}(0)]^2 \int_s \frac{d^4q}{(2\pi)^4} \frac{f_L(\nu, Q^2) + 2(\nu^2/Q^2) f_T(\nu, Q^2)}{Q^2(Q^4 - 4m^2\nu^2)}$$

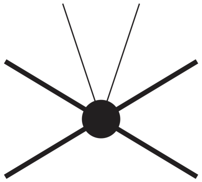
$f_L = O(p^{-2})$, $\quad f_T = O(p^0)$ in the VVCS amplitude

$\alpha_{E1} = 0.64\ \mathrm{fm}^3$

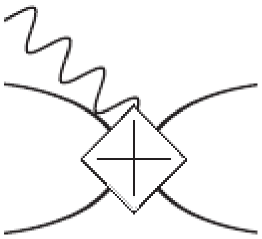
$\beta_{M1} = 0.07\ \mathrm{fm}^3$

$\text{longitudinal} = O(p^{-2})$, $\quad \text{transverse} = O(p^2)$ in TPE

- Transverse contribution to TPE starts only at N4LO
- N4LO: ΔE_{nl} needs to be regularised, an unknown lepton-NN LEC



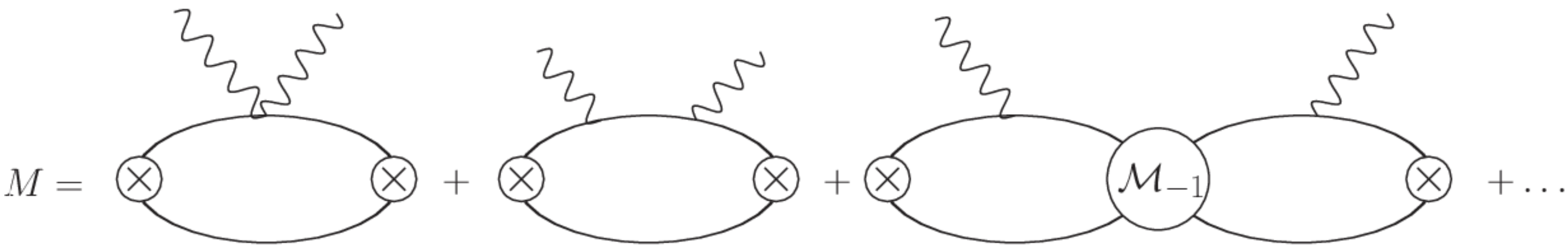
- We go up to N3LO in f_L, and up to (relative) NLO in f_T [cross check]
- One unknown LEC at N3LO in f_L
  - important for the charge form factor
  - extracted from the H-D isotope shift and proton R_E



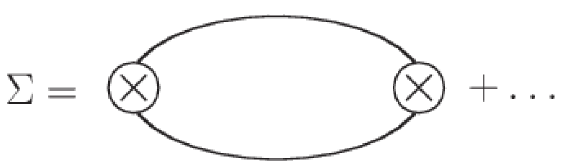
Amplitude with Deuterons

- The reaction amplitude is given by the LSZ reduction

$$T = M \left[\frac{\mathrm{d}\Sigma(E)}{\mathrm{d}E} \bigg|_{E=E_d} \right]^{-1}$$



– irreducible VVCS graphs (here full LO for f_L; crossed not shown)



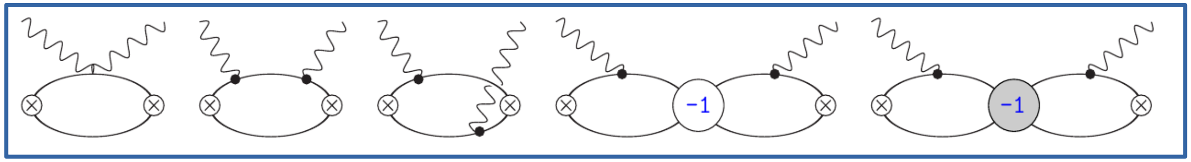
– deuteron self-energy (here at LO)

- The expression for the residue is very simple up to N3LO:

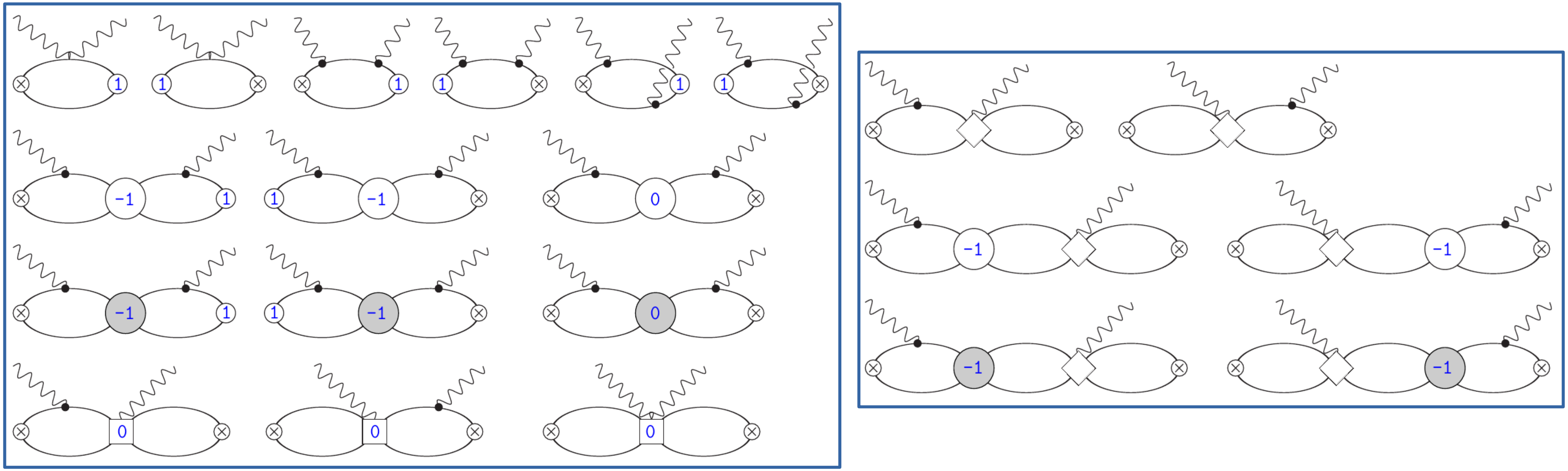
$$\left[\frac{\mathrm{d}\Sigma(E)}{\mathrm{d}E} \bigg|_{E=E_d} \right]^{-1} = \frac{8\pi\gamma}{M^2} \left[1 + (Z-1) + 0 + 0 + \dots\right]$$

Deuteron VVCS: Feynman Graphs

LO



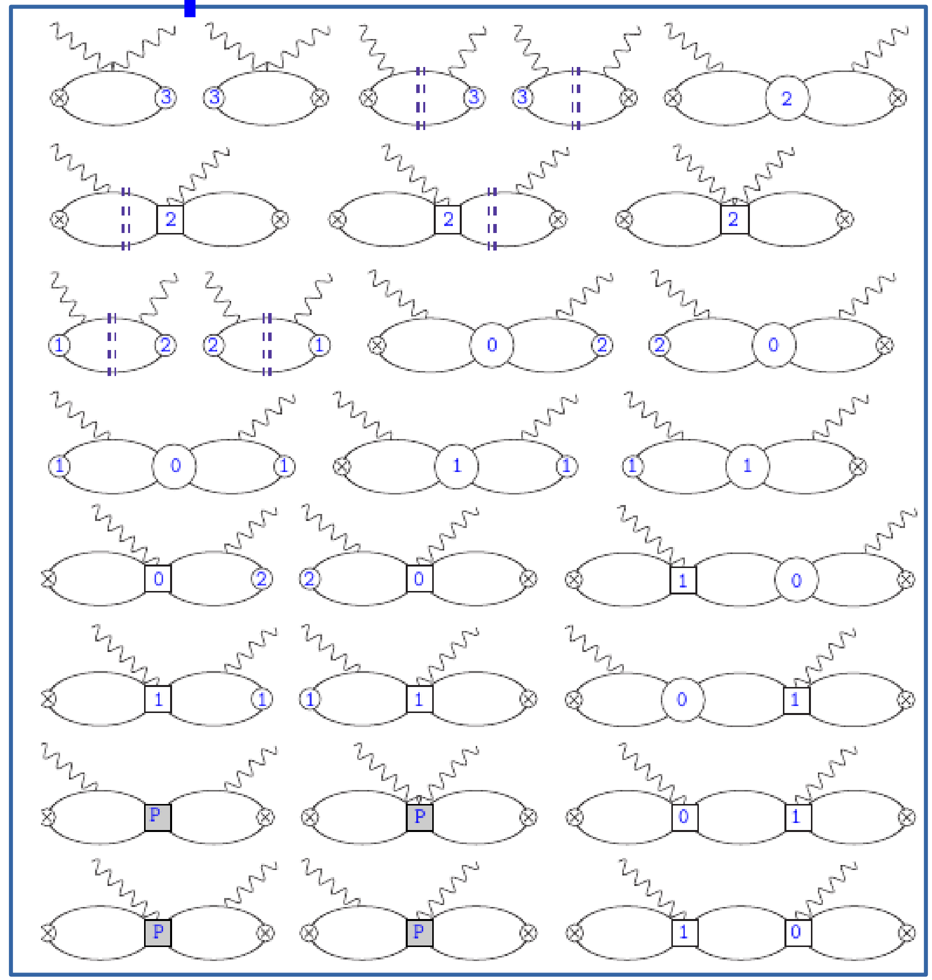
NLO



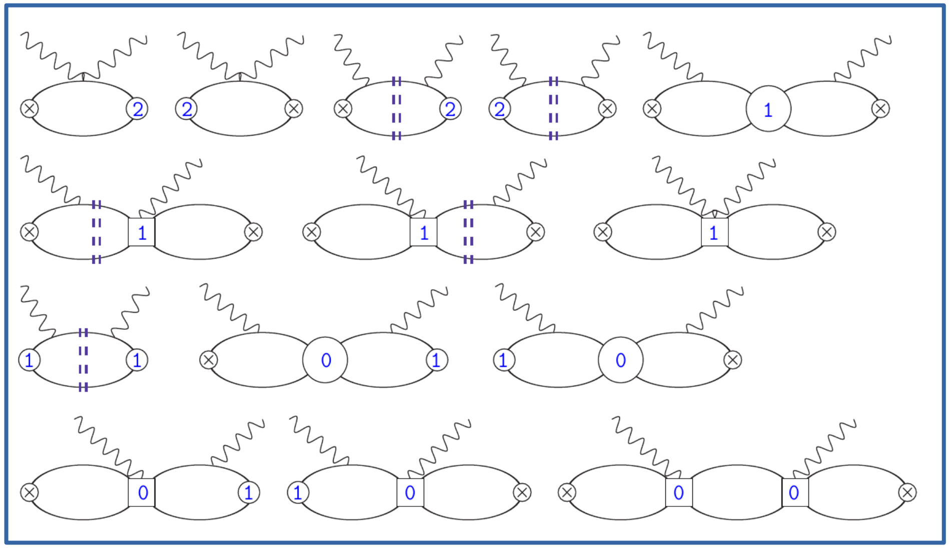
- Amplitudes are calculated analytically (dimreg+PDS) Kaplan, Savage, Wise (1998)
- Checks:
  - ➔ the sum of each subgroup (+ respective crossed graphs) is gauge invariant
  - ➔ regularisation scale dependence has to vanish

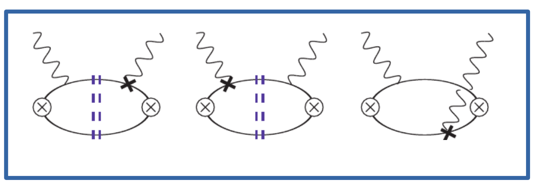
Deuteron VVCS: Feynman Graphs

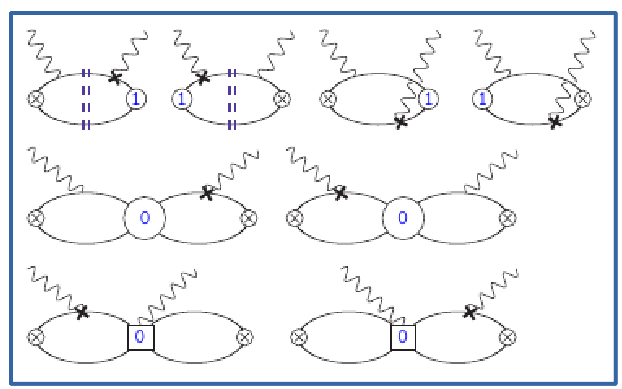
N3LO

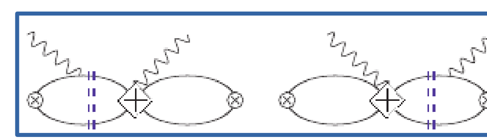


NNLO







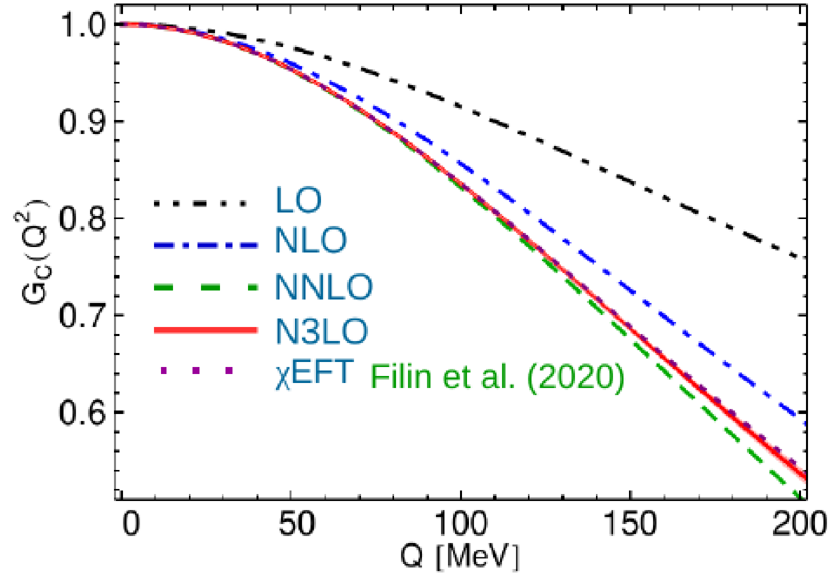


Many interesting results obtained from the VVCS amplitude, e.g., the deuteron (generalised) polarisabilities

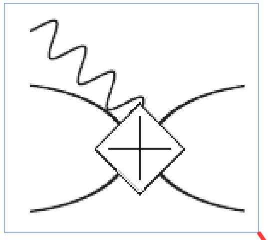
VL, Hiller Blin, Pascalutsa (2021)

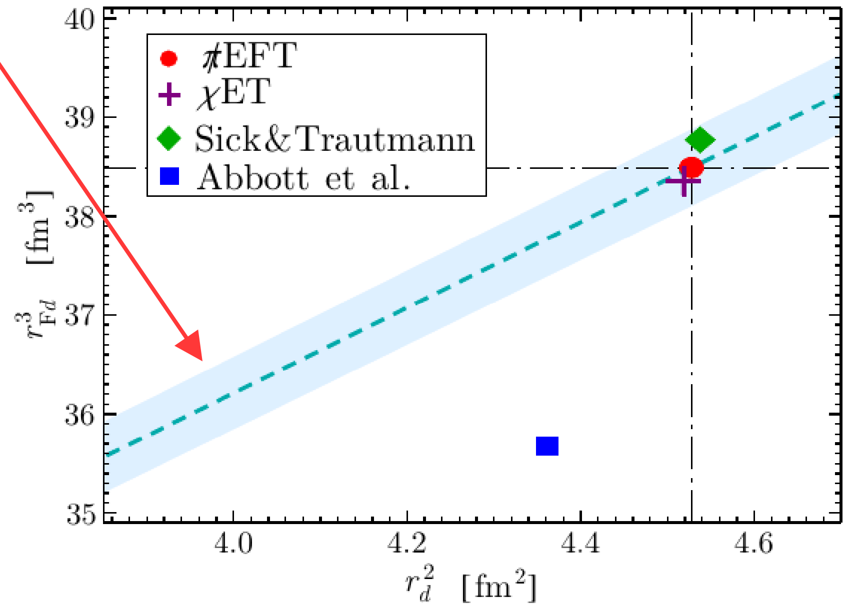
Deuteron Charge Form Factor and TPE in μD

- The deuteron charge form factor obtained from the residue of the VVCS amplitude
- The result is consistent with χEFT
- Correlation between R_F and R_E
  - generated by the N3LO LEC



VL, Hiller Blin, Pascalutsa (2021)

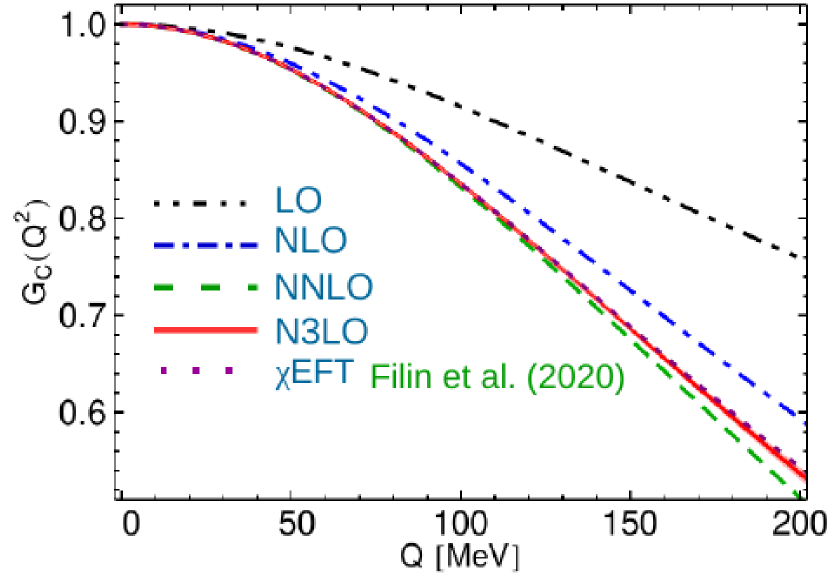




VL, Hagelstein, Pascalutsa (2022)

Deuteron Charge Form Factor and TPE in μD

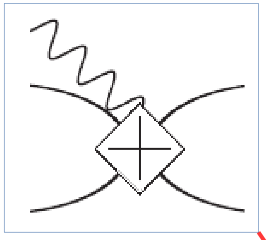
- The deuteron charge form factor obtained from the residue of the VVCS amplitude
- The result is consistent with χEFT
- Correlation between R_F and r_C
  - generated by the N3LO LEC



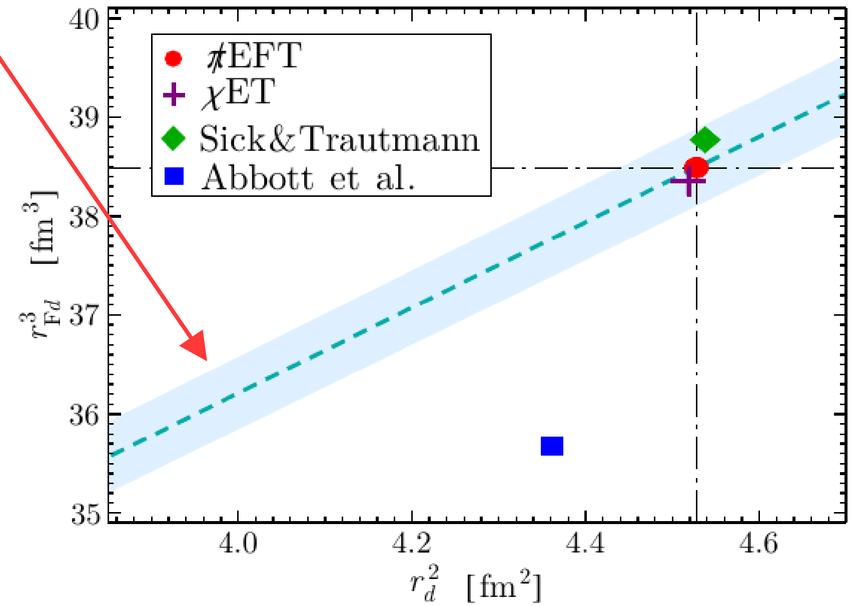
VL, Hiller Blin, Pascalutsa (2021)

$$R_F^3 = \frac{48}{\pi} \int_0^\infty \frac{dQ}{Q^4} \left[G_C^2(Q^2) - 1 - 2G_C'(0)\, Q^2 \right]$$

$$= \frac{3}{80\gamma^3} \{ Z\,[5 - 2Z(1 - 2\ln 2)] - 320/9\; r_0^2\gamma^2\,[Z(1 - 4\ln 2) - 2 + 2\ln 2] + 80(Z-1)^3\, l_1^{C0_S} \}$$



$$r_C^2 = \frac{1}{8\gamma^2} + \frac{Z-1}{8\gamma^2} + 2r_0^2 + \frac{3(Z-1)^3}{\gamma^2}\, l_1^{C0_S}$$



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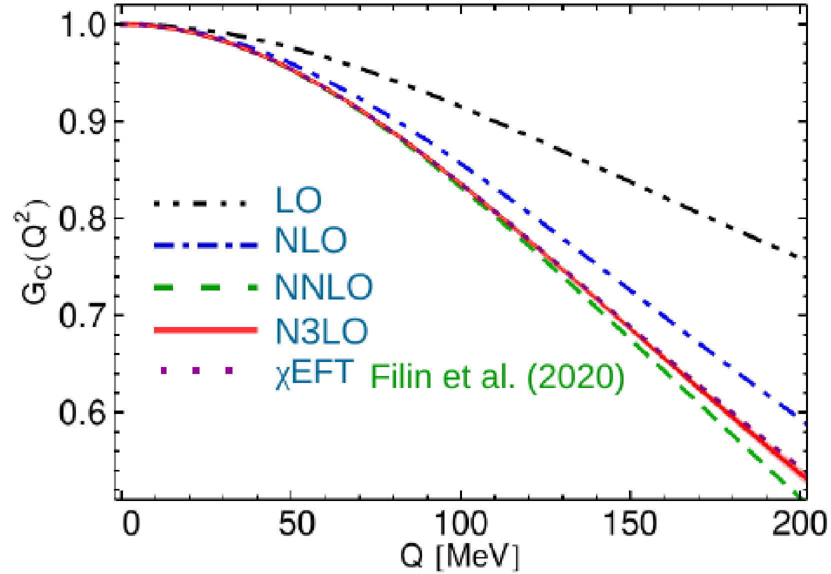
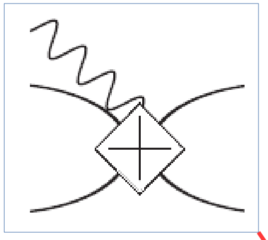


VL, Hiller Blin, Pascalutsa (2021)

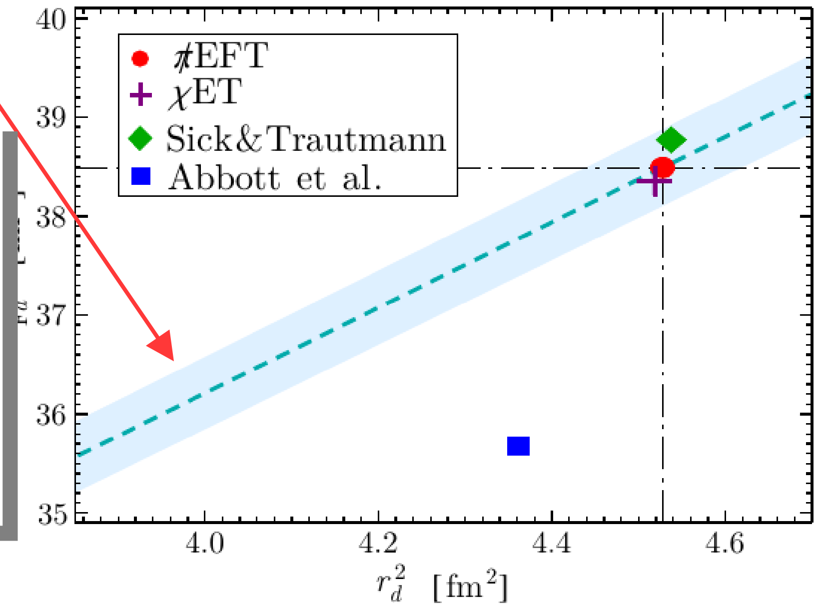
$$R_F^3 = \frac{48}{\pi}\int_0^\infty \frac{dQ}{Q^4}\left[G_C^2(Q^2) - 1 - 2G_C'(0)\,Q^2\right]$$

$= 3$



- Benchmark: EFTs work better at low Q than at least some empirical parametrizations
- Not only r_C but also higher derivatives need to be reproduced correctly!
- Agreement with χEFT vindicates both EFTs

$r_C^2 =$



VL, Hagelstein, Pascalutsa (2022)

TPE in μD: Higher-Order Corrections

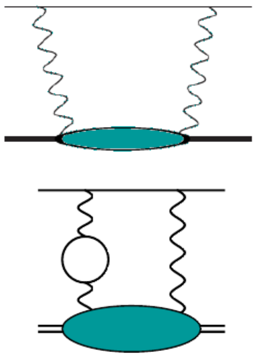
$$\Delta E_{2S} = \Delta E_{2S}^{\text{elastic}} + \Delta E_{2S}^{\text{inel}} = -1.955(19)\text{meV}$$

- Higher-order in α terms are important in D
  - Coulomb $[\mathcal{O}(\alpha^6 \log \alpha)]$

    taken from elsewhere $\Delta E_{2S}^{\text{Coulomb}} = 0.2625(15)\,\text{meV}$
  - eVP $[\mathcal{O}(\alpha^6)]$ Kalinowski (2019)

    reproduced in pionless EFT $\Delta E_{2S}^{\text{eVP}} = -0.027\ \text{meV}$

non-forward



- Single-nucleon terms at N4LO in pionless EFT and higher
  - insert empirical FFs in the LO+NLO VVCS amplitude
  - polarisability contribution (inelastic+subtraction)
    - inelastic: *ed* scattering data Carlson, Gorchtein, Vanderhaeghen (2013)
    - subtraction: nucleon subtraction function from χEFT

      VL, Hagelstein, Pascalutsa, Vanderhaeghen (2017)
  - in total: small but sizeable: $\Delta E_{2S}^{\text{hadr}} = -0.032(6)\ \text{meV}$

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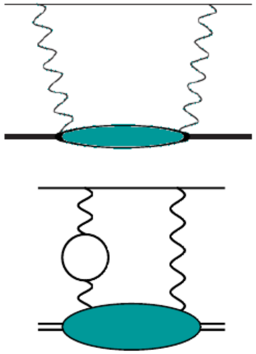

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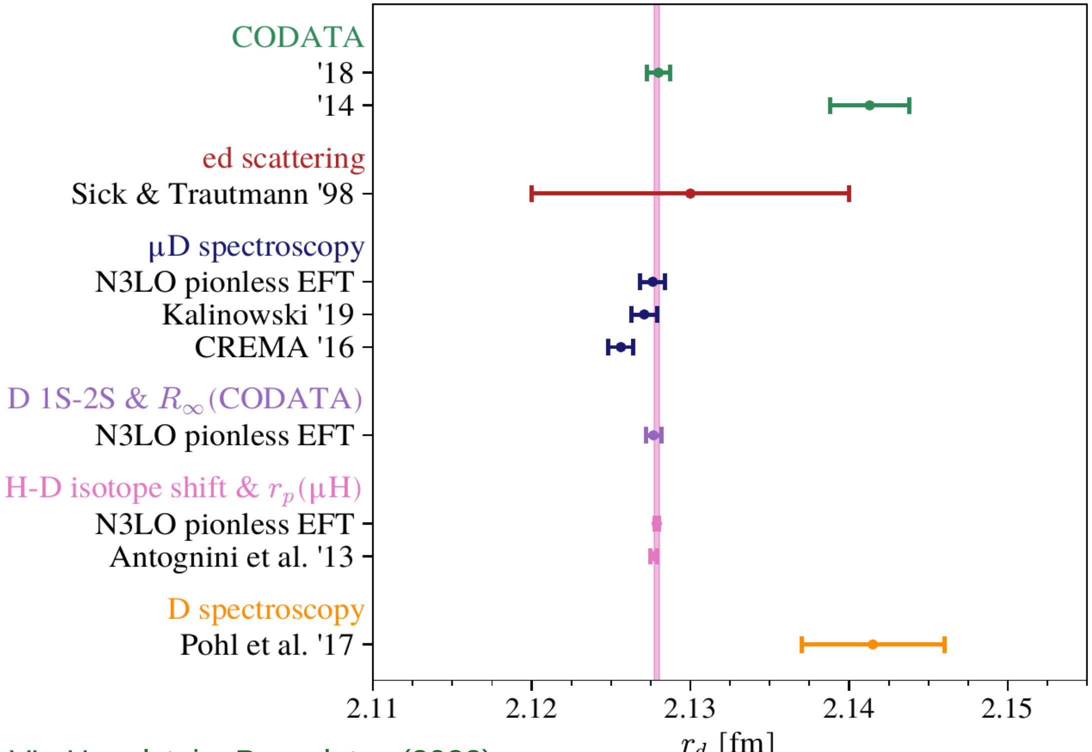
  - in total: small but sizeable: $\Delta E_{2S}^{\text{hadr}} = -0.032(6)\ \text{meV}$

$$\Delta E_{2S}^{2\gamma} = \Delta E_{2S}^{\text{elastic}} + \Delta E_{2S}^{\text{inel}} + \Delta E_{2S}^{\text{hadr}} + \Delta E_{2S}^{\text{eVP}} + \Delta E_{2S}^{\text{Coulomb}} = -1.752(20)\ \text{meV}$$

Deuteron Charge Radius and TPE in μD

- μD, D, and H-D isotope shift all consistent with one another
- Agreement with the very precise empirical value of 2γ exchange

	$E_{2S}^{2\gamma}$ [meV]
Theory prediction	
Krauth et al. '16 [5]	−1.7096(200)
Kalinowski '19 [6, Eq. (6) + (19)]	−1.740(21)
$\not\pi$EFT (this work)	−1.752(20)
Empirical (μH + iso)	
Pohl et al. '16 [3]	−1.7638(68)
This work	−1.7585(56)



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- Agreement with other calculations [most of those evaluate via structure functions (using χEFT/model NN interactions)]

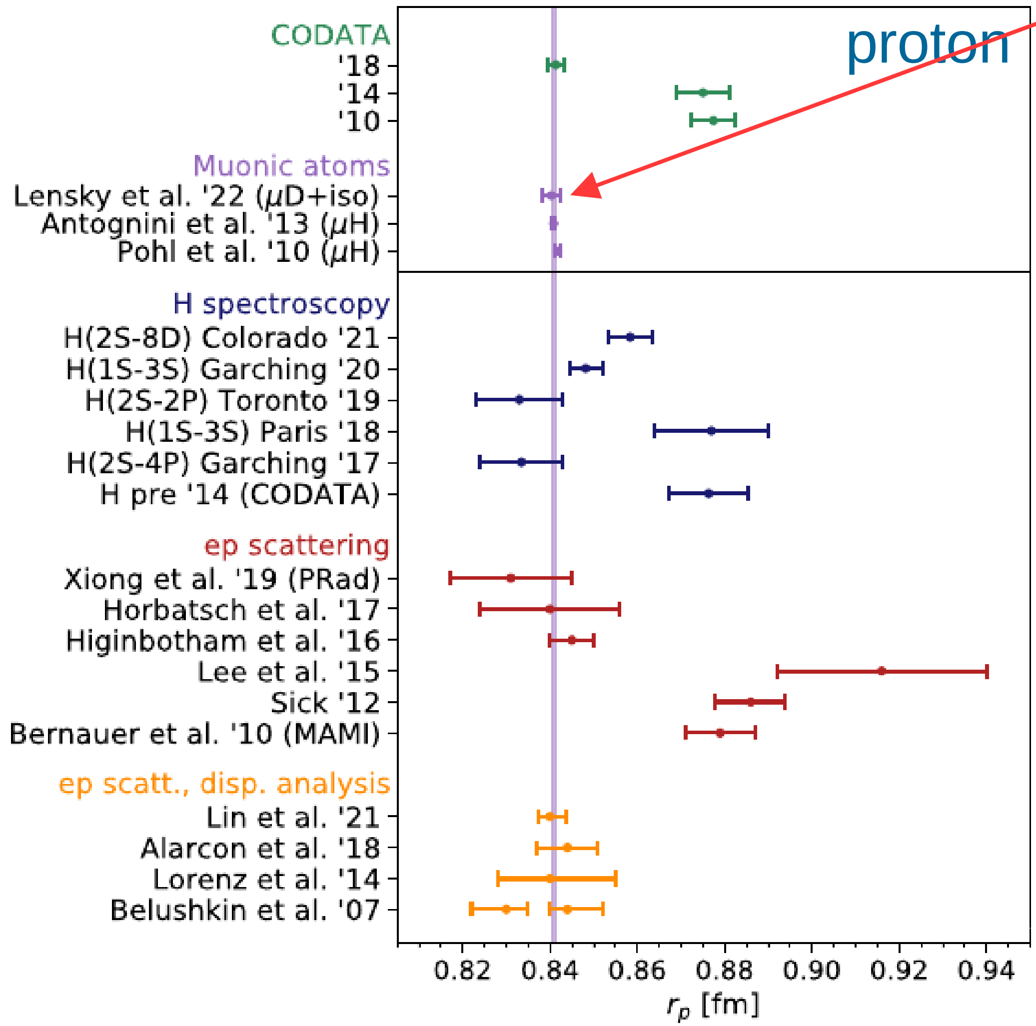
	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
E_{QED}	point nucleus	206.0344(3)	228.7740(3)	1644.348(8)	1668.491(7)
$\mathcal{C}\, r_C^2$	finite size	$-5.2259\, r_p^2$	$-6.1074\, r_d^2$	$-103.383\, r_h^2$	$-106.209\, r_\alpha^2$
E_{NS}	nuclear structure	0.0289(25)	1.7503(200)	15.499(378)	9.276(433)
E_L(exp)	experiment[a]	202.3706(23)	202.8785(34)	1258.612(86)	1378.521(48)

Proton and Deuteron Radii and Isotope Shift

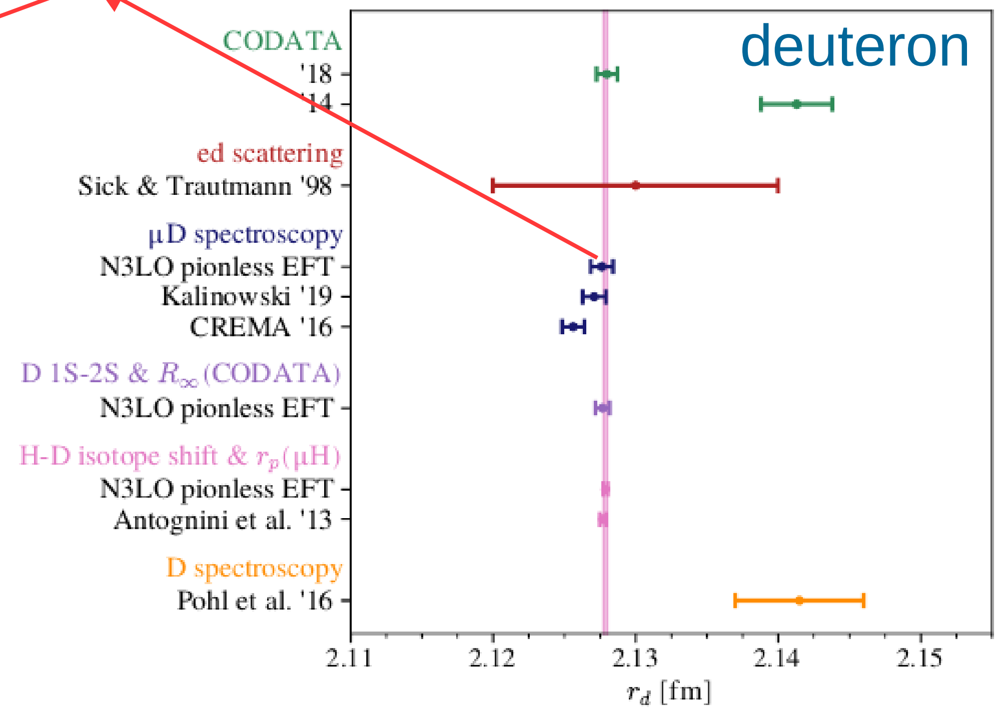
- H-D isotope shift: $E(H, 1S-2S) - E(D, 1S-2S)$

$$r_d^2 - r_p^2 = 3.820\,61(31)\ \mathrm{fm}^2$$

Jentschura et al. (2011)



Antognini, Hagelstein, Pascalutsa (2022)



VL, Hagelstein, Pascalutsa (2022)

- Muonic Deuterium and H-D isotope shift are consistent with the small proton radius

Summary and Outlook

- The mass of the muon sets a new scale that changes a lot of properties of muonic atoms compared to ordinary atoms
- Muonic (H-like) atoms and ions are important both due to their sensitivity to charge radii and their connection to nuclear/hadron physics (~TPE)
- EFTs often produce better results for TPE than data-driven approach
- Single-nucleon effects are sizeable, more importaint in heavier nuclei
- Higher-order radiative corrections are also becoming important
- μH: doing rather well, but need to shrink the TPE uncertainty by a factor of ~2 or more. How would it be possible (Lamb shift/HFS)? Is it time to revisit the HFS rescaling result?
- μD (and μHe^+): can one shrink the theory uncertainty?
- What would be a reasonable strategy for future theory calculations?

Thank You

- Many thanks to my colleagues and collaborators

  V. Pascalutsa, F. Hagelstein, J. McGovern, M. Birse, B. Acharya, S. Bacca, M. Gorchtein, K. Pachucki, ...

- And thank you for your attention!