ARXIV:2309.15929 ,
ARXIV:2309.07343 ,
ARXIV:2402.13307 ,
ARXIV:2402.14769 .

Coulomb & Radiative Corrections To β-Decay In EFT

ONGOING WORK

PETER VANDER GRIEND (UKY/FNAL), ZEHUA CAO (UKY)

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ELECTROWEAK PHYSICS INTERSECTIONS | CALASERENA RESORT | SEPT. 2024

Caltech

Neutrino Theory Network

PART 1

EFT & β DECAY

- Motivation & relevance for **fundamental physics**.
- Necessary **precision**, and requisite **loop orders**.

PART 2

FERMI FUNC.

- **Point-like** EFT of nuclei and leptons.
- The **Fermi function** from loops.

PART 3

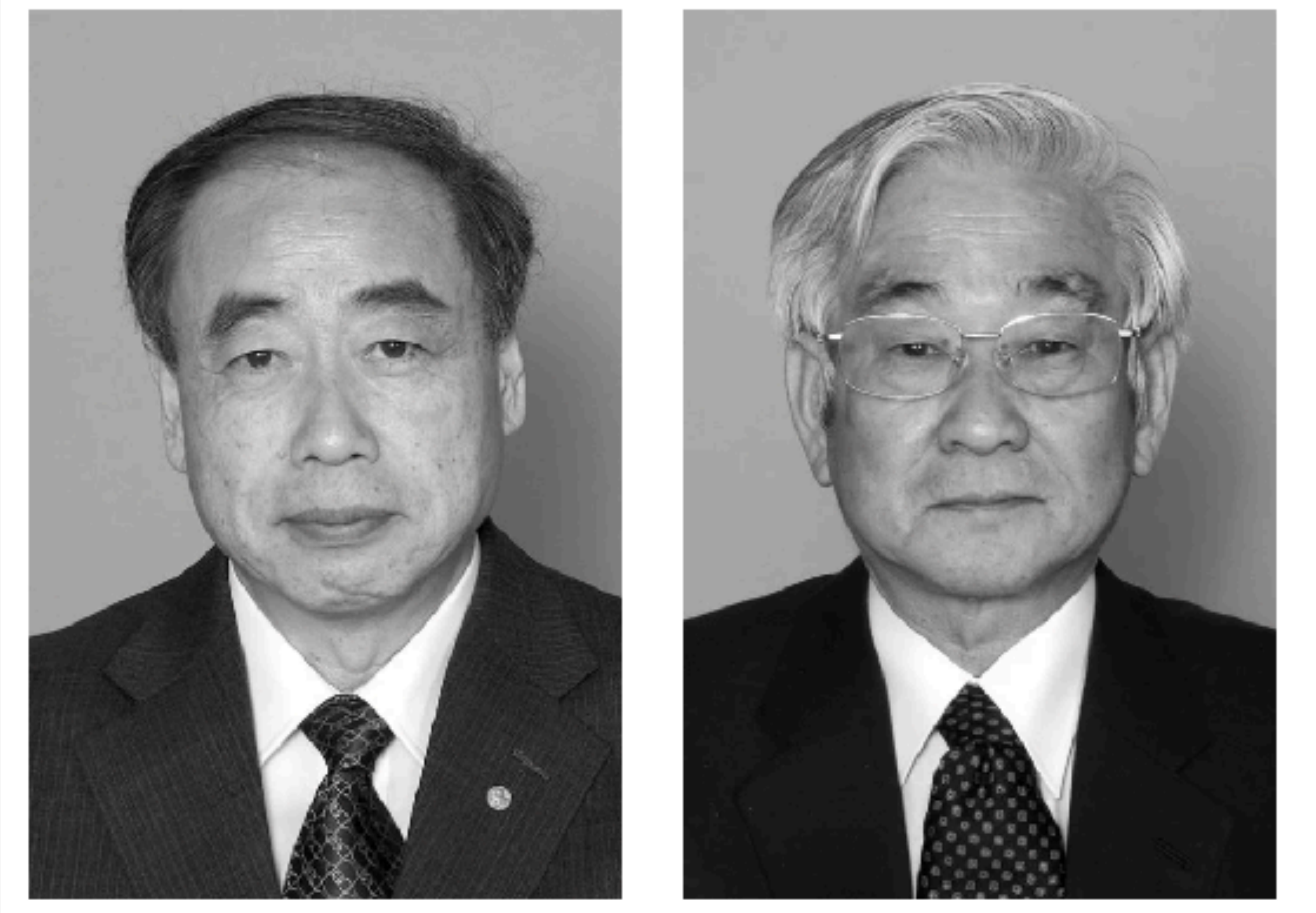
RAD. CORR.

- Structure of **radiative corrections** from EFT.
- Renormalization group **resummation of logarithms**.

Quark Mixing In The SM

FUNDAMENTAL CONSTANTS OF NATURE

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$



CKM ≡ CABIBBO–KOBAYASHI–MASKAWA

CKM Unitarity

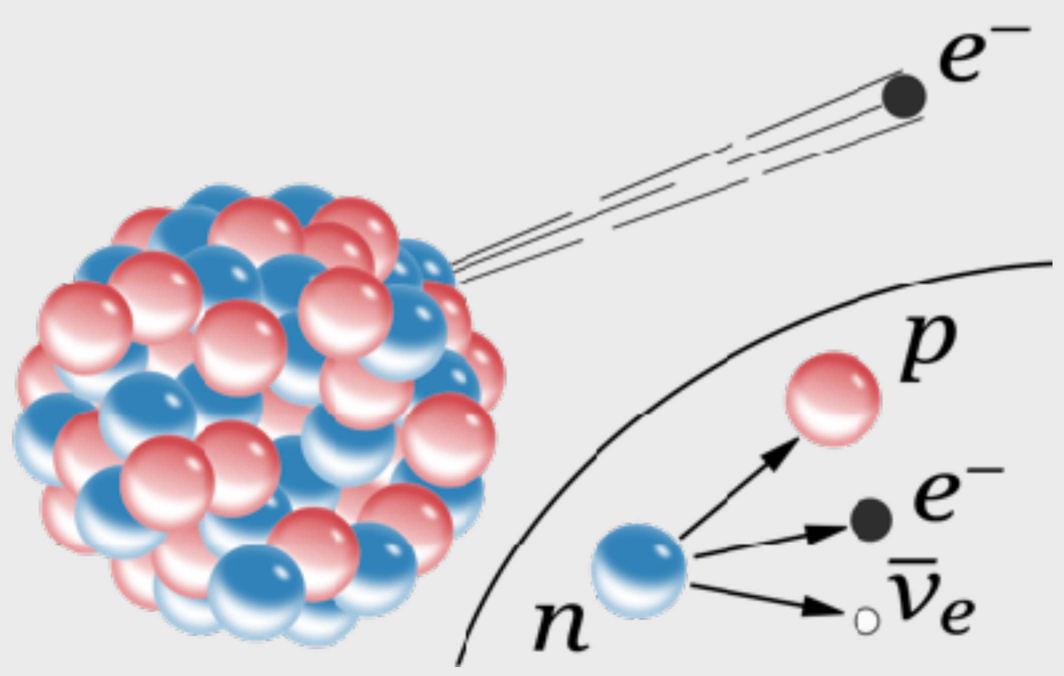
FIRST ROW UNITARITY

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

IN WOLFENSTEIN NOTATION

$$1 - \lambda_{ud}^2 + \lambda_{us}^2 + O(\lambda^6) = 1$$

$|V_{ud}|^2$



$|V_{us}|^2$

$$\frac{\Gamma(K \to \mu\nu(\gamma))}{\Gamma(\pi \to \mu\nu(\gamma))}$$

- Percent-level accuracy in Kaon decay demands 100 ppm accuracy in $0^+ \to 0^+$ beta decays

CKM Unitarity

FIRST ROW UNITARITY

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

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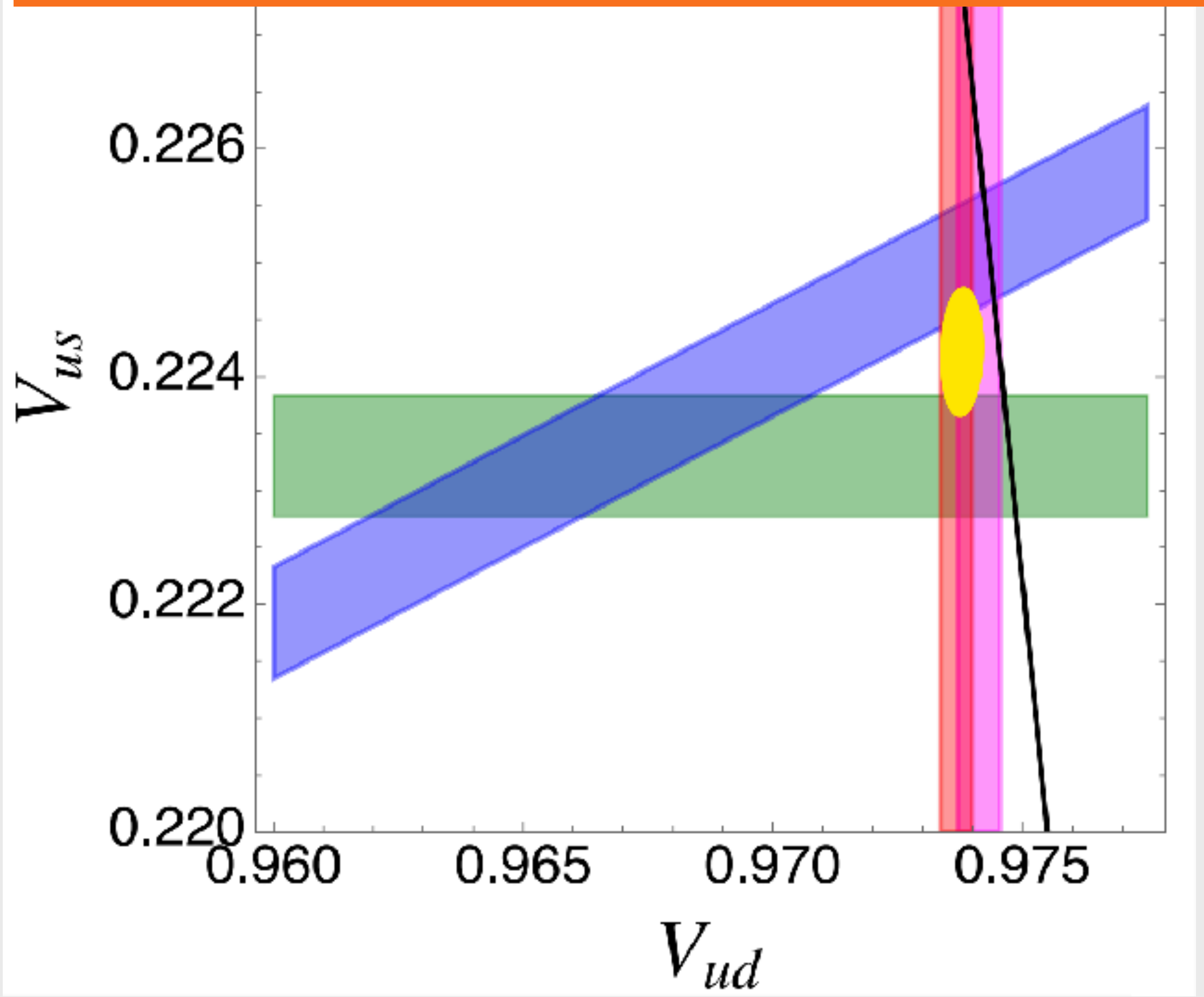
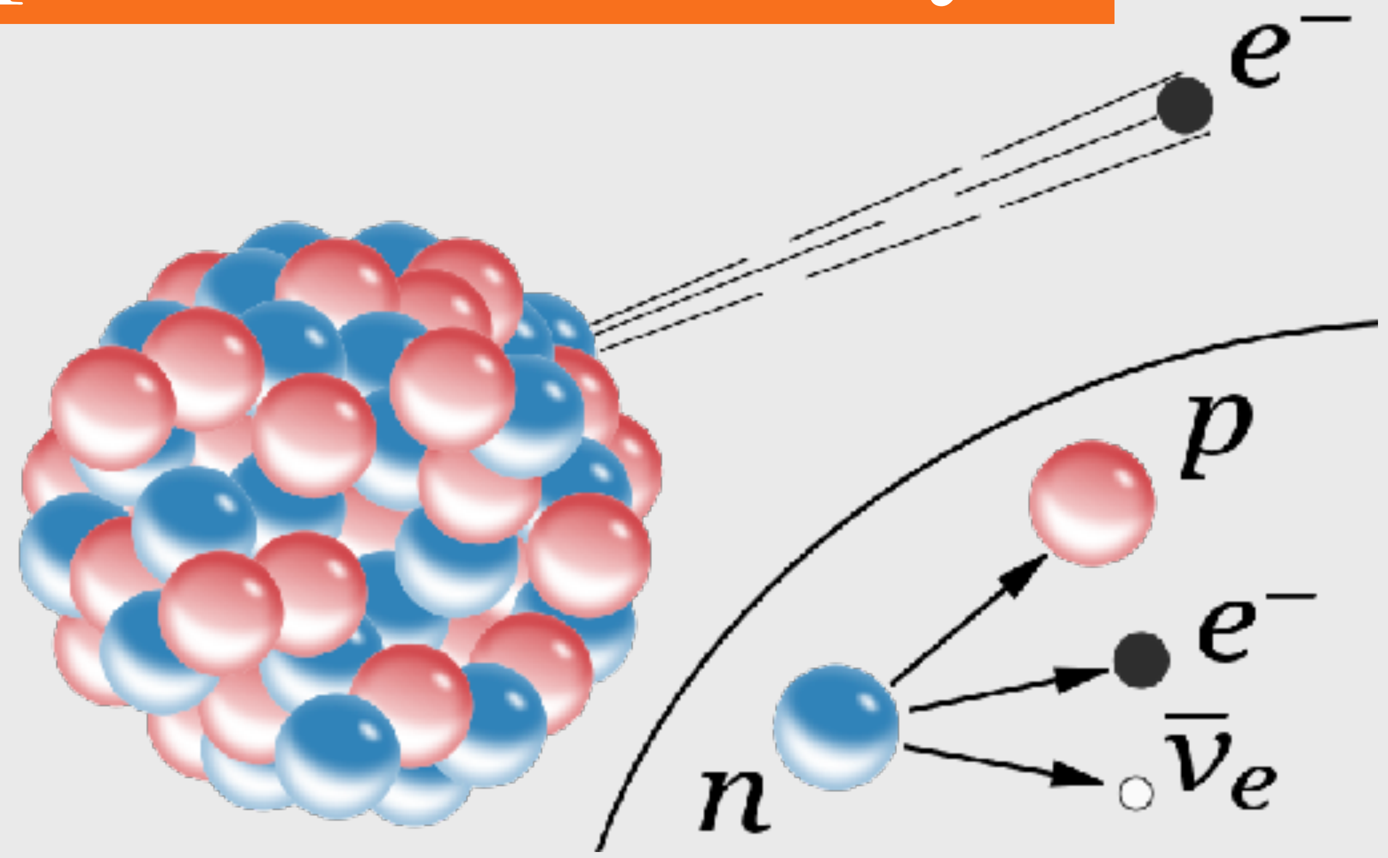
CIRIGLIANO +++ ARXIV:2208.11707



- Tension in first-row CKM unitarity.
- ***If*** theory is under control: new physics discovered!

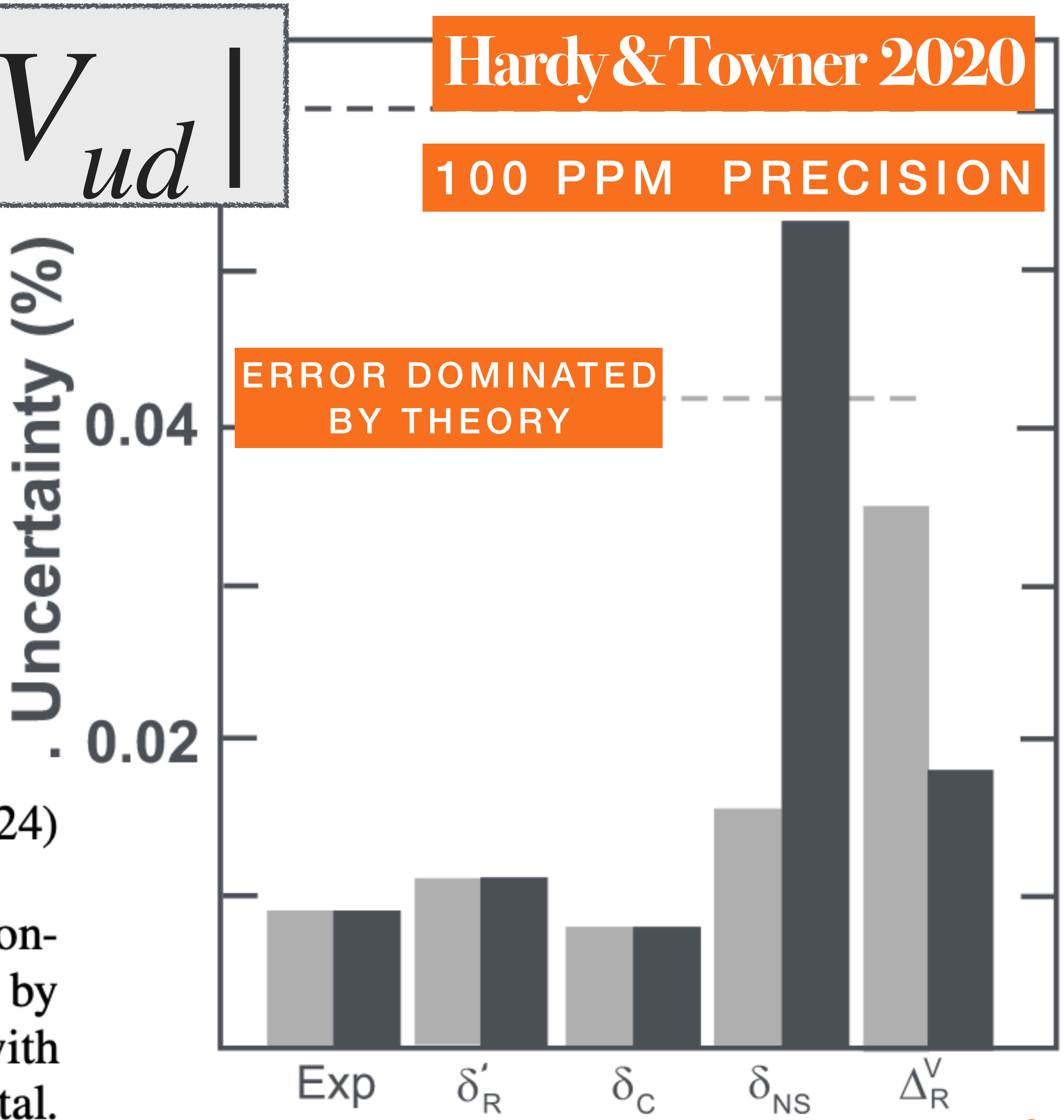
How To Measure $|V_{ud}|$

Superallowed Decays



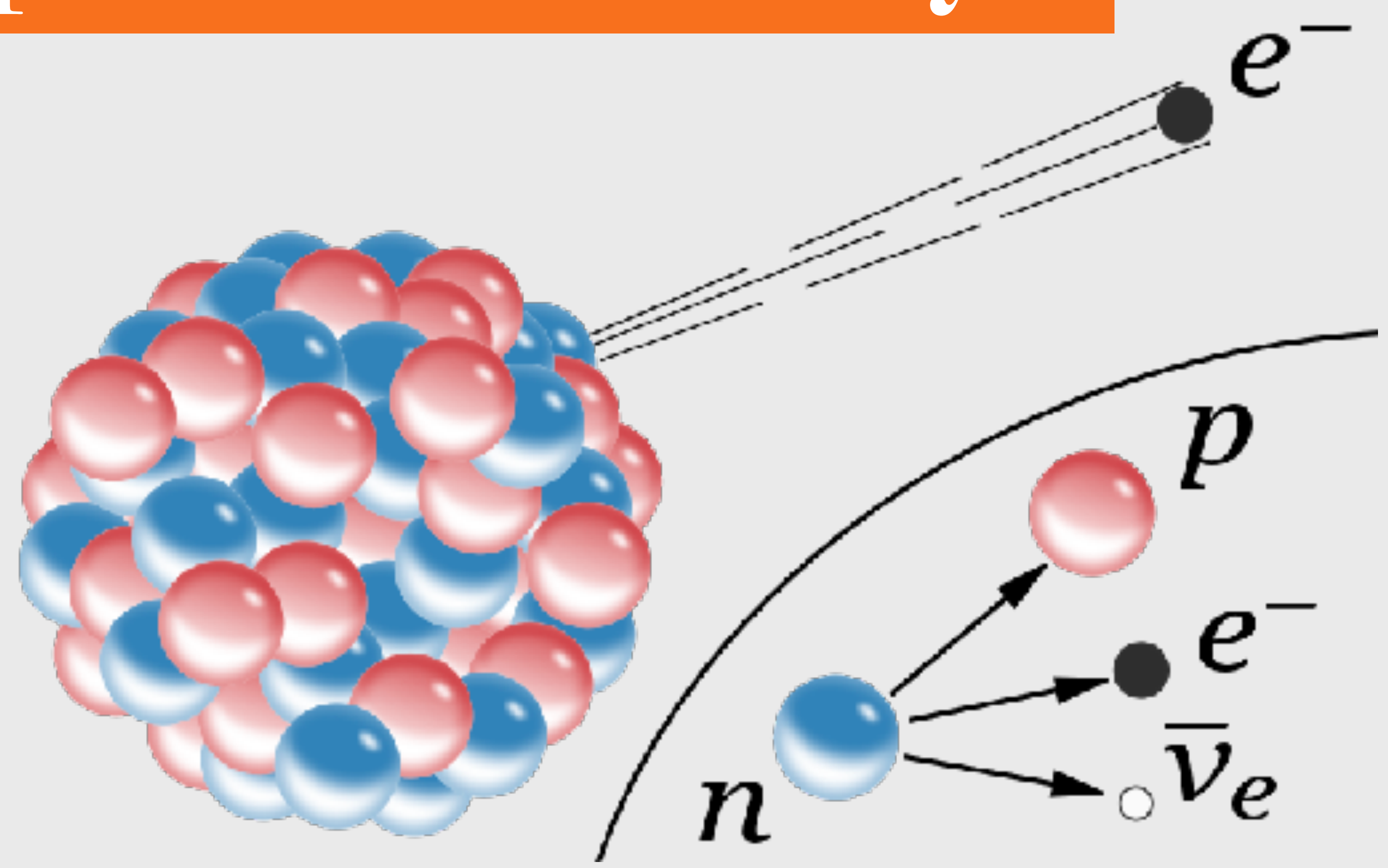
$$|V_{ud}|^2 = 0.94815 \pm 0.00060. \quad (24)$$

The uncertainty attached to $|V_{ud}|^2$ in Eq. (24) includes contributions from many sources but is completely dominated by those originating from the theoretical correction terms, with experiment contributing a mere 0.00009 to the 0.00060 total.



How To Measure $|V_{ud}|$

Superallowed Decays



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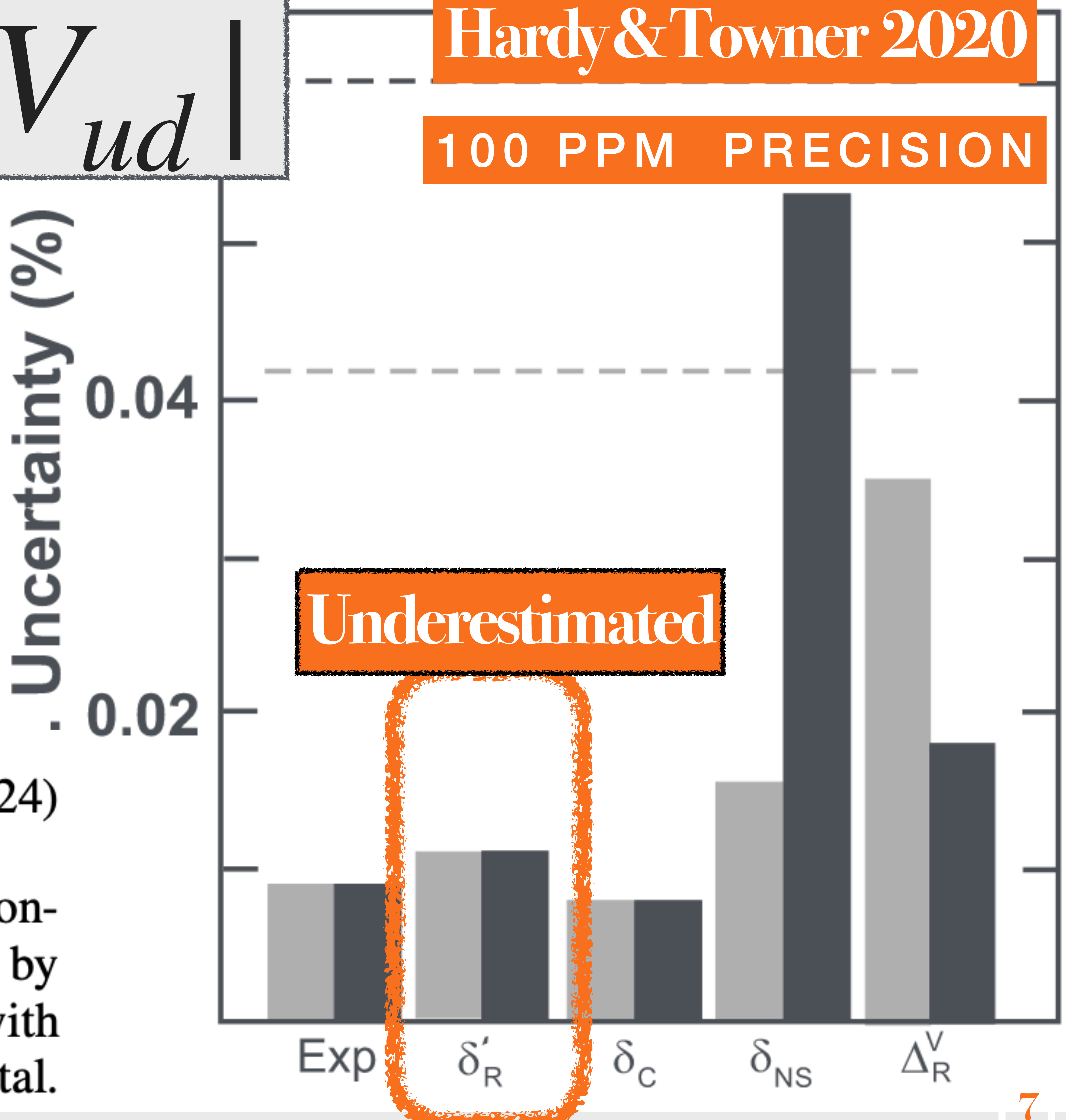
Hardy & Towner 2020

100 PPM PRECISION

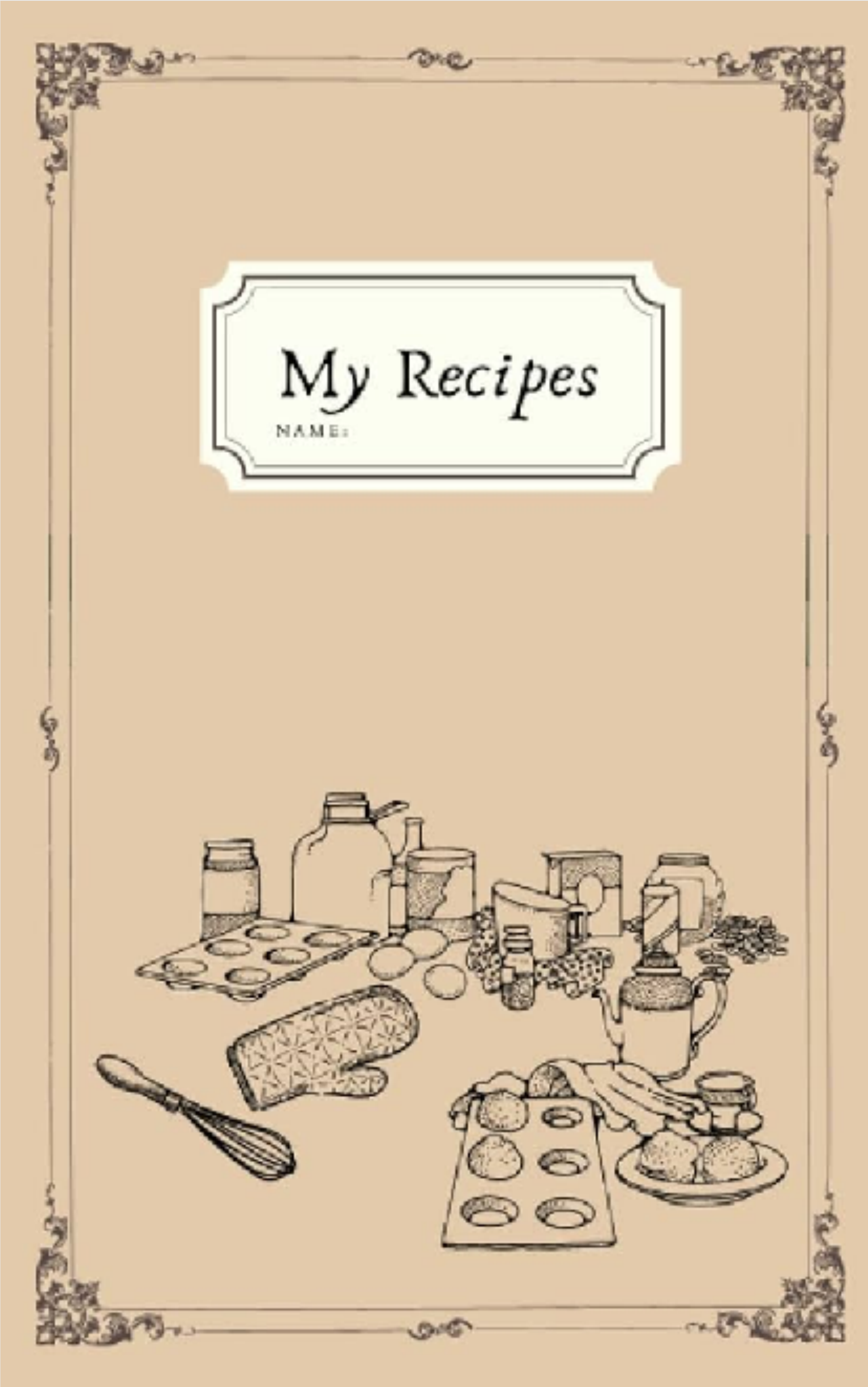


Historical Approach

$$\mathcal{F}t \equiv ft(1+\delta_R')(1+\delta_{\rm NS}-\delta_C) = \frac{K}{2G_V^2\left(1+\Delta_R^V\right)},$$

- The "ft" value includes the Fermi function (Dirac w.f.) .
- RCs are ***assumed*** to factorize (ansatz) from Fermi function.
- RCs are computed in the "independent particle model".

Towner & Hardy's Recipe



- Theorist's assignment of uncertainty on Δ_R.
- Use Sirlin & Zuchini + "heuristic estimate" for $\delta_R(Z)$. Assigned error is $\frac{1}{3}\delta_{\mathrm{HE}}^{(3)}(Z)$.
- Constrain $\delta_A(Z) \equiv \delta_{\mathrm{NS}} - \delta_C$ by demanding that the set of $\mathscr{F}t$ values agree (i.e., Z-independent).
- Average errors on δ_A treating them as statistical.

New Approach With EFT

$$\mathcal{F}t \equiv ft(1+\delta_R')(1+\delta_{NS}-\delta_C) = \frac{K}{2G_V^2(1+\Delta_R^V)}$$

Factorization Theorem At Leading Power + Corrections Of $\mathcal{O}\left((pR)^2\right)$

Long Distance

Short Distance

$$\Gamma = \int \mathrm{d}\Pi_e \mathrm{d}\Pi_\nu \left\langle |\mathcal{M}|^2 \right\rangle \times |C|^2 \;(2\pi)\delta(\Sigma E)$$

Matrix Element

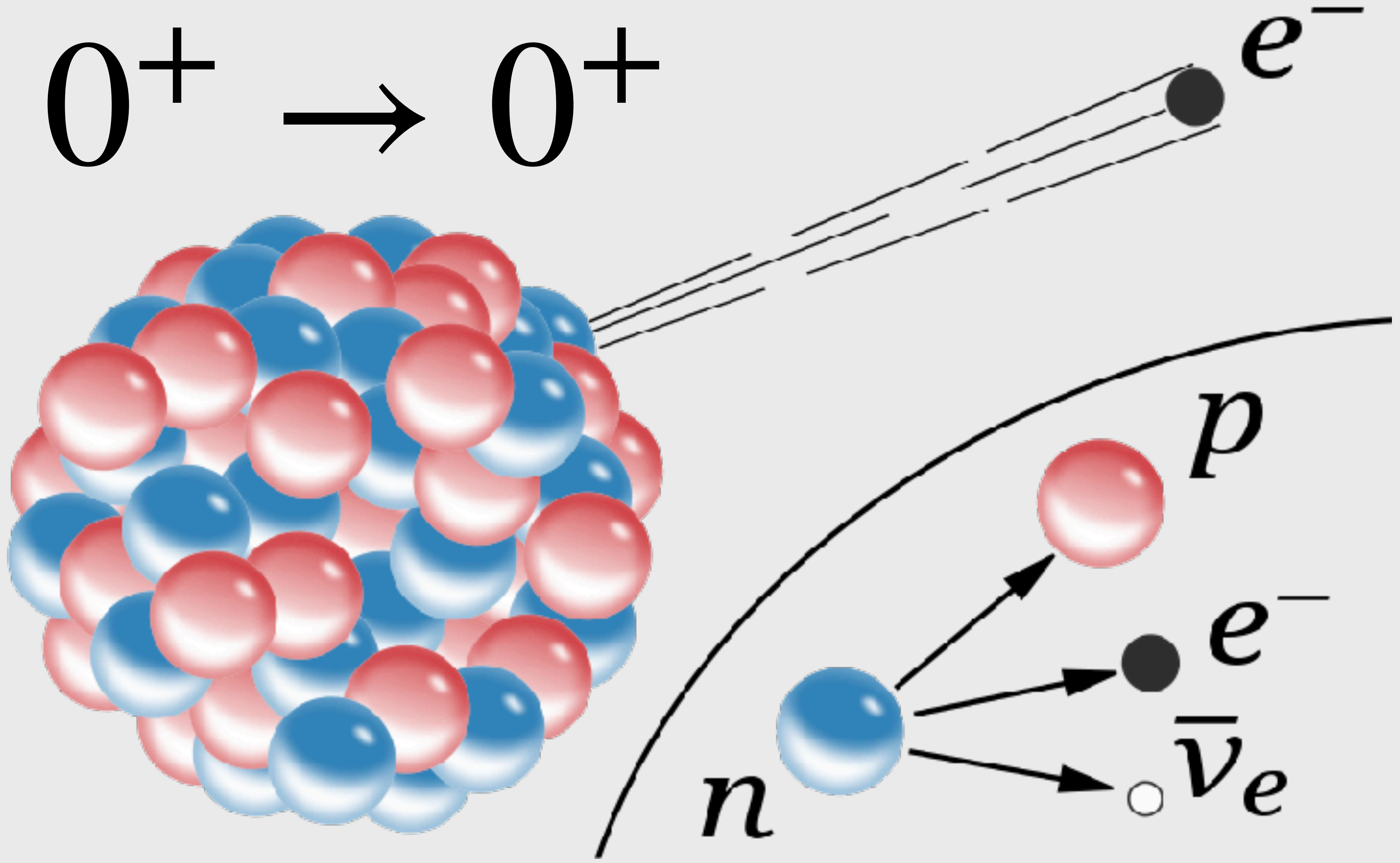
Wilson Coefficient

Calculate Matrix Element To High Order

- Scales run from m_e to m_W.
- Need control over corrections in low-energy theory ***at least*** at

$O(Z^2\alpha^3)$ i.e. 3+ loops

- **Fermi function emerges from summation of diagrams.**



- Precision goal: 100 ppm

Tower Of EFTs

W-boson mass	80 GeV
Nucleon mass	1 GeV
Fermi Momentum	200 MeV
Nuclear Radius	50 MeV
Lepton Energy	10 MeV

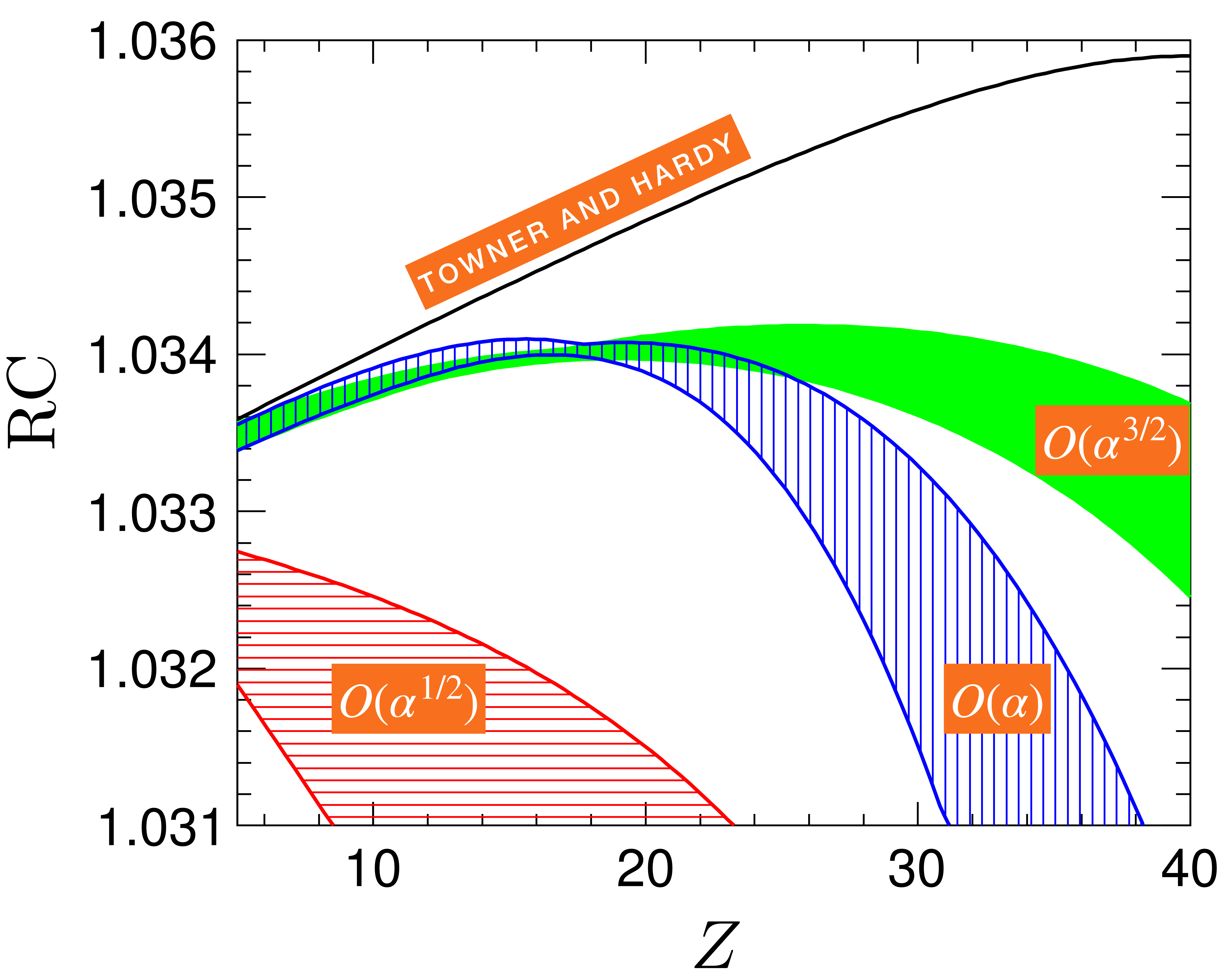
SEE WOUTER'S TALK FROM MONDAY (W-boson mass through Nuclear Radius)

$\}\, \mathrm{EFT}_{\text{long dist.}}$ (Lepton Energy)

TODAY'S TALK

Impact For Flavour Physics

- New analysis allow RG-resummation of logarithms.
- Consistent treatment of $Z^2\alpha^3$ and higher order corrections.
- Relevant at the level required for tests of CKM unitarity.



COUNTING $Z \sim \log \sim 1/\sqrt{\alpha}$

Impact For Flavour Physics

SHIFTING δ_3

transition	$(\Delta a) \times Z^2\alpha^3 \log(\Lambda/m)$
$^{14}O \to {}^{14}N$	-1.1×10^{-4}
$^{26m}Al \to {}^{26}Mg$	-3.2×10^{-4}
$^{34}Cl \to {}^{34}S$	-5.6×10^{-4}
$^{34}Ar \to {}^{34}Cl$	-6.3×10^{-4}
$^{38m}K \to {}^{38}Ar$	-7.1×10^{-4}
$^{42}Sc \to {}^{42}Ca$	-8.7×10^{-4}
$^{46}V \to {}^{46}Ti$	-10.5×10^{-4}
$^{50}Mn \to {}^{50}Cr$	-12.5×10^{-4}
$^{54}Co \to {}^{54}Fe$	-14.6×10^{-4}

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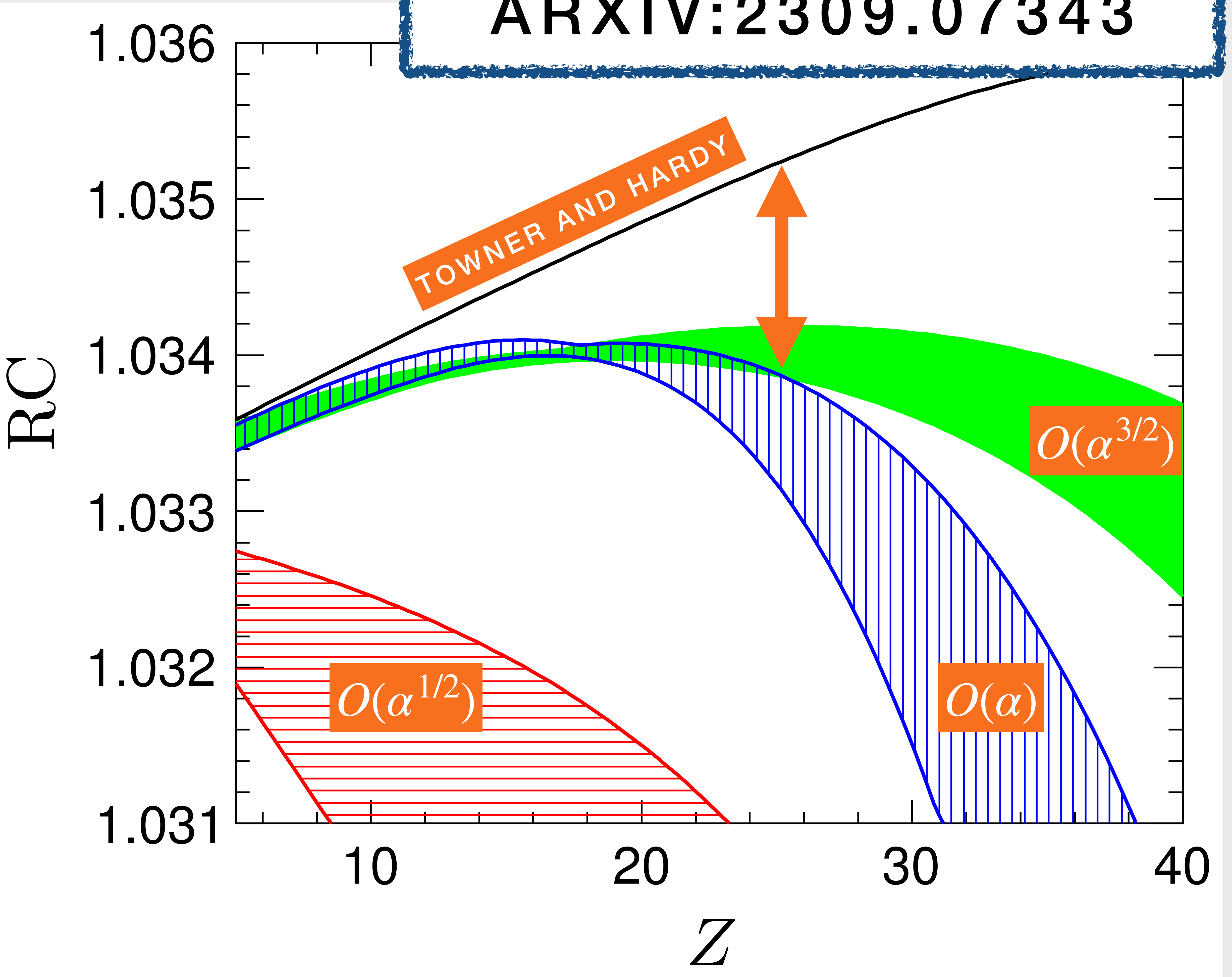

COUNTING $Z \sim \log \sim 1/\sqrt{\alpha}$

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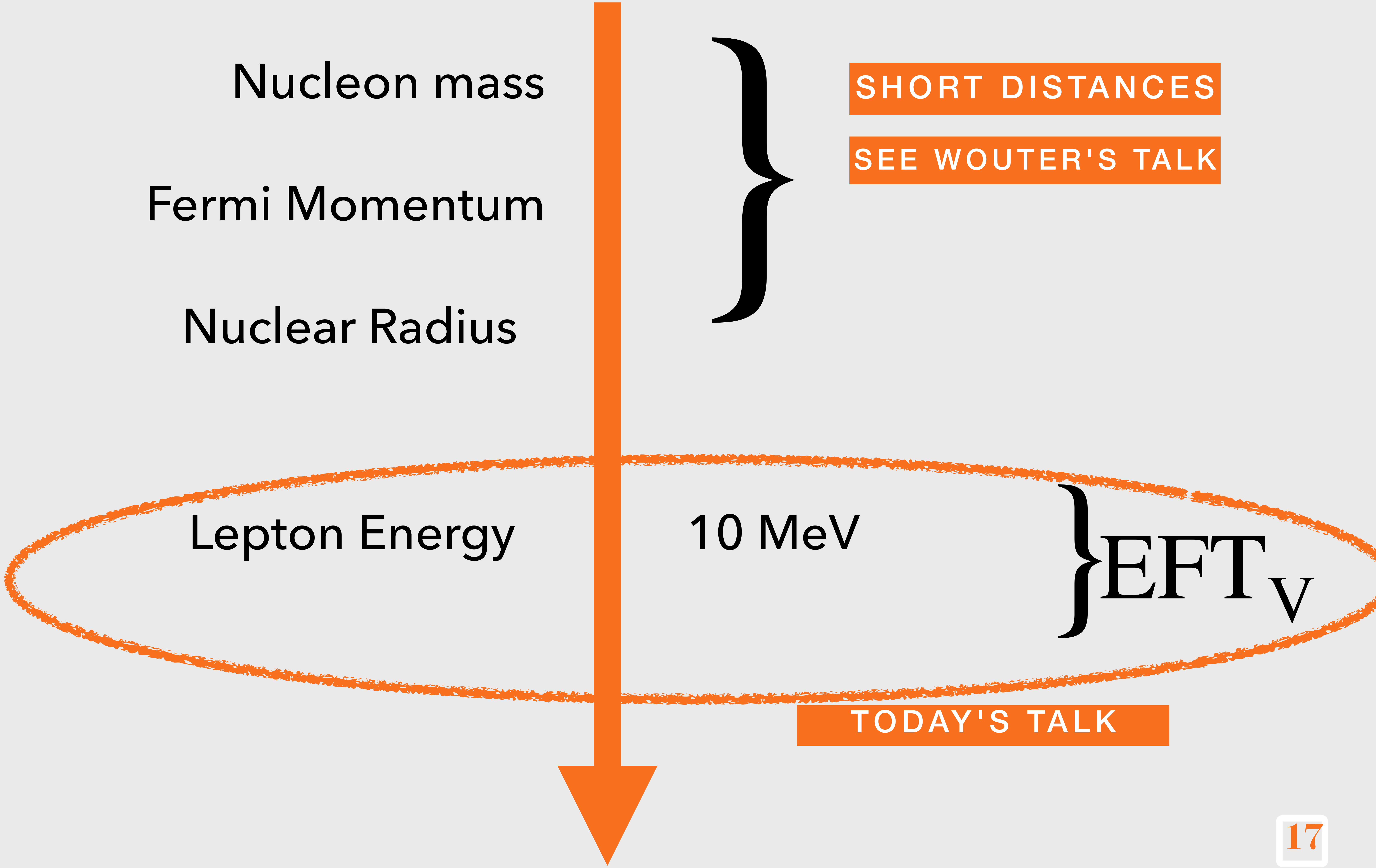
Point-Like EFT Of Nuclei

A Lagrangian For Low-Energy Beta Decay

EFT For $0^+ \to 0^+$

- Largest corrections come from long distance scales.
- Need to work to higher orders in perturbation theory.



EFT For $0^+ \to 0^+$

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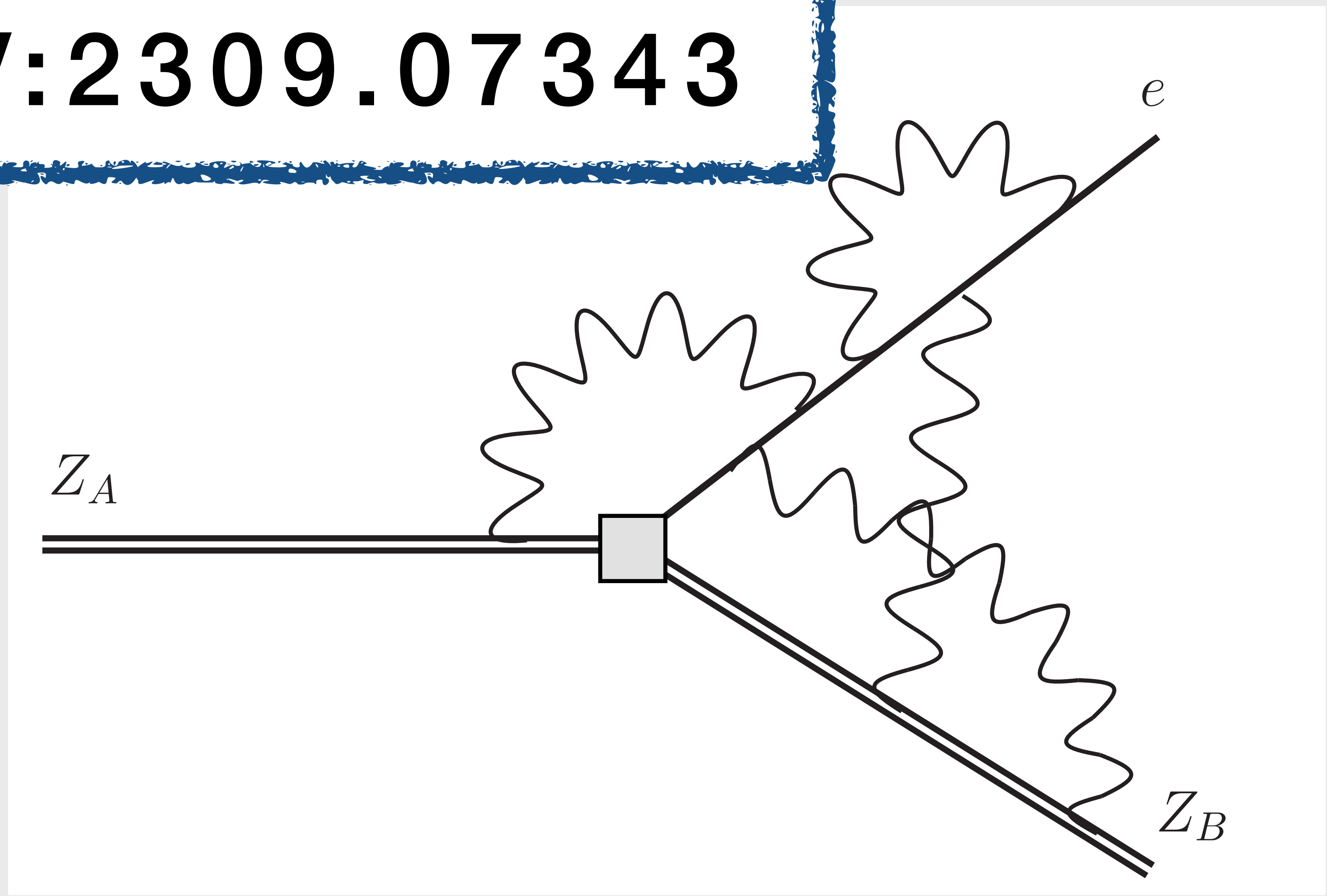


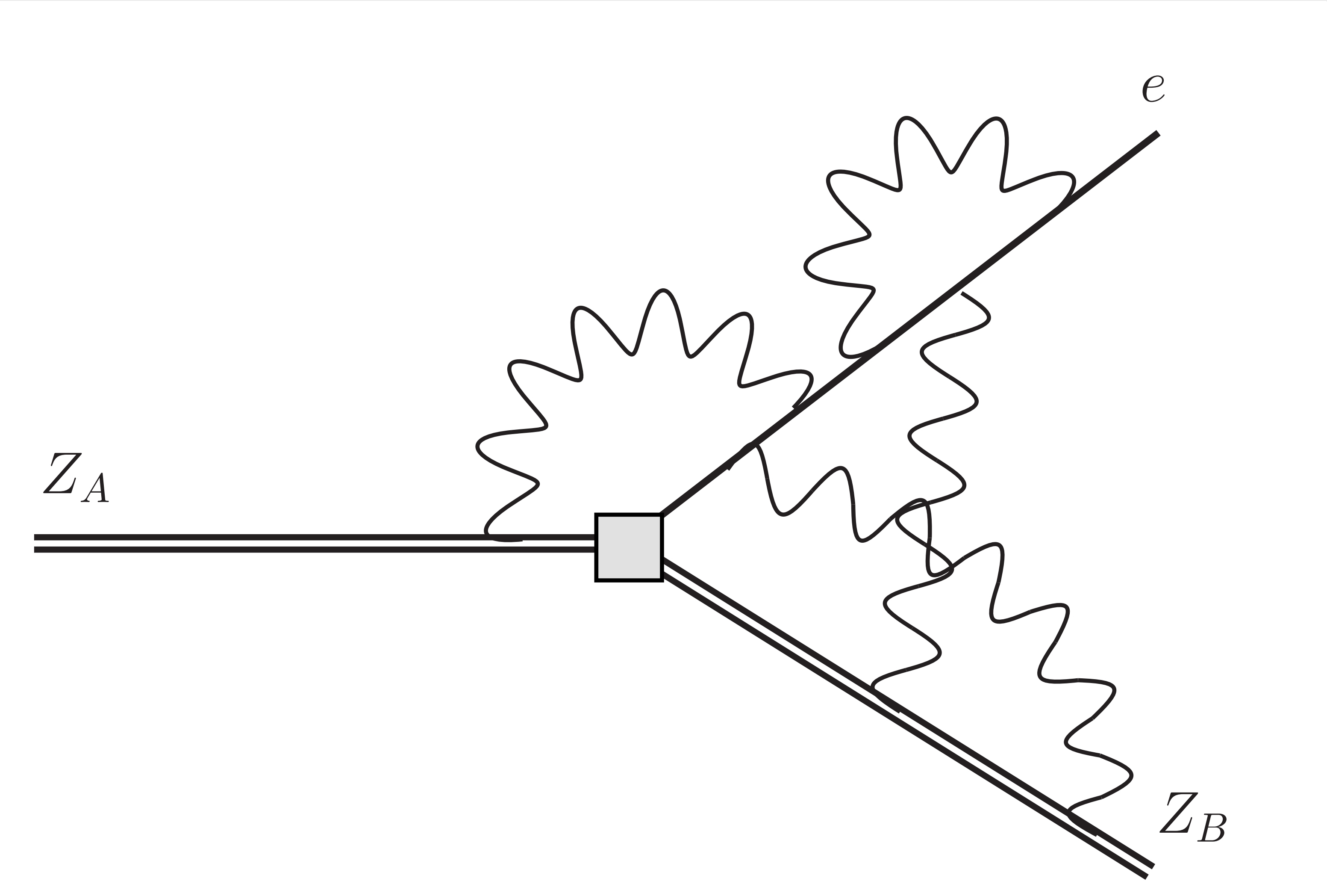
$$\mathcal{L} = h_A^\dagger (v\cdot D) h_A + h_B^\dagger (v\cdot D) h_B$$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{e}(\gamma_\mu D^\mu + m)e + \bar{\nu}\gamma^\mu \partial_\mu \nu$$

$$+C(\mu)\times\left[\bar{e}\gamma_\mu P_L \nu\right]\times\left[h_B^\dagger v^\mu h_A\right]$$

Heavy Sources + QED + Weak Current

EFT For $0^+ \to 0^+$



$\mathscr{L} =$ Heavy Nuclei

Quantum Electrodynamics

Weak Interaction

Heavy Particle EFTs

$$\frac{2M}{(p+k)^2 - M^2} \rightarrow \frac{1}{v \cdot k}$$

$$v = p/M$$

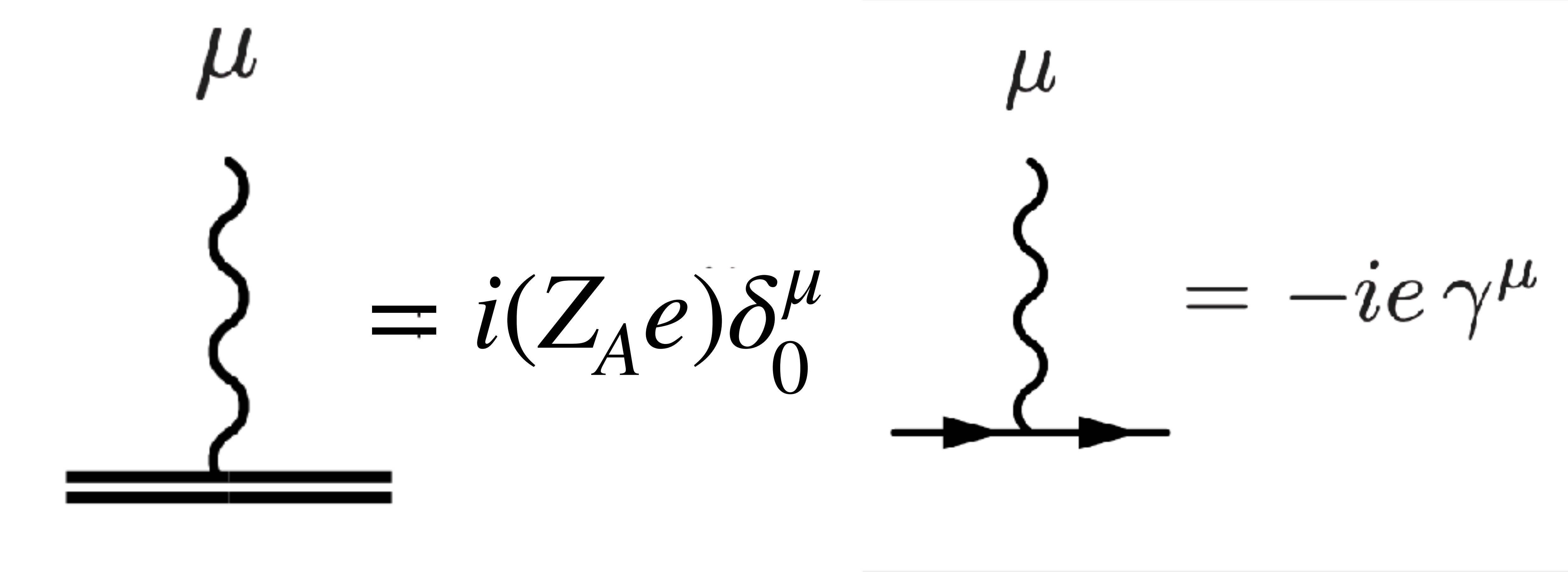
- This simplifies amplitudes.
- Heavy mass never appears.

Heavy Particle EFTs

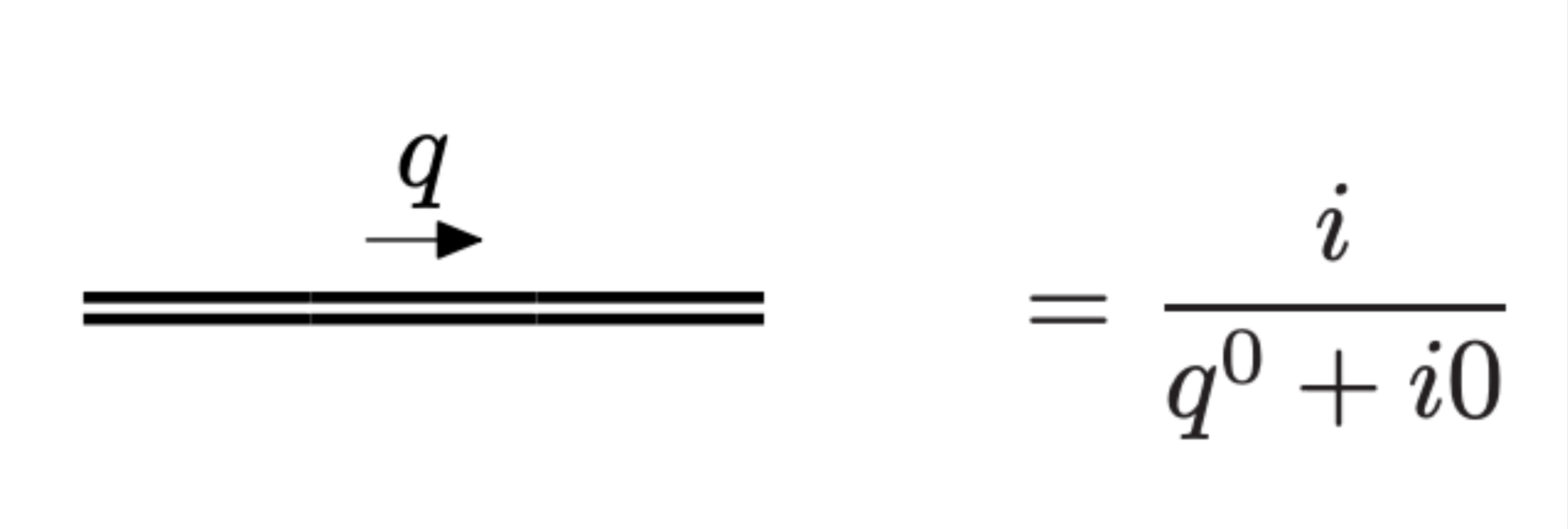
$$\mathscr{L} = h_v^\dagger (v \cdot D) h_v$$

Simplifications

$$v_\mu \quad \text{VS} \quad \gamma_\mu$$

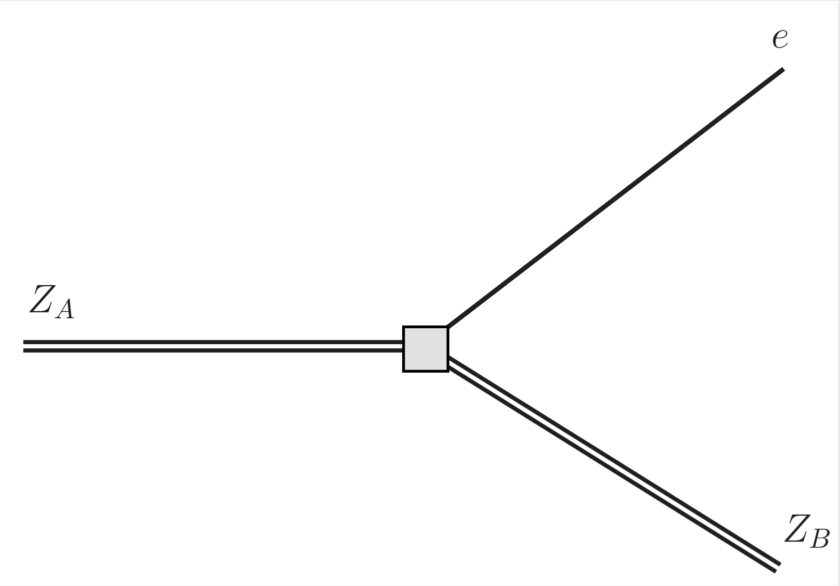


$$v \cdot q \quad \text{VS} \quad q^2 - m^2$$



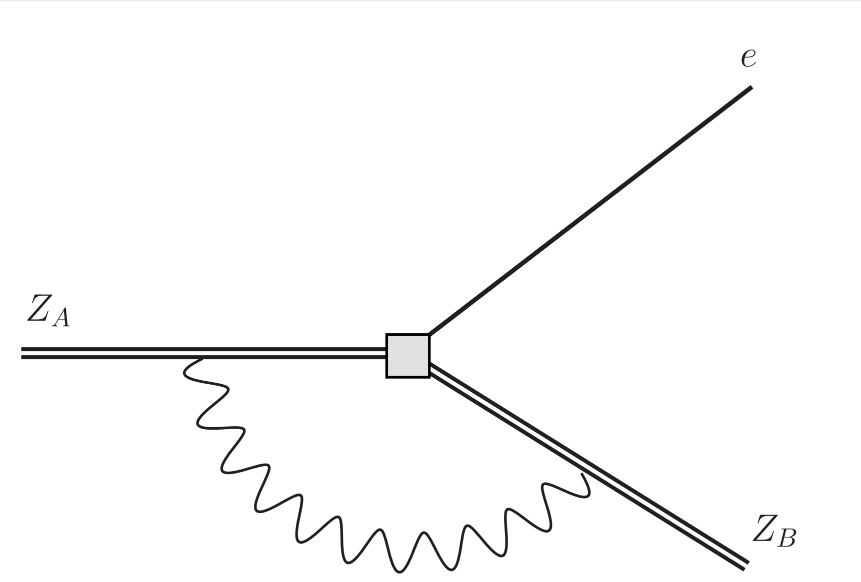
Now We Just Compute Diagrams

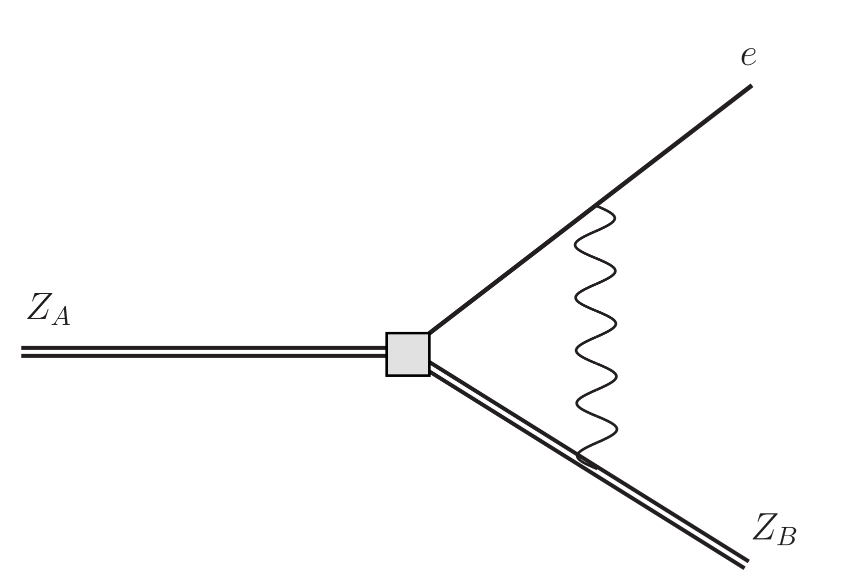
WAVEFUNCTION RENORMALIZATION NOT SHOWN

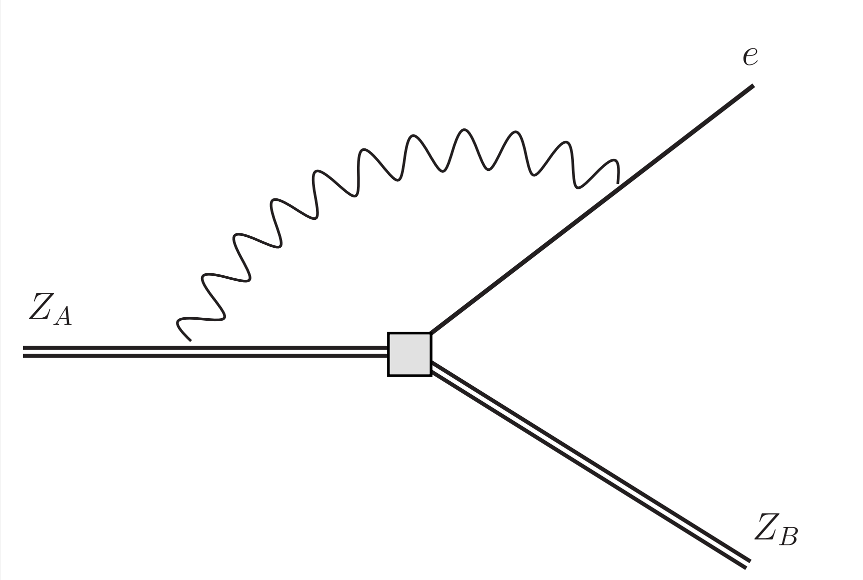


TREE-LEVEL

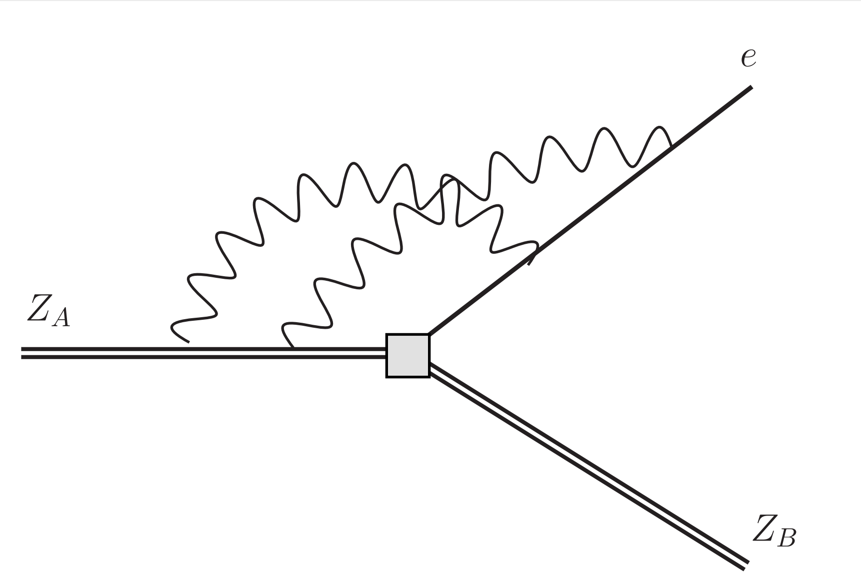
ONE LOOP

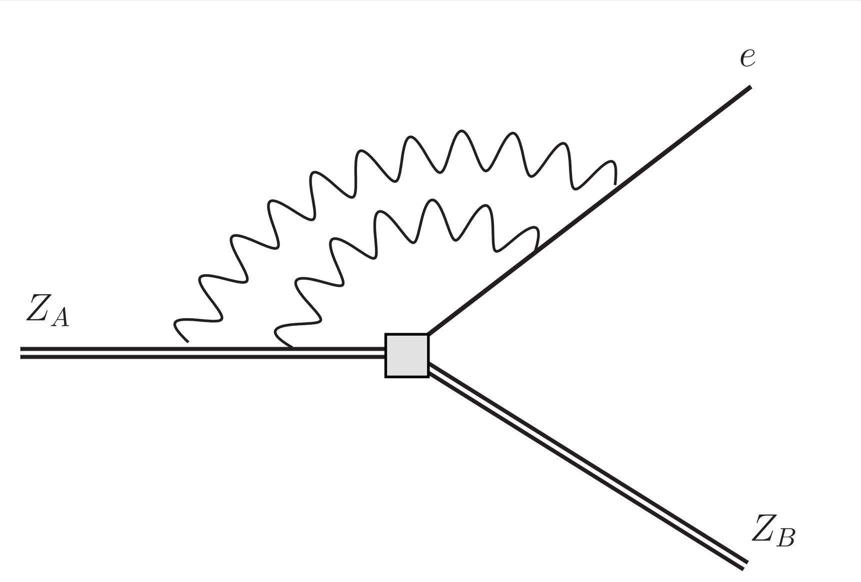


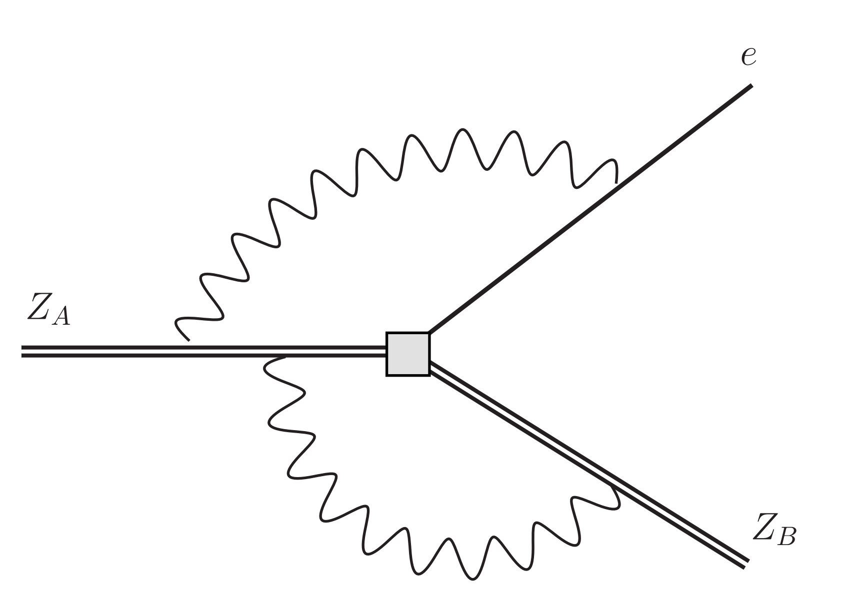


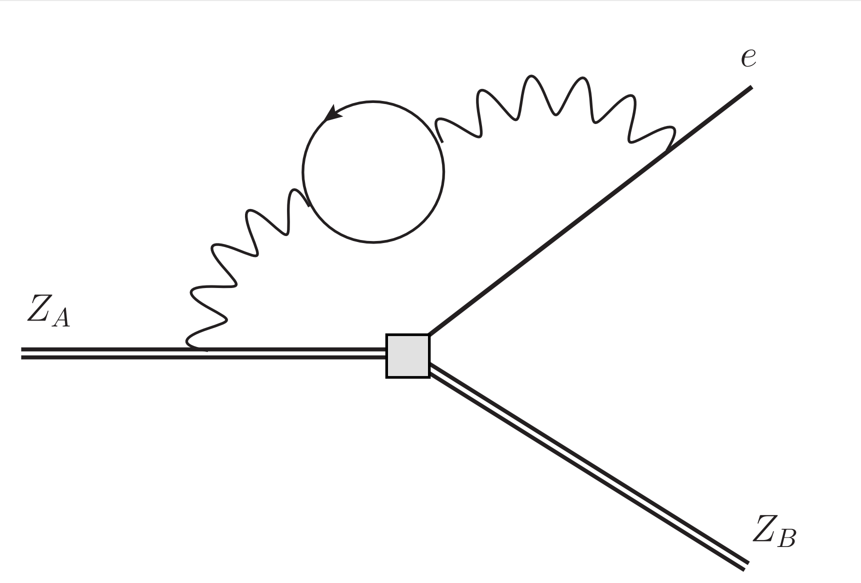


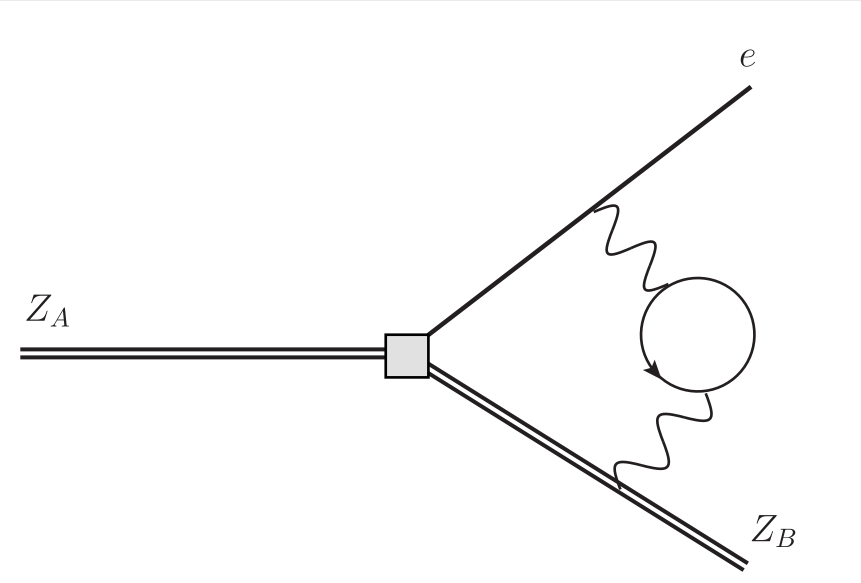
TWO LOOP

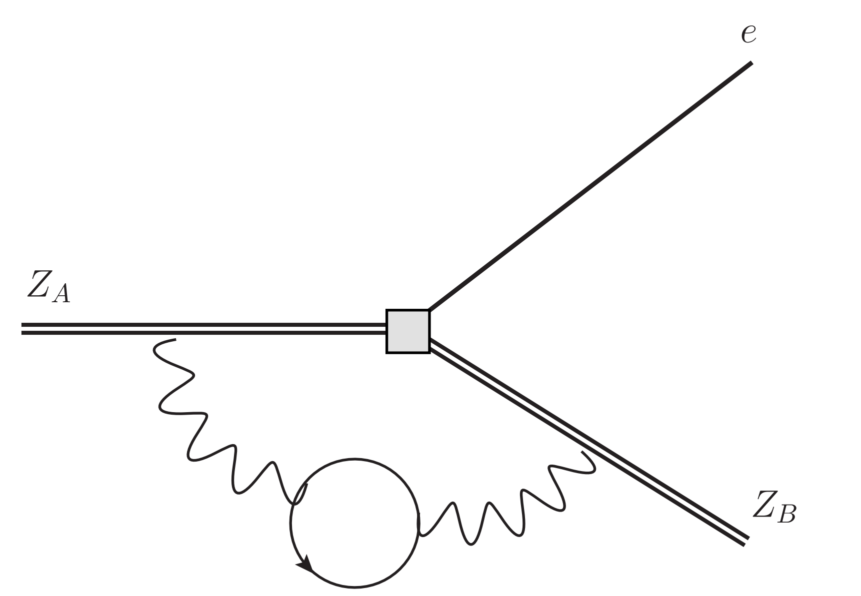


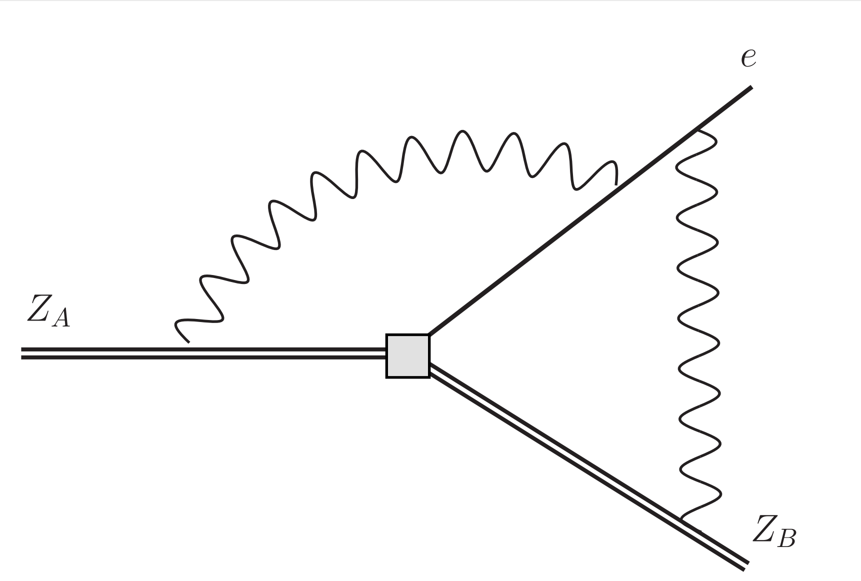


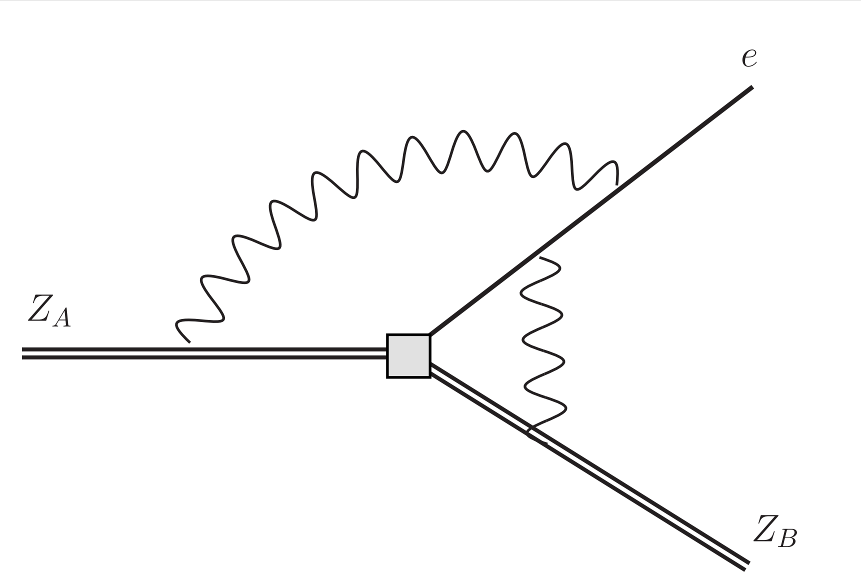












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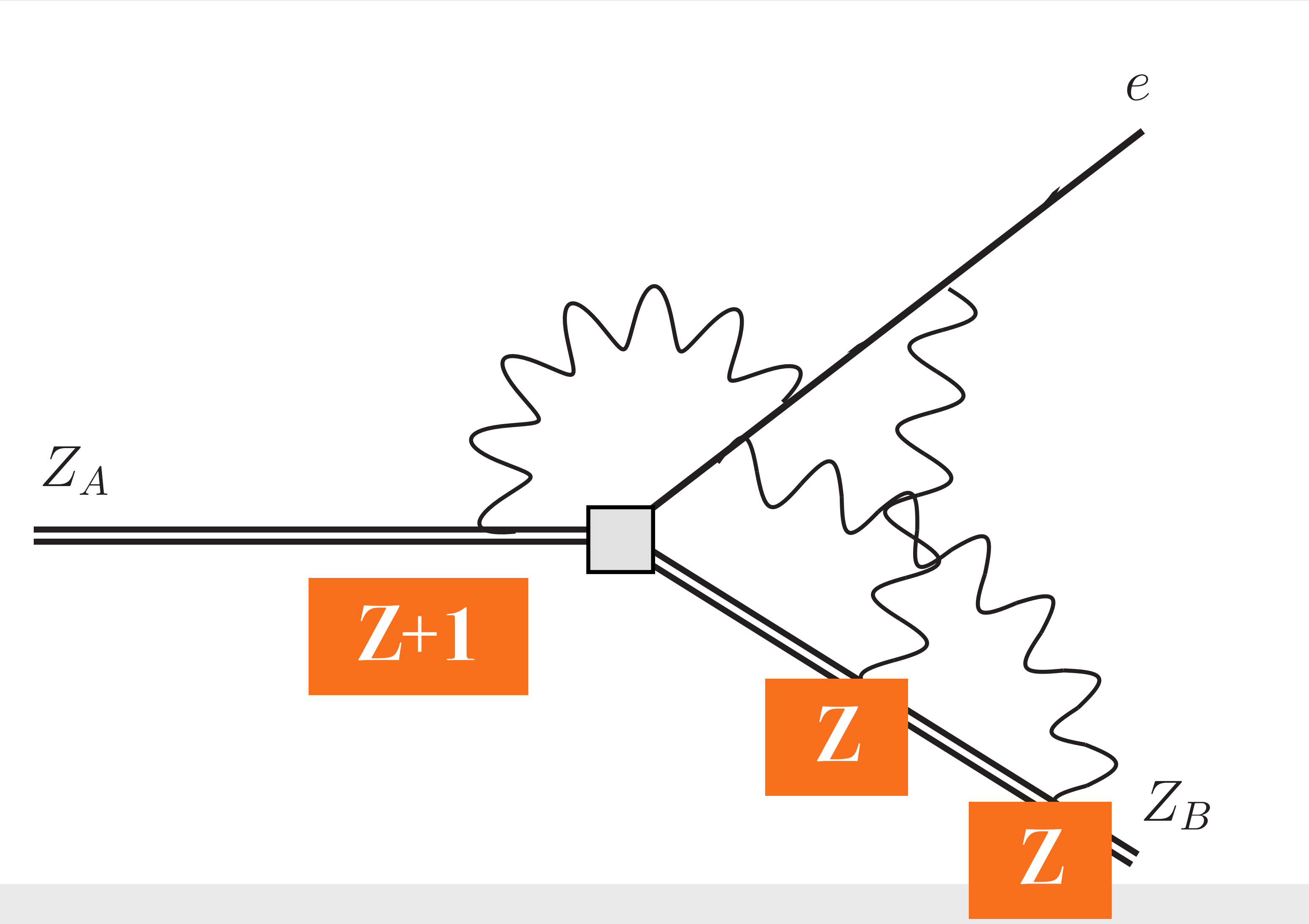
Sketch Of The Problem

$$\mathcal{F}t \equiv ft(1+\delta'_R)(1+\delta_{NS}-\delta_C) = \frac{K}{2G_V^2(1+\Delta_R^V)}$$

Fermi-Function



``Outer'' Corrections

$$(Z)(Z+1)^2e^6 = Z^3e^6 + 2Z^2e^6 + Ze^6$$

Fermi-Function

- Keeping track of factors of Z is non-trivial

Eikonal Identities

How Coulomb Physics Emerges Diagramatically

Number Of Diagrams Grows Factorially

TREE-LEVEL

- 1 diagram.

ONE LOOP

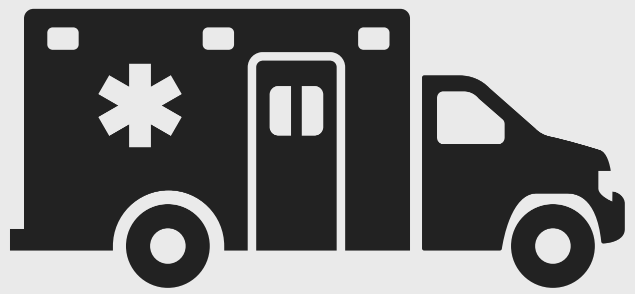
- 3 diagrams.

TWO LOOP

- 21 diagrams.

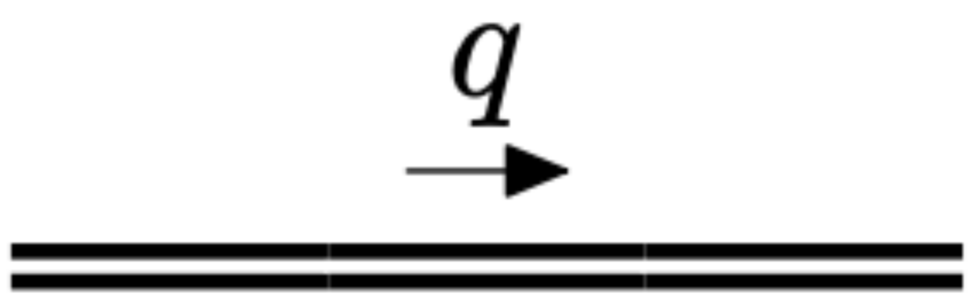
THREE LOOP

- 144 diagrams.

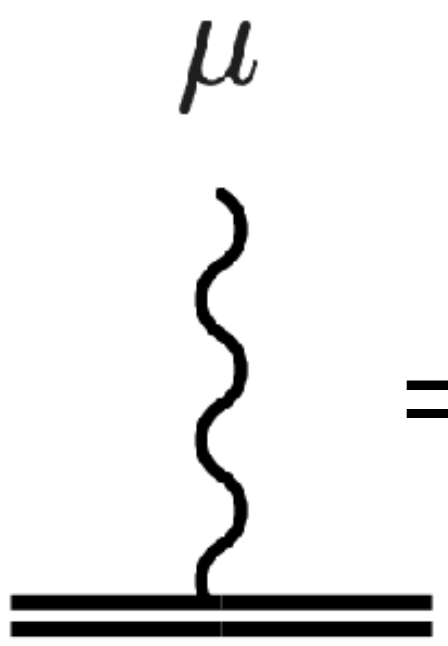


- For the Fermi function we need 4+ loops.
- This is not feasible by brute force.

Solution: Make Use Of Simplified Feynman Rules



$$= \frac{i}{q^0 + i0}$$



$$= i(Z_A e)\delta^{\mu}_{0}$$

Number Of Diagrams Grows Factorially

TREE-LEVEL

- 1 diagram.

ONE LOOP

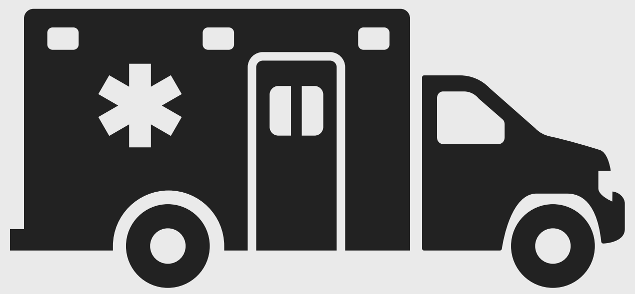
- 3 diagrams.

TWO LOOP

- 21 diagrams.

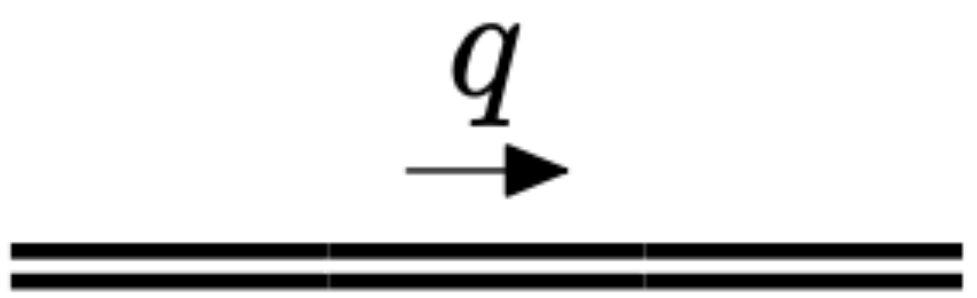
THREE LOOP

- 144 diagrams.

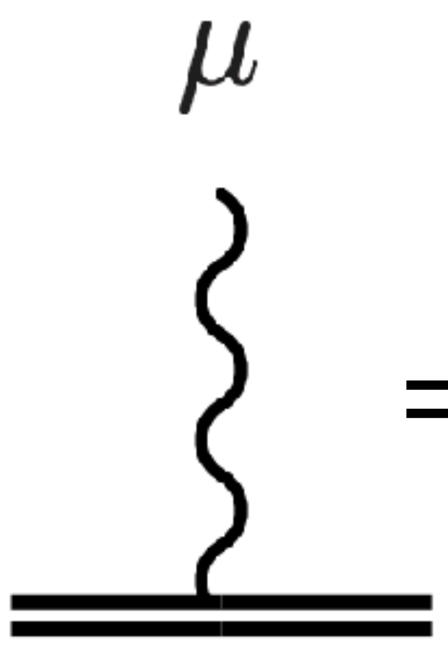


Reduce Number Of Diagrams
Avoid Difficult Integrals

Solution : Make Use Of Simplified Feynman Rules



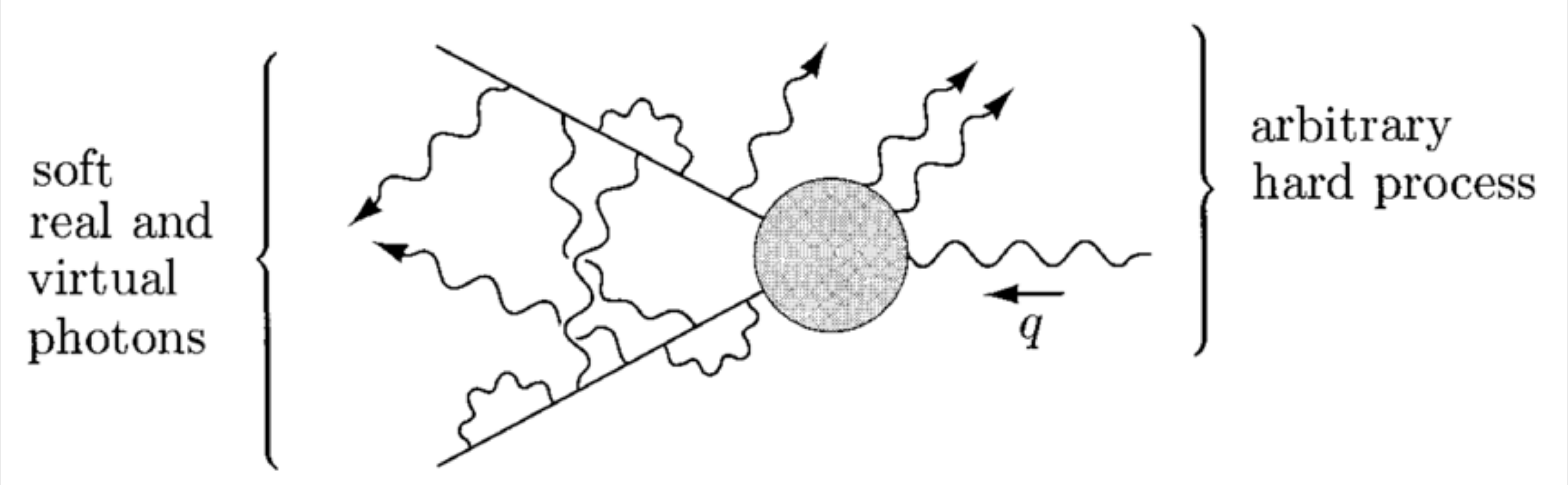
$$= \frac{i}{q^0 + i0}$$

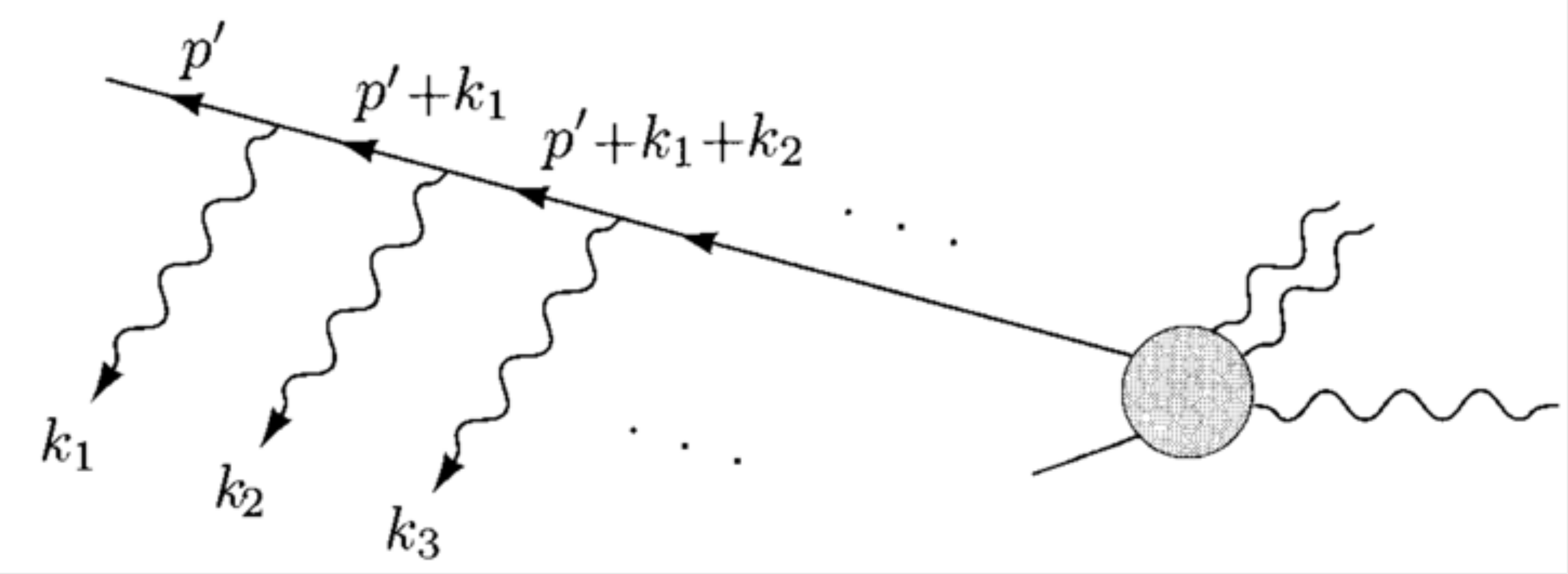


$$= i(Z_A e)\delta_0^\mu$$

Eikonal Identities

- Theory simplifies when we take the $M \to \infty$ limit (see e.g. YFS 1961)





- For heavy-particles in initial and final state, we get **Coulomb physics**

$$\frac{1}{v \cdot q + \mathrm{i}0} + \frac{1}{-v \cdot q + \mathrm{i}0} = (2\pi \mathrm{i})\delta(v \cdot q)$$

Charged Currents

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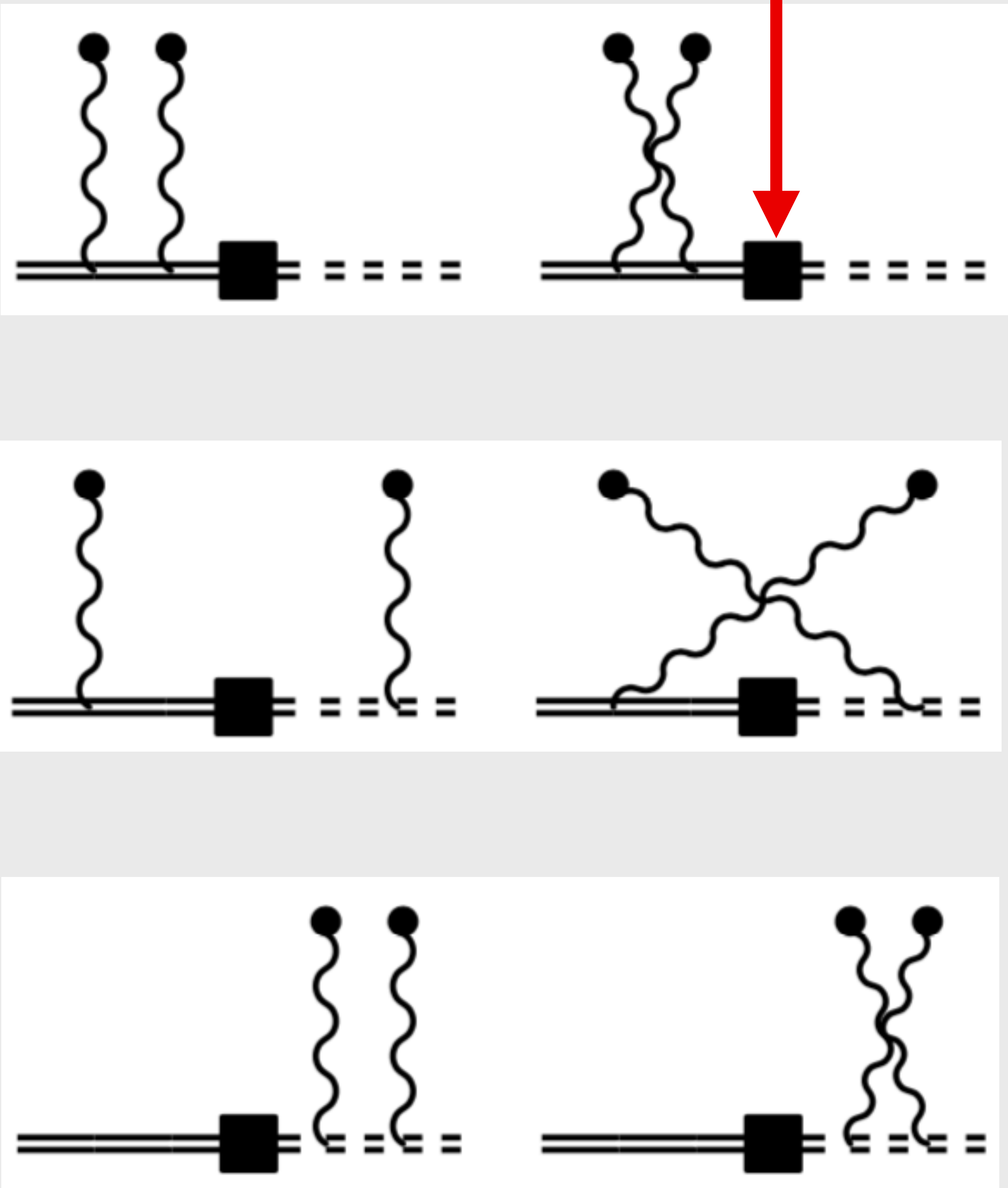
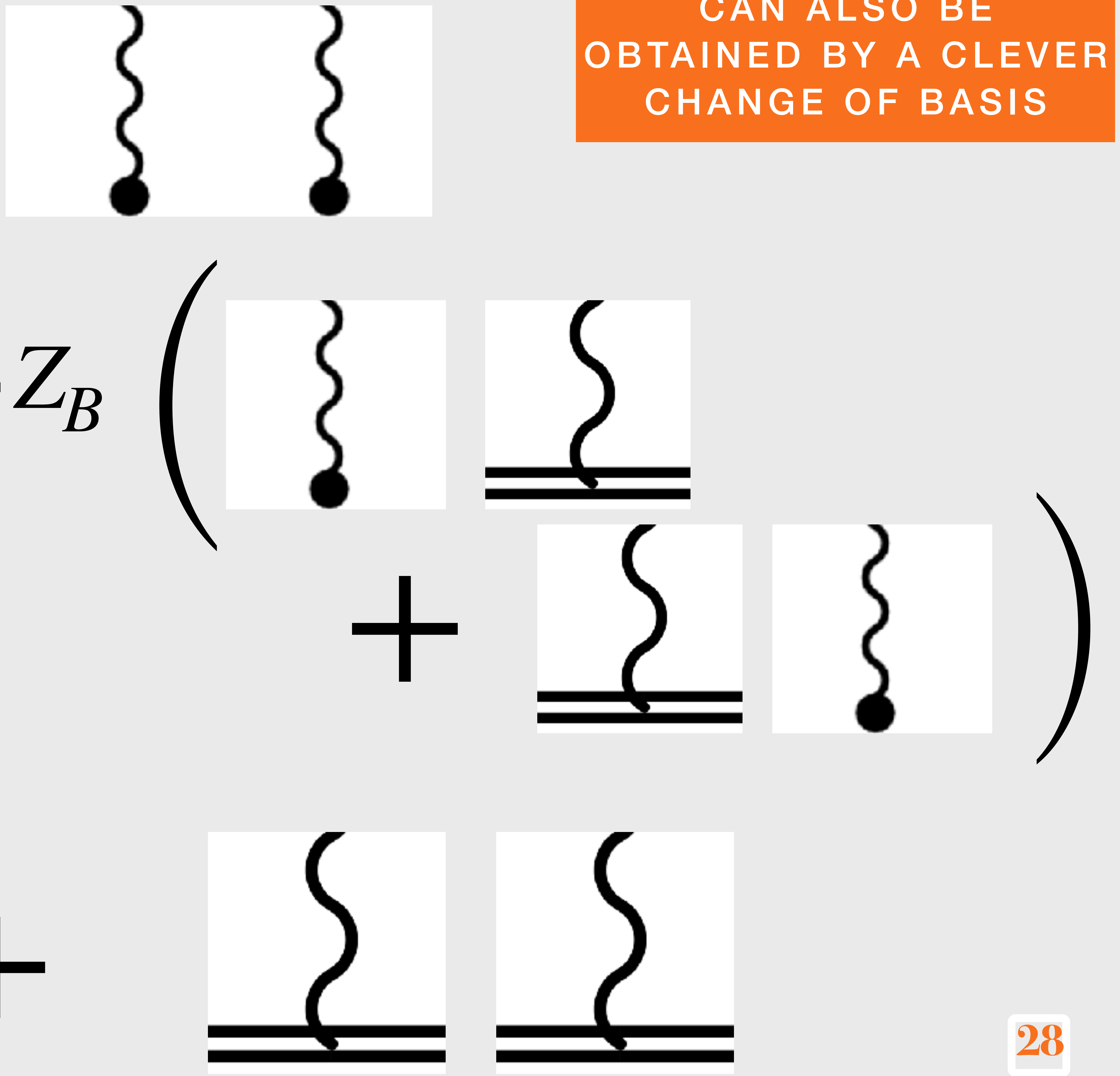
WEAK CURRENT

Z_A^2, $Z_A Z_B$, Z_B^2



Z_B^2 $+Z_B$ (+) +



CAN ALSO BE OBTAINED BY A CLEVER CHANGE OF BASIS

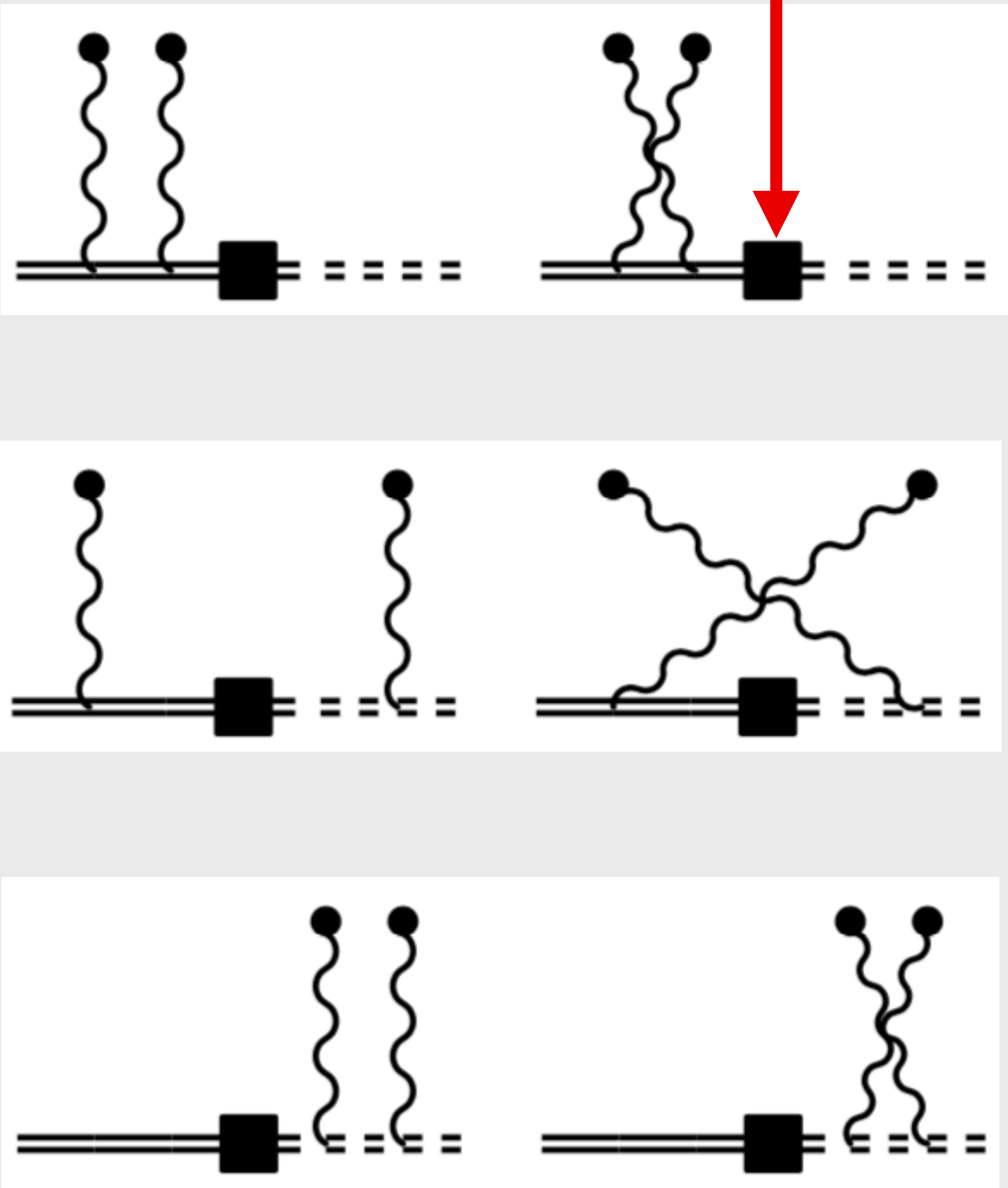
Charged Currents

WEAK CURRENT

Z_A^2

$Z_A Z_B$

Z_B^2



New Result

$$Z_B^2 \left(2 \times \text{ Coulomb} \right) + Z_B \left(\text{Coulomb} + \text{Eikonal} \right) + \left(2 \times \text{ Eikonal} \right)$$

SEE BACKUP SLIDES FOR EQUATIONS

Equivalent Feynman Rules

TREE-LEVEL

- 1 diagram.

ONE LOOP

- 2 diagrams.

TWO LOOP

- 5 diagrams.

THREE LOOP

- 10 diagrams.

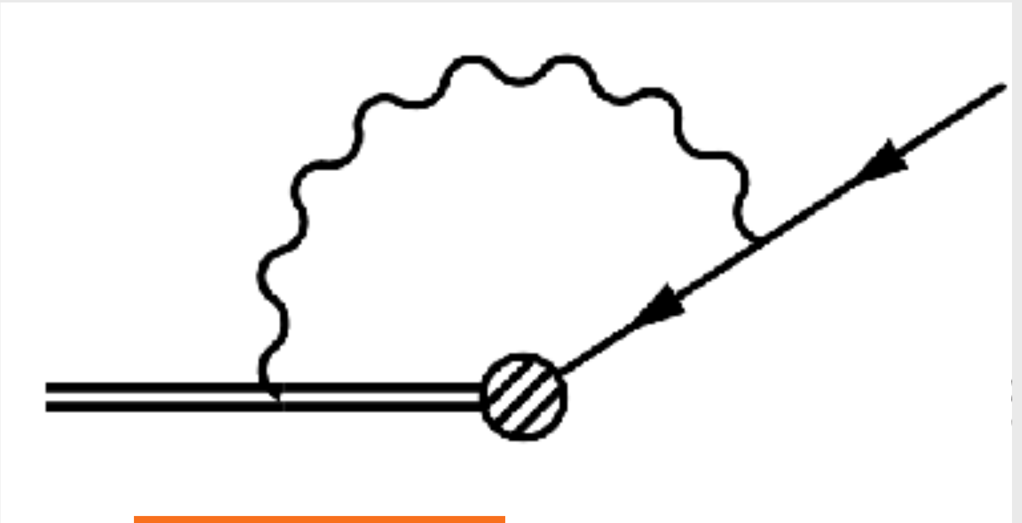
1 NUCLEUS WITH UNIT CHARGE

+ A BACKGROUND COULOMB FIELD

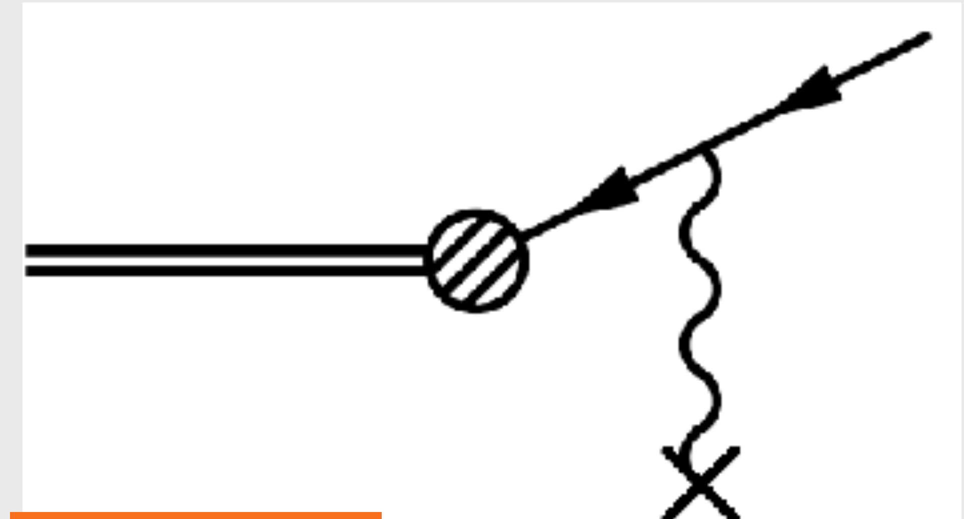
$= iZe\,\delta_0^\mu\, 2\pi\delta(q^0)$

$= ie\,\delta_0^\mu$

ONE LOOP



$\mathcal{O}(\alpha)$



$\mathcal{O}(Z\alpha)$

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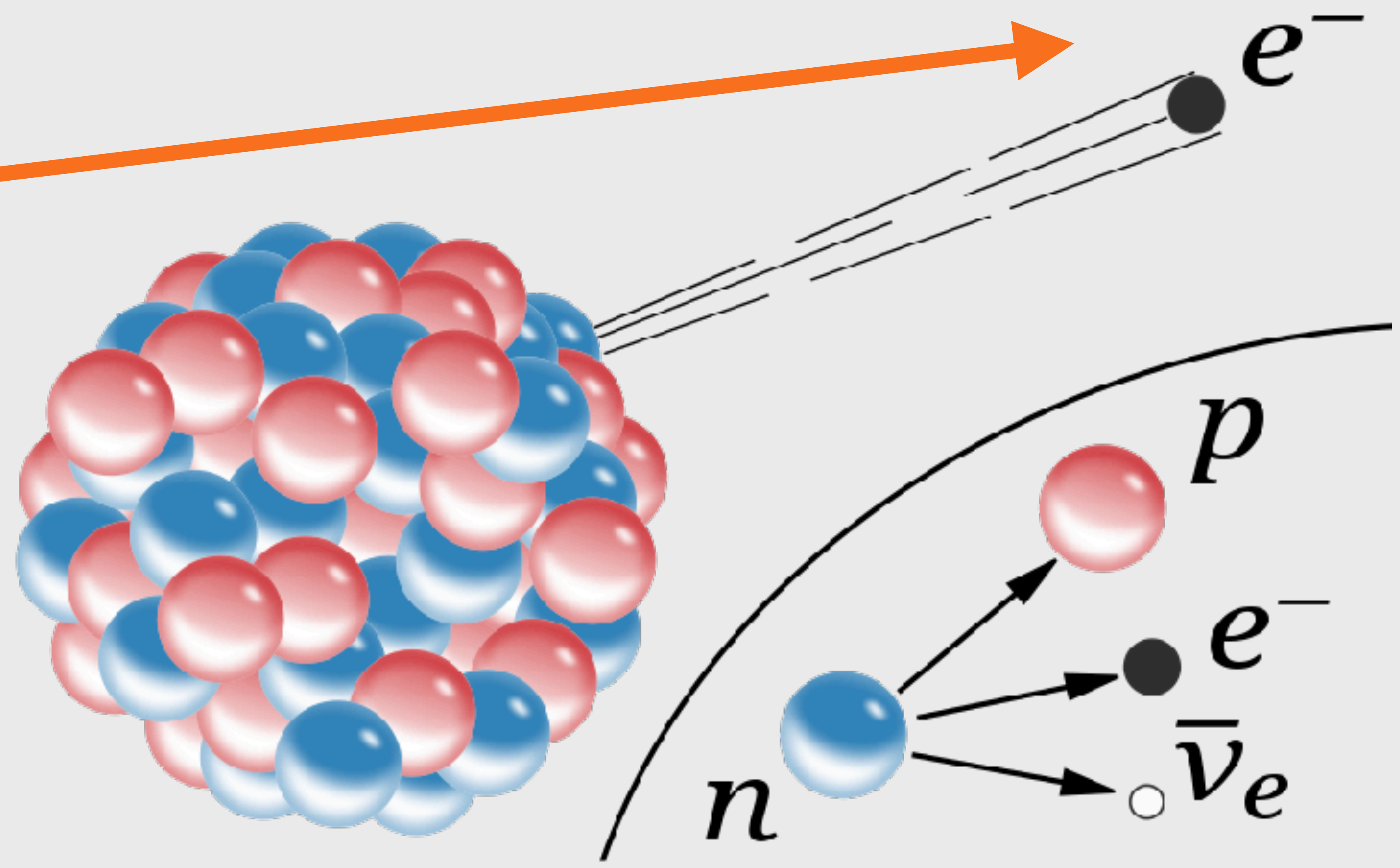
Field Theory Of The Fermi Function

Leading–Z Resummation

Fermi Function

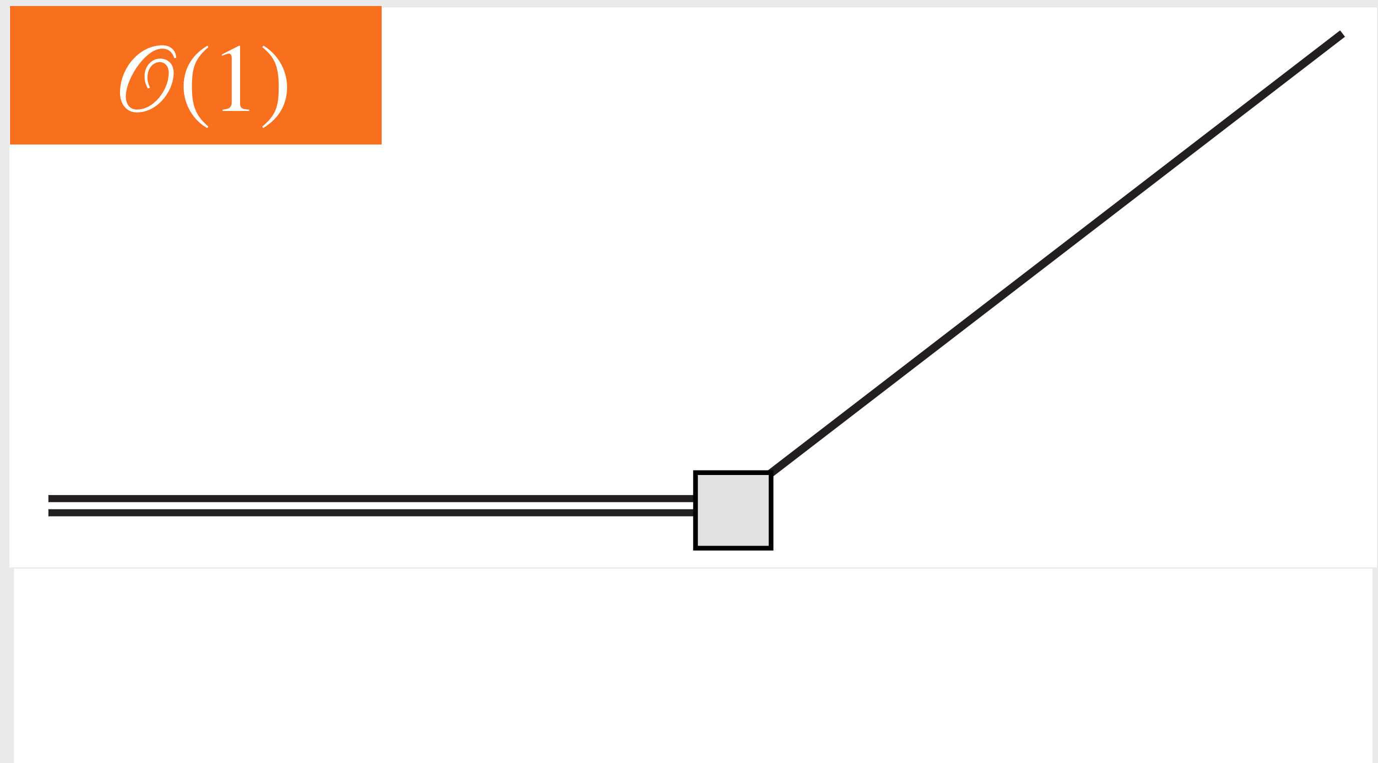
ATTRACTED TO NUCLEUS

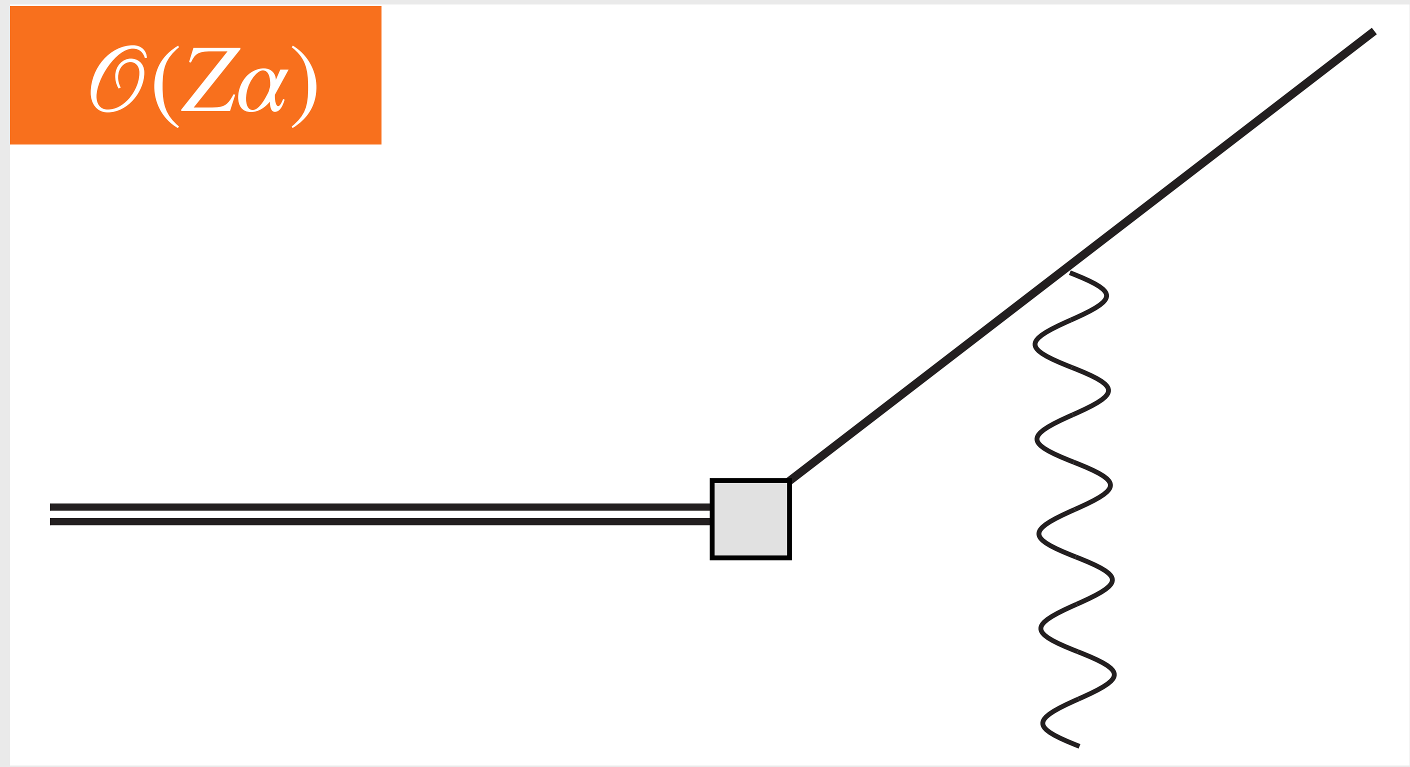


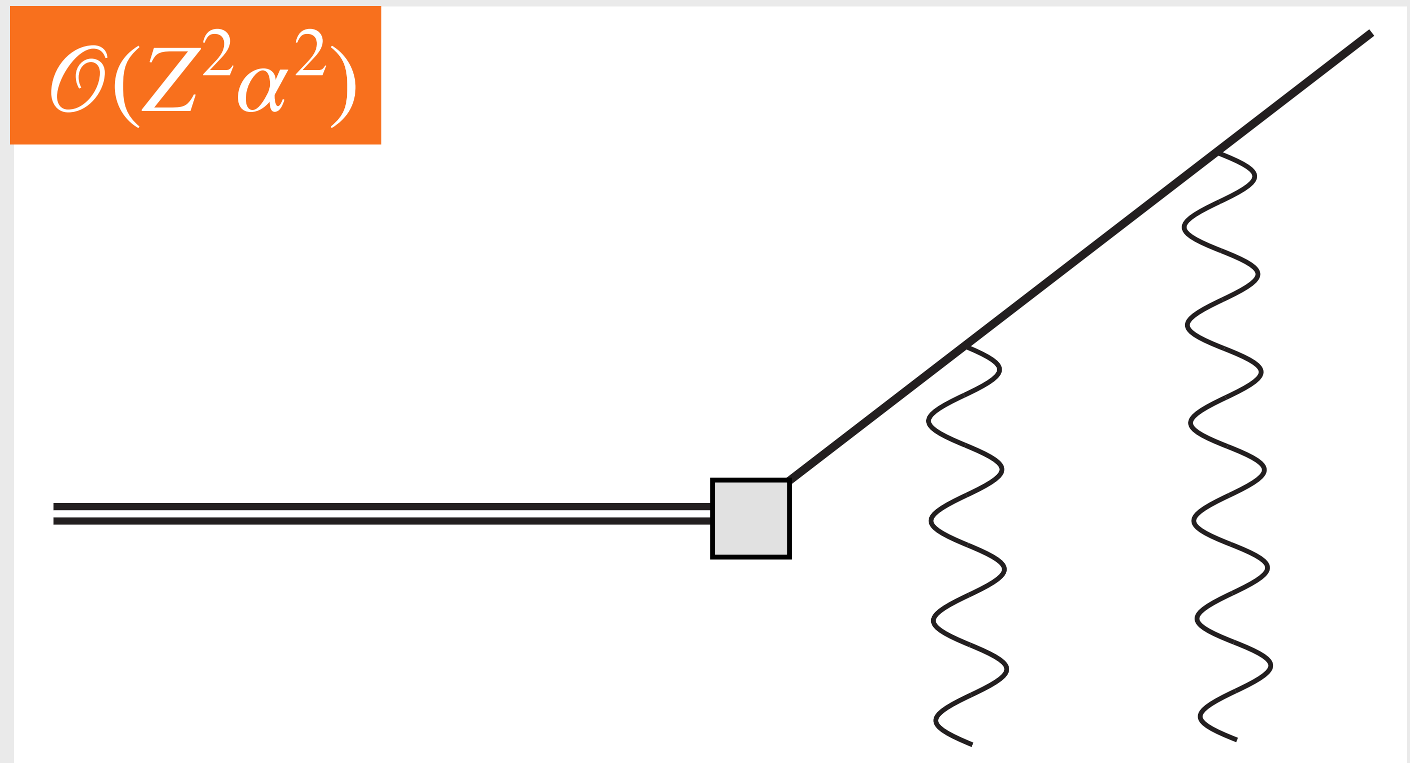
- Largest effects are a series in $Z\alpha$
- Historically done with finite-distance regulator

$$\langle e^- | \bar{\psi}(\mathbf{x}) | 0 \rangle \sim \left(\frac{1}{|\mathbf{x}|} \right)^{\nu} \qquad \nu = \sqrt{1 - Z^2\alpha^2} - 1$$

Diagrammatic Expansion $\mathscr{M}_H$







$+ \dots$

- With modified Feynman rules counting Z is easy.
- Keep only the "leading-in-Z" terms.

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Wavefunctions And Feynman Diagrams

- One can try to explicitly compute loops, but it is hard work.
- Can extract information from Dirac Equation with a Coulomb field.

Wavefunction Satisfies Lippmann-Schwinger Equation

$$|\psi_p^{(\pm)}\rangle = |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \dots$$

- One-to-one correspondence between loops and expansion of the Dirac Coulomb wavefunction.

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Factorization Of Dirac Wavefunction

WHAT WE WANT

$$\mathscr{M} = \mathscr{M}_S(\mu_S)\,\mathscr{M}_H(\mu_S, \mu_H)\,\mathscr{M}_{\mathrm{UV}}(\mu_H, \Lambda)$$

SAME

DIFFERENT

$$\Psi(\mathbf{x}) = \mathscr{M}_S(\mu_S)\,\mathscr{M}_H(\mu_S, \mu_H)\,\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

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All Orders Calculation

SEE BACKUP SLIDES FOR EQUATIONS

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

- Finite distance $\mathbf{x}$ acts as regulator.
- Can be computed in the $p_e,\ m_e \to 0$ limit.
- All orders in $Z\alpha$ solution can be obtained.

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Extraction Of Hard Matrix Element

SEE BACKUP SLIDES FOR EQUATIONS

$$\Psi(\mathbf{x}) = \mathscr{M}_S(\mu_S)\mathscr{M}_H(\mu_S, \mu_H)\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

KNOWN TO ALL ORDERS IN $Z\alpha$

ARXIV:2309.15929

Extraction Of Hard Matrix Element

SEE BACKUP SLIDES FOR EQUATIONS

$$\Psi(\mathbf{x}) = \mathscr{M}_S(\mu_S)\mathscr{M}_H(\mu_S, \mu_H)\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

KNOWN TO ALL ORDERS IN $Z\alpha$

$$\mathscr{M}_H(\mu_S, \mu_H) = \frac{\Psi(\mathbf{x})}{\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})\mathscr{M}_S(\mu_S)}$$

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PART 1

EFT & β DECAY

- Motivation & relevance for **fundamental physics**.
- Necessary **precision**, and requisite **loop orders**.

PART 2

FERMI FUNC.

- **Point-like** EFT of nuclei and leptons.
- The **Fermi function** from loops.

PART 3

RAD. CORR.

- Structure of **radiative corrections** from EFT.
- Renormalization group **resummation of logarithms**.

Long-Distance Radiative Corrections

Defining What We Mean By Outer Corrections

Factorization Theorem

ARXIV:2309.07343

- Amplitude depends on Wilson coefficient and matrix element.

$$\mathrm{d}\Gamma \propto |C(\mu)|^2 |\mathscr{M}|^2(\mu) + \mathcal{O}\left((pR)^2\right)$$

- Implies that all ***short-distances*** factorize from ***long-distances***.

$$\mathcal{F}t \equiv ft(1+\delta'_R)(1+\delta_{\mathrm{NS}}-\delta_C) = \frac{K}{2G_V^2(1+\Delta_R^V)}$$

EFT Definition Of `Outer' Corrections

$$\tilde{F}(Z,E) = \left[|\mathscr{M}|^2(\mu) \right]_{\text{leading}-Z\alpha}$$

$$(1+\tilde{\delta}_R) = \frac{\langle |\mathscr{M}|^2(\mu) \rangle}{\langle \tilde{F}(Z,E) \rangle}$$

ARXIV:2309.07343

THIS IS NOT A "FACTORIZATION THEOREM".
JUST A CONVENTIONAL DEFINITION

EFT Definition Of `Outer' Corrections

$$(1+\delta'_R) := \left[\frac{C(\mu_L)/C(\mu_H)}{\exp\left[\left(1-\sqrt{1-Z^2\alpha^2}\right)\ \log(\mu_H/\mu_L)\right]}\right]^2 \left(\frac{\int \mathrm{d}\Pi \qquad \langle|\mathcal{M}_H|^2\rangle}{\int \mathrm{d}\Pi\ F(Z,E)\times\frac{4\eta}{(1+\eta)^2}}\right)\Bigg|_{\mu=\mu_L}$$

$$(1+\tilde{\delta}_R) = \frac{\langle|\mathscr{M}|^2(\mu)\rangle}{\langle\tilde{F}(Z,E)\rangle}$$

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THIS IS NOT A "FACTORIZATION THEOREM".
JUST A CONVENTIONAL DEFINITION

RG Analysis & Anomalous Dim.

Resumming Logs

Relati

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$$d\Gamma \propto |C(\mu)|^2 |\mathscr{M}|^2(\mu)$$

No Large Logs

$$= C(\mu_H) \left[\frac{|C(\mu_L)|^2}{|C(\mu_H)|^2} \right] |\mathscr{M}|^2(\mu_L)$$

Calculate With Renormalization Group

Resummation With RG + EFT

- Need beta function in QED
- Need anomalous dimension

$$\left[\frac{|C(\mu_L)|^2}{|C(\mu_H)|^2}\right] = \exp\left[\int \frac{\gamma(Z,\alpha)}{\beta(\alpha)} d\alpha\right]$$

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Factorize & Run

$$\mathscr{M} = C(\mu)\ \mathscr{M}_H(\mu, p)$$

Nuclear Radius
50 MeV

$$C(\Lambda)$$

RG EVOLUTION

Electron Energy
5 MeV

$$C(\mu)\ \mathscr{M}_H(\mu, p)$$

Anomalous Dimension

ARXIV:2402.14769

$$\gamma_C = \frac{\mathrm{d}C(\mu)}{\mathrm{d}\log\mu}$$

SOLVE DIRAC EQ'N

SYMMETRY IN MASSLESS LIMIT

$(Z, Z-Q, Q) \leftrightarrow (Z+Q, Z, -Q)$

$$\gamma_C = \alpha\left(Z\gamma^{(1,1)} + \gamma^{(1,0)}\right) + \alpha^2\left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)}\right)$$

$$+ \alpha^3\left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)}\right) + \ldots$$

Z=0 REDUCES TO HEAVY-LIGHT CURRENT IN HQET

TAKE FROM HQET LIT.

SOLVE DIRAC EQ'N

↕ SYMMETRY

	α^0	α^1	α^2	α^3	α^4
Z^0	0	$\gamma^{(1,0)}$ ✓	$\gamma^{(2,0)}$ ✓	$\gamma^{(3,0)}$ ✓	$\gamma^{(4,0)}$ ✓
Z^1	–	0	$\gamma^{(2,1)}$	$\gamma^{(3,1)}$	$\gamma^{(4,1)}$
Z^2	–	–	$\gamma^{(2,2)}$ ✓	$\gamma^{(3,2)}$ ✓	$\gamma^{(4,2)}$
Z^3	–	–	–	0	$\gamma^{(4,3)}$
Z^4	–	–	–	–	$\gamma^{(4,4)}$ ✓

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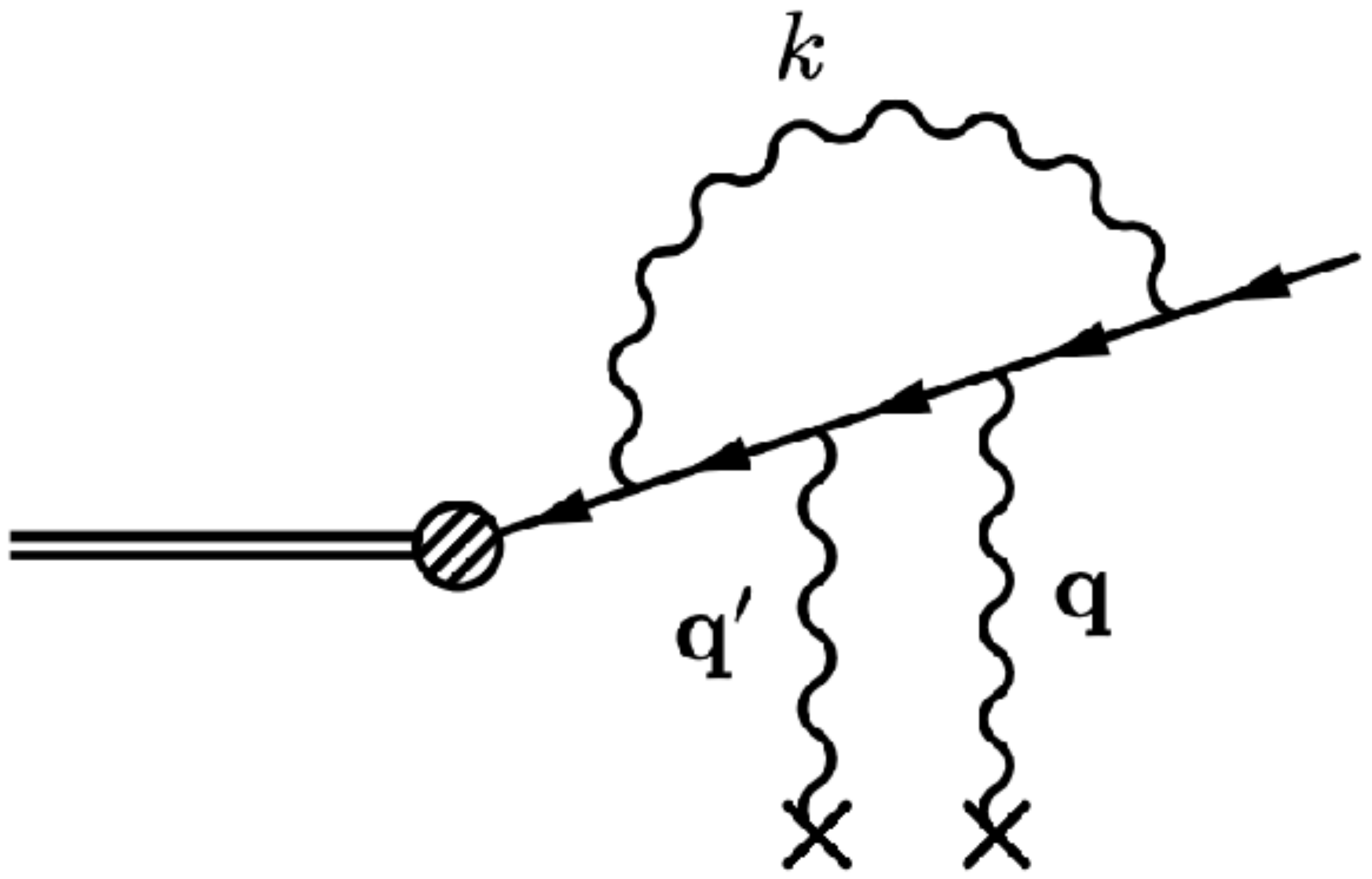
NEW INPUT!

ARXIV:2402.14769

USE EIKONAL ALGEBRA TO REDUCE DIAGRAMS

$$\gamma_2^{(1)} = 16\pi^2 \left(6 - \frac{\pi^2}{3}\right)$$



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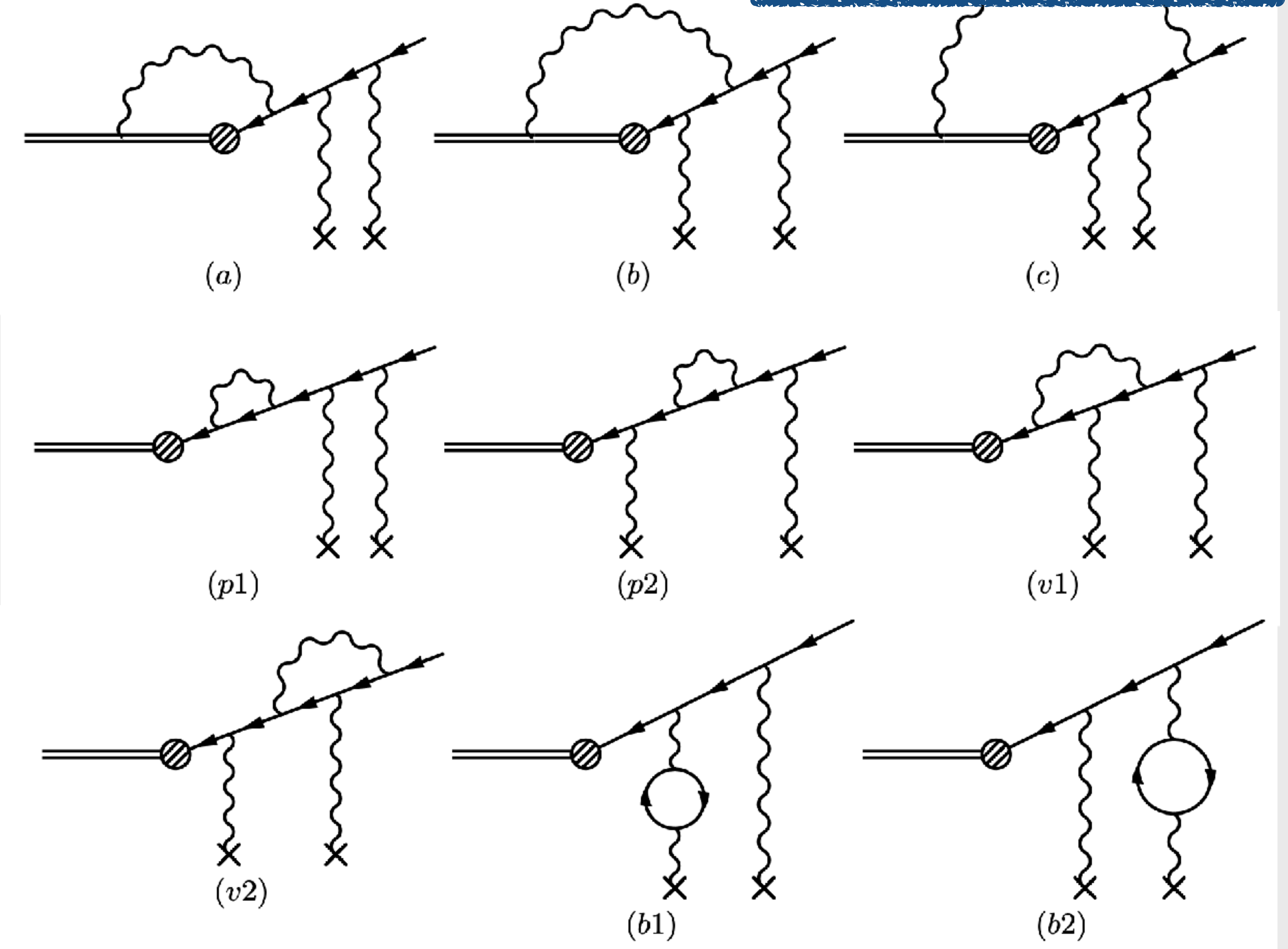

MIXED EUCLIDEAN + LORENTZIAN INTEGRALS

New Result For Anomalous Dimension

Loops / Z^n	1-loop	2-loop	3-loop	4-loop
Z^0	$\gamma_0^{(1)} = -3$	$\gamma_1^{(2)} = -16\zeta_2 + \frac{5}{2} + \frac{10}{3}n_e$	$\gamma_2^{(3)} =$ GROZIN 2003	$\gamma_3^{(4)} =$ GROZIN 2023
Z^1	$\gamma_0^{(0)} = 0$	$\gamma_1^{(1)} = \gamma_2^{(2)}$	$\gamma_2^{(2)} = \gamma_2^{(1)}$	$\gamma_3^{(3)} = \gamma_3^{(2)} - \gamma_3^{(0)}$
Z^2	–	$\gamma_1^{(0)} = -8\pi^2$	$\gamma_2^{(1)} = 16\pi^2\left(6 - \frac{\pi^2}{3}\right)$	$\gamma_3^{(2)} =$?
Z^3	–	–	$\gamma_2^{(0)} = 0$	$\gamma_3^{(1)} = 2\gamma_3^{(0)}$
Z^4			–	$\gamma_3^{(0)} = -32\pi^4$

RESUMMATION COMPLETE THROUGH 3-LOOPS!

NEW INPUT

ARXIV:2402.14769

Resummation With RG + EFT

$$(1+\delta'_R) := \left[\frac{C(\mu_L)/C(\mu_H)}{\exp\left[\left(1-\sqrt{1-Z^2\alpha^2}\right)\;\log(\mu_H/\mu_L)\right]}\right]^2 \left(\frac{\int \mathrm{d}\Pi \qquad \langle|\mathcal{M}_H|^2\rangle}{\int \mathrm{d}\Pi\;\, F(Z,E)\times\frac{4\eta}{(1+\eta)^2}}\right)_{\mu=\mu_L}$$

Contains $\log(pR)$ Enhancements

ARXIV:2309.07343

- Introduce power counting scheme

$$Z\alpha \sim \sqrt{\alpha} \qquad \alpha\log(pR) \sim \sqrt{\alpha}$$

Resummation With RG+EFT

$$(1+\delta'_R) := \underbrace{\left[\frac{C(\mu_L)/C(\mu_H)}{\exp\left[\left(1-\sqrt{1-Z^2\alpha^2}\right)\ \log(\mu_H/\mu_L)\right]}\right]^2}\left(\frac{\int \mathrm{d}\Pi \qquad \langle|\mathcal{M}_H|^2\rangle}{\int \mathrm{d}\Pi\ F(Z,E)\times\frac{4\eta}{(1+\eta)^2}}\right)_{\mu=\mu_L}$$

ARXIV:2309.07343

$$Z\alpha \sim \sqrt{\alpha} \qquad \alpha\log(pR) \sim \sqrt{\alpha}$$

- Known up to $\sim O(\alpha^2)$

e.g., $Z^3\alpha^4\log^2(pR) \sim \alpha^2$

- Known in EFT to $\sim O(\alpha)$
- Can estimate with results from Sirlin & Zuchinni (1987) at $O(Z\alpha^2) \sim \alpha^{3/2}$

Impact For Flavour Physics

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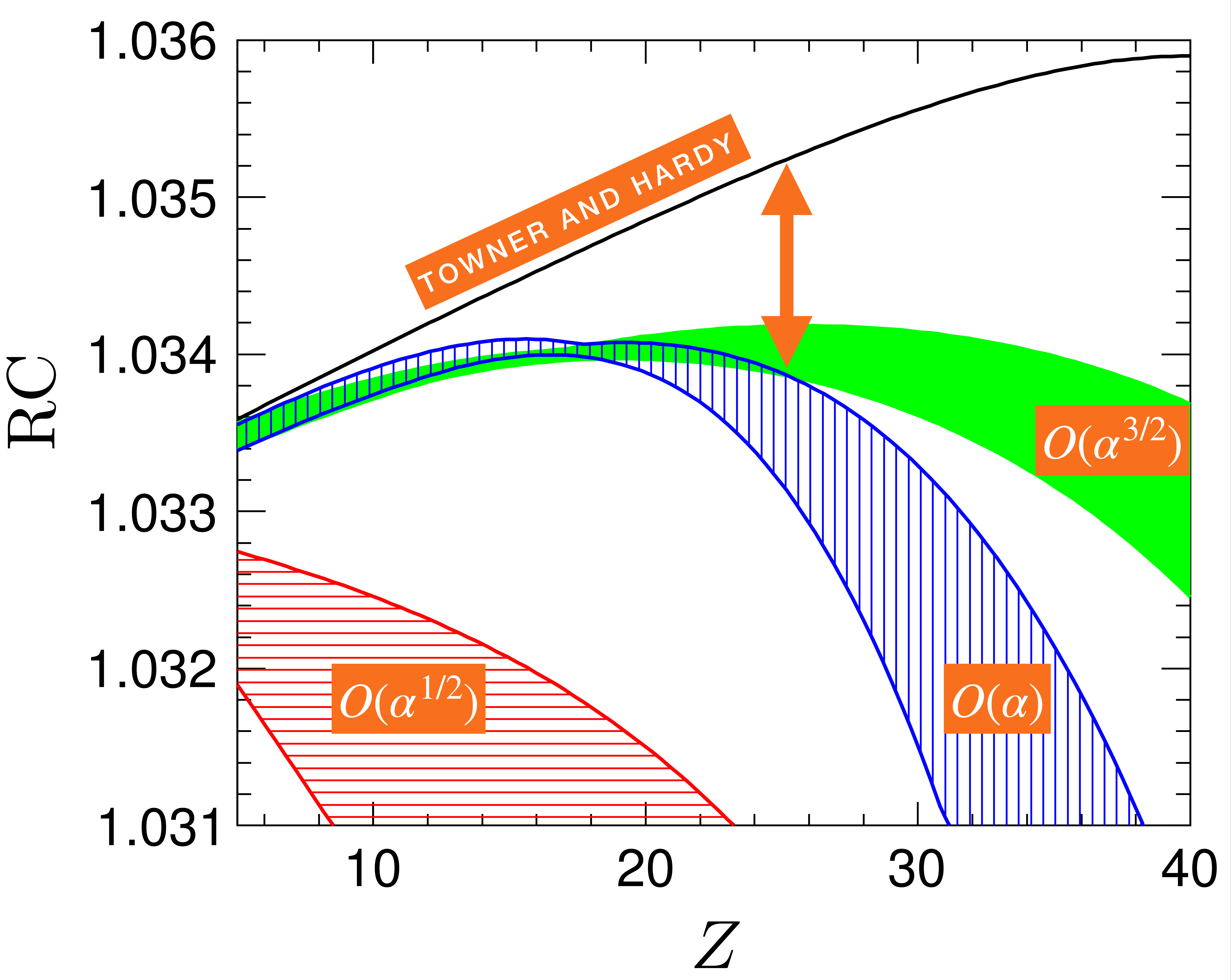
SHIFTING δ_3

transition	$(\Delta a) \times Z^2 \alpha^3 \log(\Lambda/m)$
$^{14}O \to {}^{14}N$	-1.1×10^{-4}
$^{26m}Al \to {}^{26}Mg$	-3.2×10^{-4}
$^{34}Cl \to {}^{34}S$	-5.6×10^{-4}
$^{34}Ar \to {}^{34}Cl$	-6.3×10^{-4}
$^{38m}K \to {}^{38}Ar$	-7.1×10^{-4}
$^{42}Sc \to {}^{42}Ca$	-8.7×10^{-4}
$^{46}V \to {}^{46}Ti$	-10.5×10^{-4}
$^{50}Mn \to {}^{50}Cr$	-12.5×10^{-4}
$^{54}Co \to {}^{54}Fe$	-14.6×10^{-4}



COUNTING $Z \sim \log \sim 1/\sqrt{\alpha}$

Conclusions & Outlook

Summary

ARXIV:2309.15929 ,
ARXIV:2309.07343 ,
ARXIV:2402.13307 ,
ARXIV:2402.14769 .

- Factorization + eikonal algebra + elbow grease.
- First calculation of logarithmically enhanced $Z^2\alpha^3$ corrections. Disagreement with Sirlin's guess.
- Shift in outer radiative corrections bigger than ascribed error in Towner & Hardy.
- Shifts answer towards first-row unitarity.

Take Home Messages

ARXIV:2309.15929 ,
ARXIV:2309.07343 ,
ARXIV:2402.13307 ,
ARXIV:2402.14769 .

- Calculations performed in the low-energy point-like EFT are model independent & universal.
- Fermi function and outer radiative corrections come from same scale $|\mathbf{q}_\gamma| \sim |\mathbf{p}_e|$ and don't factorize.
- Factorization theorems help constrain properties of amplitudes. Useful for beta decay.

Questions For Discussion

- How large is the error when using the Sirlin & Zucchini calculation for $Z\alpha^2$?
- To what order are radiative corrections needed at next order in (pR)?
- Does the shift in δ_3 propagate into nuclear structure in Towner & Hardy?

Backup Slides

Wavefunctions & Diagramatics

Wavefunctions And Feynman Diagrams

- Coulomb effects historically handled with ``distorted waves''

- What are the equivalent effects in Feynman diagrams?

Use Lippmann-Schwinger Equation!

$$|\psi_p^{(\pm)}\rangle = |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \ldots$$

Wavefunctions And Feynman Diagrams

- Coulomb effects historically handled with ``distorted waves''

- What are the equivalent effects in Feynman diagrams?

Loop With A Phase Factor!

$$\langle x \,|\, \psi_p^{(\pm)} \rangle = e^{i\mathbf{p}\cdot\mathbf{x}} \left(1 + \int \frac{d^3Q}{(2\pi)^3} \frac{1}{2\mathbf{P}\cdot\mathbf{Q} + \mathbf{Q}^2 \pm i\varepsilon} \frac{Z\alpha}{\mathbf{Q}^2} \, e^{i\mathbf{Q}\cdot\mathbf{x}} + \ldots \right)$$

Two-Loop Expressions At $\mathcal{O}(Z^2\alpha^2)$

Brute Force 2-Loop Calculation

- Compute Coulomb corrections explicitly through 2-loops.
- Dim-reg + renormalization. Well defined amplitude.

$$\mathcal{M}_H(\mu_S,\mu_H) = 1 + \frac{Z\alpha}{\beta}\left[i\left(\log\frac{2p}{\mu_S} - \frac{i\pi}{2}\right) + \frac{i}{2}\left(\frac{m}{E}\gamma^0 - 1\right)\right] + \left(\frac{Z\alpha}{\beta}\right)^2 \left\{\frac{-\pi^2}{12} - \frac{1}{2}\left(\log\frac{2p}{\mu_S} - \frac{i\pi}{2}\right)^2 - \frac{1}{2}\left(\log\frac{2p}{\mu_S} - \frac{i\pi}{2}\right)\left(\frac{m}{E}\gamma^0 - 1\right) + \left[\frac{5}{4} - \frac{1}{2}\left(\log\frac{2p}{\mu_H} - \frac{i\pi}{2}\right)\right]\beta^2\right\} + \mathcal{O}(\alpha^3)\,,$$

- No obvious pattern. Resummation impossible by brute force.

Eikonal Algebra Identity

New Result

$$\langle B(v)\,|\,J_{\mu_1}(q_1)\ldots\mathcal{O}\ldots J_{\mu_N}(q_N)\,|\,A(v)\rangle = v_{\mu_1}\ldots v_{\mu_N}G(q_1\ldots q_N)$$

$$G(q_1\ldots q_N) = Z^n\prod_{i=1}^{N}(2\pi \mathrm{i})\delta(v\cdot q_i)$$

$$+Z^{n-1}\sum_{j}\frac{1}{v\cdot q_i}\prod_{i\neq j}(2\pi \mathrm{i})\delta(v\cdot q_i)$$

$$+Z^{n-2}\sum_{k}\sum_{j\neq k}\frac{1}{v\cdot q_k}\frac{1}{v\cdot q_j}\prod_{i\neq j,k}(2\pi \mathrm{i})\delta(v\cdot q_i)$$

$$+\ldots$$

Fermi Function

LOG(2PR)

$$(1+\delta_R') := \left[\frac{C(\mu_L)/C(\mu_H)}{\exp\left[\left(1-\sqrt{1-Z^2\alpha^2}\right)\log(\mu_H/\mu_L)\right]}\right]^2 \left(\frac{\int \mathrm{d}\Pi \qquad \langle|\mathcal{M}_H|^2\rangle}{\int \mathrm{d}\Pi\ F(Z,E)\times\frac{4\eta}{(1+\eta)^2}}\right)_{\mu=\mu_L}$$

- We can define "outer" radiative corrections in the EFT
- Factorize into a RG-running piece, and a low-energy matrix element.
- Fermi function has been factored out.

Explicit Expressions For Fermi Function

Factorization Of Dirac Wavefunction

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

CLOSED FORM INTEGRALS AT ARBITRARILY HIGH ORDER

$$\mathcal{I}_1^{(n)} = \left[\prod_{j=1}^{n-1} C(\nu_j)\right] \times \frac{\Gamma(d-\nu_n-1)}{(4\pi)^d\Gamma(\nu_n)} B(\tfrac{d}{2}-1, 1+\tfrac{d}{2}-\nu_n)\left(\frac{\mathbf{x}^2}{4}\right)^{\nu_n+1-d},$$

$$\mathcal{I}_2^{(n)} = \left[\prod_{j=1}^{n} C(\nu_j)\right]\left[\frac{2\Gamma(\frac{d}{2}-\nu_{n+1}+1)}{(4\pi)^{d/2}\Gamma(\nu_{n+1})}\right]\left[\frac{\mathbf{x}^2}{4}\right]^{\nu_{n+1}-(d+1)/2} \times \frac{i\gamma_0\boldsymbol{\gamma}\cdot\mathbf{x}}{2|\mathbf{x}|}.$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

BARE AMPLITUDE MAY BE SUMMED TO ALL ORDERS

$$F_1^{\text{bare}} = 2^{\frac{1}{4\epsilon}-\frac{1}{2}} \left(\frac{\sqrt{\tilde{g}}}{\epsilon}\right)^{1-\frac{1}{2\epsilon}} \Gamma\left(\frac{1}{2\epsilon}\right) J_{\frac{1}{2\epsilon}-1}\left(\frac{\sqrt{8}\sqrt{\tilde{g}}}{\epsilon}\right),$$

$$(Z\tilde{\alpha})^{-1} F_2^{\text{bare}} = 2^{\frac{1}{4\epsilon}} \left(\frac{\sqrt{\tilde{g}}}{\epsilon}\right)^{-\frac{1}{2\epsilon}} \Gamma\left(1+\frac{1}{2\epsilon}\right) J_{\frac{1}{2\epsilon}}\left(\frac{\sqrt{8}\sqrt{\tilde{g}}}{\epsilon}\right).$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

RESULT CAN BE RENORMALIZED AT ALL-ORDERS IN $Z\alpha$

$$\mathcal{M}^R_{\mathrm{UV}}(\mu) = (\mu r e^{\gamma_{\mathrm{E}}})^{\eta-1}\frac{1+\eta}{2\sqrt{\eta}}\left[1 + \frac{Z\alpha}{1+\eta}\frac{i\gamma_0\boldsymbol{\gamma}\cdot\mathbf{x}}{|\mathbf{x}|}\right],$$

$$\eta = \sqrt{1-(Z\alpha)^2}$$

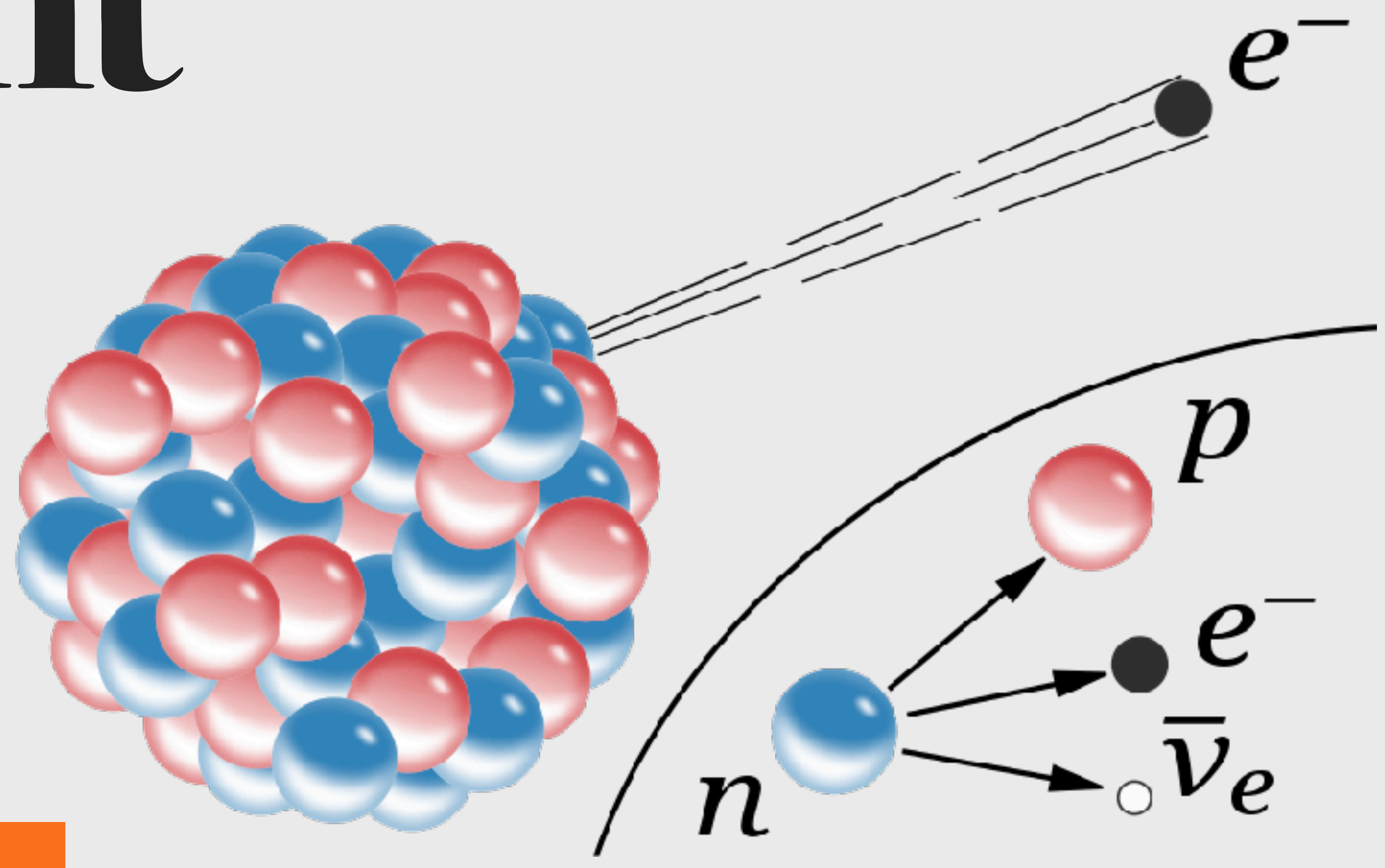
All-Orders Hard Matrix Element

$$\mathcal{M}_H(\mu_S,\mu_H) = \mathrm{e}^{\frac{\pi\xi}{2}+i\xi\left(\log\frac{2p}{\mu_S}-\gamma_{\mathrm{E}}\right)-i(\eta-1)\frac{\pi}{2}}\frac{2\Gamma(\eta-i\xi)}{\Gamma(2\eta+1)}$$

$$\sqrt{\frac{\eta-i\xi}{1-i\xi\frac{m}{E}}}\sqrt{\frac{E+\eta m}{E+m}}\sqrt{\frac{2\eta}{1+\eta}}\left(\frac{2p e^{-\gamma_{\mathrm{E}}}}{\mu_H}\right)^{\eta-1}\times\left[\frac{1+M^*}{2}+\frac{1-M^*}{2}\gamma^0\right]$$

- $\eta = \sqrt{1-Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E+m)(1+\mathrm{i}\xi m/E)/(E+\eta m)$

Coulomb Enhancement



- Largest effects are a series in $Z\alpha$

MS-BAR RENORMALIZED

UNIVERSAL RESULT FOR QED

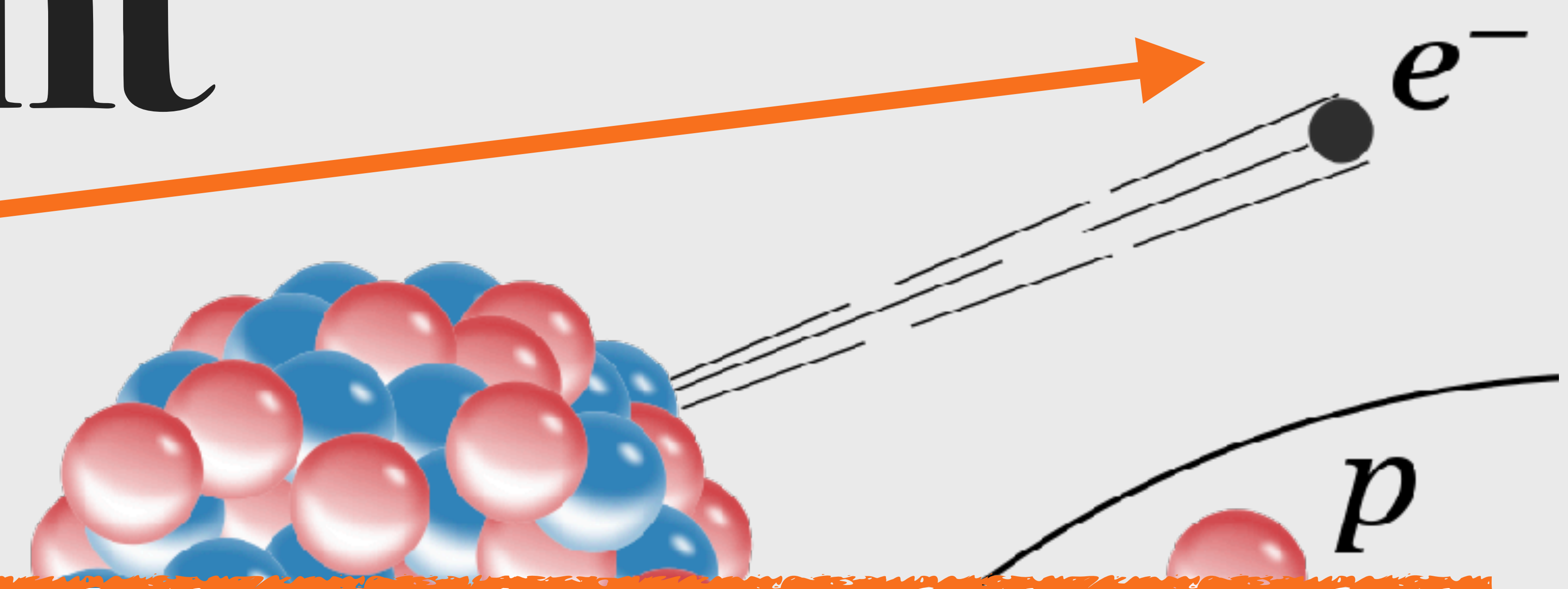
ALL ORDERS IN $Z\alpha$

$$\mathcal{M}_H(\mu_S,\mu_H) = e^{\frac{\pi\xi}{2} + i\xi\left(\log\frac{2p}{\mu_S} - \gamma_E\right) - i(\eta-1)\frac{\pi}{2}} \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta+1)} \sqrt{\frac{\eta - i\xi}{1 - i\xi\frac{m}{E}}} \sqrt{\frac{E+\eta m}{E+m}} \sqrt{\frac{2\eta}{1+\eta}} \left(\frac{2pe^{-\gamma_E}}{\mu_H}\right)^{\eta-1} \times \left[\frac{1+M^*}{2} + \frac{1-M^*}{2}\gamma^0\right]$$

- $\eta = \sqrt{1 - Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E+m)(1 + i\xi m/E)/(E+\eta m)$

Coulomb Enhancement

ATTRACTED TO NUCLEUS



- Well defined EFT matrix element. Can be evolved with RG to re-sum logs.

UNIVERSAL RESULT FOR QED

ALL ORDERS IN $Z\alpha$

$$\mathcal{M}_H(\mu_S,\mu_H) = e^{\frac{\pi\xi}{2} + i\xi\left(\log\frac{2p}{\mu_S} - \gamma_E\right) - i(\eta-1)\frac{\pi}{2}} \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta+1)} \sqrt{\frac{\eta - i\xi}{1 - i\xi\frac{m}{E}}} \sqrt{\frac{E+\eta m}{E+m}} \sqrt{\frac{2\eta}{1+\eta}} \left(\frac{2pe^{-\gamma_E}}{\mu_H}\right)^{\eta-1} \times \left[\frac{1+M^*}{2} + \frac{1-M^*}{2}\gamma^0\right]$$

- $\eta = \sqrt{1 - Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E+m)(1 + i\xi m/E)/(E+\eta m)$

Properties Of The Anomalous Dimension

Anomalous Dimension

$$\gamma_{\mathrm{C}} = \frac{\mathrm{d}C(\mu)}{\mathrm{d}\log\mu}$$

$$\gamma_{\mathrm{C}} = \alpha\left(Z\gamma^{(1,1)} + \gamma^{(1,0)}\right) + \alpha^2\left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)}\right)$$
$$+ \ \alpha^3\left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)}\right) + \ldots$$

Anomalous Dimension

$$\gamma_C = \frac{\mathrm{d}C(\mu)}{\mathrm{d}\log\mu}$$

SOLVE DIRAC EQ'N

- Subtlety: Divergent as $x \to 0$
- New result: All orders result in the $\overline{\mathrm{MS}}$-scheme (good for RG).

$$\gamma_C = \alpha\left(Z\gamma^{(1,1)} + \gamma^{(1,0)}\right) + \alpha^2\left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)}\right)$$

$$+ \ \alpha^3\left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)}\right) + \ldots$$

Anomalous Dimension

SOLVE DIRAC EQ'N

$$\gamma_C = \frac{dC(\mu)}{d\log\mu}$$

$$\gamma_C = \alpha\left(Z\gamma^{(1,1)} + \gamma^{(1,0)}\right) + \alpha^2\left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)}\right)$$

$$+\ \alpha^3\left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)}\right) + \ldots$$

Z=0 REDUCES TO HEAVY-LIGHT CURRENT IN HQET

Anomalous Dimension

$$\gamma_C = \frac{dC(\mu)}{d\log\mu}$$

SOLVE DIRAC EQ'N

SYMMETRY IN MASSLESS LIMIT

$(Z, Z-Q, Q) \longleftrightarrow (Z+Q, Z, -Q)$

$$\gamma_C = \alpha\left(Z\gamma^{(1,1)} + \gamma^{(1,0)}\right) + \alpha^2\left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)}\right)$$

$$+ \ \alpha^3\left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)}\right) + \ldots$$

Z=0 REDUCES TO HEAVY-LIGHT CURRENT IN HQET

TAKE FROM HQET LIT.

SOLVE DIRAC EQ’N

SYMMETRY

	α^0	α^1	α^2	α^3	α^4
Z^0	0	$\gamma^{(1,0)}$ ✓	$\gamma^{(2,0)}$ ✓	$\gamma^{(3,0)}$ ✓	$\gamma^{(4,0)}$ ✓
Z^1	–	0	$\gamma^{(2,1)}$	$\gamma^{(3,1)}$	$\gamma^{(4,1)}$
Z^2	–	–	$\gamma^{(2,2)}$ ✓	$\gamma^{(3,2)}$ ✓	$\gamma^{(4,2)}$
Z^3	–	–	–	0	$\gamma^{(4,3)}$
Z^4	–	–	–	–	$\gamma^{(4,4)}$ ✓

NEW INPUT!

Ratio Of Wilson Coefficients

$$Z \sim L \sim \alpha^{-1/2}$$

$$\log\left(\frac{C(\mu_L)}{C(\mu_H)}\right) = \frac{\gamma_0^{(1)}}{2\beta_0}\left\{\left[\log\frac{a_H}{a_L} + \frac{Z^2\gamma_1^{(0)}}{\gamma_0^{(1)}}(a_H - a_L)\right] + \left[\frac{Z\gamma_1^{(1)}}{\gamma_0^{(1)}}(a_H - a_L)\right]\right.$$
$$\left. + \left[\left(\frac{\gamma_1^{(2)}}{\gamma_0^{(1)}} - \frac{\beta_1}{\beta_0}\right)(a_H - a_L) + \left(\frac{Z^2\gamma_2^{(1)}}{\gamma_0^{(1)}} - \frac{\beta_1}{\beta_0}\frac{Z^2\gamma_1^{(0)}}{\gamma_0^{(1)}}\right)\frac{1}{2}(a_H^2 - a_L^2) + \frac{Z^4\gamma_3^{(0)}}{\gamma_0^{(1)}}\frac{1}{3}(a_H^3 - a_L^3)\right]\right\}$$