Dispersive Determination of $\eta^{(\prime)}$ Transition Form Factors

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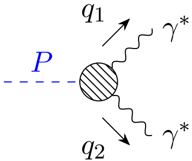
η and η' transition form factors

- Pseudoscalar ($P = \pi^0,\ \eta,\ \eta'$) transition form factors defined by

$$i \int \mathrm{d}^4 x\, e^{i q_1 \cdot x} \langle 0 \big| T\{ j_\mu(x) j_\nu(0) \} \big| P(q_1+q_2) \rangle$$
$$= \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$$



- Normalization related to di-photon decays governed by chiral anomaly:

$$\Gamma(P \to \gamma\gamma) = \frac{\pi \alpha_{\mathrm{em}}^2 M_P^3}{4} \big| F_{P\gamma^*\gamma^*}(0,0) \big|^2$$

- For pion: low-energy theorem predicts its value

  Bell, Jackiw 1969; Adler 1969; Bardeen 1969

- For η and η': complicated by η–η' mixing

  Feldmann, Kroll, Stech 1998–2000;

  Escribano, Gonzàlez-Solís, Masjuan, Sánchez-Puertas 2016

Factorization breaking in the η and η' TFFs

- Past approaches: Application of VMD form factor in the low-energy regime

$$F_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \frac{1}{q_1^2 - M_V^2} \times \frac{1}{q_2^2 - M_V^2}$$

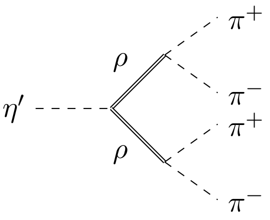
- For high energies ($|q_1^2|, |q_2^2| \to \infty$) pQCD predicts Walsh, Zerwas 1972

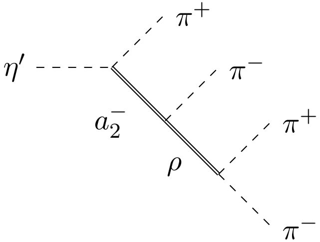
$$F_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \frac{1}{q_1^2 + q_2^2}$$

- No factorization in the singly-virtual TFFs present
- Model-independent description of intermediate energy regime with factorization breaking of paramount importance for control over uncertainties
- Exp. study (BaBar 2018) showed for $|q_1^2| = |q_2^2| \in [6.5,\, 45]\text{GeV}^2$ VMD factorization is breaking down

Formalism for doubly-virtual representations

- Start from $\eta' \to 2(\pi^+\pi^-)$ amplitude
  - describe decay via two rho resonances by hidden local symmetry (HLS) model Guo, Kubis, Wirzba 2012
  - left-hand-cut contribution due to a_2 exchange by phenomenological Lagrangian models





Final-state interaction

- in HLS amplitude: introduce pair-wise pion rescattering by replacing ρ propagators by Omnès functions
- in a_2 exchange amplitude $\Rightarrow$ inhomogenous Omnès problem

A Solution strategy for inhom. Omnès problem

Coupled integral equation(s):

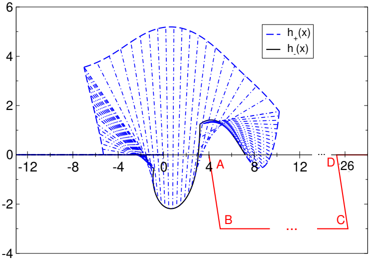
- $1 \to 3$ decay amplitude:

$$A(s) = \Omega(s)\left[P_n(s) + \frac{s^n}{\pi}\int \mathrm{d}\mu(x)\,\frac{\hat{A}(x)}{x - s - i\epsilon}\right]$$

- with 'hat'-function given by angular averages:

$$\hat{A}(x) = \frac{1}{2\kappa}\sum_{\ell} C_\ell(x,\kappa)\int_{-1}^{1}\mathrm{d}z\, z^\ell A(h(x,z))$$

- Approach by
  Gasser and Rusetsky, 2018
  - deform path of dispersion integral
  - applied by them to $\eta \to 3\pi \Rightarrow$

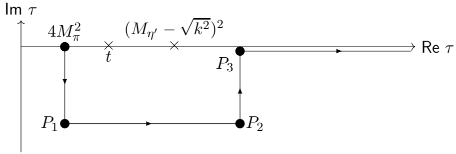


Inhomogeneous Omnès problem in $\eta' \to 2(\pi^+\pi^-)$

- Solution (P-wave) expressed in twice subtracted dispersion integral

$$f_1(t,k^2) = \left[P(t) + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}\tau}{\tau^2}\frac{\hat{G}(\tau,k^2)\sin\delta_1^1(\tau)}{(\tau - t - i\epsilon)|\Omega(\tau)|}\right]\Omega(t) + \hat{G}(t,k^2)$$

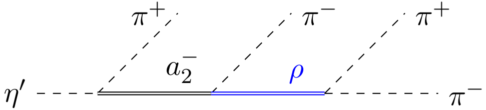
- Inhomogeneity $\hat{G}$ known for phenomenological model, but challenges direct evaluation due to singularity structure



- deform path of integration into complex plane (inspired by ideas of Gasser, Rusetsky 2018)

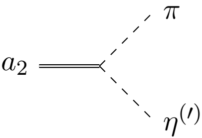
Inhomogeneity function

Left-hand cut contribution from phenomenological model:



$+$ 7 permutations

- ρ propagators replaced by Omnès functions
- projected onto $\pi^+\pi^-$–P-wave
- coupling pinned down from exp. widths for:



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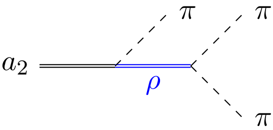

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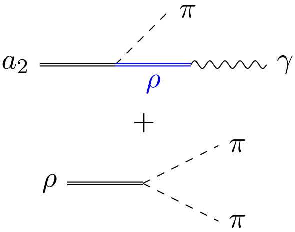
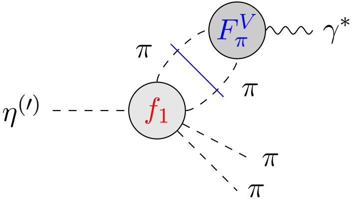


Towards a TFF representation

1st step

- unitarity condition:

$$\mathrm{Im}\, M(\eta^{(\prime)} \to \pi^+\pi^-\gamma^*) \sim \int \mathrm{d}\Phi_2\, M(\eta^{(\prime)} \to 2(\pi^+\pi^-)) M(\pi^+\pi^- \to \gamma^*)$$

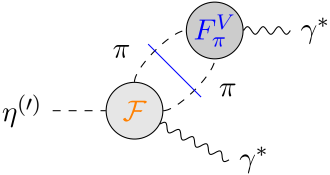


- fix subtraction constants from fit to pion spectra in real photon decays
- a_2 induced LHC leads to curvature effect

2nd step

- apply another (unsubtracted) dispersion relation

⇒ double-spectral representation of isovector doubly-virtual TFF



Putting the pieces together

Construct TFF from four ingredients:

$$F_{\eta^{(\prime)}\gamma^*\gamma^*} = F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*} + F^{(I=0)}_{\eta^{(\prime)}\gamma^*\gamma^*} + F^{\text{eff}}_{\eta^{(\prime)}\gamma^*\gamma^*} + F^{\text{asym}}_{\eta^{(\prime)}\gamma^*\gamma^*}$$

Isospin 1

- Dispersive piece: offers low-energy description
- reproduces low-energy cuts and singularities
  - additionally, left-hand cut contribution

Isospin 0

- Small; Description of narrow low-energy resonances

Effective Pole Term

- Parameterize higher intermediate states
- Full saturation of normalization sum rule
- Describe high-energy singly-virtual data

pQCD piece

- Induces leading-twist behavior of TFF ($\mathcal{O}(1/Q^2)$ asymptotics)

Analytic HLbL: η/η' poles in WP and beyond

	$a_\mu^{\eta-\text{pole}} \times 10^{11}$	$a_\mu^{\eta'-\text{pole}} \times 10^{11}$
CA [Masjuan, Sánchez-Puertas 2017]	$16.3\,(1.4)$	$14.5\,(1.9)$
DS [Eichmann et al. 2019]	$15.8\,(1.2)$	$13.3\,(0.9)$
DS [Raya et al. 2020]	$14.7\,(1.9)$	$13.6\,(0.8)$

- CA result in WP $a_\mu^{\text{PS}-\text{poles}} = 93.8^{+4.0}_{-3.6} \times 10^{-11}$
- Dispersive analysis: reconstruction of η/η' TFFs by incorporating all the lowest-lying singularities
- Aim: TFFs with fully-controlled uncertainty estimates
  - Dispersive input, normalization, singly/doubly virtual asymptotics
- Propagation to $g-2$ pole contributions

Status of calculation

- Starting point $\eta' \to 2(\pi^+\pi^-)$. Subsequent application of dispersion relations.
  - Factorization-breaking effects due to $a_2(1320)$ exchange included
  - Fix subtraction constant in intermediate step by fitting to $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ data
  - Finalizing curvature parameters due to a_2-exchange in the underlying representation

→ Evaluation of TFFs and uncertainty propagation to $a_\mu^{\eta-\text{pole}}/a_\mu^{\eta'-\text{pole}}$ ([still] in progress)

- **Final results (finally/hopefully) before plenary meeting this fall**