# Status update: $\pi^0 \to \gamma^*\gamma^*$ transition form factor on CLS ensembles

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The transition form factor is extracted from matrix elements

$$M_{\mu\nu}(p,q_1) = i\int \mathrm{d}^4x \mathrm{e}^{iq_1\cdot x}\langle 0|T\{J_\mu(x)J_\nu(0)\}|\pi^0(p)\rangle = \epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2),$$

where $J_\mu$ is the electromagnetic current. $q_1$ and $q_2$ are the four-momenta associated with the two currents, and $p$ is the four-momentum of the pion.

The Euclidean matrix elements read

$$M_{\mu\nu} = (i^{n_0})M^E_{\mu\nu},\ M^E_{\mu\nu} = -\int_{-\infty}^{\infty}\mathrm{d}\tau \mathrm{e}^{\omega_1\tau}\int \mathrm{d}^3x \mathrm{e}^{-i\vec{q}_1\cdot\vec{x}}\langle 0|T\{J_\mu(\vec{x},\tau)J_\nu(\vec{0},0)\}|\pi^0(p)\rangle,$$

and defining $\widetilde{A}_{\mu\nu}(\tau)$, the matrix elements can be obtained by integration

$$M^E_{\mu\nu}(p,q_1) = \frac{2E_\pi}{Z_\pi}\int_{-\infty}^{\infty}\mathrm{d}\tau \mathrm{e}^{\omega_1\tau}\widetilde{A}_{\mu\nu}(\tau),$$

where $\tau$ is the time separation between the two EM currents.

# Lattice correlators

$\widetilde{A}_{\mu\nu}(\tau)$ is connected to a 3-point correlator calculated on the lattice by

$$C^{(3)}_{\mu\nu}(\tau, t_\pi) \equiv a^6 \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{x}, t_i) J_\nu(\vec{0}, t_f) P^\dagger(\vec{z}, t_0)\rangle \mathrm{e}^{i\vec{p}\cdot\vec{z}} \mathrm{e}^{-i\vec{q}_1\cdot\vec{x}}$$

$$\widetilde{A}_{\mu\nu}(\tau) \equiv \lim_{t_\pi \to +\infty} \mathrm{e}^{E_\pi(t_f - t_0)} C^{(3)}_{\mu\nu}(\tau, t_\pi),$$

where $t_\pi$ is the time separation between the pion and the closest EM current.

For convenience we define a scalar funtion $\widetilde{A}^{(1)}(\tau)$:

$$\widetilde{A}_{0k}(\tau) = (\vec{q}_1 \times \vec{p}) \widetilde{A}^{(1)}(\tau)$$

$$\epsilon'^k \widetilde{A}_{kl}(\tau) \epsilon^l = -i(\vec{\epsilon}' \times \vec{\epsilon}) \cdot \left( \vec{q}_1 E_\pi \widetilde{A}^{(1)}(\tau) + \vec{p} \frac{\mathrm{d}\widetilde{A}^{(1)}(\tau)}{\mathrm{d}\tau} \right)$$

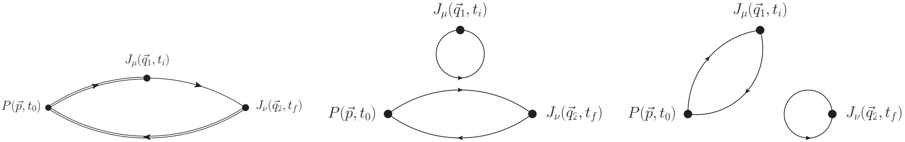
In the moving frame $(p_z \neq 0)$ we also define $\widetilde{A}_{12}(\tau) \equiv -iE_\pi p_z \widetilde{A}^{(2)}(\tau)$.

## CLS $N_f = 2+1$ ensembles PRD**100** (2019) 034520

- non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions
- tree-level improved Lüscher-Weisz gauge action
- four lattice spacings, multiple pion masses, large volumes ($M_\pi L \geq 4$)

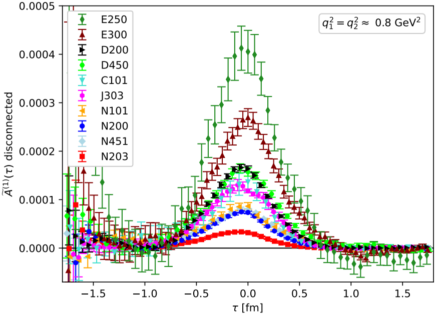
| ID | $\beta$ | $L^3 \times T$ | $a$/fm | $\kappa_l$ | $\kappa_s$ | $M_\pi$/MeV | $M_\pi L$ | $N_{\mathrm{conf}}$ |
|---|---|---|---|---|---|---|---|---|
| H101 | 3.40 | $32^2 \times 96$ | 0.08636 | 0.136760 | 0.13675962 | 416 | 5.8 | 1000 |
| H102 | | $32^2 \times 96$ | | 0.136865 | 0.13654934 | 354 | 5.0 | 1900 |
| H105 | | $32^2 \times 96$ | | 0.136970 | 0.13634079 | 281 | 3.9 | 2800 |
| N101 | | $48^2 \times 128$ | | 0.136970 | 0.13634079 | 280 | 5.9 | 1600 |
| C101 | | $48^2 \times 96$ | | 0.137030 | 0.13622204 | 224 | 4.7 | 2200 |
| S400 | 3.46 | $32^2 \times 128$ | 0.07634 | 0.136984 | 0.13670239 | 349 | 4.3 | 1700 |
| N401 | | $48^2 \times 128$ | | 0.137062 | 0.13654808 | 286 | 5.3 | 950 |
| H200 | 3.55 | $48^2 \times 96$ | 0.06426 | 0.137000 | 0.137000 | 419 | 4.4 | 2000 |
| N202 | | $48^2 \times 128$ | | 0.137000 | 0.137000 | 411 | 6.4 | 900 |
| N203 | | $48^2 \times 128$ | | 0.137080 | 0.13684028 | 346 | 5.4 | 1500 |
| N200 | | $48^2 \times 128$ | | 0.137140 | 0.13672086 | 284 | 4.4 | 1700 |
| D200 | | $64^2 \times 128$ | | 0.137200 | 0.13660175 | 200 | 4.2 | 1100 |
| E250 | | $96^2 \times 192$ | | 0.137232867 | 0.136536633 | 129 | 4.0 | 800 |
| N300 | 3.70 | $48^2 \times 128$ | 0.04981 | 0.137000 | 0.137000 | 342 | 5.1 | 1200 |
| N302 | | $48^2 \times 128$ | | 0.137064 | 0.13687218 | 343 | 4.2 | 1100 |
| J303 | | $64^2 \times 192$ | | 0.137123 | 0.13675466 | 258 | 4.2 | 650 |

# Disconnected contribution




In addition to the quark-line connected diagram, there are contributions from two quark-line disconnected diagrams that have to be calculated.

- The quark loops are computed using stochastic all-to-all methods, while the two-point functions are computed using point sources.
- The dependence of the disconnected piece on the pion mass is clearly visible

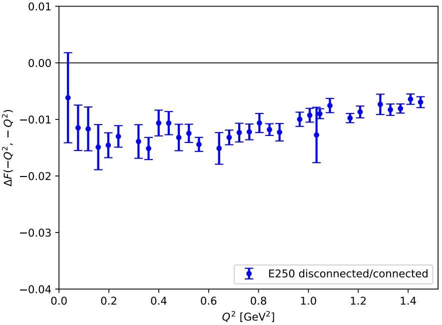


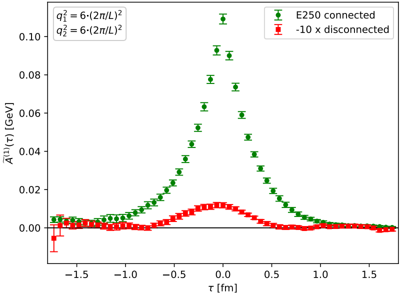
# Disconnected contribution

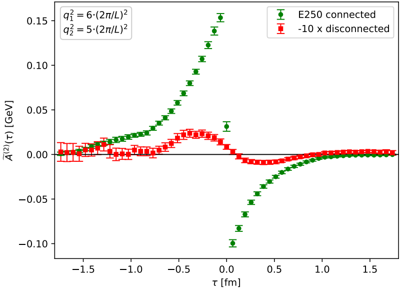
- We find the disconnected contribution

$$\Delta F(-Q_1^2,-Q_2^2) = \frac{\mathcal{F}^{\text{disc}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}{\mathcal{F}^{\text{conn}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}$$

is at most at a few percent level.



# Modeling the tail

Recall that $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{e}^{\omega_1 \tau} \widetilde{A}_{\mu\nu}(\tau)$.

We want to model $\widetilde{A}_{\mu\nu}(\tau)$ at large $|\tau|$ to get the tail contribution.

- Lowest Meson Dominance (LMD)

$$\widetilde{A}_{\mu\nu}^{\mathrm{LMD}} = \frac{Z_\pi}{4\pi E_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tilde{\omega} \frac{\left(P_{\mu\nu E}\tilde{\omega} + Q^E_{\mu\nu}\right)\left(\alpha M_V^4 + \beta(q_1^2 + q_2^2)\right)}{\left(\tilde{\omega} - \tilde{\omega}_1^{(+)}\right)\left(\tilde{\omega} - \tilde{\omega}_1^{(-)}\right)\left(\tilde{\omega} - \tilde{\omega}_2^{(+)}\right)\left(\tilde{\omega} - \tilde{\omega}_2^{(-)}\right)} \mathrm{e}^{-i\tilde{\omega}\tau}$$

$$\text{with} \quad P^E_{\mu\nu} = i\epsilon_{\mu\nu 0 i} p^i, \qquad \tilde{\omega}_1^{(\pm)} = \pm i\sqrt{M_V^2 + |\vec{q}_1|^2}$$

$$Q^E_{\mu\nu} = \epsilon_{\mu\nu i 0} E_\pi q_1^i - i\epsilon_{\mu\nu i j} q_1^i p^j, \qquad \tilde{\omega}_2^{(\pm)} = -i\left(E_\pi \mp \sqrt{M_V^2 + |\vec{q}_2|^2}\right)$$

  This gives an explicit expression for $\widetilde{A}_{\mu\nu}^{\mathrm{LMD}}$, which we use to fit our data using $\alpha$, $\beta$ and $M_V$ as fit parameters.

- Vector Meson Dominance (VMD): Set $\beta = 0$ in the LMD model

# Parameterizing the form factor: $z$-expansion

After obtaining the transition form factor at several virtualities $(q_1^2, q_2^2) \equiv (-Q_1^2, -Q_2^2)$, we parameterize it using a conformal mapping

$$z_k = \frac{\sqrt{t_{\rm cut} + Q_k^2} - \sqrt{t_{\rm cut} - t_0}}{\sqrt{t_{\rm cut} + Q_k^2} + \sqrt{t_{\rm cut} - t_0}}, \text{ with } t_{\rm cut} = 4m_\pi^2,\ t_0 = t_{\rm cut}\left(1 - \sqrt{1 + \frac{Q_{\rm max}^2}{t_{\rm cut}}}\right).$$

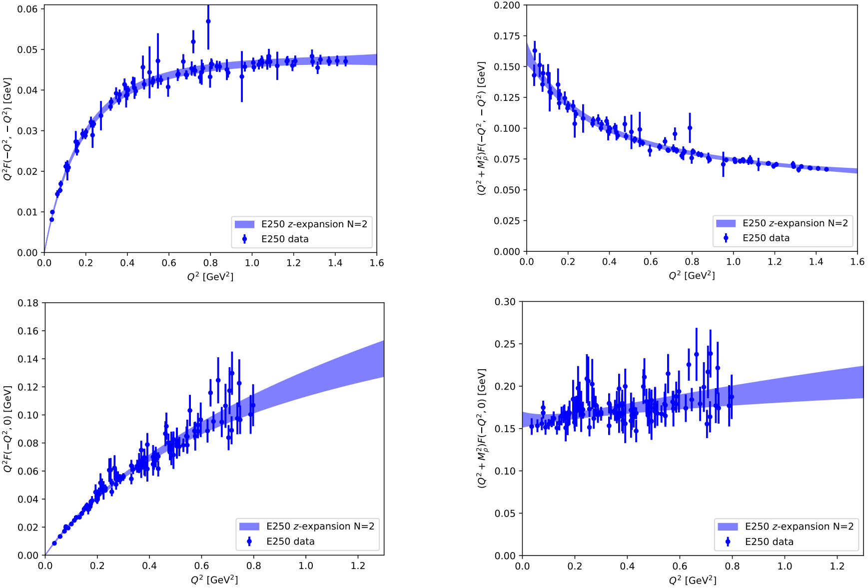
The form factor is then written as an expansion in $z_1$ and $z_2$:

$$P(Q_1^2, Q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) = \sum_{n,m=0}^{N} c_{nm} \left(z_1^n + (-1)^{N+n}\frac{n}{N+1} z_1^{N+1}\right)\left(z_2^m + (-1)^{N+m}\frac{m}{N+1} z_2^{N+1}\right),$$

where the coefficients $c_{nm} = c_{mn}$, the fit parameters, are symmetric.

$P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2 + Q_2^2}{M_V^2}$ is the vector meson pole with $M_V = 775$ MeV.

# Results: Transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$



Thank you!
Any questions?

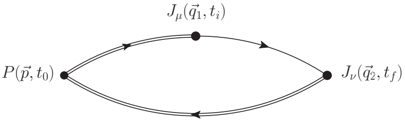
# Backup slides

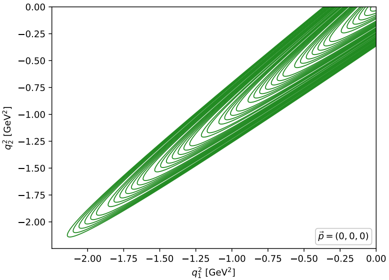
# Photon virtualities

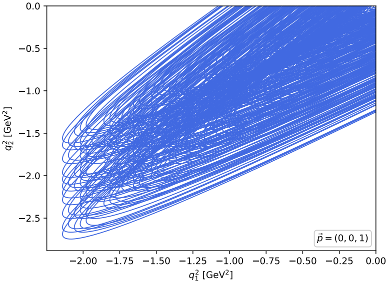
- Use both the rest frame of the pion, $\vec{p} = (0,0,0)$, and a moving frame $\vec{p} = (0,0,1)$ (in units of $2\pi/L$)
- The four-momenta associated with the EM currents are

$$q_1 = (\omega_1, \vec{q}_1)$$
$$q_2 = (E_\pi - \omega_1, \vec{p} - \vec{q}_1)$$

- Each curve in the plot represents a fixed value of $\vec{q}_1$ and $\vec{p}$
- $\omega_1$ is a free parameter (this tracks the curve from one end to another)



# Partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$

Recall the relation between the partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$ and the transition form factor:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\pi \alpha_e^2 m_{\pi^0}^3}{4} \mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(0,0)$$

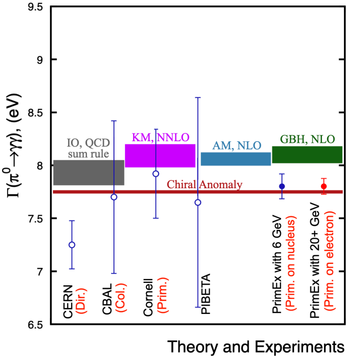



Figure from JLab whitepaper arXiv:2306.09360.