Muon g-2 Theory Initiative
Spring 2024 meeting

Pion pole’ s contribution to HLbL

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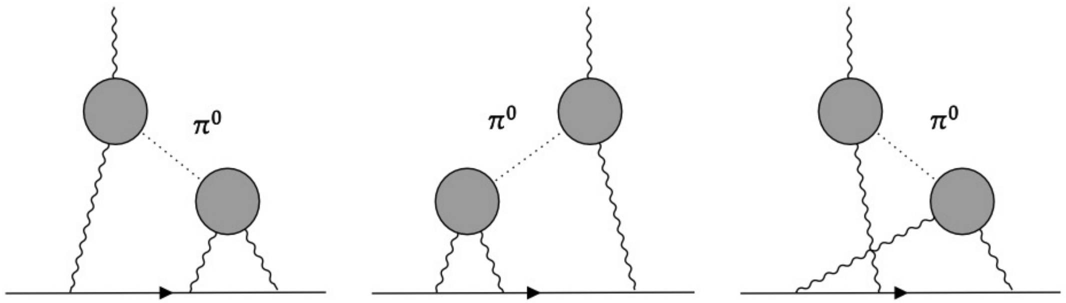
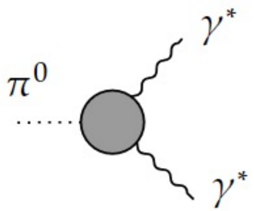
On behalf of RBC-UKQCD Collaboration

Starting point



$$a_\mu^{\pi^0-\text{pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau$$
$$w_1(Q_1,Q_2,\tau)\mathcal{F}_{\pi^0\gamma^*\gamma}(Q_1^2,0)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_2^2,Q_3^2)$$
$$+w_2(Q_1,Q_2,\tau)\mathcal{F}_{\pi^0\gamma^*\gamma}(Q_3^2,0)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_3^2)$$

w_1, w_2 are known weight functions



$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)$ Transition form factor (TFF) are the inputs

Question: How to calculate TFF for arbitrary momentum Q_1^2, Q_2^2

Methodology

- In the continuum theory, TFF is defined in Euclidean space as

$$\varepsilon_{\mu\nu\alpha\beta}Q^{\alpha}Q'^{\beta}\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2,Q'^2) = i\int d^4x\, e^{-i(Q-P/2)\cdot x}\mathcal{H}_{\mu\nu}(x)$$

$Q=(iE,\vec{Q})$ is arbitrary 4-momentum for one off-shell photon

$\mathcal{H}_{\mu\nu}(x) = \langle 0|T\{J_\mu(\frac{x}{2})J_\nu(-\frac{x}{2})\}|\pi(P)\rangle$ is the hadronic function

- Step 1: Lorentz decomposition

$$\mathcal{H}_{\mu\nu}(x) = \varepsilon_{\mu\nu\alpha\beta}x^{\alpha}P^{\beta}H(x^2,P\cdot x)$$

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2,Q'^2) = i\int d^4x\,\omega(Q,P,x)H(x^2,P\cdot x)$$

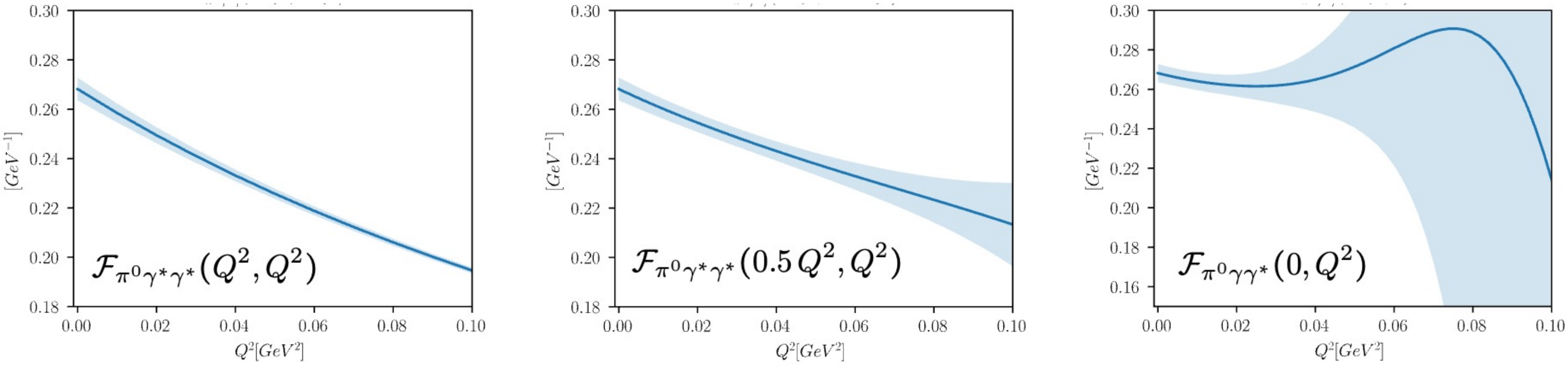
Tian Lin (PKU)
2nd year PhD student

- Step 2: Spatial rotation average
  - In pion rest frame $P=(im_\pi,\vec{0})$, $H(x^2,P\cdot x)$ is invariant under spatial rotation
  - Thus, one can perform spatial rotation average for ω. It only depends on $|\vec{Q}|$ and $|\vec{x}|$, rather than the angle

$$\omega(Q,P,x) = -e^{(E-\frac{1}{2}m_\pi)t}\frac{|\vec{x}|}{|\vec{Q}|}j_1(|\vec{Q}||\vec{x}|)$$

Problem

- Results for TFF



- Very noisy for $(Q_1^2, Q_2^2) = (0, Q^2)$ at large Q^2, $Q^2 = 2|\vec{Q}|m_\pi - m_\pi^2$
- When $|E| = |\vec{Q}|$ becomes large

$$\omega(Q,P,x) = -e^{(E-\frac{1}{2}m_\pi)t}\frac{|\vec{x}|}{|\vec{Q}|}j_1(|\vec{Q}||\vec{x}|) \Rightarrow \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, Q'^2) = i\int d^4x\, \omega(Q,P,x) H(x^2, P\cdot x)$$

Time: exponential growth

Space: significant oscillation

enhance the noise from $H(x^2, P\cdot x)$

Make things simple

- In the chiral limit, $P = (im_\pi, \vec{0}) = (0, \vec{0})$

  - $H(x^2, P\cdot x)$ is independent of $P\cdot x$

$$H(x^2, P\cdot x) \quad \Rightarrow \quad H(x^2, 0)$$

  - One can perform SO(4) average

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2) = i\int d^4x \boxed{\frac{2}{Q^2}\left(-\frac{J_1(Qx)}{Qx} + J_0(Qx) - J_2(Qx)\right)} H(x^2, 0) \qquad Q = Q_1 - \frac{1}{2}P$$

No exponential growth for large x

The simple case inspires the solution!

Physical world

- Hadronic function can be written in terms of pion structure function

SO(4) symmetric

$$H(x^2, P\cdot x) = \int_0^1 du\, e^{i(u-\frac{1}{2})P\cdot x}\, \phi_\pi(x^2,u)\, H(x^2,0)$$

Pion structure function
- Symmetric in $\phi_\pi(x^2,u) = \phi_\pi(x^2,1-u)$
- Normalization $\int_0^1 du\, \phi_\pi(x^2,u) = 1$

- At small x^2 and up to higher-twisted correction, $\phi_\pi(x^2,u)$ is equivalent to pion distribution amplitude

Bali et. al., PRD 98 (2018) 094507

- TFF can be written as

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*} = -2i\int_0^1 du \int d^4x\, \frac{J_2(Qx)}{Q^2}\, \phi_\pi(x^2,u)\, H(x^2,0)$$

→ Lattice input

No S/N problem

Introduce unknown structure function?

Gegenbauer expansion

- Short summary

lattice input (positive) — structure function — analytical, known

$$a_\mu^{\pi^0\text{-pole}} = \int d^4x_1 \int d^4x_2 \int du_1 \int du_2\, H(x_1^2,0)H(x_2^2,0)\,\phi_\pi(x_1^2,u_1)\phi_\pi(x_2^2,u_2)\,\rho(x_1,x_2,u_1,u_2)$$

- Introduce Gegenbauer polynomials $C_{2n}(2u-1)$
  - It forms a complete polynomial basis and satisfies the orthogonal condition

$$\int_0^1 du\, u(1-u)\, C_{2n}(2u-1)\, C_{2m}(2u-1) = \delta_{nm}\frac{(n+1)(n+2)}{4(2n+3)}$$

- Perform Gegenbauer expansion for $\phi_\pi(x^2,u)$

$$\phi_\pi(x^2,u) = 6u(1-u)\sum_n a_{2n}(x^2)\, C_{2n}(2u-1)$$

Gegenbauer moment

- Consequently

$$a_\mu^{\pi^0\text{-pole}} \propto \sum_{n,m}\int d^4x_1 \int d^4x_2\, H(x_1^2,0)H(x_2^2,0)\, a_{2n}(x_1^2)a_{2m}(x_2^2)\,\rho_{2n,2m}(x_1,x_2)$$

structure information — suppress quickly as n,m increases

Gegenbauer expansion

- Short summary

$$a_\mu^{\pi^0\text{-pole}} \propto \sum_{n,m} \int d^4x_1 \int d^4x_2\, H(x_1^2,0)H(x_2^2,0)\, a_{2n}(x_1^2)a_{2m}(x_2^2)\, \rho_{2n,2m}(x_1,x_2)$$

structure information — suppress quickly as n,m increases

- Recall

$$\phi_\pi(x^2,u) = 6u(1-u)\sum_n a_{2n}(x^2)\, C_{2n}(2u-1)$$

Normalization condition $\int_0^1 du\, \phi_\pi(x^2,u) = 1$ yields $a_0(x^2) = 1$

- Dominant contribution is structure independent

$$a_\mu^{\pi^0\text{-pole}} = a_{0,0}^{\pi^0\text{-pole}} + a_{0,2}^{\pi^0\text{-pole}} + a_{2,0}^{\pi^0\text{-pole}} + \cdots$$

independent of pion structure

2nd moment, depends on pion structure

Structure functions

- Extract structure function from lattice data?
  - Boost pion with large momenta (S/N problem)
  - Inverse problem
  - Similar to the calculation of pion distribution amplitude

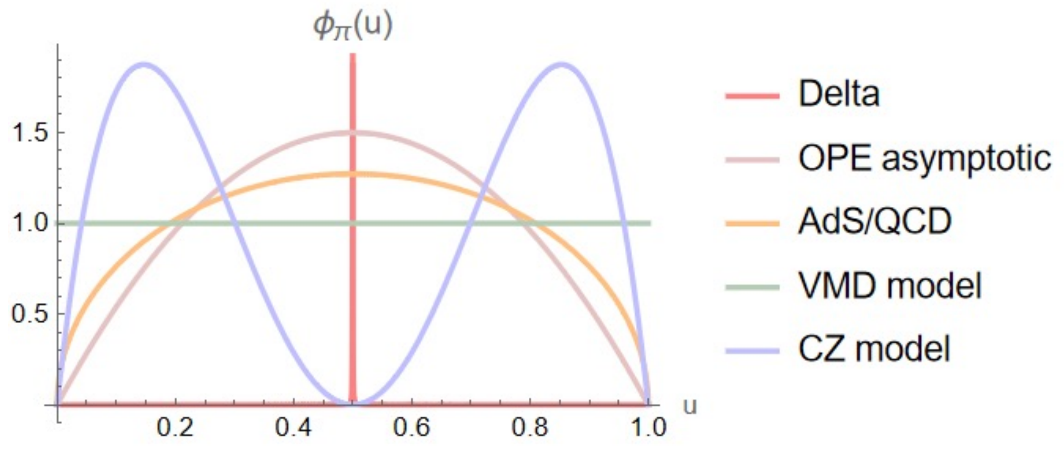
A completely new project!

- Control the structure function dependence using various models

Delta $\phi_\pi(u) = \delta(u - \frac{1}{2})$
OPE asymptotic :$\phi_\pi(u) = 6u(1-u)$
AdS/QCD:$\phi_\pi(u) = \frac{8}{\pi}\sqrt{u(1-u)}$
VMD model:$\phi_\pi(u) = 1$
CZ model: $\phi_\pi(u) = 30u(1-u)(1-2u)^2$

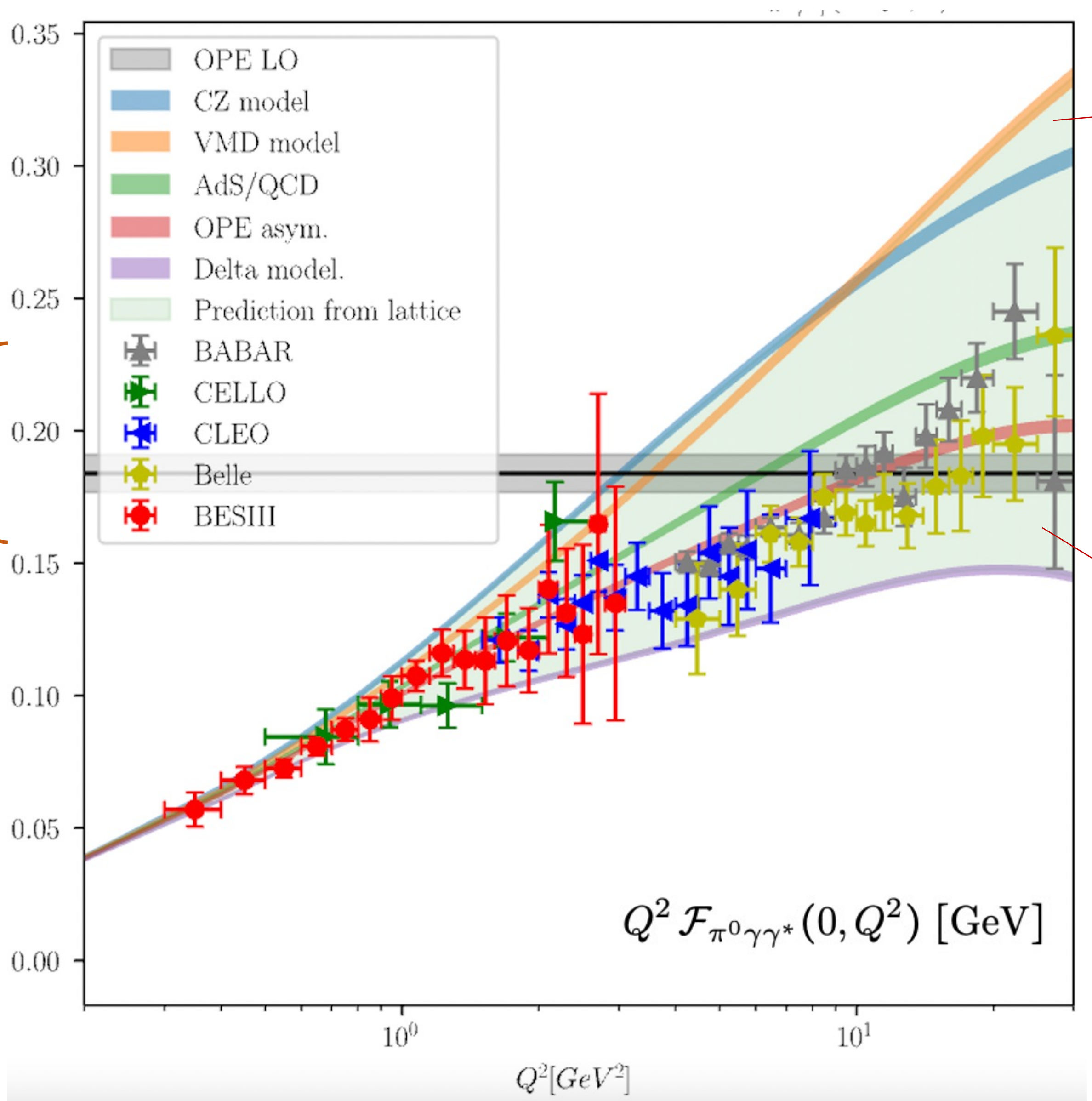
SD, OPE

LD, VMD



Transition form factor

Experimental data



Result from a given model

Lattice QCD provide $H(x^2, 0)$

Combine diff. models

Cover all exp. data!

RBC-UKQCD ensembles @ physical m_π

Ensembles	$a[\mathrm{GeV}^{-1}]$	L/a	T/a	L/fm	$m_\pi[\mathrm{MeV}]$	$m_\pi L$	# of confs
24D	1.015	24	64	4.7	141.56(22)	3.3	253
32D	1.015	32	64	6.2	141.38(20)	4.5	63
32Df	1.378	32	64	4.7	142.89(40)	4.7	69
48I	1.730	48	96	5.5	139.60(16)	5.5	112
64I	2.359	64	128	5.4	135.33(20)	5.4	65

Domain wall fermion + Iwasaki gauge action (+DSDR)

Systematic effects from structure func.

Model	a_2
CZ model	0.667
VMD model	0.389
AdS/QCD	0.146
OPE asym.	0
Delta model	-0.583

maximum

minimum

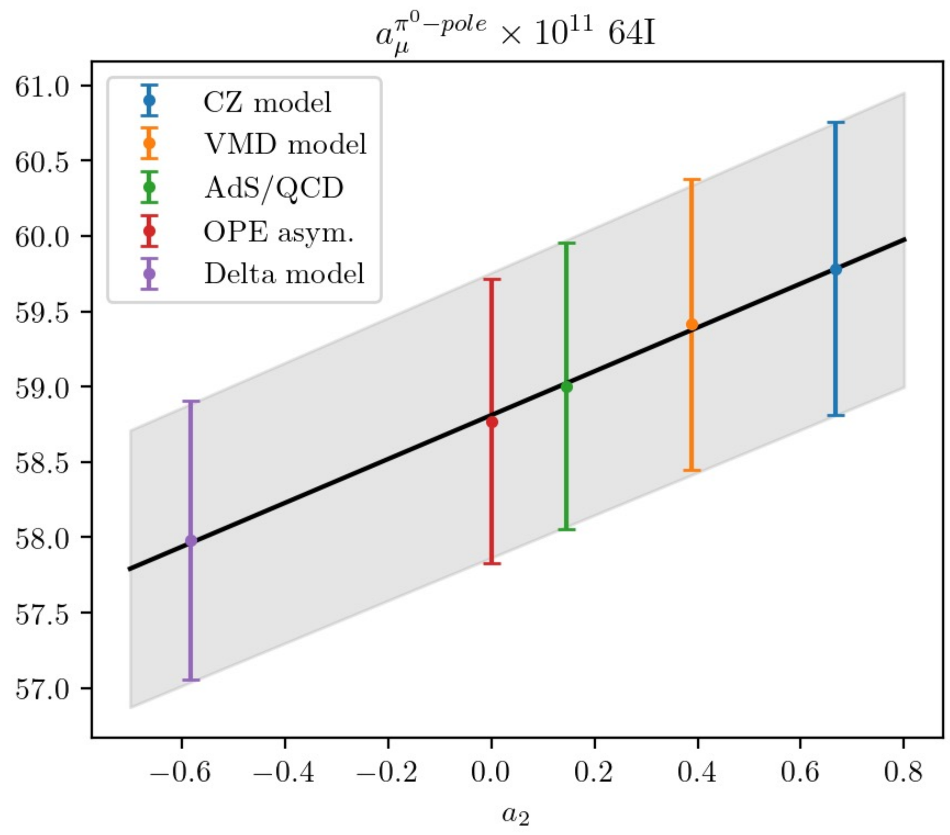
CZ model: $\phi_\pi(u) = 30u(1-u)(1-2u)^2$

VMD model: $\phi_\pi(u) = 1$

AdS/QCD: $\phi_\pi(u) = \frac{8}{\pi}\sqrt{u(1-u)}$

OPE asymptotic: $\phi_\pi(u) = 6u(1-u)$

Delta $\phi_\pi(u) = \delta(u-\frac{1}{2})$

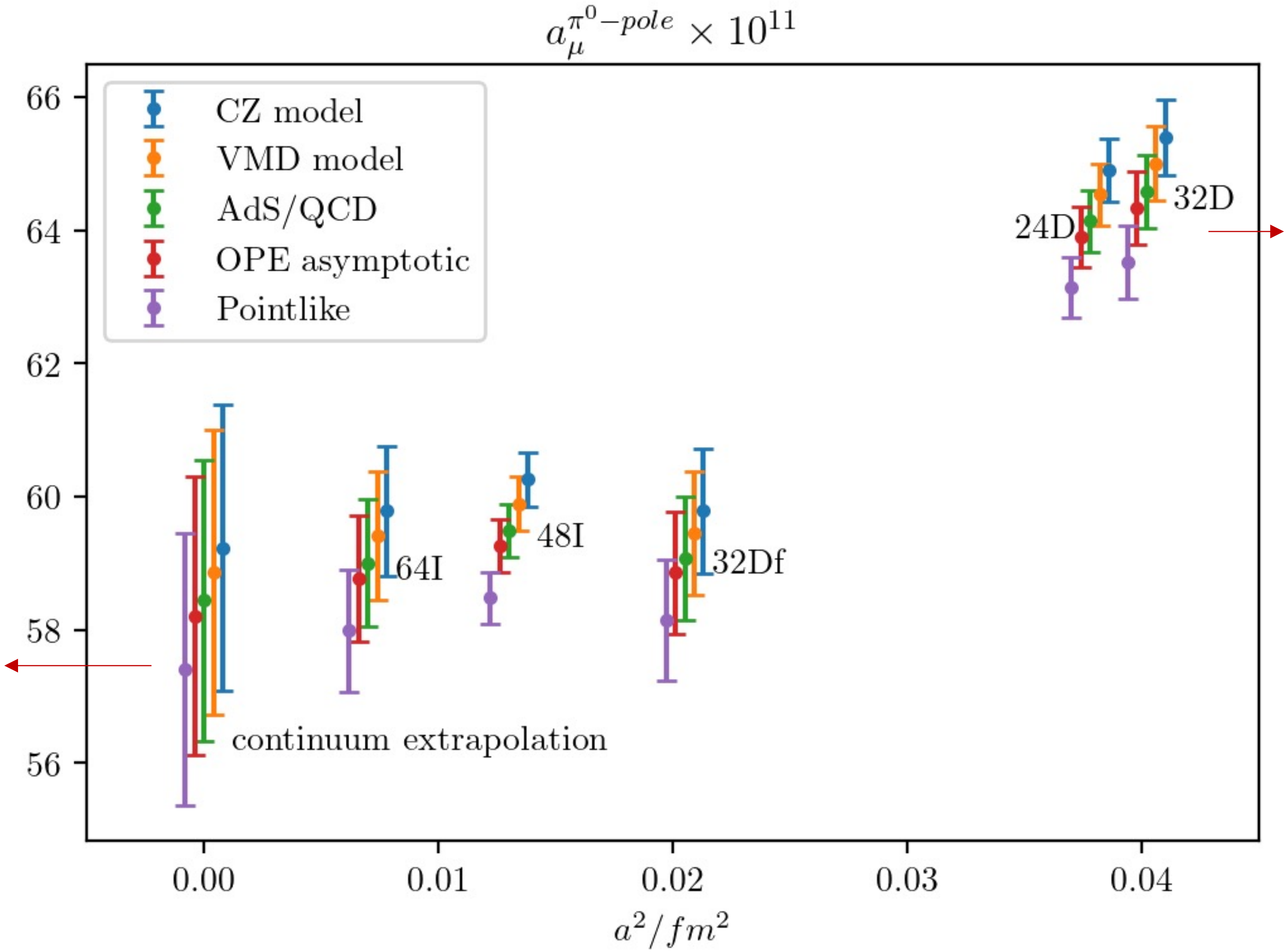


- 64I: $a_\mu^{\pi^0\text{-pole}} \times 10^{11} = 58.81(95) + 1.46(5)a_2$

0th moment, model independent ≫ 2nd moment

- Results well described by linear form on a_2, confirming higher-moment contributions are negligible

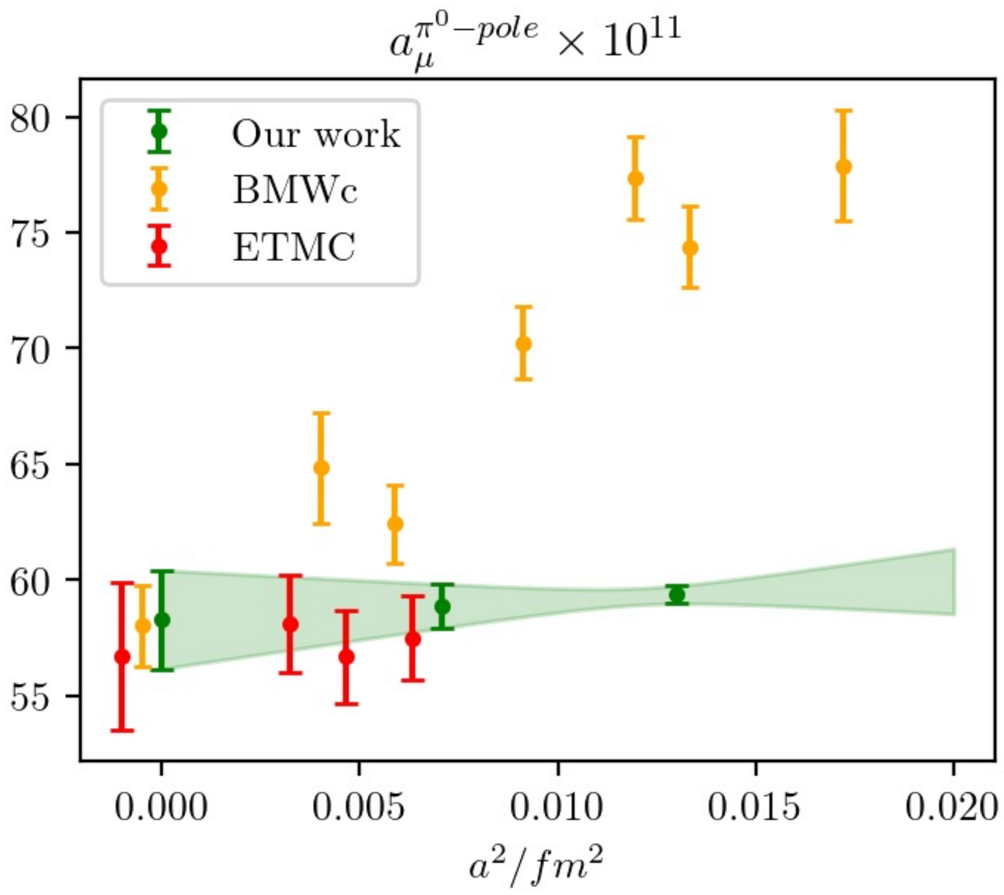
Continuum extrapolation

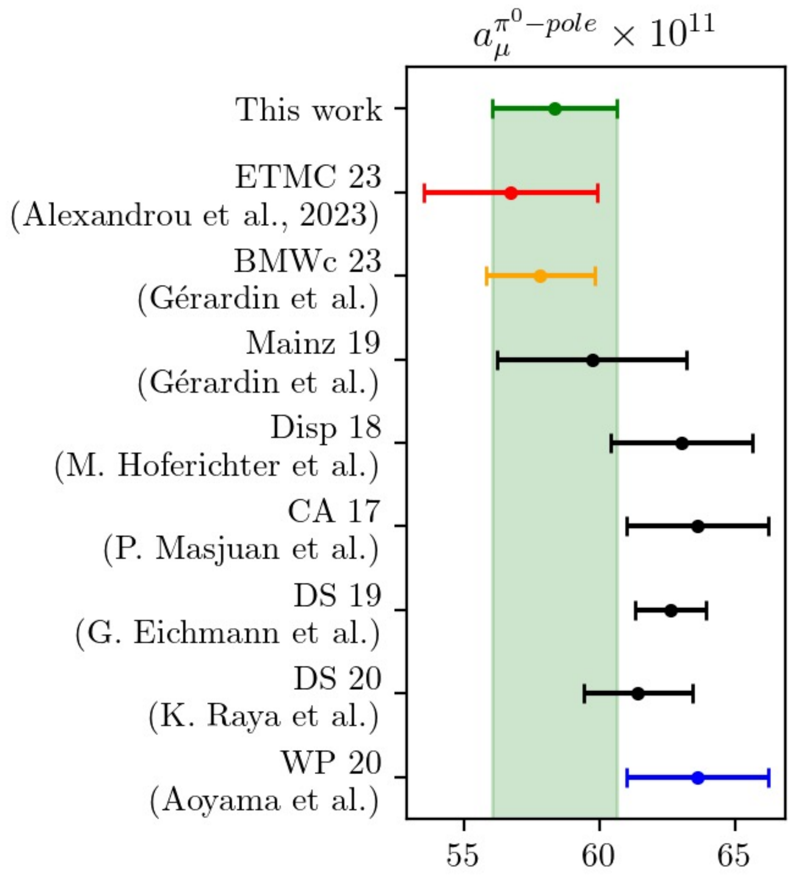


24D and 32D have same lattice spacing but different volume

Various models lead to consistent continuum limit

Summary





- A new method to calculate pion pole’ s contribution
- Statistical: pion at rest, $H(x^2,0)$ with SO(4) average ➡ accurate result
- Systematic: model dependence is well-controlled

- On going efforts: 96I, disc. diagram, error budget