
Small- x the evolution of color dipole in the NLO and the argument of coupling constant in the BK equation

Ian Balitsky

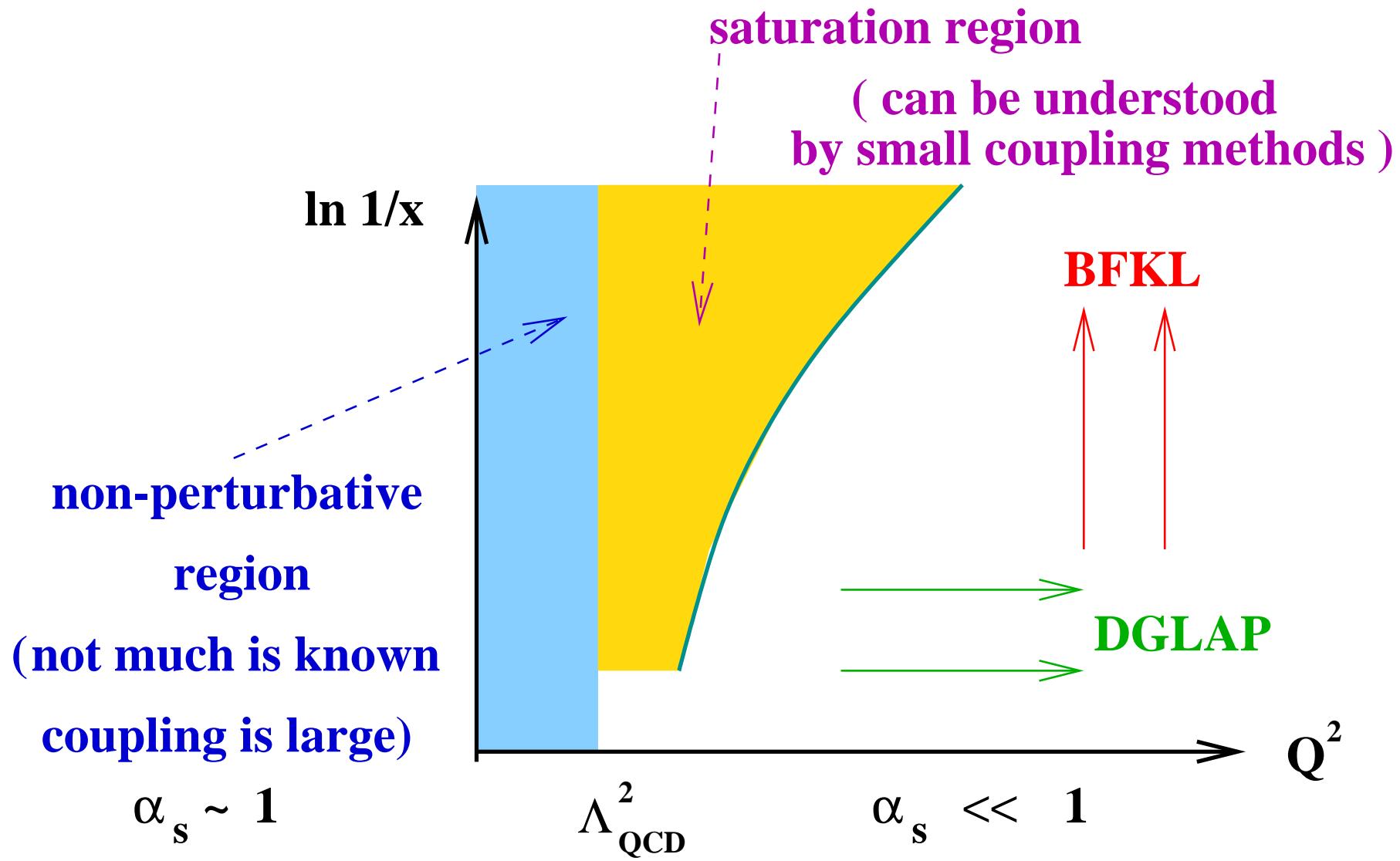
JLab & ODU

Fermilab, 29 March 2007

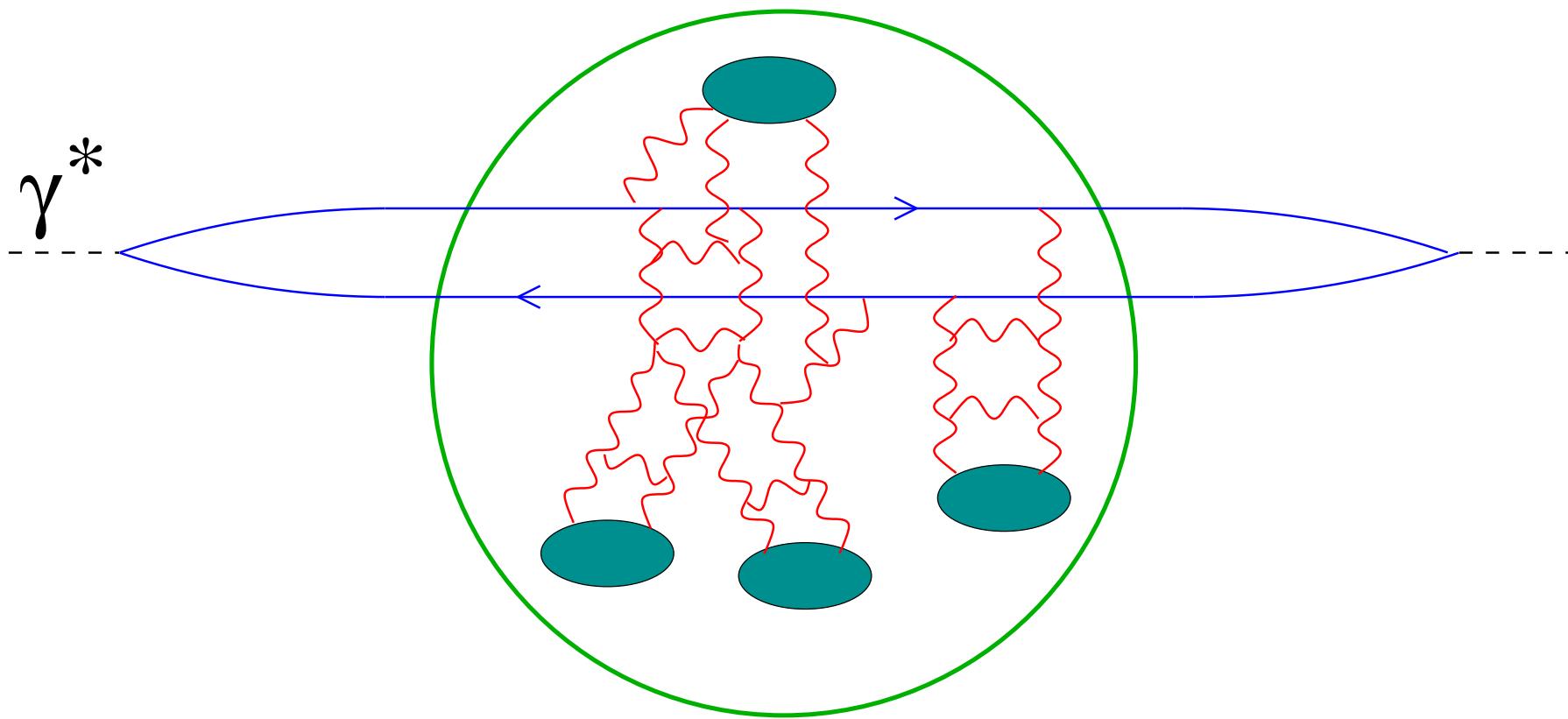
Plan

- Small-x DIS as an evolution of Wilson lines
- Non-linear evolution equation (BK eqn)
- Quark loop contribution to NLO BK.
- Can the high-energy scattering can be described in terms of dipoles?
- Bubble chain and the argument of coupling constant
- Gluon part of NLO BK
- Conclusions and outlook

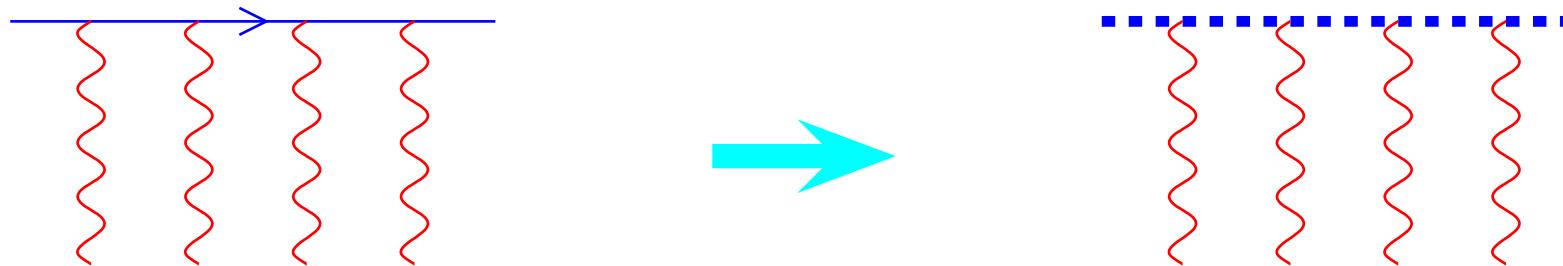
“Phase diagram” for DIS



Small- x DIS from the nucleus

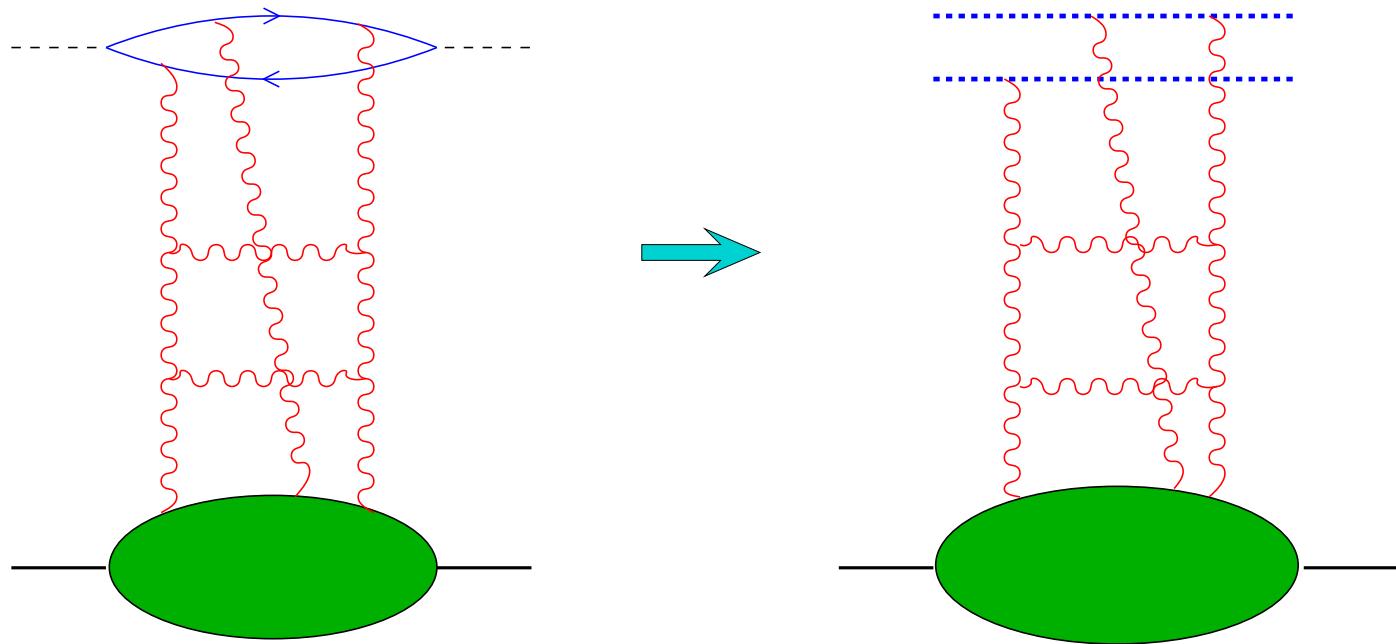


Fast quark moves along the straight line \Rightarrow



quark propagator reduces to the Wilson line collinear to quark's velocity

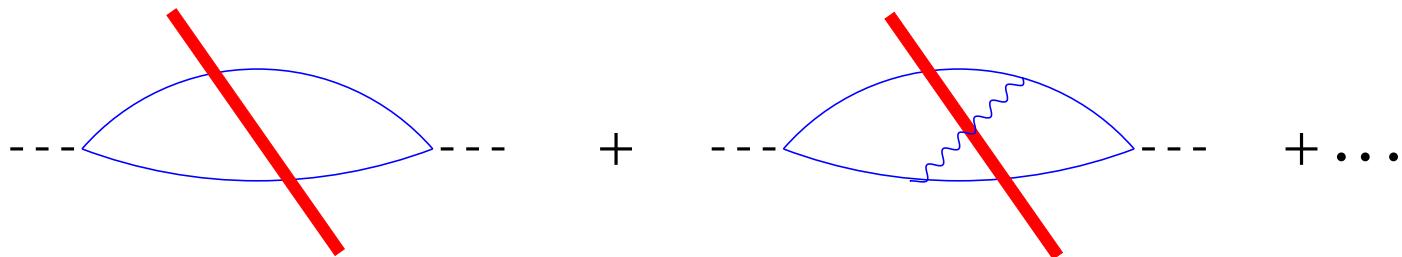
At high energies, the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (“color dipole”).



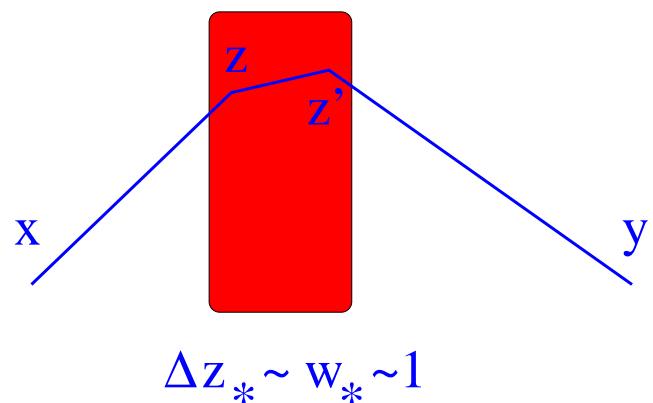
$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I(k_\perp) \langle B | \text{Tr}\{ U^{\eta_A}(k_\perp) U^{\dagger \eta_A}(-k_\perp) \} | B \rangle + \dots$$

In the spectator frame

High-speed nucleus shrinks to a “pancake” \Rightarrow



Quarks (and gluons) do not have time to deviate in the transverse direction



$$p = \alpha p_1 + \beta p_2 + p_\perp = \text{Sudakov variables}$$

$$(z | \frac{1}{p^2 + i\epsilon} | z') = \frac{s}{8\pi x_*} \int_0^\infty \frac{d\alpha}{\alpha} e^{-i\alpha(z-z')_\bullet - i\frac{(z-z')_\perp^2}{4(z-z')_*}\alpha s}$$

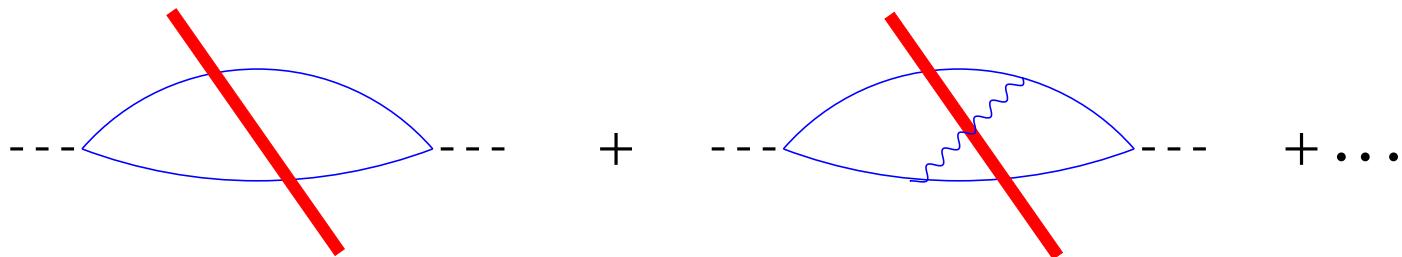
$$\Rightarrow |z - z'|_\perp \sim \sqrt{\frac{w_*}{\alpha s}} \sim \sqrt{\frac{1}{s}}$$

$$z_* \equiv z \cdot p_1, \quad z_\bullet \equiv z \cdot p_2$$

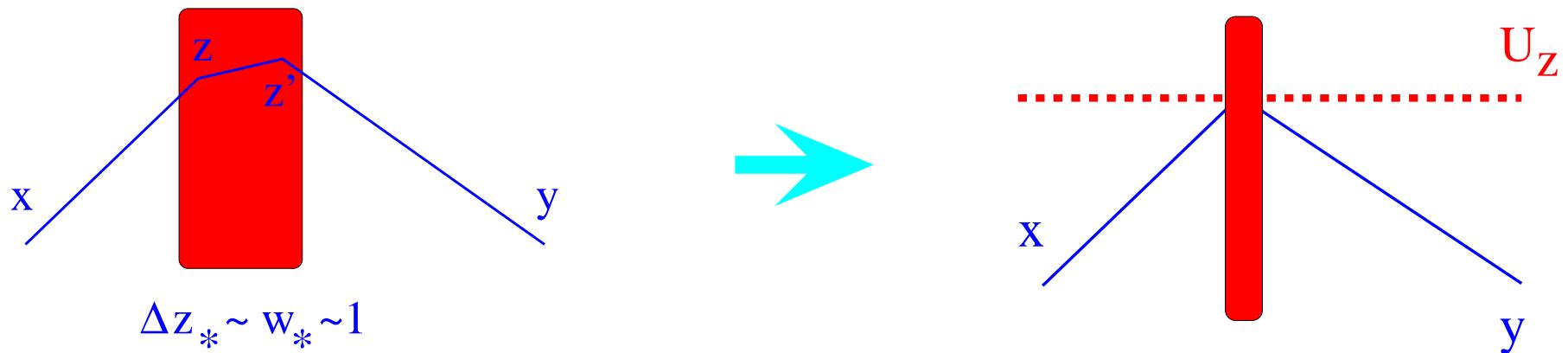
\Rightarrow Interaction $\sim U_z$ = gauge factor ordered along (a segment of) the straight line

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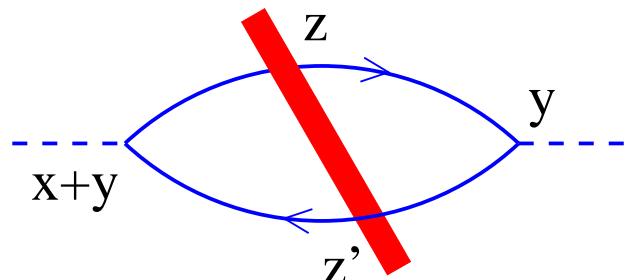


$$G(x, y) = \int dz \delta(z_*) (x | \frac{\not{p}}{p^2} | z) \not{p}_2 U_z(z | \frac{\not{p}}{p^2} | y)$$

$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$ – Wilson line

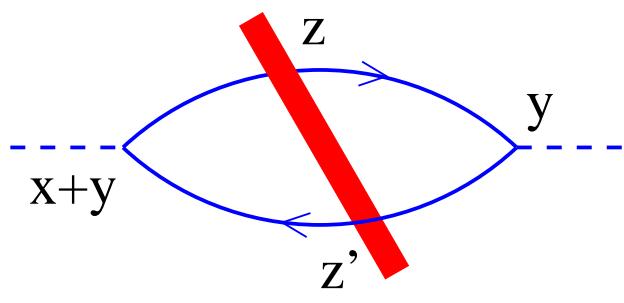
$$[x, y] \equiv P e^{ig \int_0^1 du (x-y)^\mu A_\mu (u x + (1-u)y)}$$

Feynman diagrams in a shock-wave background



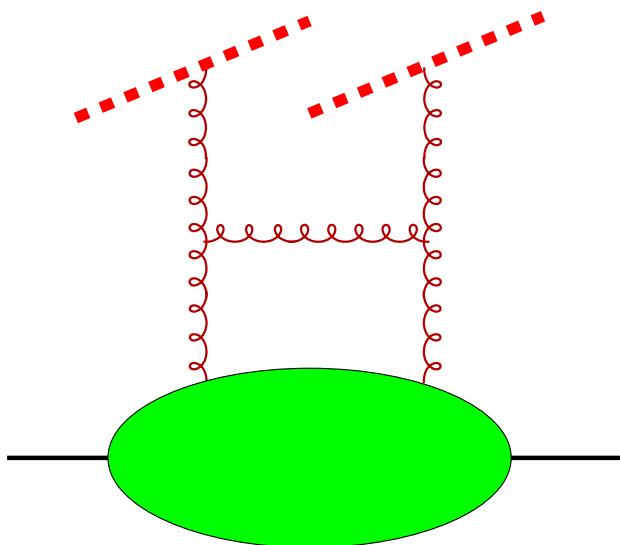
$$\begin{aligned} & \int d^4x d^4y e^{-ip_A \cdot x} \langle T\{j_A(x+y) j'_A(y)\} \rangle_{\textcolor{red}{A}} \\ = & \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \text{Tr}\{\textcolor{red}{U}(k_\perp) \textcolor{red}{U}^\dagger(-k_\perp)\} + \dots \Rightarrow \\ A(s) = & \int \frac{d^2 k_\perp}{4\pi^2} I(k_\perp) \langle B | \text{Tr}\{\textcolor{red}{U}(k_\perp) \textcolor{red}{U}^\dagger(-k_\perp)\} | B \rangle + \end{aligned}$$

Feynman diagrams in a shock-wave background



$$\begin{aligned} & \int d^4x d^4y e^{-ip_A \cdot x} \langle T\{j_A(x+y)j'_A(y)\} \rangle_A \\ &= \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \text{Tr}\{\mathcal{U}(k_\perp) \mathcal{U}^\dagger(-k_\perp)\} + \dots \Rightarrow \\ & A(s) = \int \frac{d^2k_\perp}{4\pi^2} I(k_\perp) \langle B | \text{Tr}\{\mathcal{U}(k_\perp) \mathcal{U}^\dagger(-k_\perp)\} | B \rangle + \end{aligned}$$

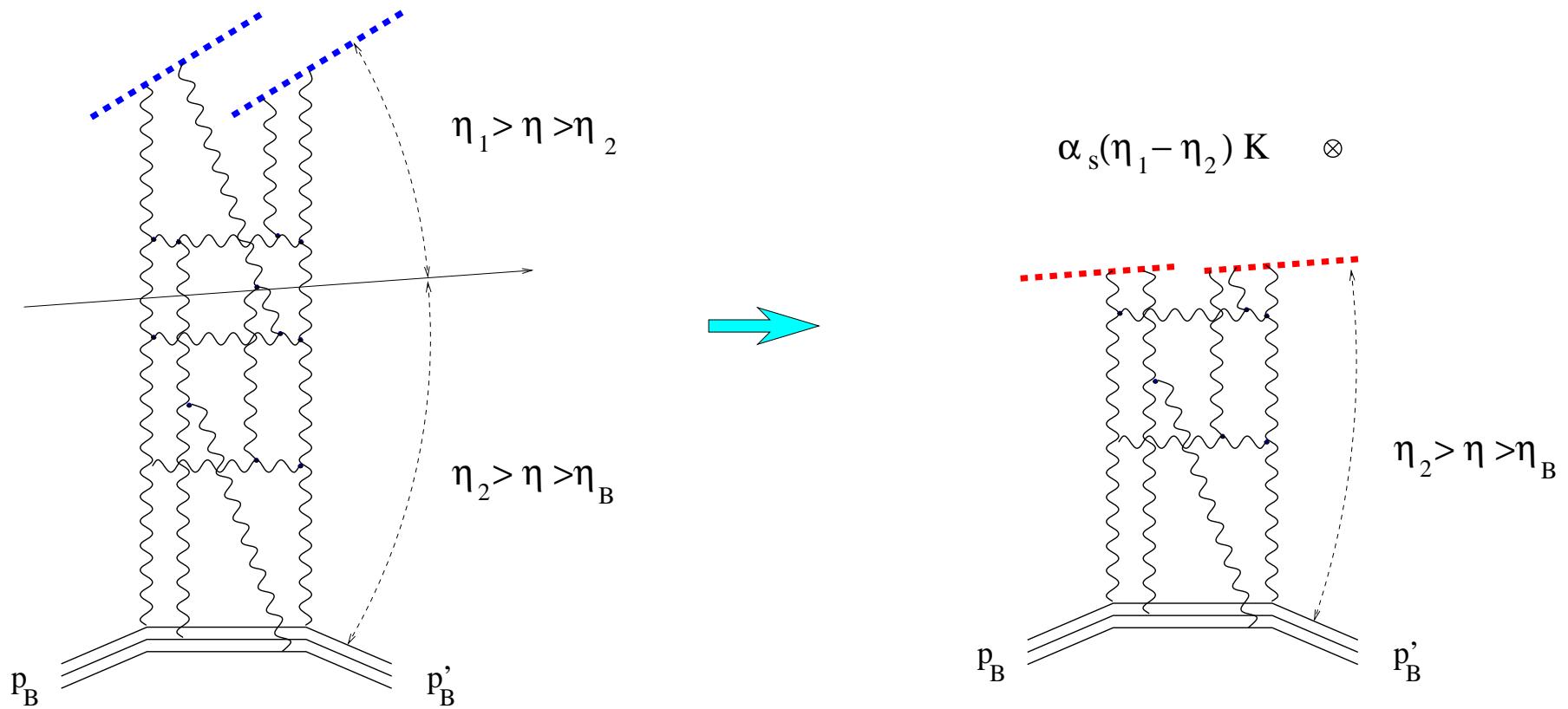
Energy dependence of the amplitude $A(s)$ is determined by the dependence of the Wilson lines on the rapidity η_A defined by the slope of the line.



$$\langle B | U_x^{\eta_A} U_y^{\dagger \eta_A} | B \rangle \sim \int_{\eta_B}^{\eta_A} d\eta$$

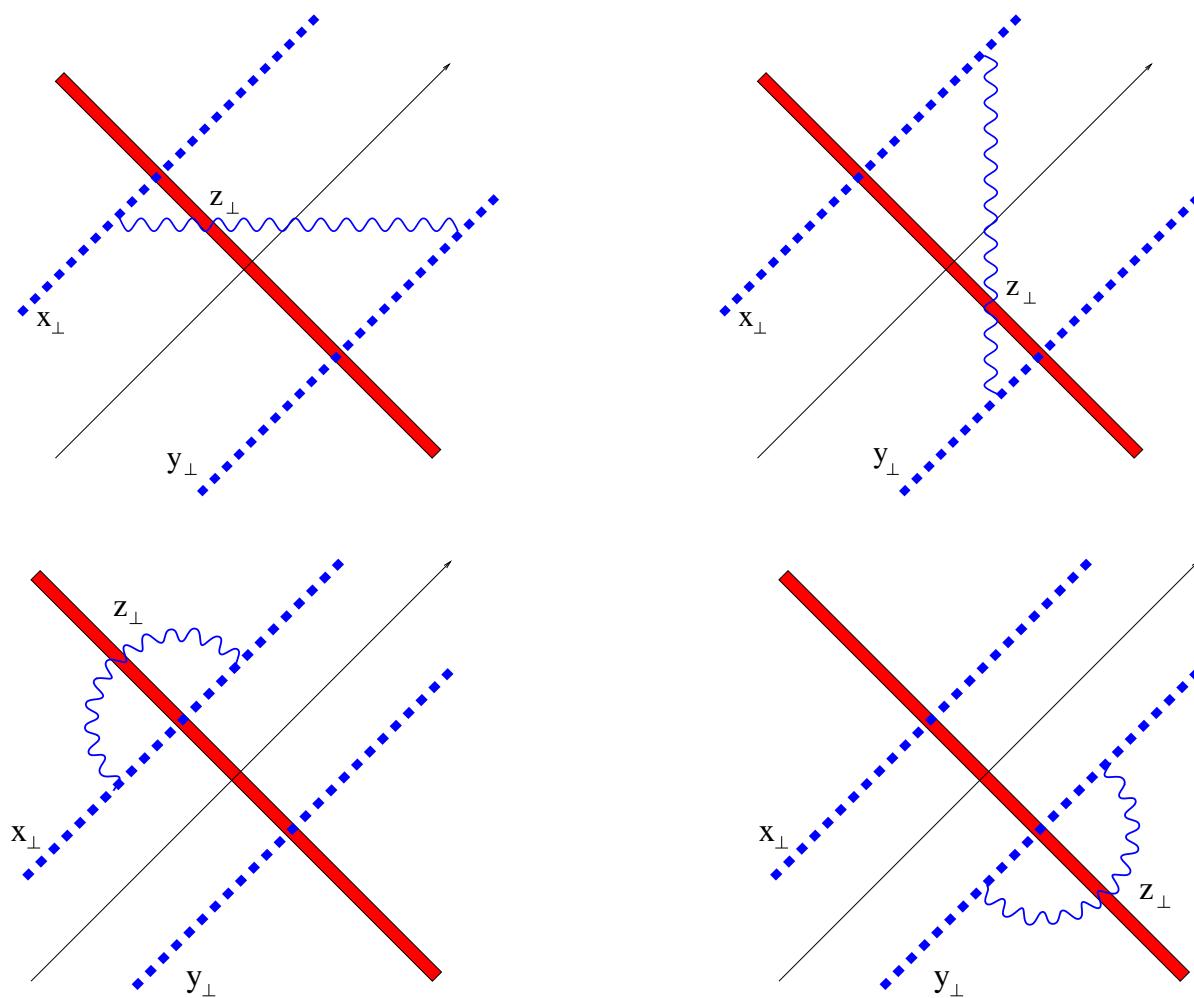
Evolution equation

To get the evolution equation, consider the dipole with the slope $\parallel \eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with the slope corresponding to η_2).



In the frame $\parallel \eta_1$ the gluons with $\eta < \eta_2$ are seen as a pancake \Rightarrow

One-loop evolution



The structure is

$[x \rightarrow z: \text{free propagation}]$
 \times
 $[U^{ab}(z_\perp) - \text{instantaneous interaction with the } \eta < \eta_2 \text{ shock wave}]$
 \times
 $[z \rightarrow y: \text{free propagation}]$

$$U_z^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} = (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2) (U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

\Rightarrow non-linear evolution

Non-linear evolution equation

$$\begin{aligned} \frac{\partial}{\partial \eta} \mathcal{U}(x_\perp, y_\perp) &= \\ -\frac{\bar{\alpha}}{4\pi} \int dz_\perp \frac{(\vec{x} - \vec{y})_\perp^2}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{z}_\perp - \vec{y}_\perp)^2} \\ &\times \left\{ \mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(z_\perp, y_\perp) - \mathcal{U}(x_\perp, y_\perp) - \mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp) \right\} \end{aligned}$$

$$\mathcal{U}(x_\perp, y_\perp) \equiv \frac{1}{N_c} (N_c - \text{Tr}\{U(x_\perp)U^\dagger(y_\perp)\})$$

LLA for DIS in pQCD \Rightarrow BFKL

LLA for DIS in sQCD \Rightarrow BK eqn

(s for semiclassical)

Non-linear evolution equation

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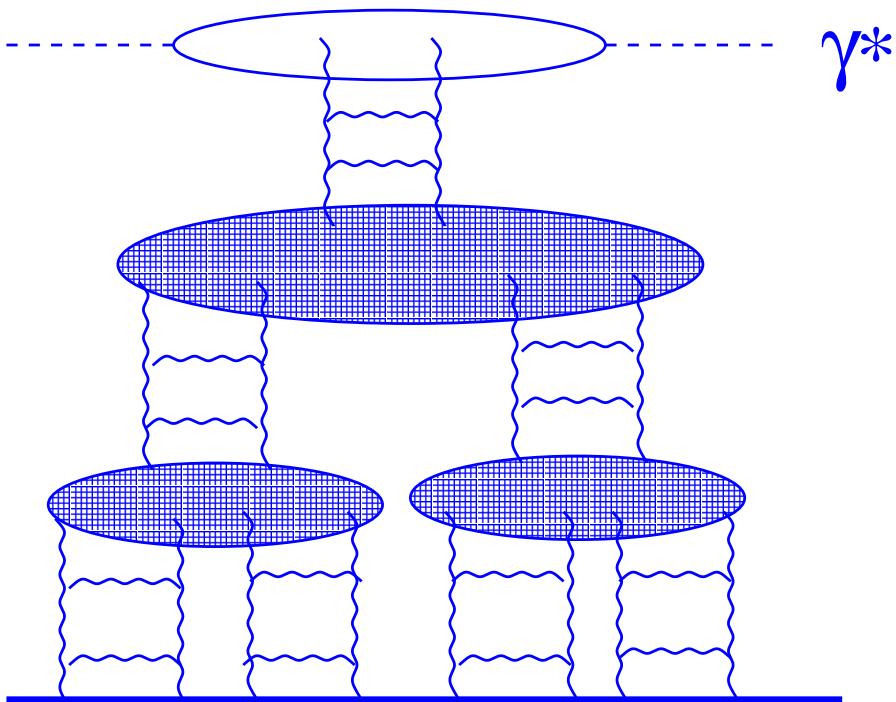
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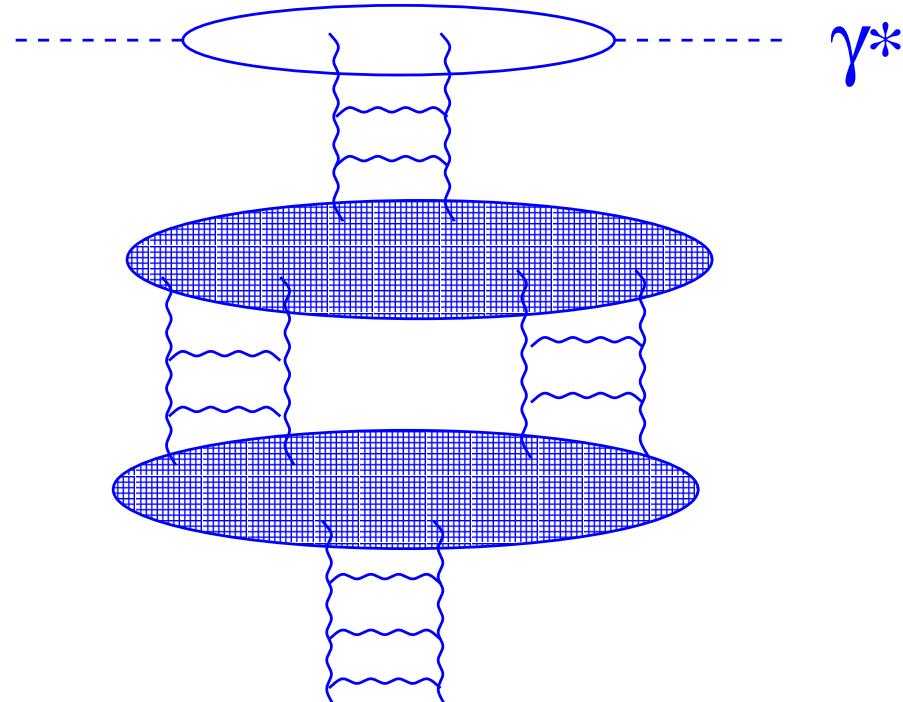
(s for semiclassical)

Example - LLA for the structure functions of large nuclei: $\alpha_s \ln \frac{1}{x} \sim 1$,
 $\alpha_s^2 A^{1/3} \sim 1$

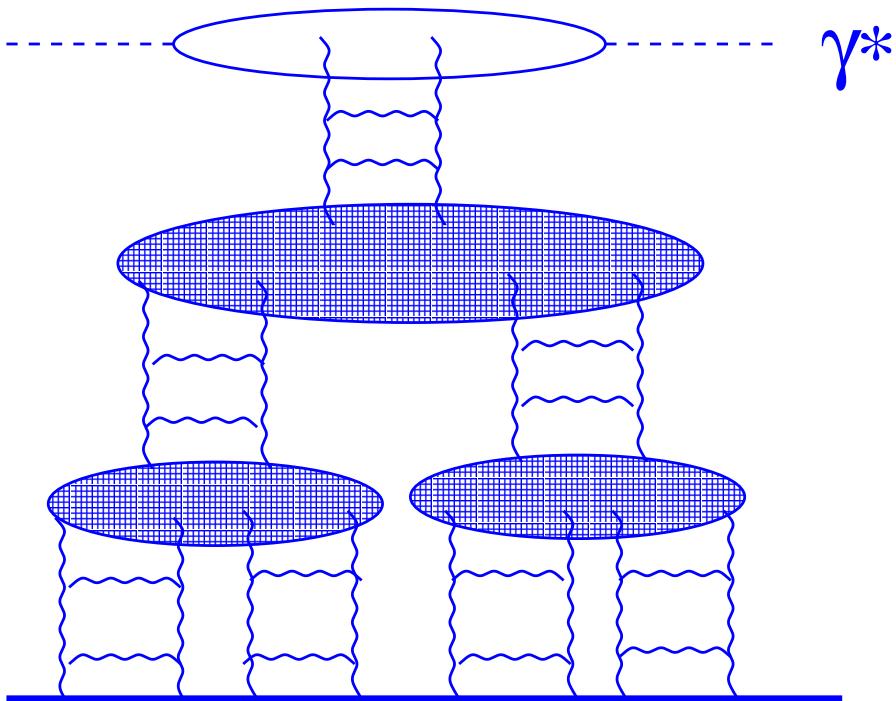
Non-linear equation sums up the
“fan” diagrams



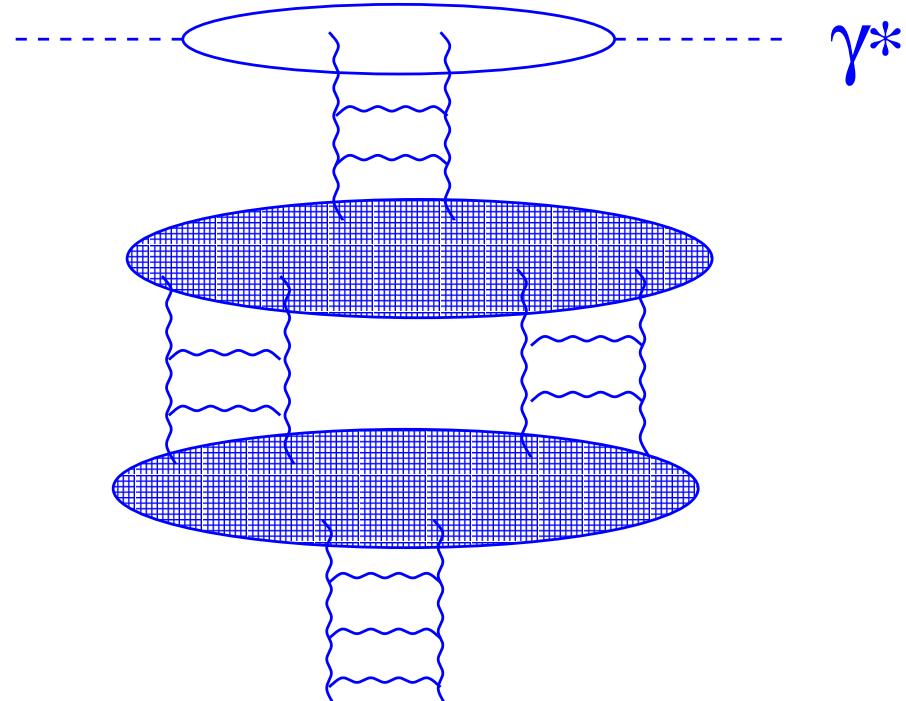
Example of the diagrams left behind
by the NL eqn: pomeron loops



Non-linear equation sums up the “fan” diagrams



Example of the diagrams left behind by the NL eqn: pomeron loops

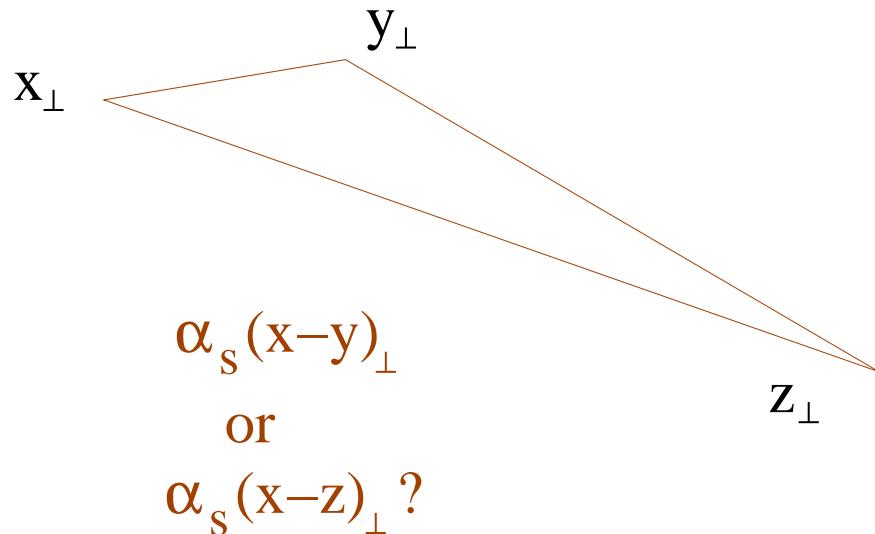
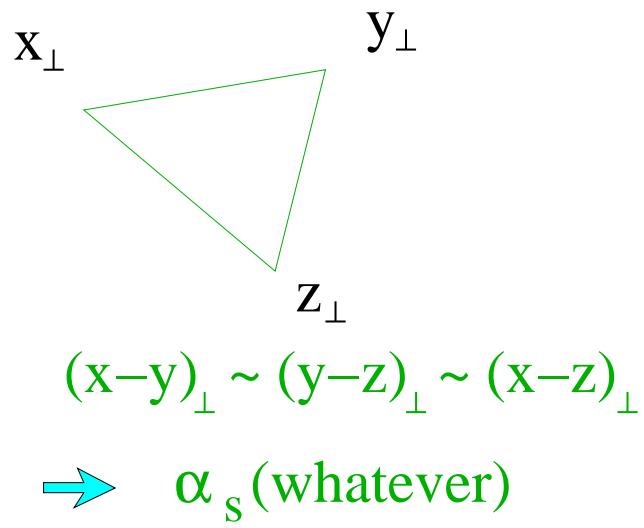


$x_B \rightarrow 0 \xrightarrow{\text{BFKL}}$ gluon density increases $\xrightarrow{\text{BK}}$ saturation $\Rightarrow \text{CGC}$

Scattering of two ions (CDG's) - pomeron loops, glasma etc.

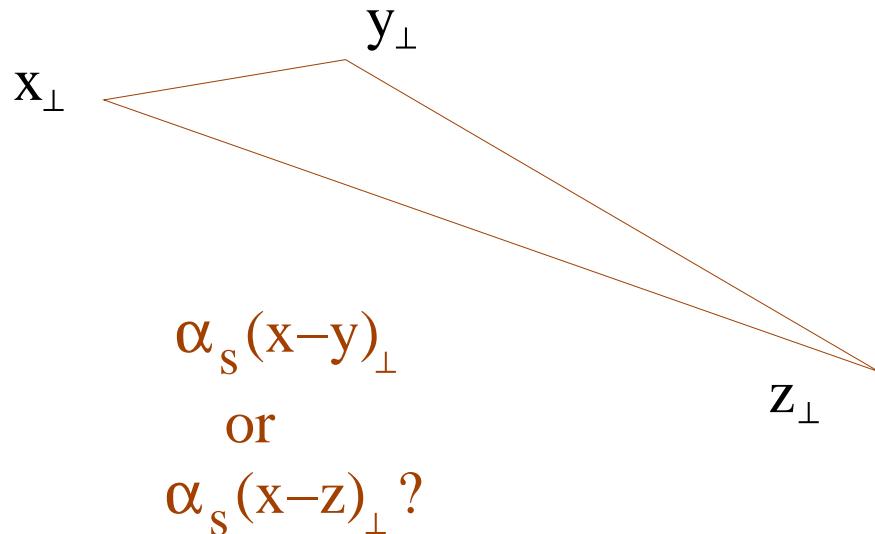
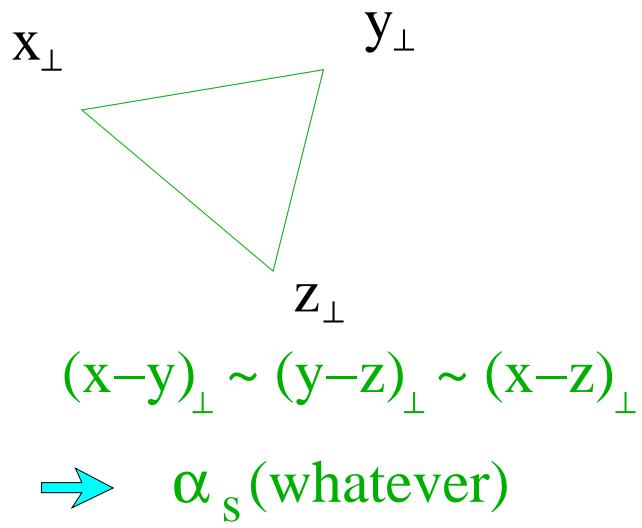
Argument of the α_s in the BK equation

$$\frac{\partial}{\partial \eta} \mathcal{U}(x_\perp, y_\perp) = \frac{\alpha_s(?_\perp)}{2\pi^2} \int dz_\perp \frac{(\vec{x} - \vec{y})_\perp^2}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{z}_\perp - \vec{y}_\perp)^2}$$
$$\times [\mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(z_\perp, y_\perp) - \mathcal{U}(x_\perp, y_\perp) - \mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)]$$



Argument of the α_s in the BK equation

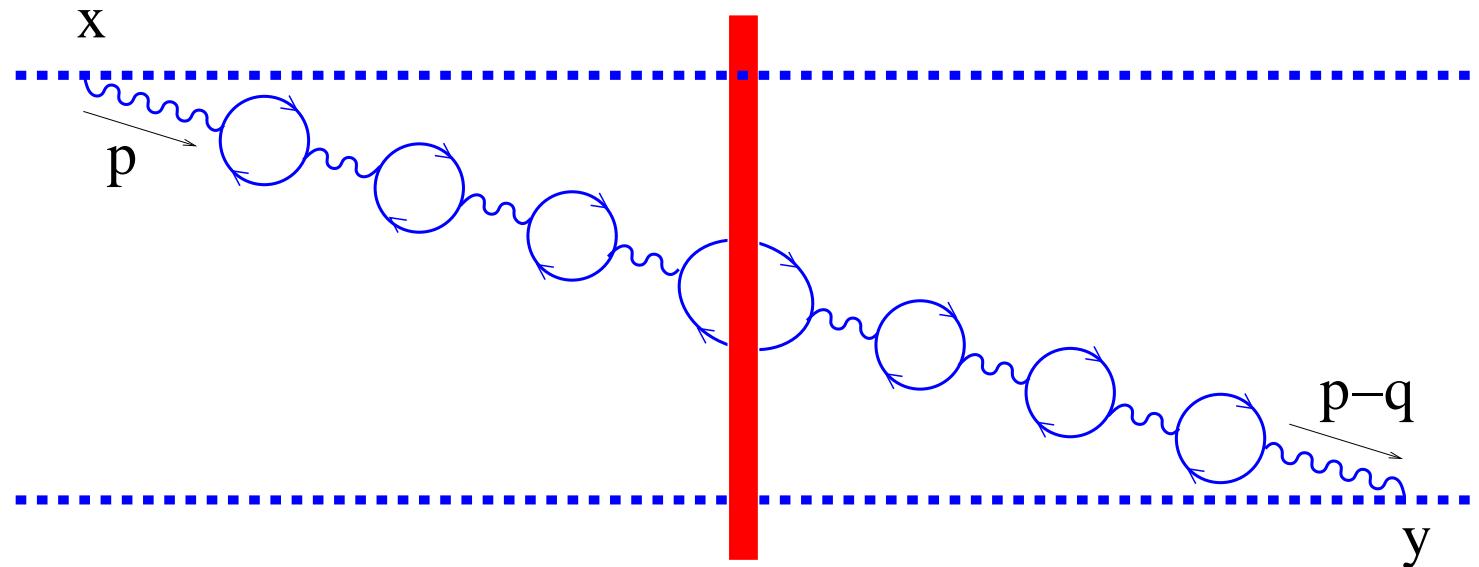
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$$\times [\mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(z_\perp, y_\perp) - \mathcal{U}(x_\perp, y_\perp) - \mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)]$$



Result: $\alpha_s = \alpha_s(|x - y|_\perp)$

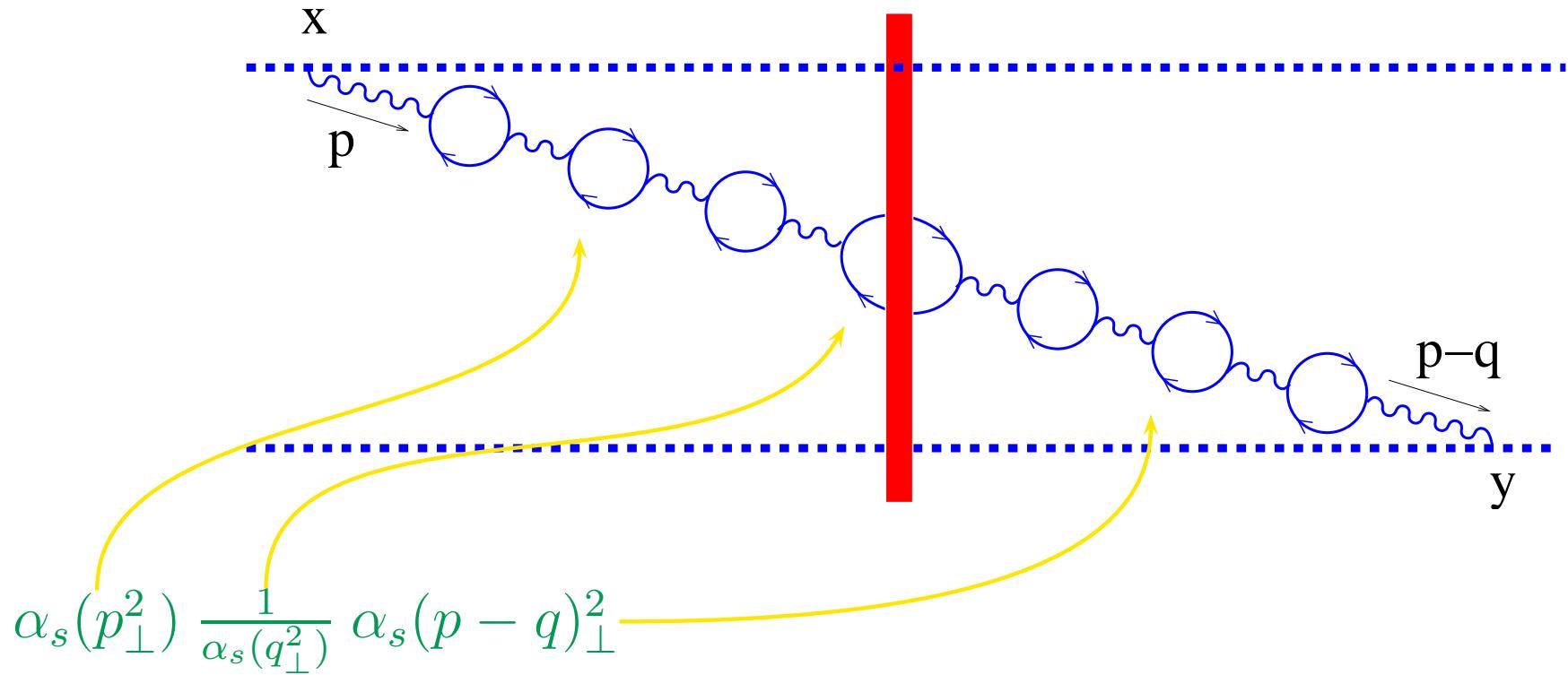
Quark bubble chain and the argument of α_s

$$\alpha_s(p_\perp^2) = \frac{\alpha_s(\mu^2)}{1 + \left(\frac{11}{3}N_c - \frac{2}{3}n_f \right) \frac{\alpha_s}{4\pi} \ln \frac{p_\perp^2}{\mu^2}}$$



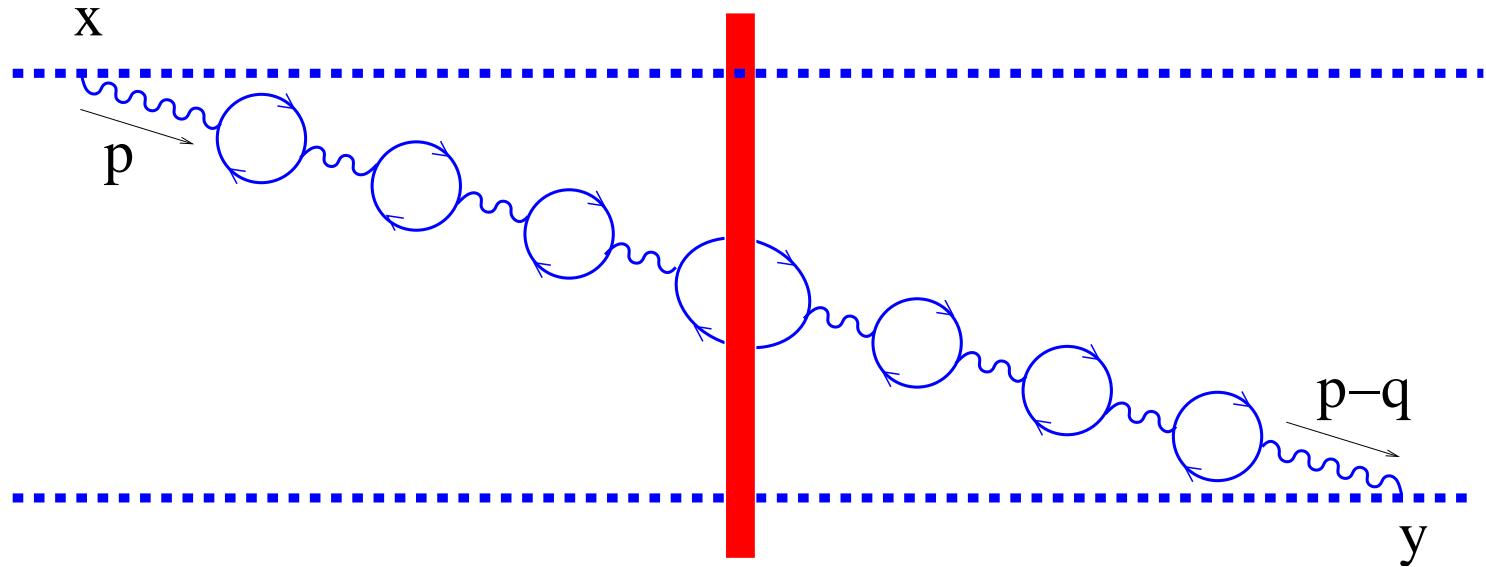
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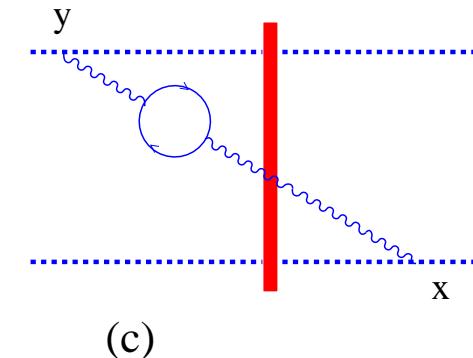
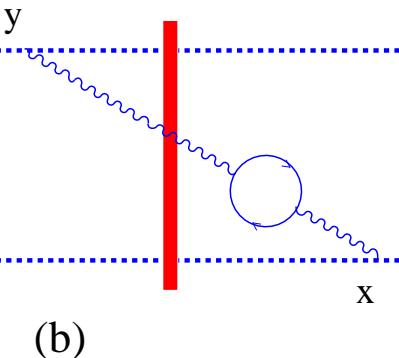
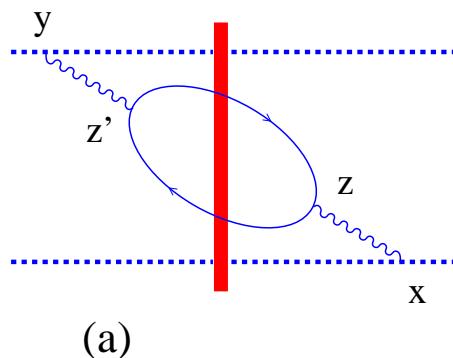
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$$\alpha_s(p_\perp^2) \frac{1}{\alpha_s(q_\perp^2)} \alpha_s(p-q)_\perp^2 \Rightarrow \alpha_s(x-y)_\perp^2 \text{ after the Fourier transformation}$$

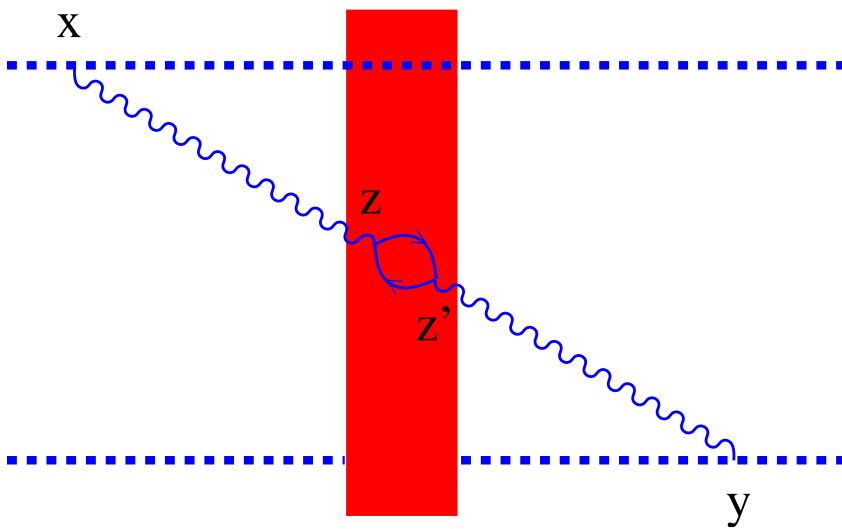
Quark contribution to the NLO kernel



$$\begin{aligned}
 (a) = & -\frac{4\alpha_s^2}{\pi} n_f \Delta \eta t^a U_x \otimes t^b U_y^\dagger \int d^d p d^d q d^d q' \frac{e^{i(p,x)_\perp - i(p-q-q',y)_\perp}}{p^2 (p-q-q')^2} \\
 & \text{Tr}\{t^a U(q) t^b U^\dagger(q')\} \int_0^1 dv du \left[- (q+q')^2 \frac{\bar{u} u \Gamma(1-\frac{d}{2}) \mu^{2-d}}{(P^2 \bar{v} v + Q^2 \bar{u} u)^{1-\frac{d}{2}}} \right. \\
 & + \frac{\Gamma(2-\frac{d}{2}) \mu^{2-d}}{(P^2 \bar{v} v + Q^2 \bar{u} u)^{2-\frac{d}{2}}} \left\{ P^2 [\bar{v} v(q, q') - \bar{u} u Q^2 + 2 \bar{u} u \bar{v} v (q^2 + q'^2)] - 2 \bar{u} u \bar{v} v (P, q)(P, q') \right. \\
 & \left. + \bar{u} u (1-2u) [\bar{v} v(q, q')(P, q+q') + \bar{v} q^2 (P, q') + v q'^2 (P, q)] + \bar{u}^2 u^2 Q^2 (q+q')^2 \right\} \left]
 \end{aligned}$$

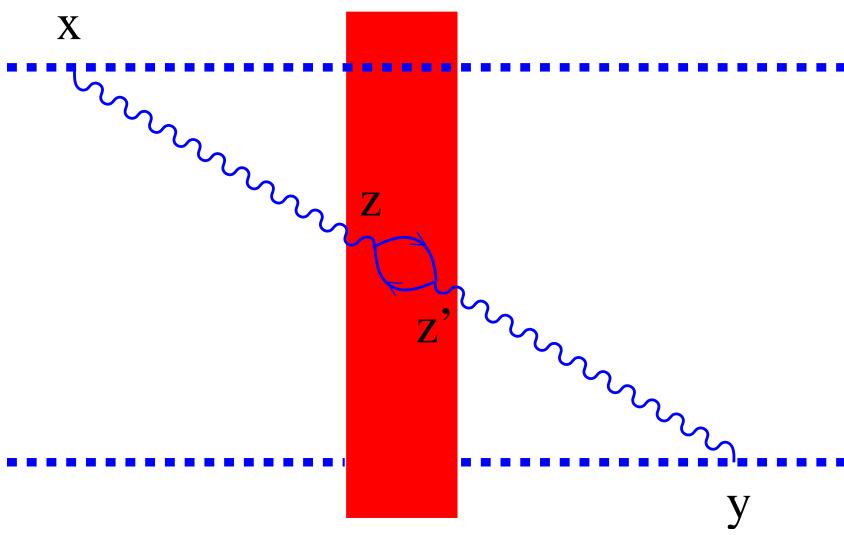
$$P = p - (q + q')u, \quad Q^2 = q_\perp^2 \bar{v} + q'^\perp v, \quad \bar{v} \equiv 1 - v, \quad d = d_\perp = 2 - \epsilon$$

A problem: quark loop inside the shock wave

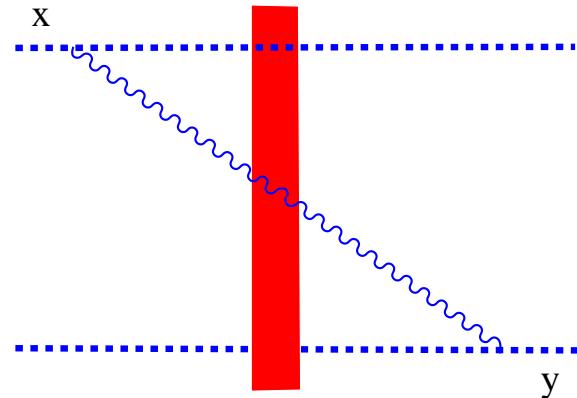


$|z - z'|_{\perp}^2 \sim \frac{1}{\alpha_s} \Rightarrow$ one can expand the quark loop near the light cone \Rightarrow the contribution is local in z_{\perp} .

A problem: quark loop inside the shock wave



Tree-level gluon propagator $G_{\bullet\bullet}(x, y) = \int dz \delta(z_*) (x| \frac{1}{p^2} |z) \frac{1}{\alpha} \partial_\perp^2 U_z(z| \frac{1}{p^2} |y)$



We need the operators up to twist 4 \Rightarrow the candidates for an additional interaction term are

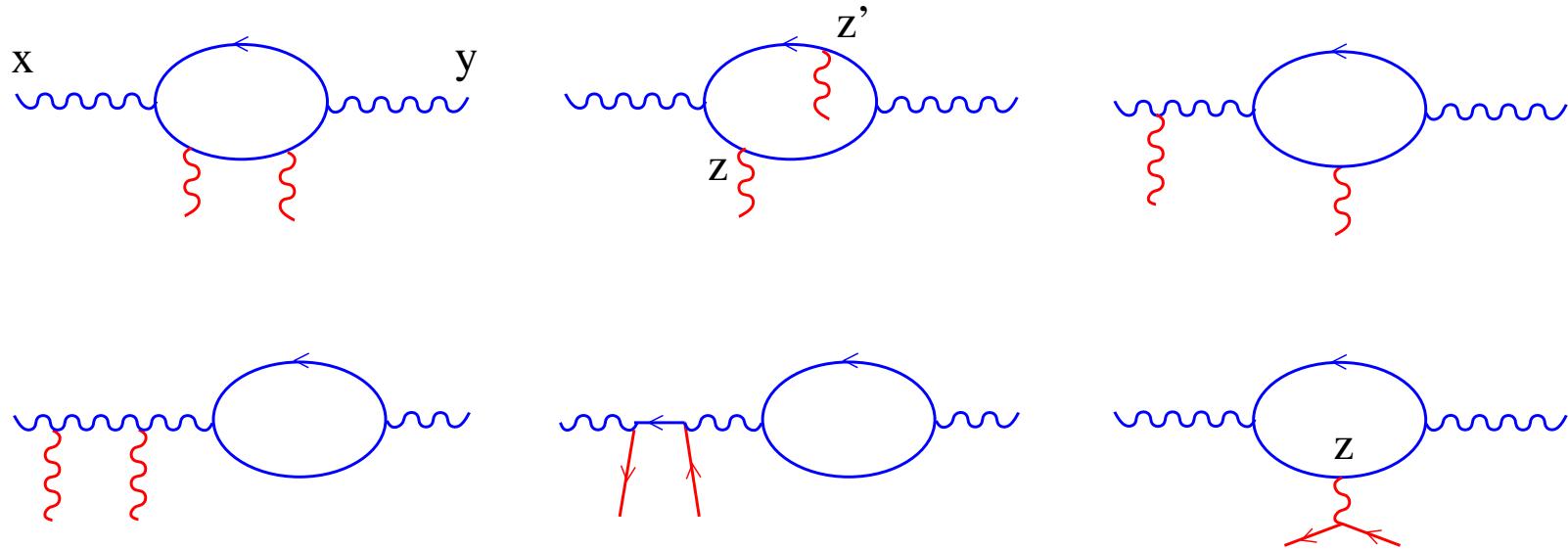
$$[GG] = \int du dv [\infty, u] G_{\bullet i}(up_1 + z_\perp) [u, v] G_\bullet^i(vp_1 + z_\perp) [v, -\infty]$$

$$[DG] = \int du [\infty, u] D^i G_{\bullet i}(up_1 + z_\perp) [u, -\infty], \quad -i[DG] + 2[GG] = \partial_\perp^2 U$$

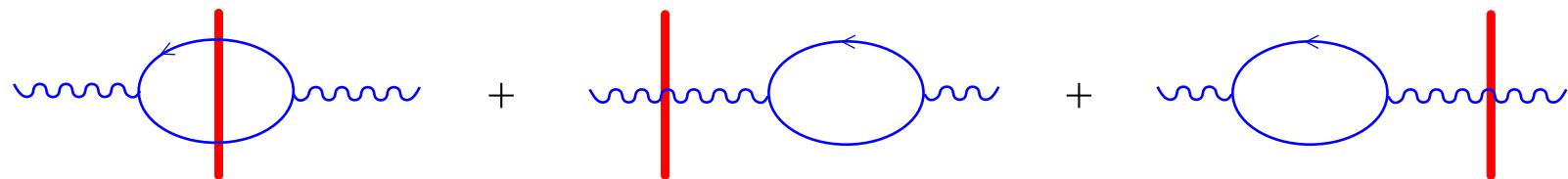
A way to fish out such extra term is to find the light-cone expansion of $U_x U_y^\dagger$ at $x_\perp \rightarrow y_\perp$ up to twist-4 terms and compare it with the expansion of the expression from the previous page.



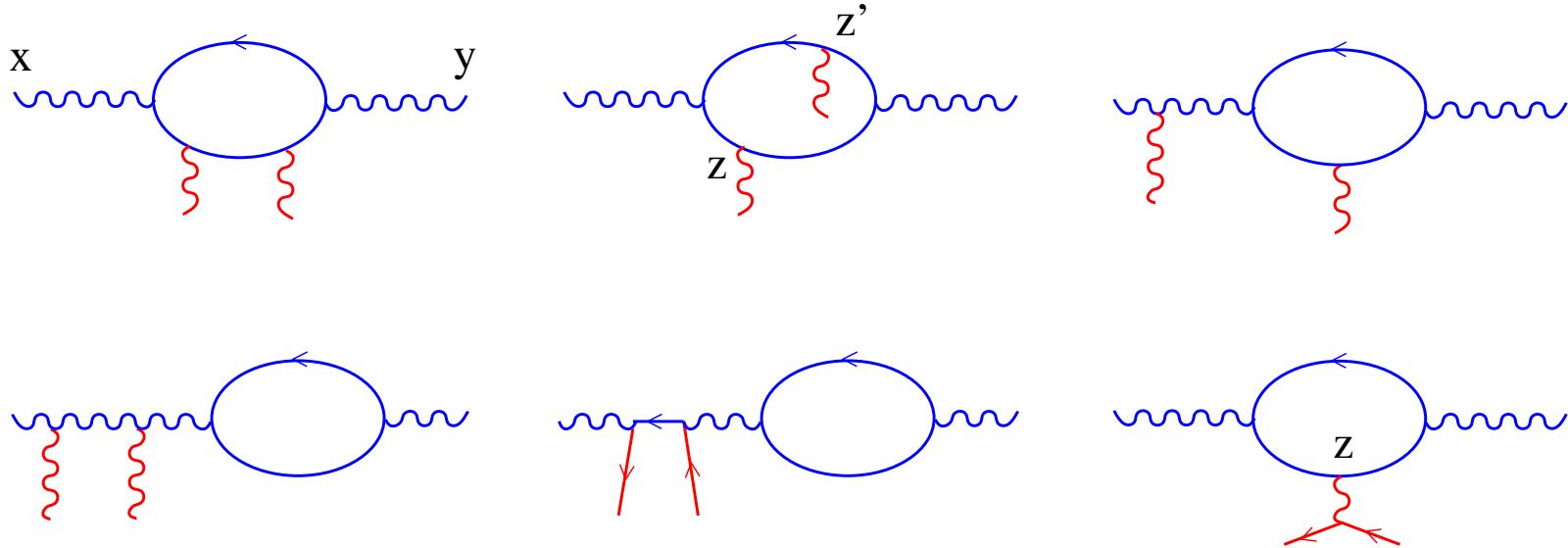
The light-cone expansion of the sum of the diagrams



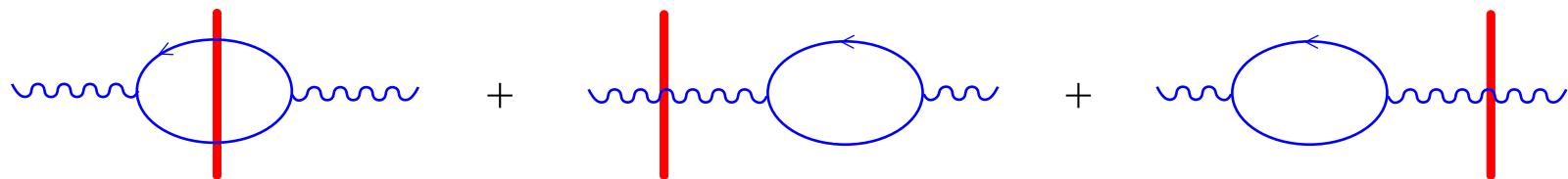
coincides with the expansion of



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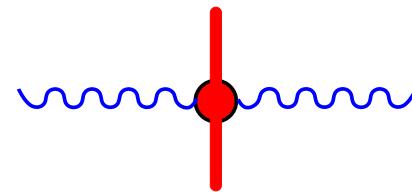


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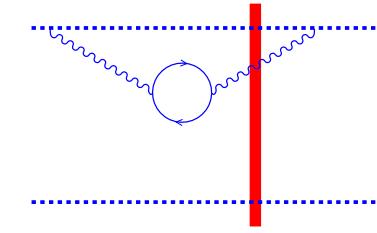
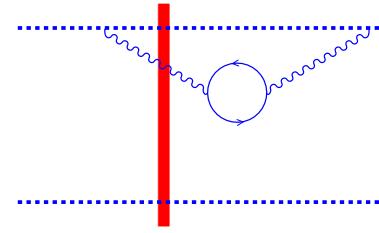
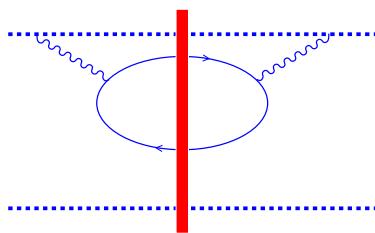
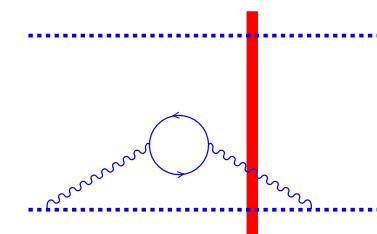
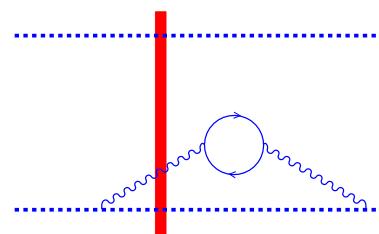
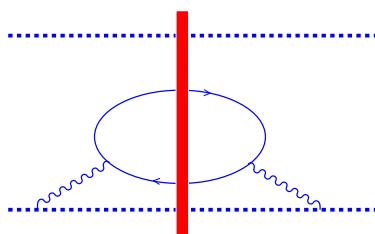
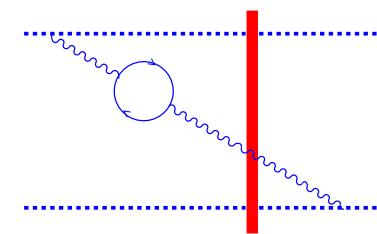
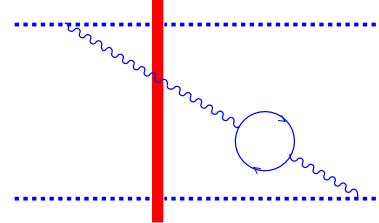
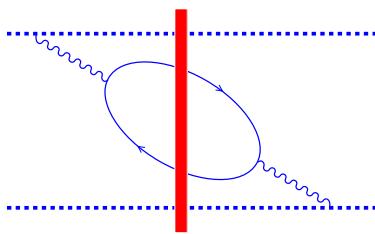
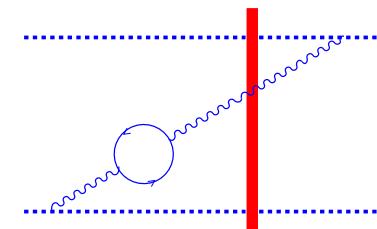
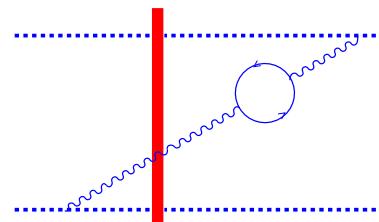
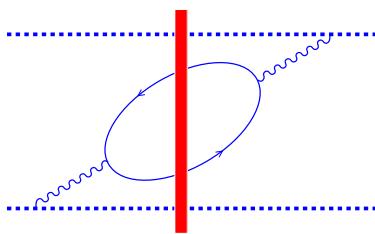
⇒

no additional vertex



at the one-loop level

Diagrams for the dipole evolution



Quark-loop contribution to NLO BK

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
&\times \left[\frac{(x-y)^2}{X^2 Y^2} \left(1 - \frac{\alpha_s n_f}{6\pi} [\ln(x-y)^2 \mu^2 + \frac{5}{3}] \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
&+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2 z d^2 z' \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_z^\dagger\} \frac{1}{(z-z')^4} \\
&\times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
\end{aligned}$$

$$X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z'$$

Quark-loop contribution to NLO BK

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
 &\times \left[\frac{(x-y)^2}{X^2 Y^2} \left(1 - \frac{\alpha_s n_f}{6\pi} [\ln(x-y)^2 \mu^2 + \frac{5}{3}] \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
 &+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2 z d^2 z' \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_z^\dagger\} \frac{1}{(z-z')^4} \\
 &\times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
 \end{aligned}$$

$$X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z'$$

$$\alpha_s(\mu) [1 - \frac{\alpha_s n_f}{6\pi} \ln(x-y)^2 \mu^2] +$$

Quark-loop contribution to NLO BK

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
&\times \left[\frac{(x-y)^2}{X^2 Y^2} \left(1 - \frac{\alpha_s n_f}{6\pi} [\ln(x-y)^2 \mu^2 + \frac{5}{3}] \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
&+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2 z d^2 z' \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_z^\dagger\} \frac{1}{(z-z')^4} \\
&\times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
\end{aligned}$$

$$X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z'$$

$$\alpha_s(\mu) [1 - \frac{\alpha_s n_f}{6\pi} \ln(x-y)^2 \mu^2] + \text{gluon loop (work in progress)}$$

Quark-loop contribution to NLO BK

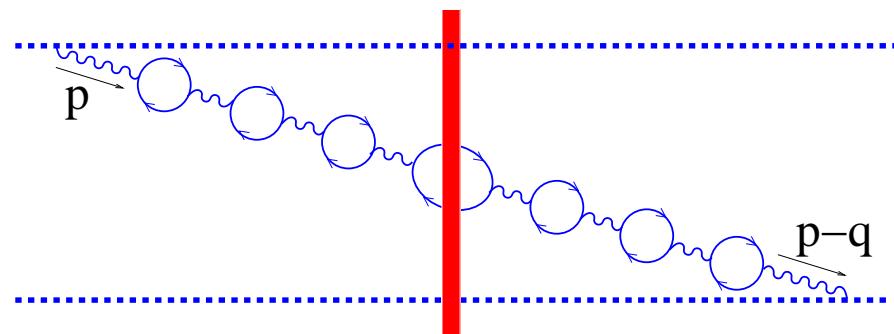
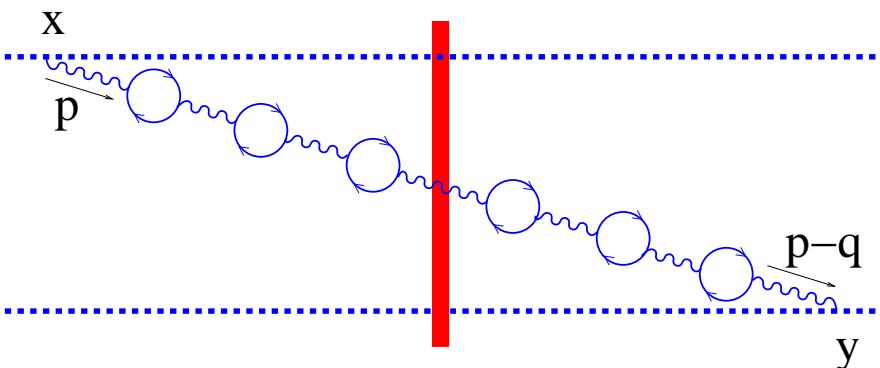
$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
&\times \left[\frac{(x-y)^2}{X^2 Y^2} \left(1 - \frac{\alpha_s n_f}{6\pi} [\ln(x-y)^2 \mu^2 + \frac{5}{3}] \right) + \frac{\alpha_s n_f}{6\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right] \\
&+ \frac{\alpha_s^2}{\pi^4} n_f \text{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2 z d^2 z' \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_z t^b U_z^\dagger\} \frac{1}{(z-z')^4} \\
&\times \left\{ 1 - \frac{X'^2 Y^2 + Y'^2 X^2 - (x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right\}
\end{aligned}$$

$$X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z'$$

$$\alpha_s(\mu) \left[1 - \frac{\alpha_s n_f}{6\pi} \ln(x-y)^2 \mu^2 \right] + \text{gluon loop (work in progress)}$$

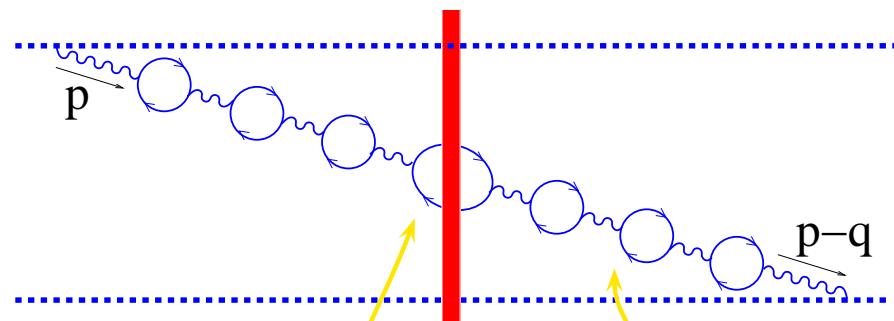
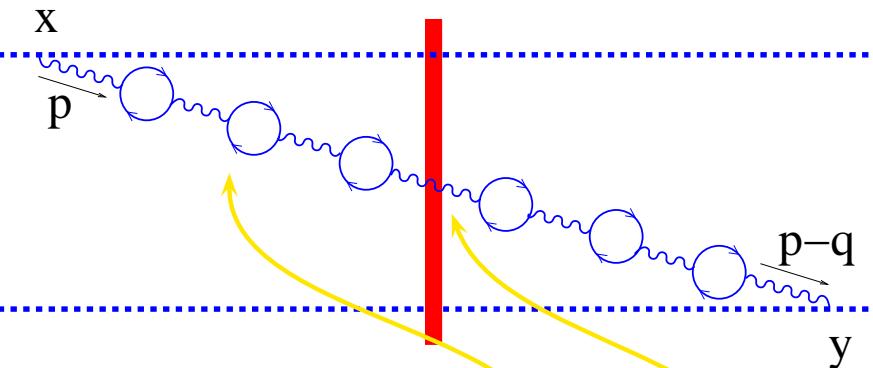
$$= \alpha_s(\mu) \left[1 + \frac{b_0 \alpha_s}{4\pi} \ln(x-y)^2 \mu^2 \right] = \color{red} \alpha_s(|x-y|)$$

Bubble chain and the argument of coupling constant



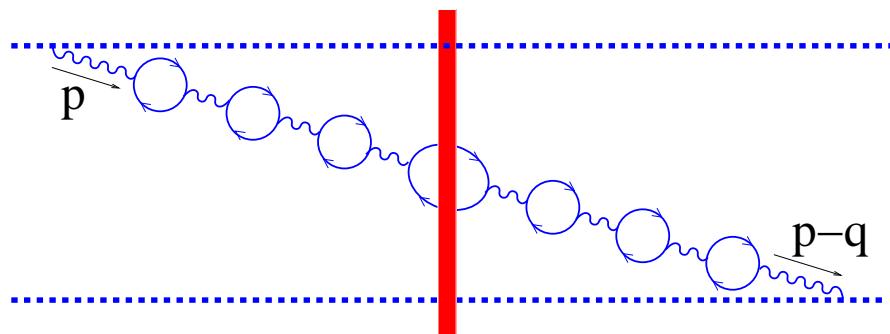
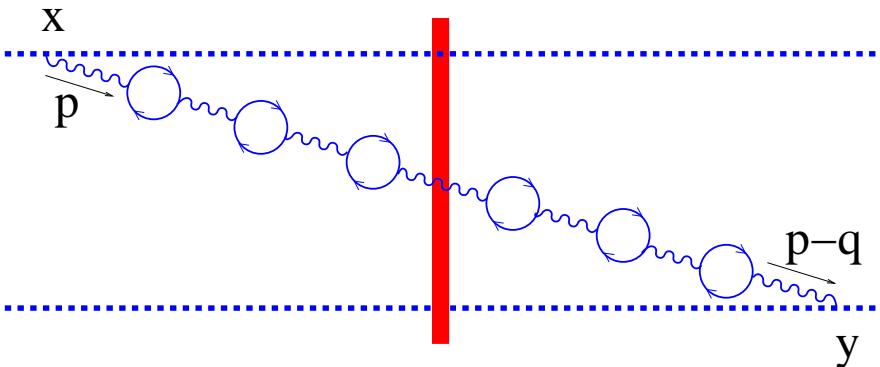
Leading log accuracy: $\alpha(\mu) \ll 1, \alpha(\mu) \ln \frac{p_\perp^2}{\mu^2} \sim 1$

Bubble chain and the argument of coupling constant



$$\begin{aligned}
 LLA &\Rightarrow \frac{\alpha_s(p_\perp^2)}{p_\perp^2} \left[\frac{1}{\alpha_s(\mu^2)} + \left(\frac{11}{3}N_c - \frac{2}{3}n_f \right) \ln \frac{q_\perp^2}{\mu^2} \right] \frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2} \\
 &= \frac{\alpha_s(p_\perp^2)}{p_\perp^2} \frac{1}{\alpha_s(q_\perp^2)} \frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2}
 \end{aligned}$$

Bubble chain and the argument of coupling constant



$$\begin{aligned}
 LLA &\Rightarrow \frac{\alpha_s(p_\perp^2)}{p_\perp^2} \left[\frac{1}{\alpha_s(\mu^2)} + \left(\frac{11}{3}N_c - \frac{2}{3}n_f \right) \ln \frac{q_\perp^2}{\mu^2} \right] \frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2} \\
 &= \frac{\alpha_s(p_\perp^2)}{p_\perp^2} \frac{1}{\alpha_s(q_\perp^2)} \frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2}
 \end{aligned}$$

Fourier transformation (up to \ln^2 accuracy) \Rightarrow

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s(|x-y|)}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
 &\times \left[\frac{(x-y)^2}{X^2 Y^2} + \frac{1}{X^2} \left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right]
 \end{aligned}$$

Comparison with the triumvirate of Kovchegov & Weigert

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = & \frac{\alpha_s}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 \right. \right. \\
& + \frac{b\alpha_s}{4\pi} \left(\ln(x-y)^2 \mu^2 + \frac{5}{3} \right) + \frac{b\alpha_s}{4\pi} \left(\ln(x-y)^2 \mu^2 + \frac{5}{3} \right) + \left(\frac{b\alpha_s}{4\pi} \right)^2 \ln^2(x-y)^2 \mu^2 \left. \right] \\
& \left. + \frac{b\alpha_s}{4\pi} \left(\frac{1}{X^2} \ln \frac{X^2}{Y^2} \left[1 + \frac{b\alpha_s}{4\pi} \ln(x-y)^2 \mu^2 + \frac{b\alpha_s}{4\pi} \ln X^2 \mu^2 \right] + X \leftrightarrow Y \right) \right\} \quad (\text{B} + \text{KW})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = & \frac{\alpha_s((x-y)^2 e^{5/3})}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \\
& \times \left[\frac{(x-y)^2}{X^2 Y^2} + \frac{1}{X^2} \left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] \quad (\text{B})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = & \frac{1}{2\pi^2} \int d^2 z [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \quad (\text{KW}) \\
& \times \left[\frac{1}{X^2} \alpha_s(X^2 e^{5/3}) + \frac{1}{Y^2} \alpha_s(Y^2 e^{5/3}) - \frac{2(x-z, y-z)}{X^2 Y^2} \frac{\alpha_s(X^2 e^{5/3}) \alpha_s(Y^2 e^{5/3})}{\alpha_s(R^2)} \right]
\end{aligned}$$

Gluon part of NLO BK

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
&\times \left[\frac{(x-y)^2}{X^2 Y^2} \left\{ 1 + \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \ln(x-y)^2 \mu^2 \right\} + \frac{\alpha_s N_c}{4\pi} \Phi(X^2, Y^2, X \cdot Y) \right] \\
&+ \frac{\alpha_s}{4\pi^2} \int_0^1 du \int d^2 z' \left\{ \frac{\bar{u}u}{2} \left[\frac{X_{ij}}{uX^2 + \bar{u}X'^2} - \frac{X_i Y'_j}{\bar{u}u X^2 Y'^2} - (x \leftrightarrow y) \right]^2 \right. \\
&\times [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z)] \\
&+ \left[\frac{X_{ij}}{uX^2 + \bar{u}X'^2} - \frac{X_i Y'_j}{\bar{u}u X^2 Y'^2} - (x \leftrightarrow y) \right] \left(\frac{X_i}{X^2} - \frac{Y_i}{Y^2} \right) \left(\frac{X'_j}{X'^2} - \frac{Y'_j}{Y'^2} \right) \\
&\times \left. [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} + \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\}] \right\}
\end{aligned}$$

$$X_{ij} = \frac{(z' - z)_k}{(z - z')^2} \left[(X + X')_k \delta_{ij} - \frac{2}{u} \delta_{ik} X'_j - \frac{2}{\bar{u}} X_i \delta_{jk} \right] + \frac{X_i X'_j}{u X^2} - \frac{X_i X'_j}{\bar{u} X'^2}$$

Subtraction of the (LO kernel)² : $\frac{1}{u} \rightarrow [\frac{1}{u}]_+$

$\Phi(X^2, Y^2, X \cdot Y)$ is UV and IR finite. Typical term: $\text{Li}_2(\frac{X \cdot Y + i\sqrt{X^2 Y^2 - X \cdot Y^2}}{X^2}) + \text{c.c.}$

- High-energy scattering can be described in terms of dipoles (Wilson lines)
 $U_x U_y^\dagger$ - no new operators at the 1-loop level.
- For the creation of dipoles in the small- x evolution, the coupling constant is determined by the size of the *parent* dipole rather than the size of the produced dipoles.

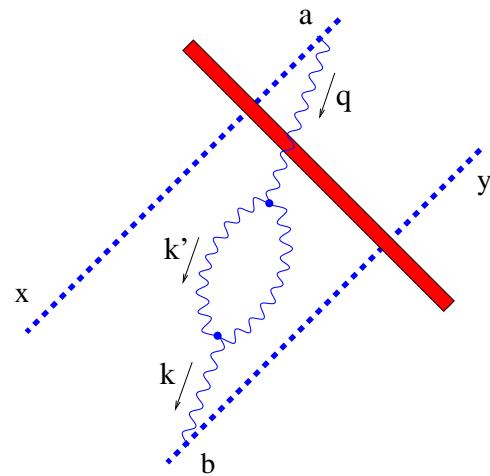
- High-energy scattering can be described in terms of dipoles (Wilson lines) $U_x U_y^\dagger$ - no new operators at the 1-loop level.
- For the creation of dipoles in the small- x evolution, the coupling constant is determined by the size of the *parent* dipole rather than the size of the produced dipoles.

Outlook

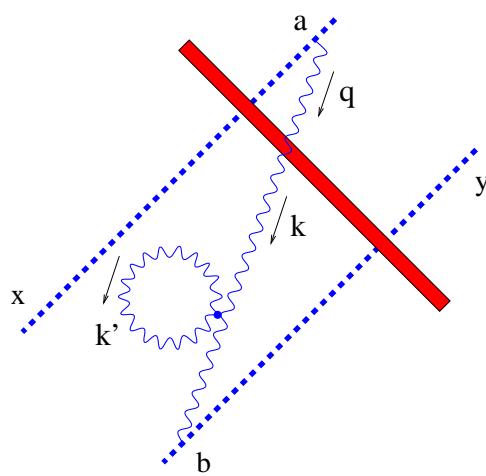
- Finish the calculation of gluon loops.

The calculation of gluon part is done in collaboration with my student G. Chirilli.

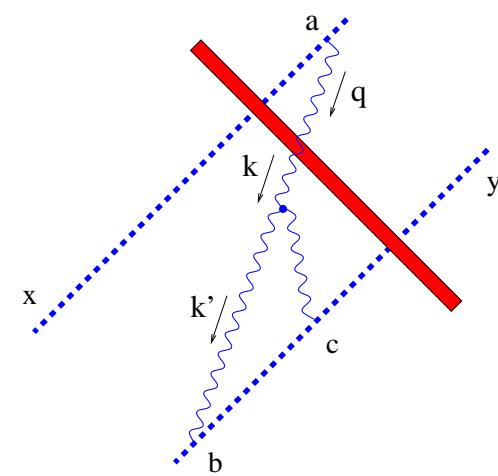
Diagrams for the running coupling (and Φ)



(a)



(b)



(c)

