Momentum Broadening in Strongly Coupled $\mathcal{N} = 4$ Yang-Mills Theory, revisited

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Why $\mathcal{N} = 4$ SYM?

what do we hope to learn?

- Quark-gluon plasma produced in heavy ion collisions is a strongly coupled fluid
  - The fundamental degrees of freedom of this fluid are quark and gluon fields.
- We don't have tools to systematically study the real time dynamics of strongly coupled quantum field theories such as QCD.

  ☆ Exception: theories with a known holographic dual.

- Supersymmetric $\mathcal{N} = 4$ Yang-Mills is a theory that contains fermionic and gluon fields, where calculations at strong coupling $\lambda = g^2 N_c \to \infty$ (provided $N_c \to \infty$) are feasible.

  $\Longrightarrow$ We hope to learn about physical features of strongly coupled fluids!

Classic results of strongly coupled $\mathcal{N} = 4$ SYM

that we will address today

- The heavy quark drag force:

$$F = \frac{\pi}{2}\sqrt{\lambda}T^2 v\gamma \equiv \eta_D p\ , \qquad \text{with } p = M\gamma v\ .$$

- The heavy quark diffusion coefficient, both for longitudinal and transverse momentum:

$$\kappa_T = \pi\sqrt{\lambda}T^3\gamma^{1/2}\ , \qquad \kappa_L = \pi\sqrt{\lambda}T^3\gamma^{5/2}\ .$$

- The jet quenching parameter

$$\hat{q} = \frac{\pi^{3/2}\Gamma(3/4)}{\Gamma(5/4)}\sqrt{\lambda}T^3\ .$$

Wilson loops and momentum broadening

a heuristic derivation

- Roughly speaking, the amplitude for a hard particle to transition from a state with momentum $\mathbf{p}$ to a state with momentum $\mathbf{p}+\mathbf{k}$

$$\langle \mathbf{p}+\mathbf{k}\,|_{\text{out}}\,|\,\mathbf{p}\rangle_{\text{in}} = \int d^3x\, e^{-i\mathbf{x}\cdot\mathbf{k}}\, W_{[x_f,x_i]}$$

where W is a Wilson line.

- This means that the momentum broadening probability is given by

$$P(\mathbf{k}) \propto \int d^3L\, e^{i\mathbf{k}\cdot\mathbf{L}}\, \langle W[C]\rangle_T(\mathbf{L})\ ,$$

where $W[C]$ is a long, rectangular Wilson loop characterized by a velocity v.

Wilson loops

the configurations of interest

How to calculate Wilson loops
in strongly coupled $\mathcal{N} = 4$ SYM

- The AdS/CFT correspondence provides a way to calculate the expectation value of Wilson loops at finite temperature:

$$\langle W[\mathscr{C}]\rangle = \exp\left\{-(\mathscr{S}_{\mathrm{NG}}[\Sigma(\mathscr{C})] - \mathscr{S}_0)\right\} \,,$$

where $\mathscr{S}$ is the action of a string (with boundary conditions set by $\mathscr{C}$)

$$\mathscr{S}_{\mathrm{NG}}[\Sigma] = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma\, d\tau \sqrt{\det\left(g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right)} \,,$$

in a higher dimensional spacetime with a black hole with temperature T

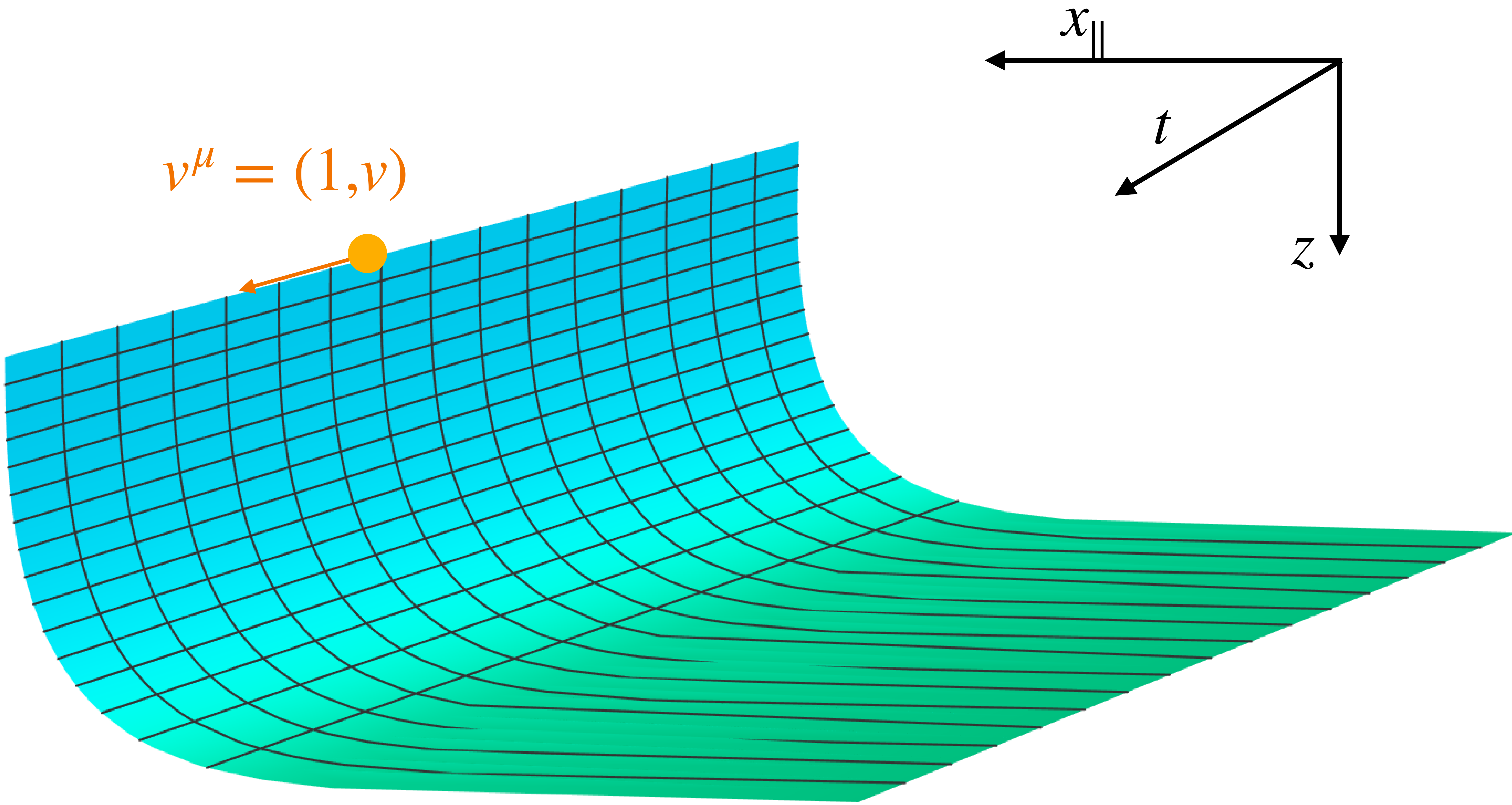
$$ds^2 = \frac{1}{z^2}\left[f(z)\, d\tau^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)} + z^2 d\Omega_5^2\right] \,, \qquad f(z) = 1 - (\pi T z)^4 \,.$$

What had been done so far

The trailing string [Gubser hep-th/0605182; HKKKY hep-th/0605158]

The energy-momentum flow down the string gives the drag force.

Fluctuations on top of this configuration give the broadening coefficients [Gubser hep-th/0612143; Casalderrey-Solana & Teaney hep-ph/0605199, hep-th/0701123]



What had been done so far

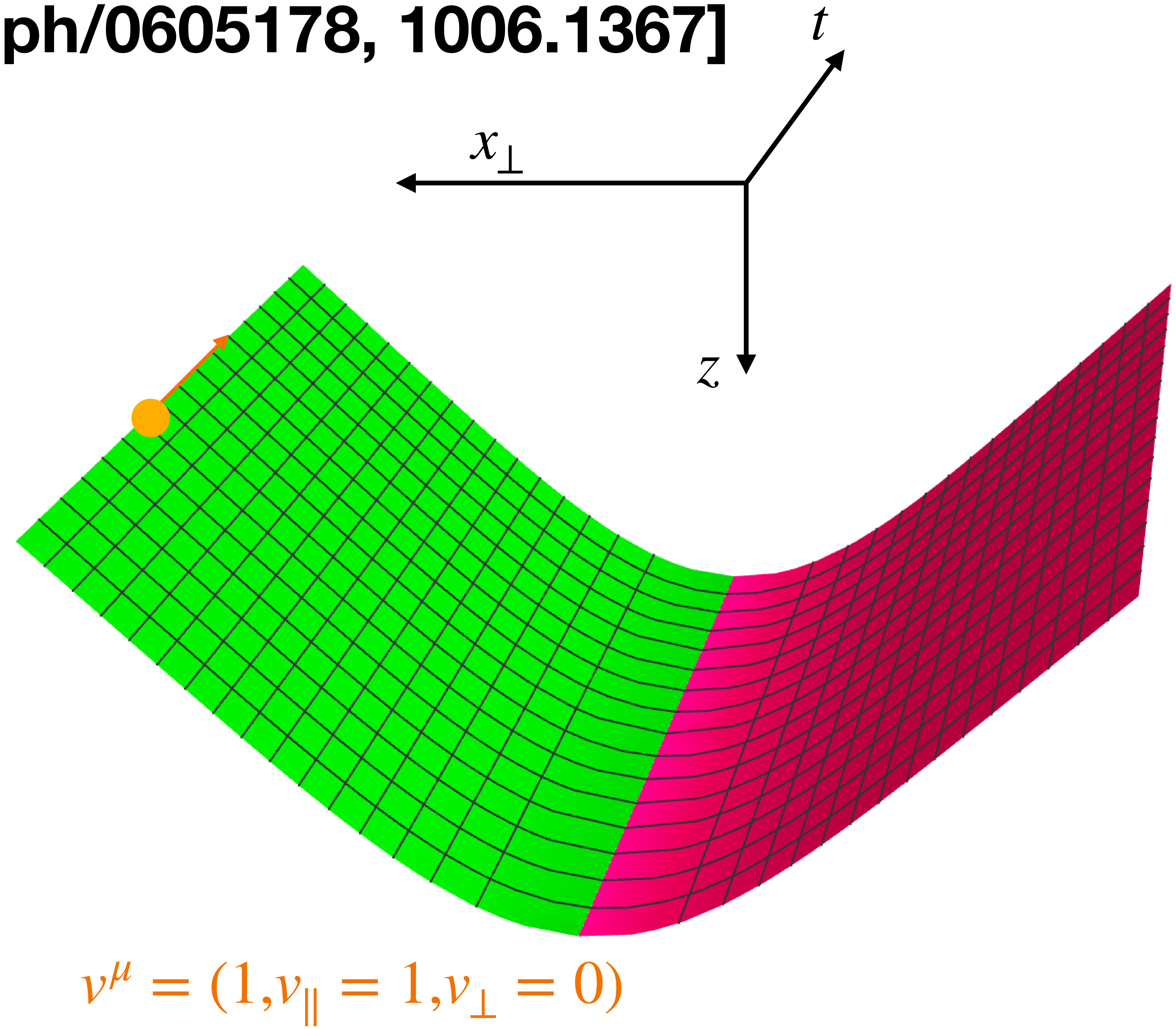
The LRW configuration [hep-ph/0605178, 1006.1367]

This configuration was used to calculate $\hat{q}$.

Note: the $x_{\parallel}$ coordinate is omitted on the right (no separation).

There is no momentum flow down the string into the horizon.

$\Longleftrightarrow$ no energy loss.



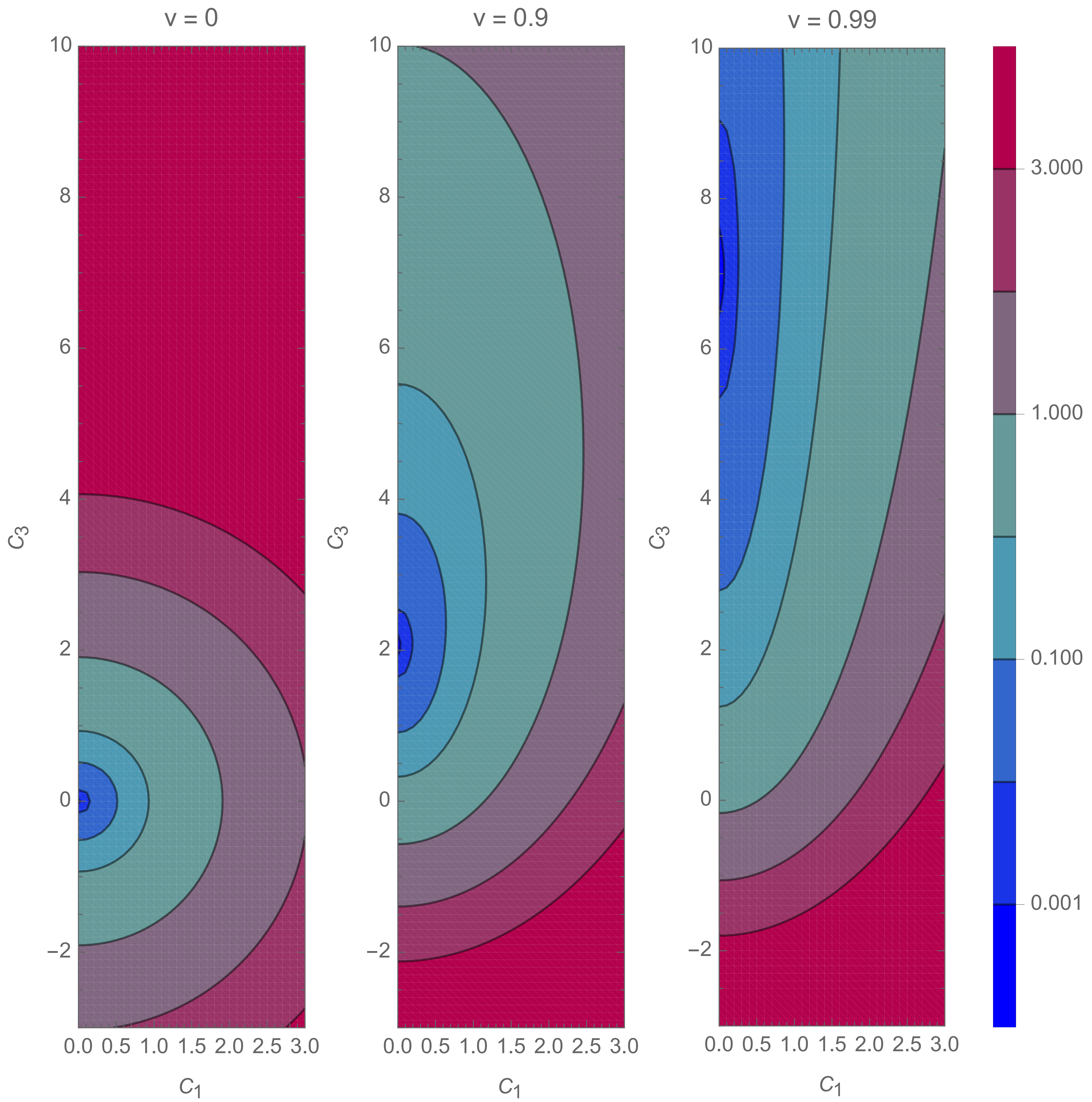
Two string configurations

different kinematic regimes

- These are two specific kinematic regimes in which:
  - the trailing string describes the most likely value of heavy quark energy loss in the $M \to \infty$ limit. Small fluctuations in the configuration describe momentum fluctuations of a heavy quark that loses energy as it propagates.
  - the LRW configuration describes momentum broadening of a hard particle that does not lose any energy.
- $P(\mathbf{k})$ should contain all of this information: both the most likely values and the conditional distributions.

The broadening probability distribution

without further ado



We obtain

$$P(\mathbf{k}) \propto \exp\left[-\sqrt{\lambda}Tt \times \tilde{S}_{\text{tot}}\left(\frac{2\mathbf{k}}{\sqrt{\lambda}Tt}\right)\right].$$

On the left, we plot

$$\tilde{S}_{\text{tot}}(\mathbf{C})\,, \qquad \text{with } \mathbf{C} = \frac{2\mathbf{k}}{\sqrt{\lambda}tT}\,.$$

Let us look at each value of v individually. We take $k_3 > 0$ to be the momentum lost.

$v = 0$

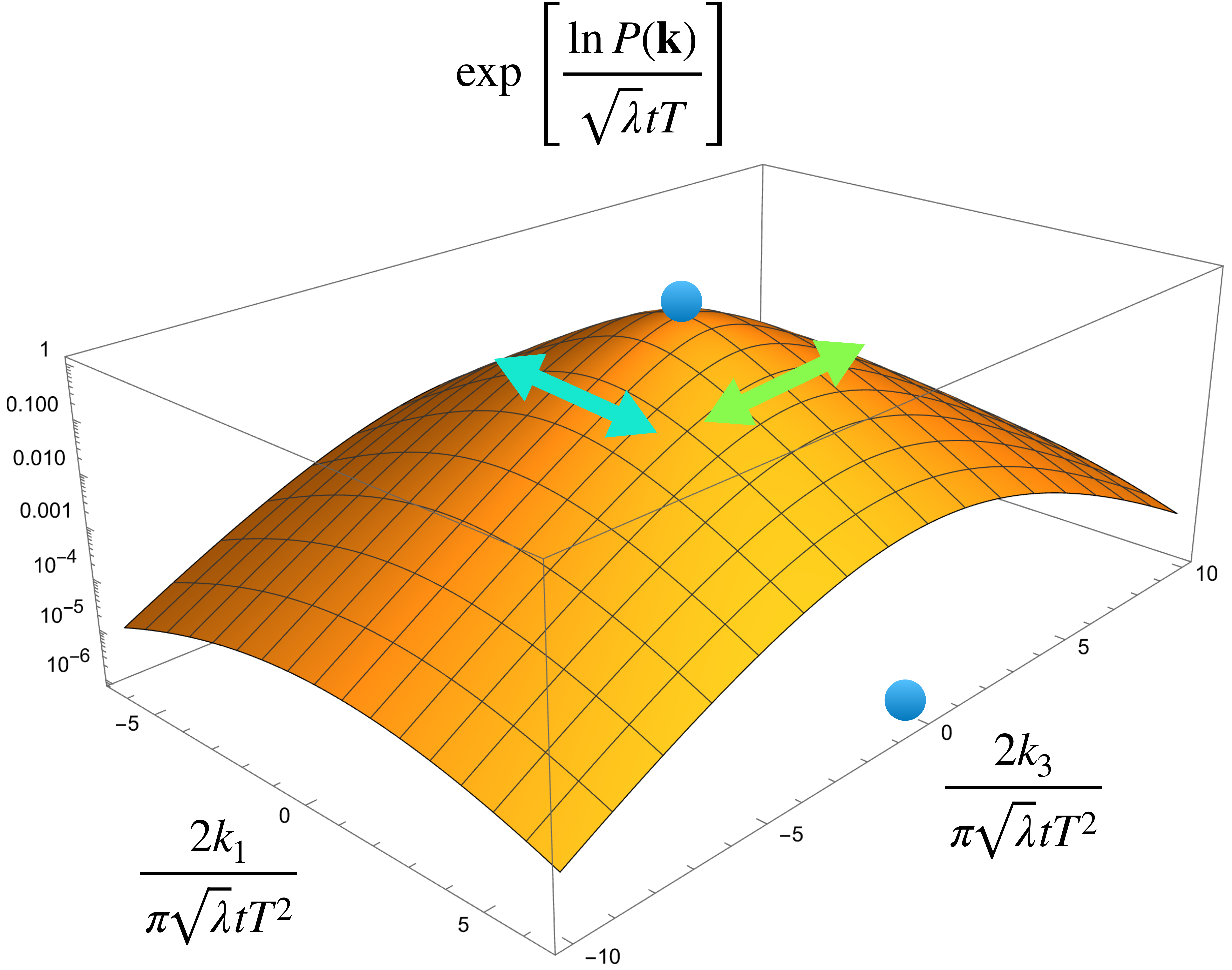
: most likely momentum change

$\Longrightarrow$ most likely value of energy loss

$\langle k_1 \rangle = \langle k_2 \rangle = \langle k_3 \rangle = 0$

: $\kappa_T = \pi\sqrt{\lambda}T^3$

: $\kappa_L = \pi\sqrt{\lambda}T^3$



$$v = 0.9$$

: most likely momentum change

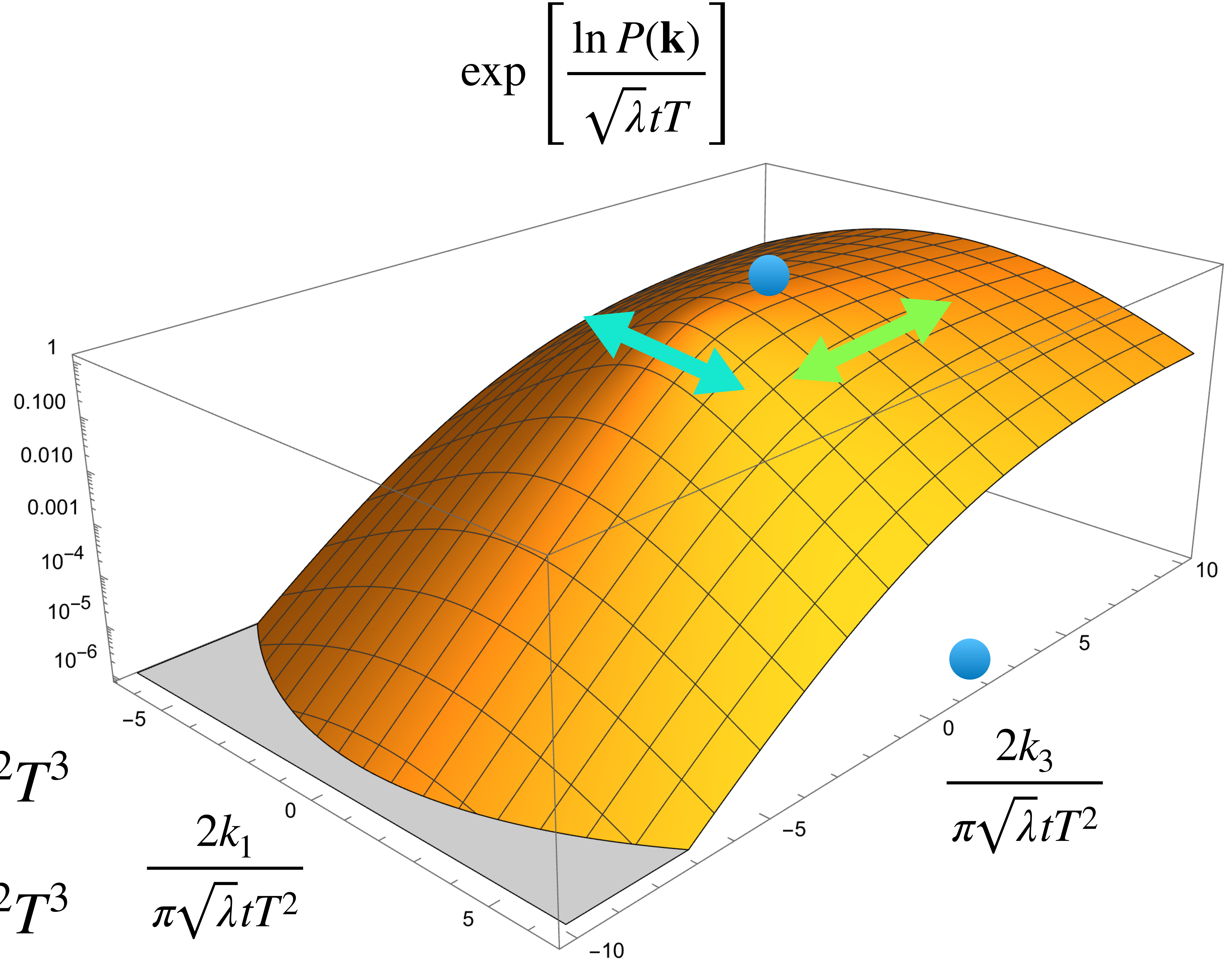
$\Longrightarrow$ most likely value of energy loss

$$\langle k_1 \rangle = \langle k_2 \rangle = 0$$

$$\langle k_3 \rangle = \frac{\pi\sqrt{\lambda}T^2 t}{2}\gamma v$$

: $\kappa_T = \pi\sqrt{\lambda}\gamma^{1/2}T^3$

: $\kappa_L = \pi\sqrt{\lambda}\gamma^{5/2}T^3$



$$v = 0.99$$

●: most likely momentum change

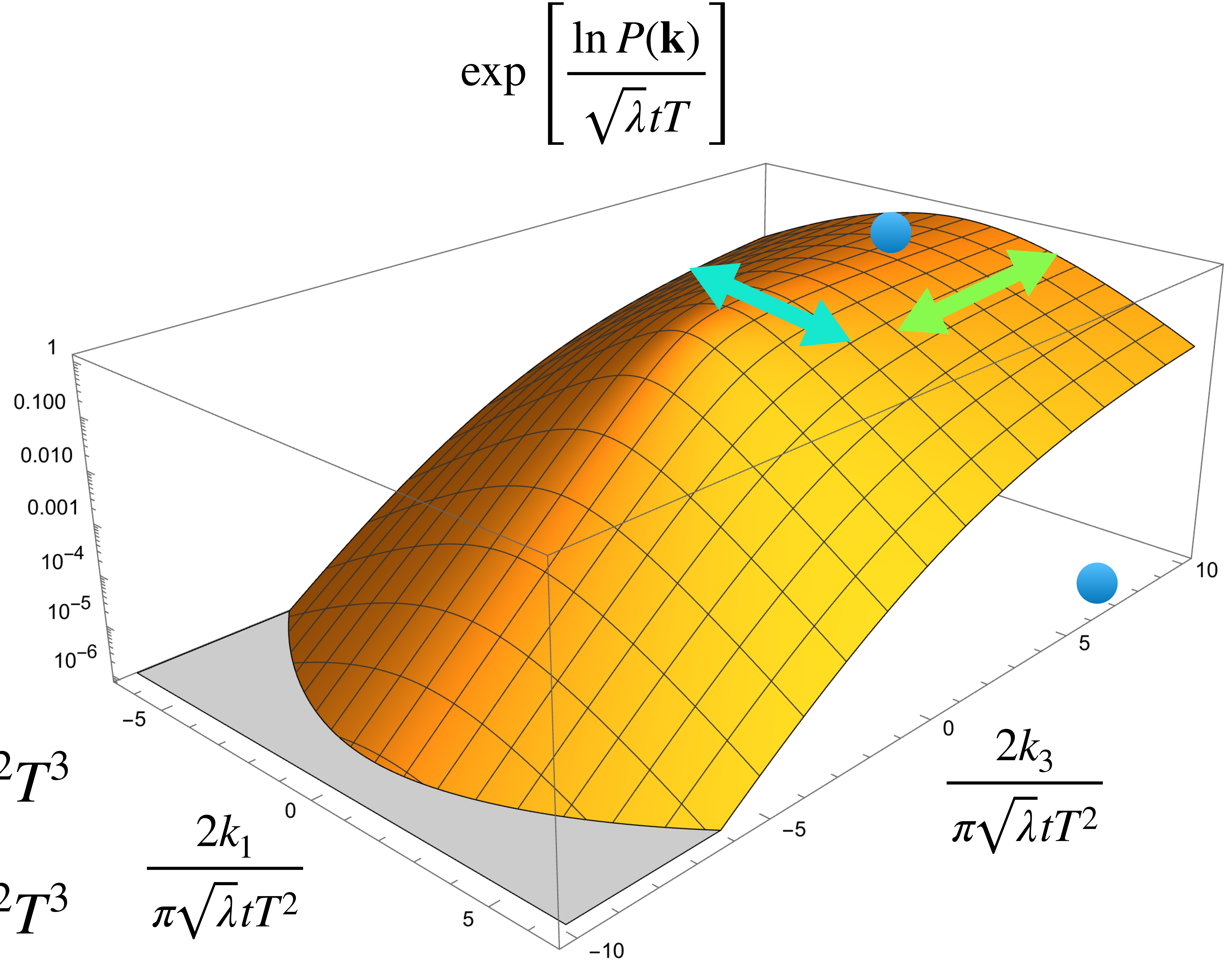
$\Longrightarrow$ most likely value of energy loss

$$\langle k_1 \rangle = \langle k_2 \rangle = 0$$

$$\langle k_3 \rangle = \frac{\pi\sqrt{\lambda}T^2 t}{2}\gamma v$$

↔ (cyan): $\kappa_T = \pi\sqrt{\lambda}\gamma^{1/2}T^3$

↔ (green): $\kappa_L = \pi\sqrt{\lambda}\gamma^{5/2}T^3$



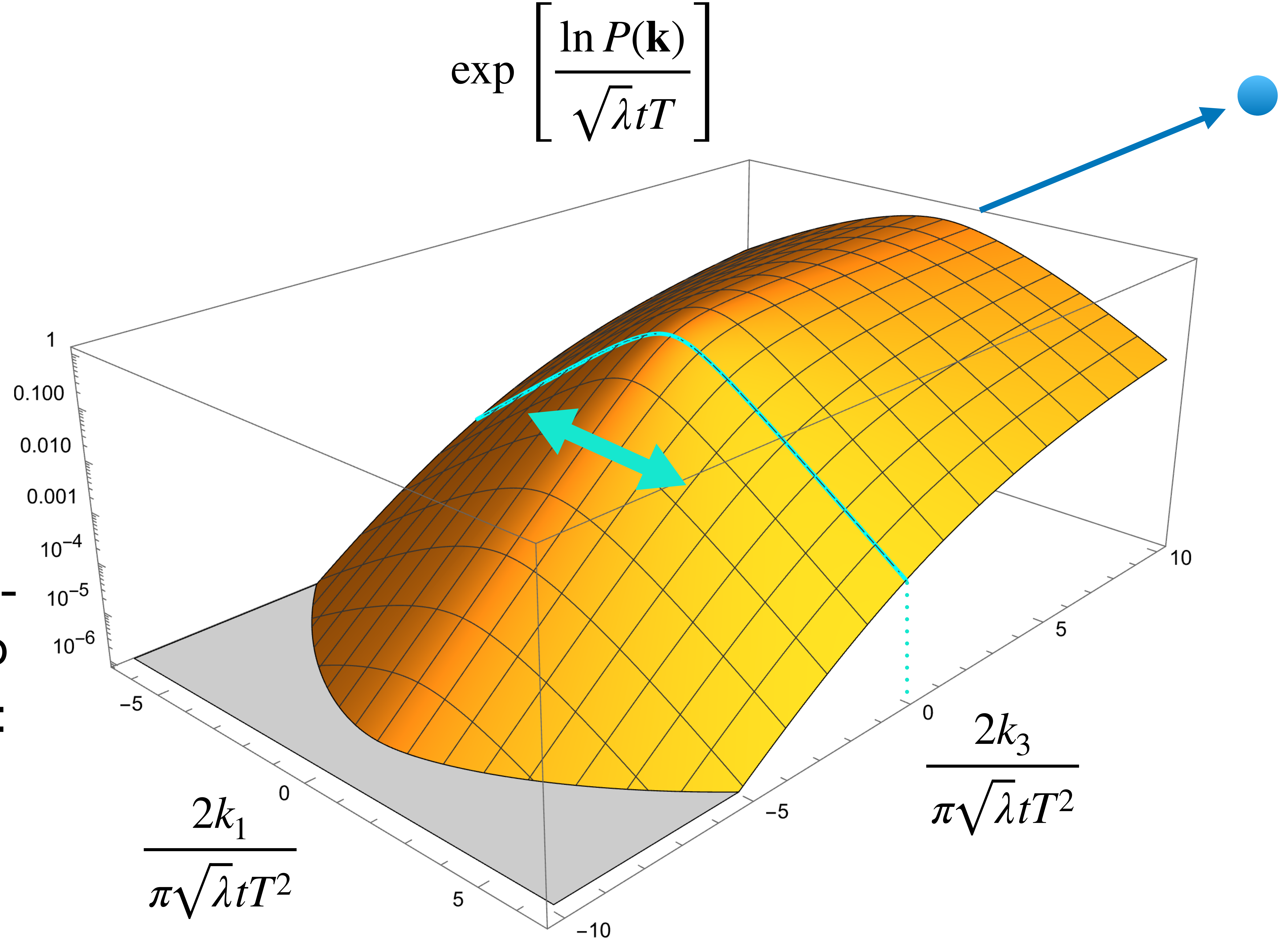
$v = 1$

●: most likely momentum change does not exist! (goes to infinity)

However, broadening is well-defined conditioned on zero momentum lost, i.e. $k_3 = 0$:

↔

$$\hat{q} = \frac{\pi^{3/2}\Gamma(3/4)}{\Gamma(5/4)}\sqrt{\lambda}T^3$$



So, every one of the classic results is there.
What else do we have access to?

Non-Gaussian features

beyond the drag and diffusion picture for heavy quarks

- We have the full $P(\mathbf{k})$ at large λ, so we can calculate the leading contribution to all of the fully connected moments of higher order. We will discuss:
  - Skewness along k_3
  - Correlation between transverse momentum broadening and energy loss
  - The 4th order moments (Kurtosis and broadening correlation)
- We find that for all of the moments the following large γ behavior holds:

$$\langle k_\perp^{2m} k_3^n \rangle_c \propto \sqrt{\lambda} t (\sqrt{\gamma}\, T)^{n+2m+1} \times \gamma^{n-1} \,.$$

Skewness and drag-broadening correlations

the third order moments for HQ transport

- We find that the distribution has a non-zero skewness (which was manifest in the figures):

$$\langle (k_3 - \langle k_3 \rangle)^3 \rangle = \frac{9v}{2} \pi \sqrt{\lambda} t \gamma^4 T^4 \ ,$$

- and a non-zero correlation between energy loss and momentum broadening:

$$\langle k_\perp^2 (k_3 - \langle k_3 \rangle) \rangle = 3 v \pi \sqrt{\lambda} t \gamma^2 T^4 \ .$$

- Note that they can be significant even at modest speeds.

Kurtosis and longitudinal-transverse broadening correlation

the fourth order moments for HQ transport

- We find the kurtosis along the transverse direction

$$\langle k_1^4 \rangle - 3\langle k_1^2 \rangle^2 = \langle k_2^4 \rangle - 3\langle k_2^2 \rangle^2 = \frac{9(8-v^2)}{8}\pi\sqrt{\lambda}t\gamma^{3/2}T^5 \, ,$$

- the kurtosis along the longitudinal direction

$$\langle (k_3 - \langle k_3 \rangle)^4 \rangle - 3\langle (k_3 - \langle k_3 \rangle)^2 \rangle^2 = \frac{9(8+19v^2)}{8}\pi\sqrt{\lambda}t\gamma^{11/2}T^5 \, ,$$

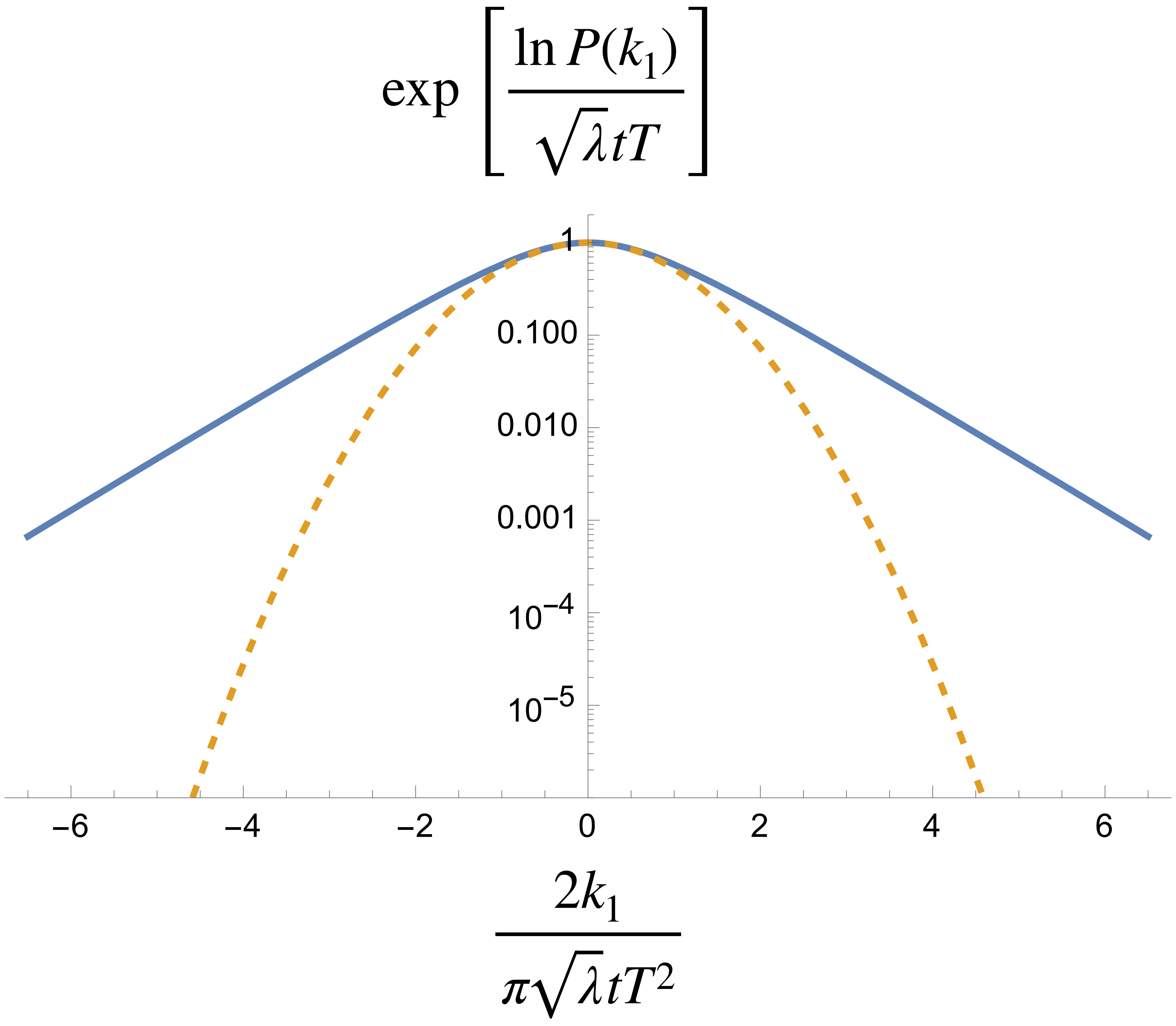
- and the cross-correlation

$$\langle k_\perp^2 (k_3 - \langle k_3 \rangle)^2 \rangle = \frac{3(8+9v^2)}{4}\pi\sqrt{\lambda}t\gamma^{7/2}T^5 \, .$$

What do we learn in the $\nu = 1$ case?

connections to BDMPS-Z

- If we calculate the conditional probability for k_1 when $k_3 = 0$, we get the momentum broadening distribution of interest in the BDMPS-Z formalism.
- The distribution is well-approximated by a Gaussian if the coupling constant $\lambda = g^2 N_c$ is large.

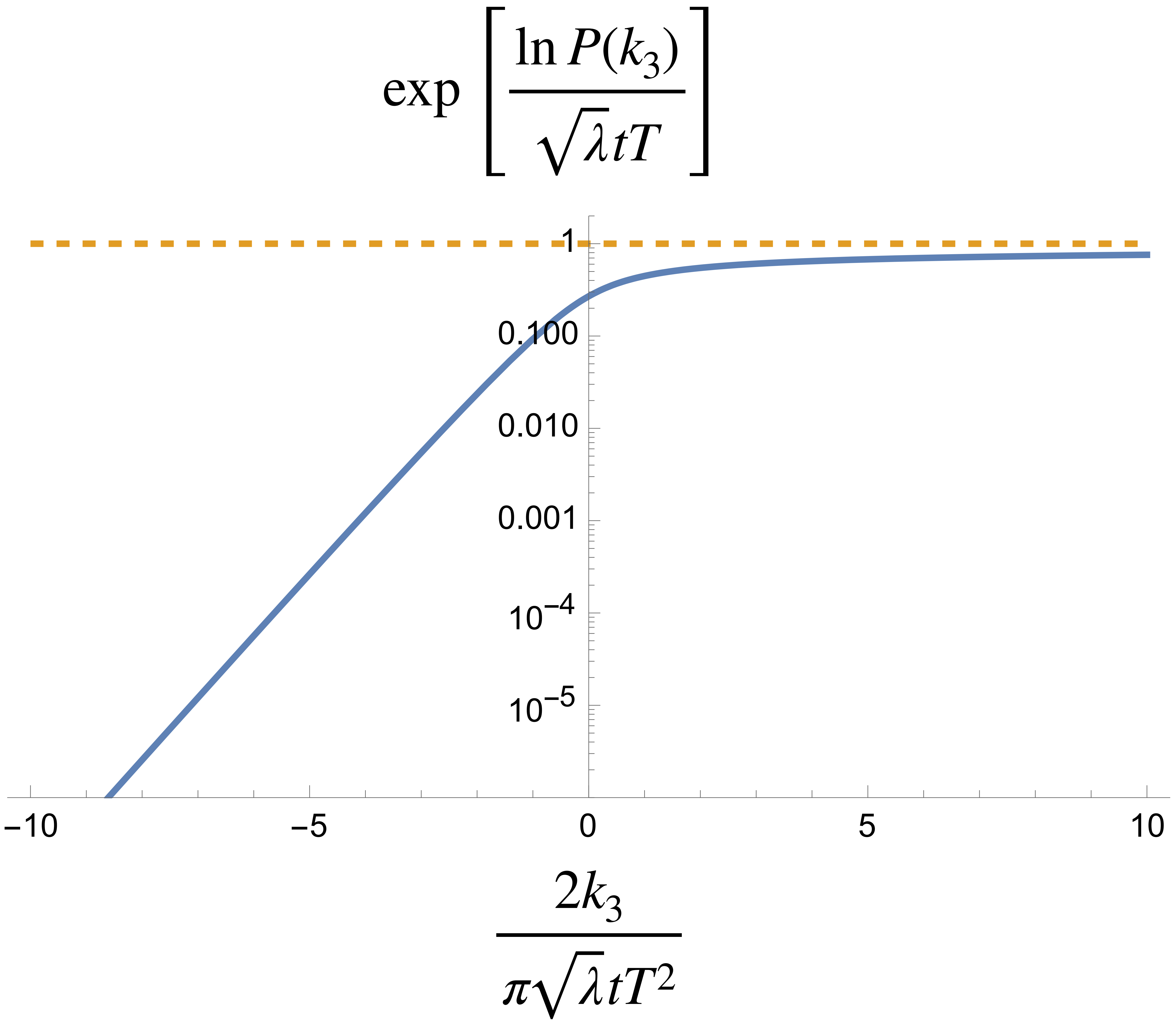


What do we learn in the $v = 1$ case?

connections to BDMPS-Z

- However, if we plot $P(k_3)$, the result is not a normalizable probability distribution.

- Note that the relation between the Wilson loop and the probability distribution for momentum change relies on $\mathbf{k}$ being a scale that is small compared with the hard scale of the problem $(E \text{ or } M)$.



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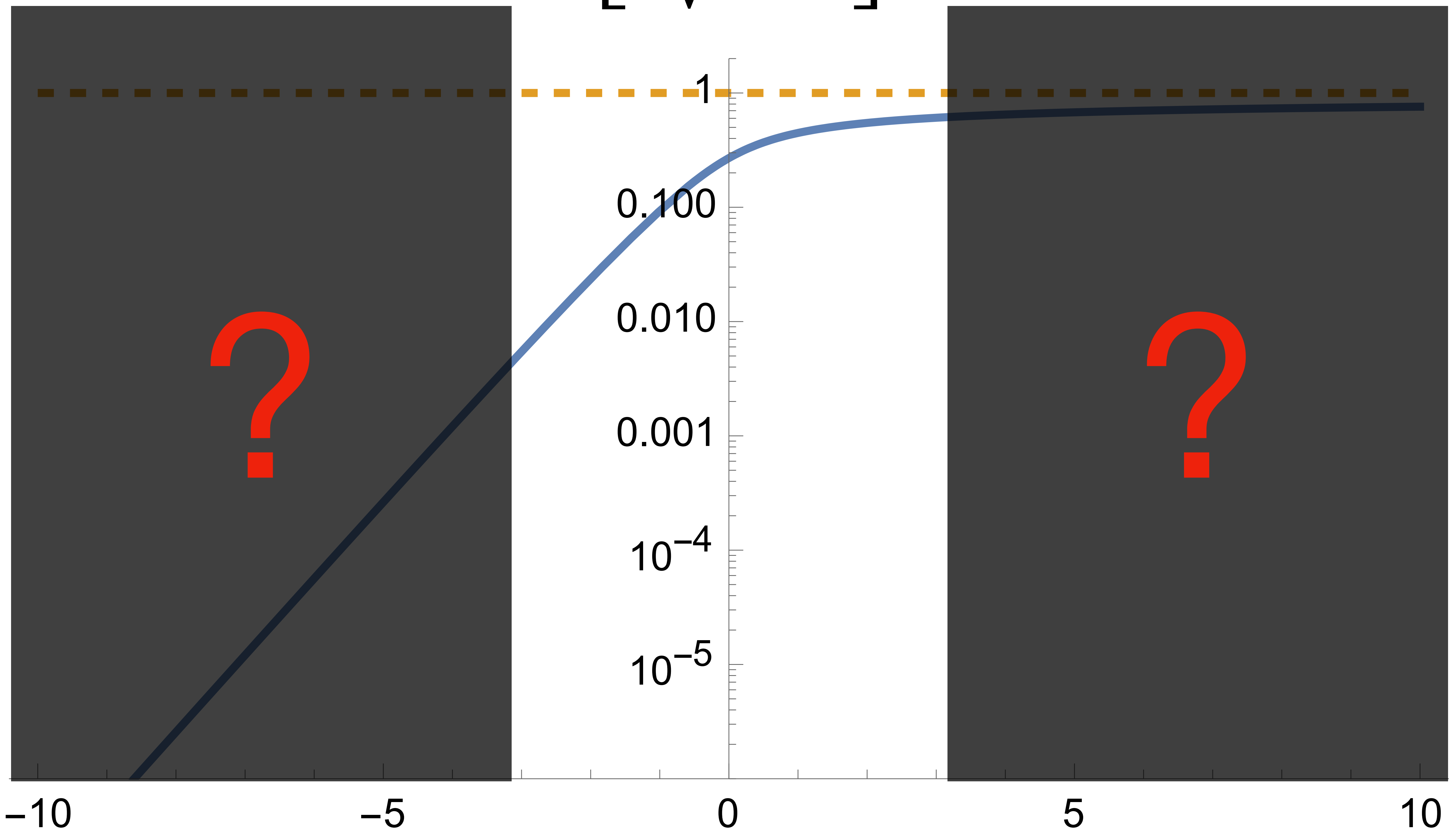
$$\exp\left[\frac{\ln P(k_3)}{\sqrt{\lambda} t T}\right]$$



$$\frac{2k_3}{\pi\sqrt{\lambda} t T^2}$$

If one wants to interpret this as a longitudinal broadening calculation, one cannot go to arbitrarily high momentum

Transverse momentum distribution at fixed energy loss

obtaining $\langle k_\perp^2 \rangle$ from conditional probabilities

It is still interesting to explore the features of this distribution:

- One may condition the above distribution at a fixed nonzero value of k_3 and obtain

$$\frac{< k_\perp^2 >_{k_3}}{t} = \frac{\hat{q}}{2}\left(1 + \left(\frac{2k_3}{\pi\sqrt{\lambda}tT^2}\right)^2\right)^{1/4},$$

or, more suggestively of a correlation between the rate of energy loss and broadening,

$$\frac{d < k_\perp^2 >_{dk_3/dt}}{dt} = \frac{\hat{q}}{2}\left(1 + \left(\frac{2}{\pi\sqrt{\lambda}T^2}\frac{dk_3}{dt}\right)^2\right)^{1/4}.$$

Summary

- We have revisited and extended the characterization of momentum broadening in $\mathcal{N} = 4$ SYM.
  - We have presented $\eta_D, \kappa_T, \kappa_L$ and $\hat{q}$ in a unified fashion.
  - We have calculated the non-Gaussian corrections to heavy quark transport, e.g., how energy loss is correlated with transverse momentum broadening.
- Prospects:
  - Non-Gaussian fluctuations in heavy quark transport
  - Jet momentum broadening