Implementation and benchmarking of experimental solenoids in Xsuite

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With input from

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Work supported by: CHART

https://xsuite.web.cern.ch

Outline

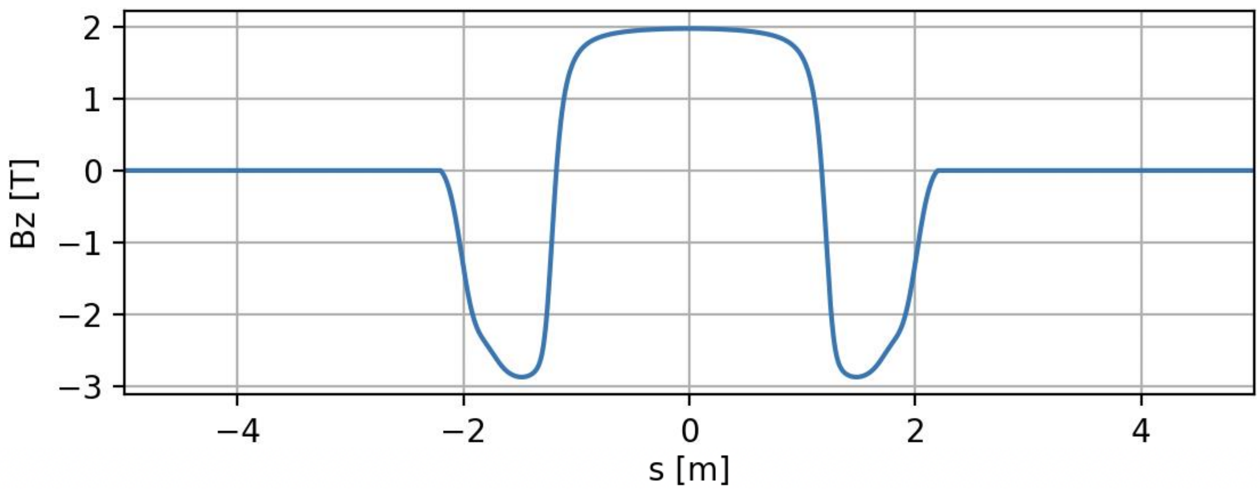
- **Solenoid modeling**
  - Uniform solenoid (constant B_s)
  - General case $B_s(s)$
  - Radial field component
  - Implementation
  - Benchmarks
  - Synchrotron radiation
  - Solenoid tilt
- **FCC-ee lattice example**

Xsuite solenoid elements allow simulating regions with a magnetic field in the form:

$$\mathbf{B}(r, s) = B_s(s)\hat{\mathbf{i}}_s + B_r(r, s)\hat{\mathbf{i}}_r$$

- **Longitudinal component** is s-dependent but independent on the transverse coordinate
  → Allows us to model fringe fields and solenoid-antisolenoid interfaces
- **No small-angle approximation** is used in the modelling such that the solenoid **can be used with arbitrary tilt** with respect to the reference frame of the incoming beam

Bz profile prepared by H. Burkhardt (map from M. Koratsinos)



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Vector potential in a region with a uniform longitudinal magnetic field can be written as(*):

$$A_x = -\frac{B_s}{2} y \qquad A_y = +\frac{B_s}{2} x$$

$$A_s = 0$$

(*) verify: $$\mathbf{B} = \nabla \times \mathbf{A} = \begin{bmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ B_s \end{bmatrix}$$

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$$A_s = 0$$

The dynamics of a particle moving in such a region is defined by the following **Hamiltonian**:

$$H = p_\zeta - \sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

Definitions:

$$\zeta = s - \beta_0 ct$$

$$p_\zeta = \frac{1}{\beta_0^2} \frac{E - E_0}{E_0}$$

$$\delta = \frac{P - P_0}{P_0}$$

Solution for constant B_s

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The dynamics of a particle moving in such a region is defined by the following **Hamiltonian**:

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where a_x and a_y are the **normalized potentials**

$$a_{x,y} = \frac{q_0}{P_0} A_{x,y}$$

and p_x and p_y are the **canonical momenta defined as:**

$$p_x = p_x^{\mathrm{kin}} + a_x$$

where the **kinetic momenta** are defined as:

$$p_x^{\mathrm{kin}} = m_0 \gamma v_x$$

Vector potential in a region with a uniform longitudinal magnetic field can be written as(*):

$$A_x = -\frac{B_s}{2} y \qquad A_y = +\frac{B_s}{2} x$$

$$A_s = 0$$

The dyn … **tonian**:

where *a* …

and p_x a …

where t…

Both canonical and kinetic momenta are **available in the xsuite output**:

	Definition	Xsuite name
Canonic momentum	$p_x = p_x^{\text{kin}} + a_x$	`px`
Kinetic momentum	$p_x^{\text{kin}} = m_0 \gamma v_x$	`kin_px`
Trajectory slope	$x' = \frac{dx}{ds}$	`kin_xprime`

$$H = p_\zeta - \sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

Hamilton's equation for the case of a uniform solenoid can be **integrated analytically** obtaining the following **map** ("corkscrew" motion):

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \leftarrow \begin{pmatrix} \cos^2(\omega L) & \frac{1}{2\omega}\sin(2\omega L) & \frac{1}{2}\sin(2\omega L) & \frac{1}{\omega}\sin^2(\omega L) \\ -\frac{\omega}{2}\sin(2\omega L) & \cos^2(\omega L) & -\omega\sin^2(\omega L) & \frac{1}{2}\sin(2\omega L) \\ -\frac{1}{2}\sin(2\omega L) & -\frac{1}{\omega}\sin^2(\omega L) & \cos^2(\omega L) & \frac{1}{2\omega}\sin(2\omega L) \\ \omega\sin^2(\omega L) & -\frac{1}{2}\sin(2\omega L) & -\frac{\omega}{2}\sin(2\omega L) & \cos^2(\omega L) \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}$$

$$\zeta \leftarrow \zeta + L\left(1 - \frac{\beta_0}{\beta}\frac{1+\delta}{\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2}}\right)$$

where:

$$\omega = \frac{q_0}{P_0}\frac{B_s}{2}$$

References:

[1] A. Wolski, Beam Dynamics in High Energy Particle Accelerators, DOI: 10.1142/13333

[2] E. Forest, Beam Dynamics: A New Attitude and Framework, Harwood Academic, 1998

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Expression of the radial component when B_s depends on s

We now consider the **more general field distribution:**

$$\mathbf{B}(r,s) = B_s(s)\hat{\mathbf{i}}_s + B_r(r,s)\hat{\mathbf{i}}_r$$

The **longitudinal and transverse components are not indep**endent since they must satisfy Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_s}{\partial s} = 0$$

From this we can obtain:

$$B_r(r,s) = -\frac{r}{2}\frac{dB_s}{ds}$$

Proof – we use the fact that B_s does not depend on r and we look for solutions in the for a solution in the form $B_r(r,s) = A(r)C(s)$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r(r,s)) = -\frac{d}{ds}B_s(s) \Rightarrow \frac{C(s)}{r}\frac{d}{dr}(rA(r)) = -\frac{d}{ds}B_s(s) \Rightarrow \frac{d}{dr}(rA(r)) = -\frac{r}{C(s)}\frac{d}{ds}B_s(s)$$

We integrate in *r* obtaining: $rA(r) = -\frac{r^2}{2C(s)}\frac{d}{ds}B_s(s)$ and we replace in the expression of B_r.

Transverse kick from the radial field component

We consider a **short section of length Δs**, and we compute the **transverse kick from the B_r component**

$$\Delta p_x^{\mathrm{kin}} = \frac{q_0}{P_0}\hat{\mathbf{i}}_x \cdot \left(\mathbf{v} \times B_r \hat{\mathbf{i}}_r\right)\Delta t = \frac{q_0}{P_0} B_r \mathbf{v} \cdot \underbrace{\left(\hat{\mathbf{i}}_r \times \hat{\mathbf{i}}_x\right)}_{= -\frac{y}{r}\hat{\mathbf{i}}_s}\Delta t$$

$$= -\frac{q_0}{P_0} B_r \frac{y}{r} v_s \Delta t = -\frac{q_0}{P_0}\frac{y}{r} B_r \Delta s$$

From before

$$B_r(r,s) = -\frac{r}{2}\frac{dB_s}{ds}$$

$$= \frac{q_0}{P_0}\frac{y}{2}\frac{dB_s}{ds}\Delta s = \frac{q_0}{P_0}\frac{y}{2}\Delta B_s$$

The transverse kick is proportional to the change in the longitudinal field component!

Transverse kick from the radial field component

We want to evaluate **the change in the canonical momentum**

$$\Delta p_x = \Delta p_x^{\mathrm{kin}} + \Delta a_x$$

$$\Delta p_x^{\mathrm{kin}} = \frac{q_0}{P_0}\frac{y}{2}\Delta B_s$$

$$A_x = -\frac{B_s}{2}y \Rightarrow \Delta a_x = -\frac{q_0}{P_0}\frac{y}{2}\Delta B_s$$

We obtain that the **change in the canonical momentum from the radial field component is zero!**

$$\Delta p_x = 0$$

This means:

- As in Xsuite we track canonical momenta, **there is no actual map required to model the effect of B_r**
- To see the deflection from the B_r component we **need to inspect the kinetic momenta which are also provided in output**

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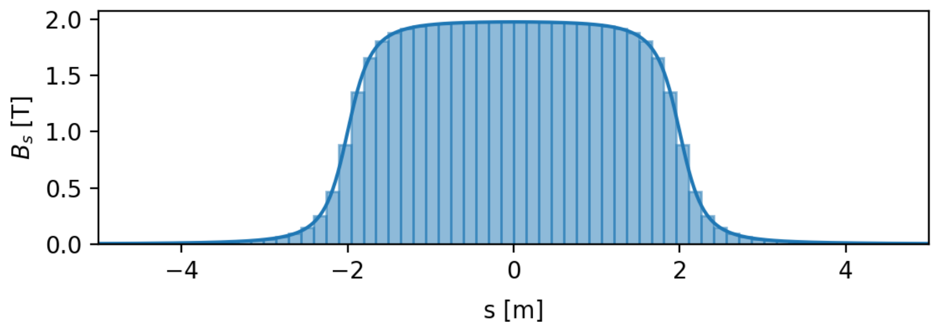
- **FCC-ee lattice example**

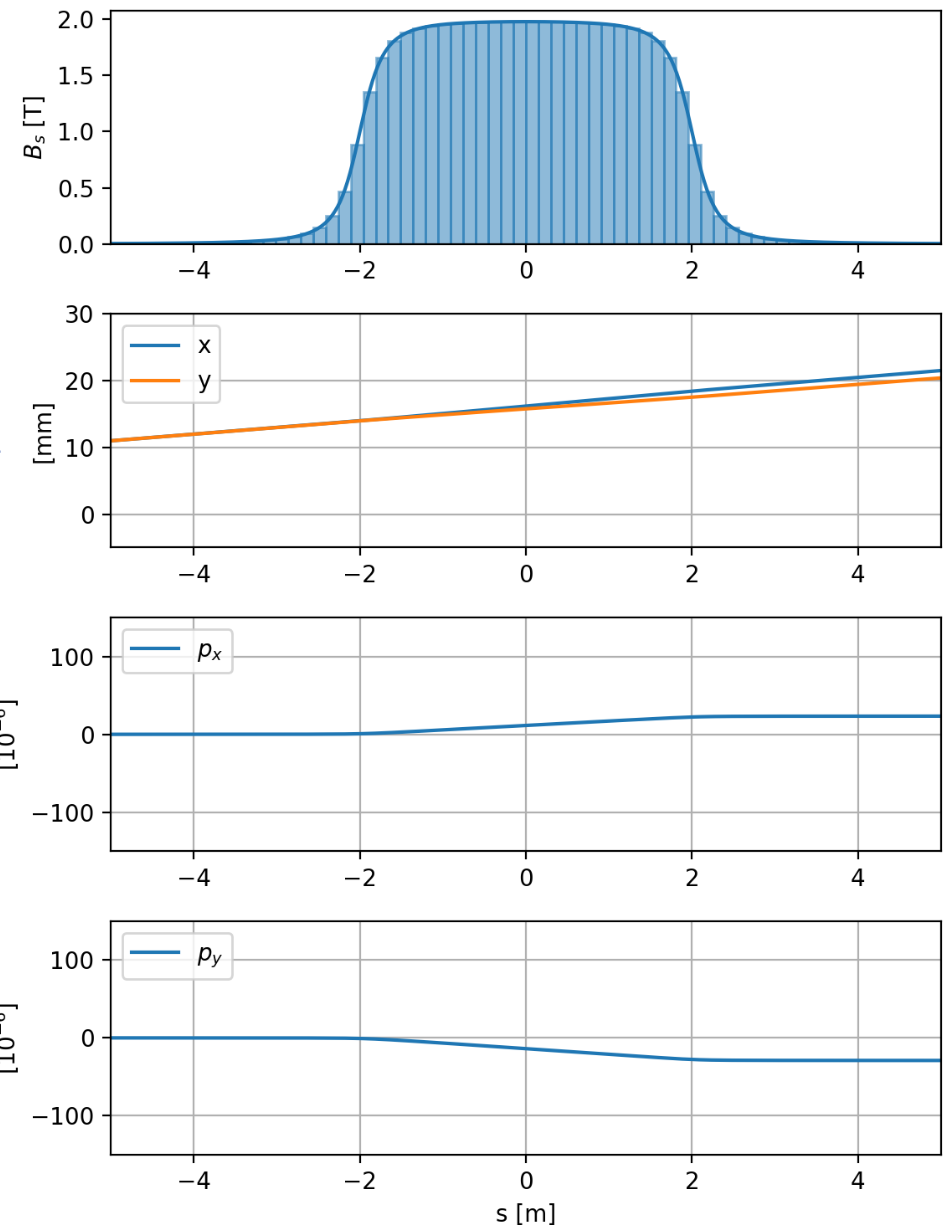
If we have a solenoid with a given $B_s(s)$ profile, we model it with a set of **thick solenoid slices with constant B_s in each slice**



If we have a solenoid with a given $B_s(s)$ profile, we model it with a set of **thick solenoid slices with constant B_s in each slice**

Test with a particle with x'= 1 mrad, y'= 1 mrad:

- If we plot the **canonical momenta** (px, py in the output) we **do not see strong deflections from the edges**



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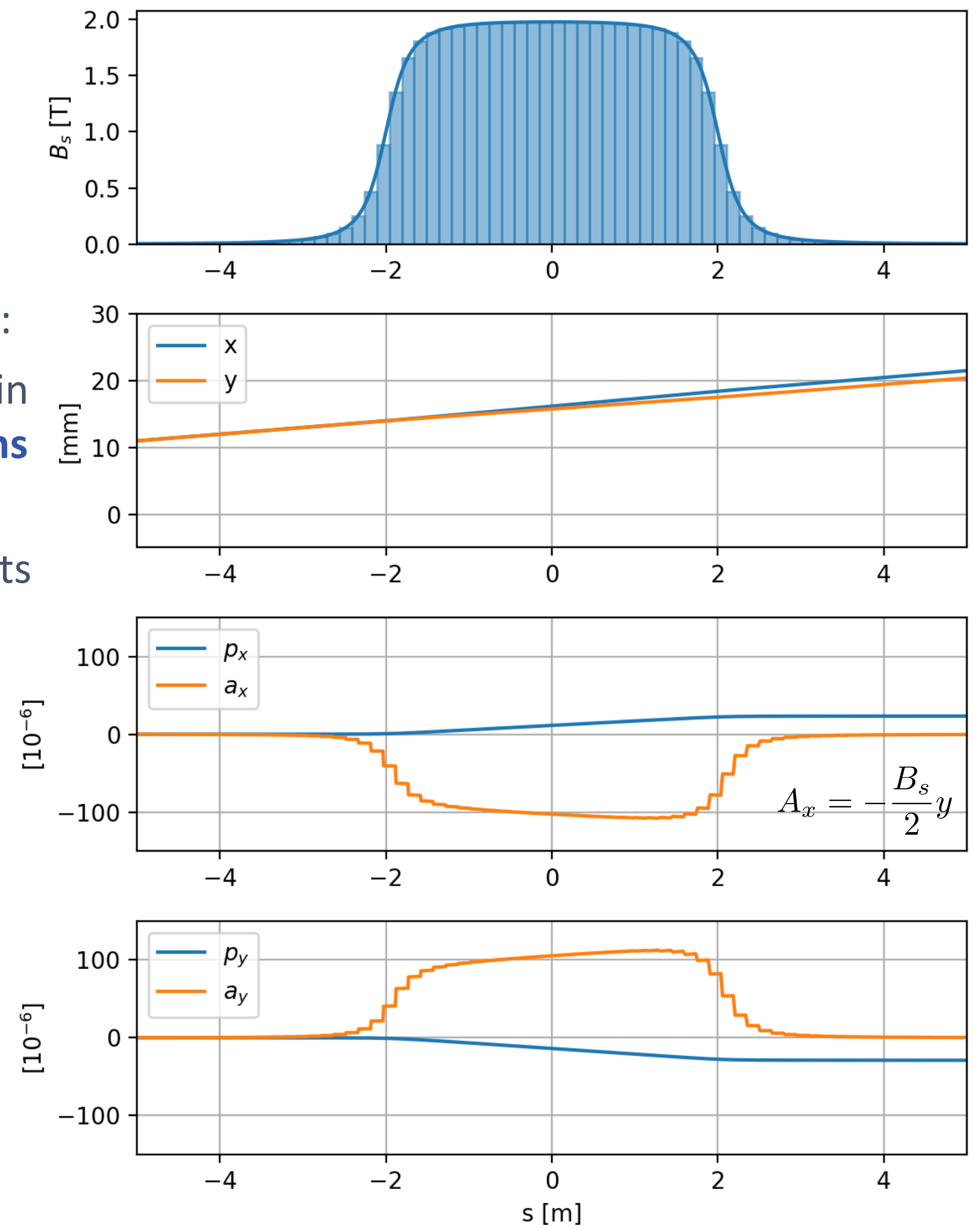
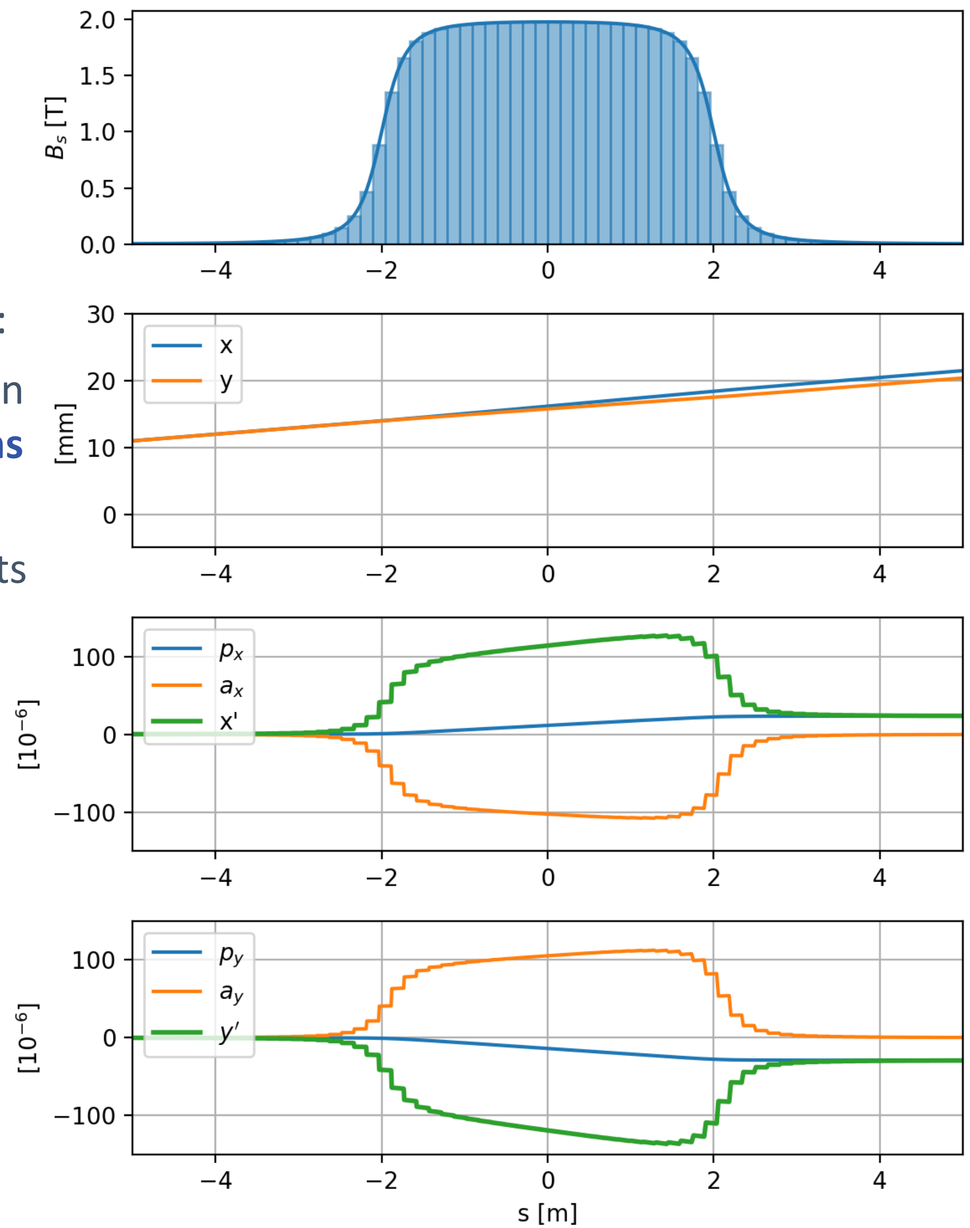
Test with a particle with x'= 1 mrad, y'= 1 mrad:

- If we plot the **canonical momenta** (px, py in the output) we **do not see strong deflections from the edges**
- The normalized **vector potential** components (ax, ay in the output) show **discrete steps where B_s changes**



Implementation

If we have a solenoid with a given $B_s(s)$ profile, we model it with a set of **thick solenoid slices with constant B_s in each slice**



Test with a particle with x'= 1 mrad, y'= 1 mrad:

- If we plot the **canonical momenta** (`px`, `py` in the output) we **do not see strong deflections from the edges**
- The normalized **vector potential** components (`ax`, `ay` in the output) show **discrete steps where B_s changes**
- The **trajectory slopes x' y'** (`kin_xprime`, `kin_yprime` in the output) show the physical deflection at the edged, since:

$$p_x^{\mathrm{kin}} = p_x - a_x$$

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Benchmark

Xsuite implementation **benchmarked against time-domain tracking within a field map**

- Based on **"Boris push" algorithm** (reused module from PyECLOUD)
- **Analytic field map** for a circular finite-length solenoid

Boris tracking scheme

$$\mathbf{v}^- = \mathbf{v}_k + \frac{q}{m}\mathbf{E}_k\frac{\Delta t}{2},$$

$$\frac{\mathbf{v}^+ - \mathbf{v}^-}{\Delta t} = \frac{q}{2mc}(\mathbf{v}^+ + \mathbf{v}^-)\times\mathbf{B}_k,$$

$$\mathbf{v}_{k+1} = \mathbf{v}^+ + \frac{q}{m}\mathbf{E}_k\frac{\Delta t}{2}.$$

References:

[1] Phys. Plasmas 20, 084503 (2013)

[2] https://www.particleincell.com/2011/vxb-rotation/

Finite-length circular solenoid field map

$$B_r(r,\theta,z) = \frac{B_0}{\pi}\sqrt{\frac{a}{rm_+}}\left(E(m_+) - \left(1-\frac{m_+}{2}\right)K(m_+)\right) - \frac{B_0}{\pi}\sqrt{\frac{a}{rm_-}}\left(E(m_-) - \left(1-\frac{m_-}{2}\right)K(m_-)\right), \quad (5)$$

$$B_\theta(r,\theta,z) = 0, \quad (6)$$

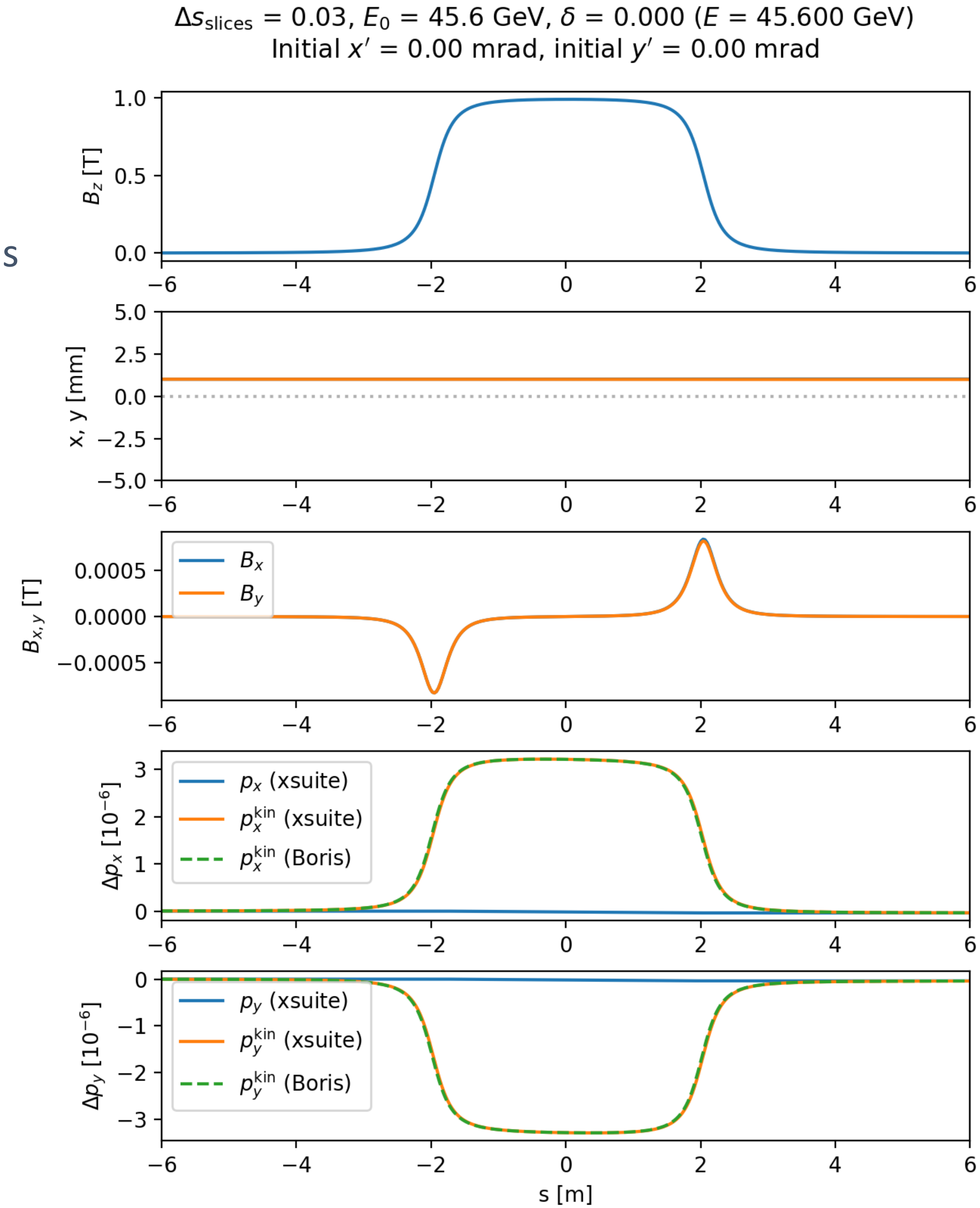
$$B_z(r,\theta,z) = \frac{B_0\zeta_+}{4\pi}\sqrt{\frac{m_+}{ar}}\left(K(m_+) + \left(\frac{a-r}{a+r}\right)\Pi(u,m_+)\right) - \frac{B_0\zeta_-}{4\pi}\sqrt{\frac{m_-}{ar}}\left(K(m_-) + \left(\frac{a-r}{a+r}\right)\Pi(u,m_-)\right). \quad (7)$$

Here, $B_0 = \mu_0 In$ in SI units. Furthermore, $u = 4ar/(a+r)^2$, $m_+ = 4ar/\left((a+r)^2+\zeta_+^2\right)$, $m_- = 4ar/\left((a+r)^2+\zeta_-^2\right)$, $\zeta_+ = z + (L/2)$, $\zeta_- = z - (L/2)$, L and a are the length and radius of the solenoid, respectively, and the complete elliptic integrals of the first, second, and third kinds are given in Appendix A. The magnitude of the magnetic field inside the solenoid in the limit of infinite length, $L \to \infty$, is $|B_0|$. A sketch depicting the geometry is shown in Fig. 5.

Reference: AIP Advances 10, 065320 (2020)

Agreement is found to be excellent in all tested cases:

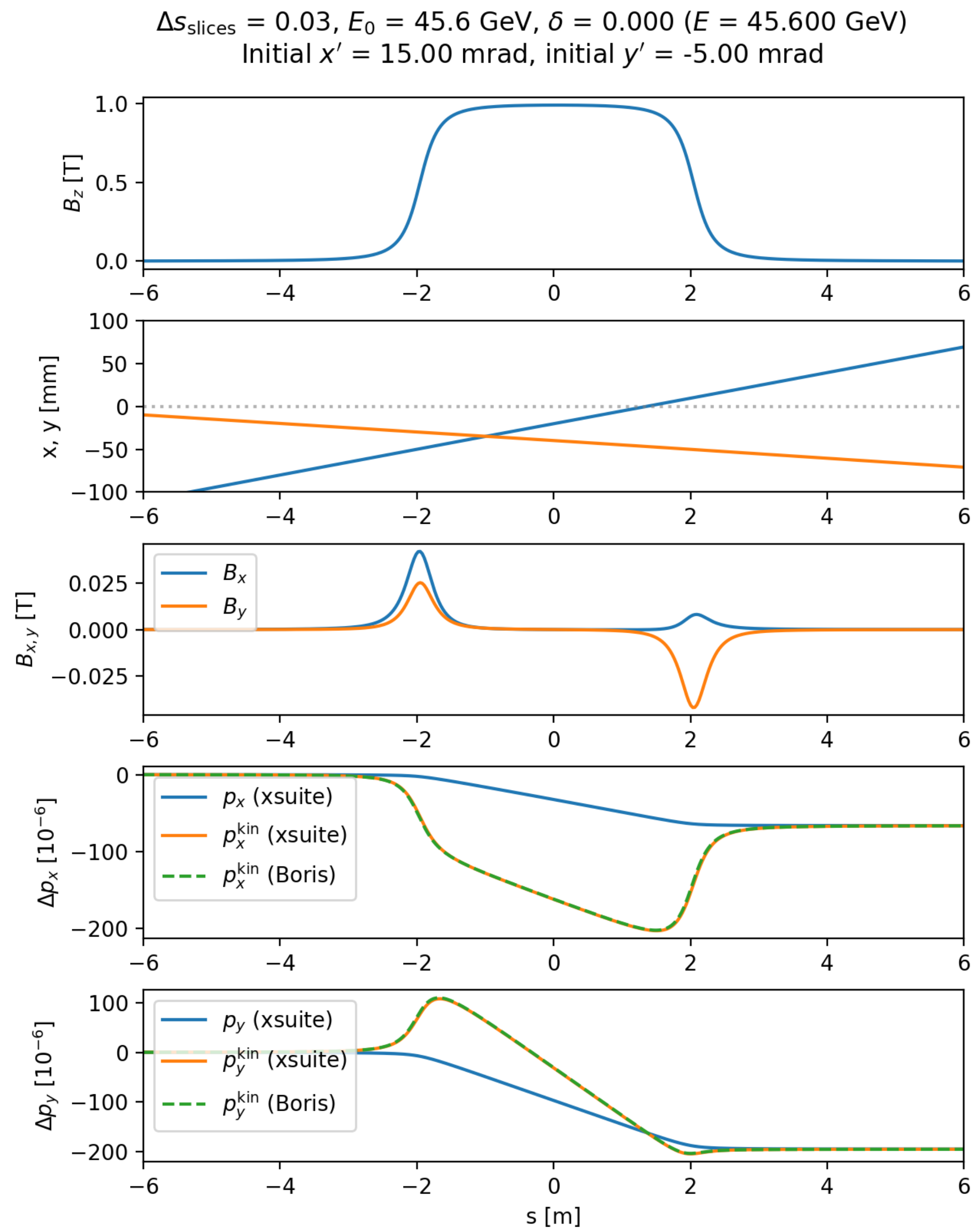
- Small distance from the solenoid axis



Benchmark

Agreement is found to be excellent in all tested cases:

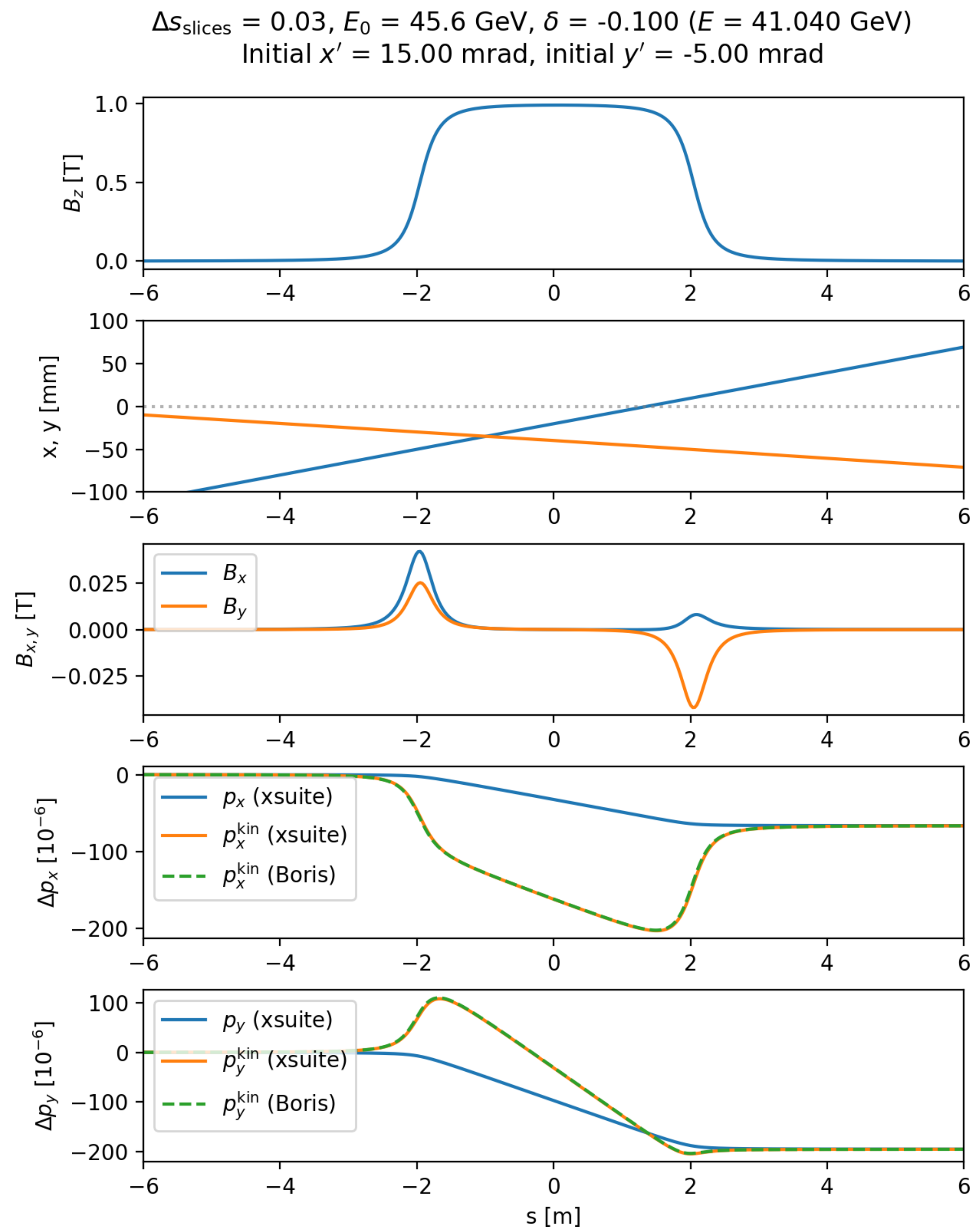
- Small distance from the solenoid axis
- **Large angles** (>10 mrad) and **large distance from the axis** (> 1 cm)



Benchmark

Agreement is found to be excellent in all tested cases:

- Small distance from the solenoid axis
- **Large angles** (>10 mrad) and **large distance from the axis** (> 1 cm)
- On and **off momentum** also for very large energy deviations



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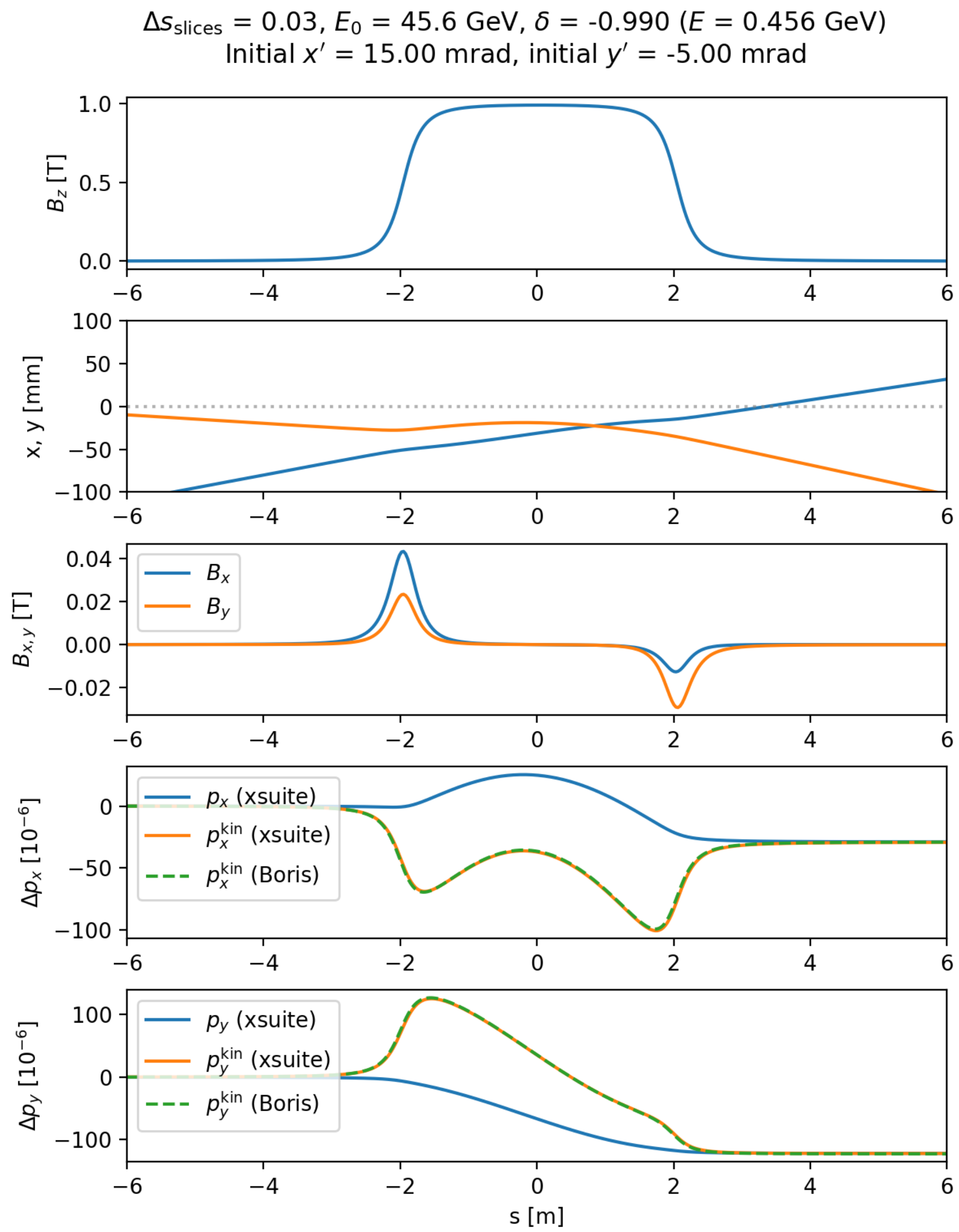

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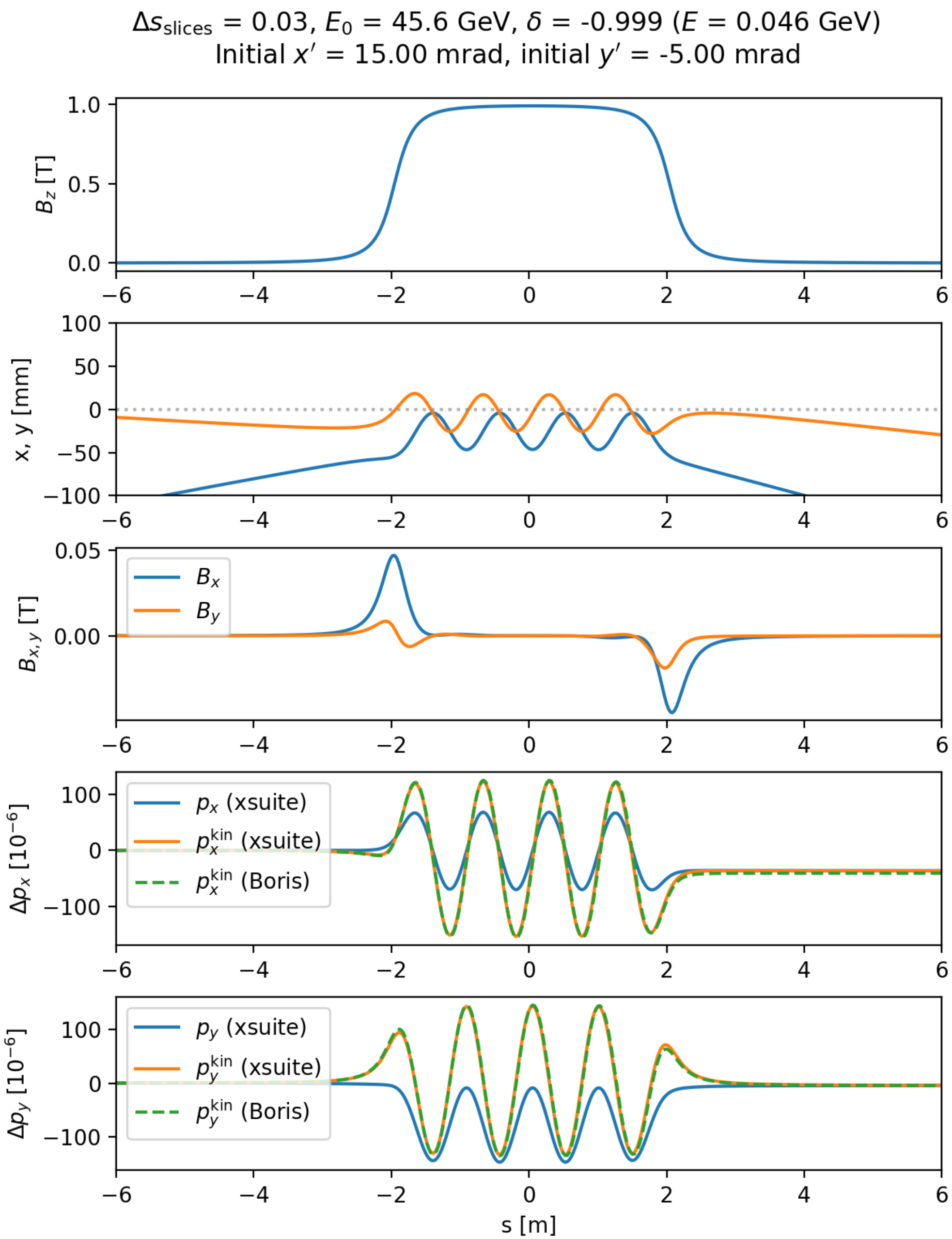

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Only the magnetic field component orthogonal to the particle trajectory contributes to the radiation. We can write it as:

$$\mathbf{B}_\perp = \mathbf{B} - \left(\hat{\mathbf{t}} \cdot \mathbf{B}\right) \hat{\mathbf{t}} \quad \Rightarrow \quad \mathbf{B}_\perp \cdot \hat{\mathbf{t}} = 0$$

For a short solenoid slice, **$B_\perp$ component can be calculated from the applied kick**:

$$\frac{\Delta \mathbf{p}^{\text{kin}}}{\Delta T} = \frac{q\beta c}{P_0} \hat{\mathbf{t}} \times \mathbf{B}_\perp \quad \xrightarrow{\mathbf{B}_\perp \cdot \hat{\mathbf{t}} = 0} \quad |\mathbf{B}_\perp| = \frac{q\beta c}{P_0} \frac{\Delta \mathbf{p}^{\text{kin}}}{\Delta T}$$

Lorentz force

We can use the **results (and the code) valid for transverse fields**, for example:

$$P_{\text{rad}} = \frac{2 r_0 q^2 \beta^2 \gamma^2 \left|\mathbf{B}_\perp\right|^2}{3 m_0} \qquad E_{\gamma c} = \frac{3 q \hbar \beta^2 \gamma^2 \left|\mathbf{B}_\perp\right|}{2 m_0} \qquad \dot{n}_{\text{rad}} = \frac{15\sqrt{3}}{8} \frac{P_{\text{rad}}}{E_{\gamma c}}$$

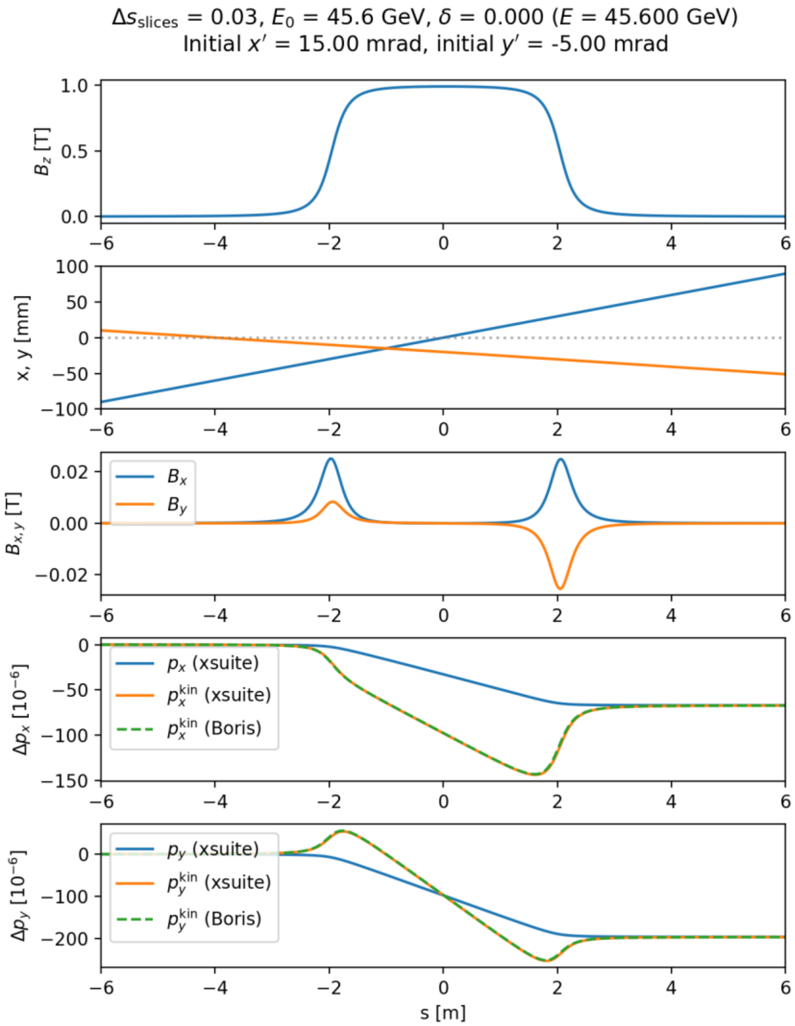
Photons are **emitted in the direction of the particle trajectory**.

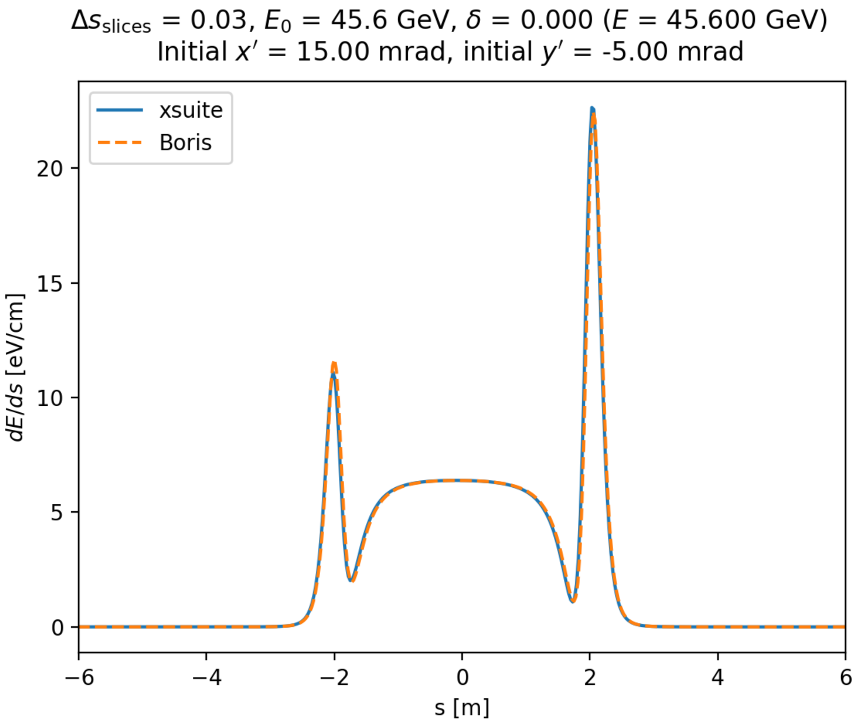
Radiation - benchmark

Radiation **benchmarked against results from Boris time-domain tracker**, using the fact the **radiated power can be simply calculated from the derivative of the velocity**:

$$P_{\mathrm{rad}} = \frac{2e^2\gamma^4\dot{\mathbf{v}}^2}{12\pi\epsilon_0 c^3}$$

(can be done in postprocessing if radiation is weak enough)





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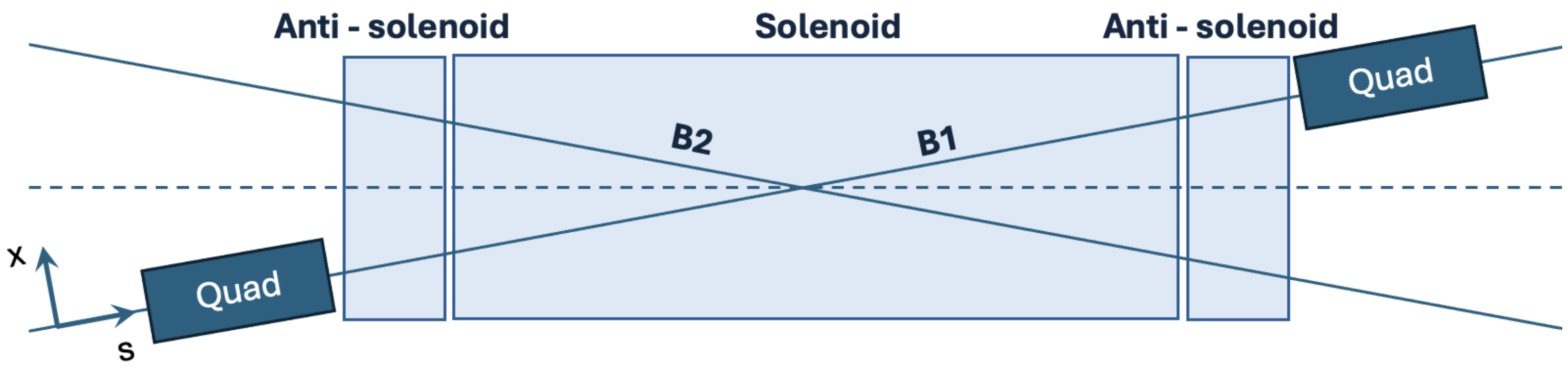
The modelling presented above **does not rely on any small-angle approximation** (the map can be used when entering the solenoid with large angles and large offsets)

→ The **solenoid tilt can be handled as done for other Xsuite elements**, i.e. by suitable reference frame transformations at entrance and exit

→ A **shift needs to be introduced also on the time coordinate** so that the the reference path length stays unchanged



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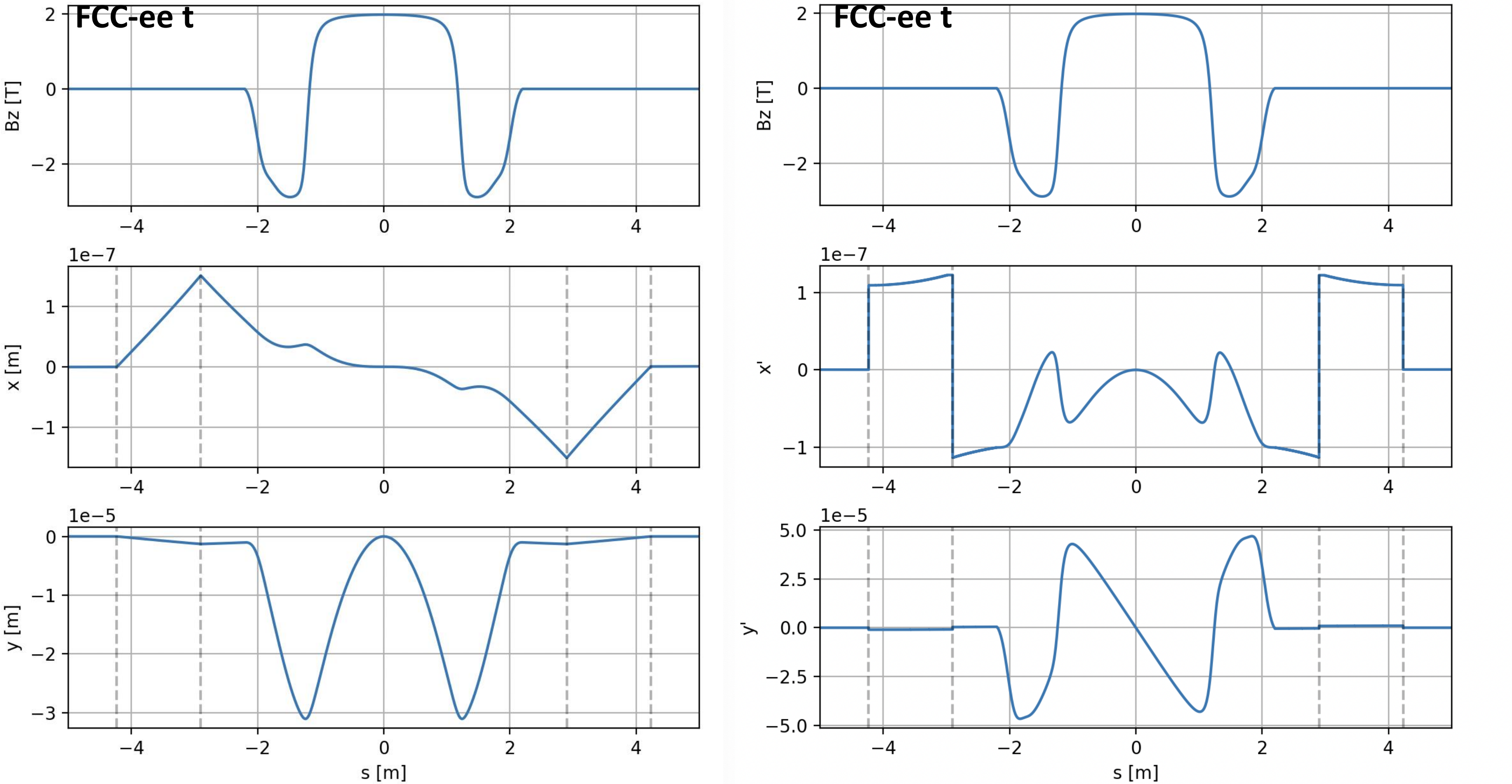
Solenoid in FCC-ee lattice

Prepared **Xsuite model of the FCC-ee ring including the experimental solenoid**

- Xsuite twiss used to quantify effect of the solenoid on orbit, optics and linear coupling
- Xsuite match module used to implement **local corrections of these effects**
  - → Described in this [interactive notebook]()

Orbit after correction



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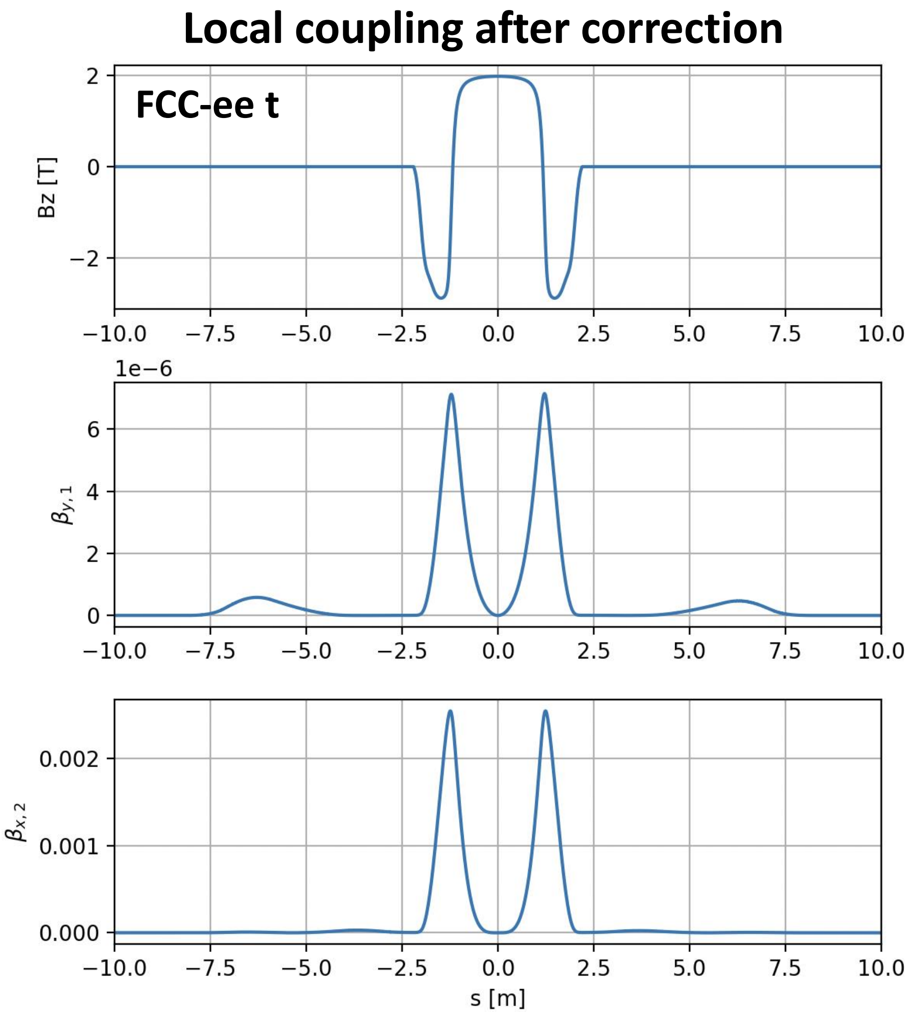
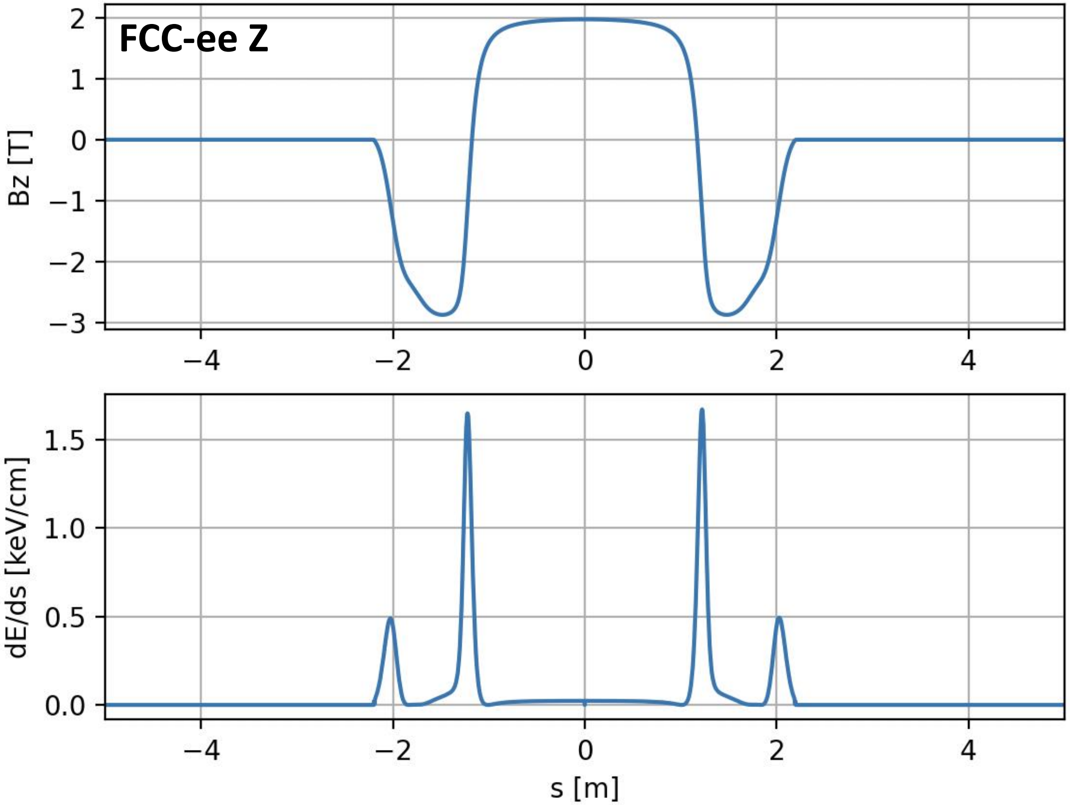


Solenoid in FCC-ee lattice

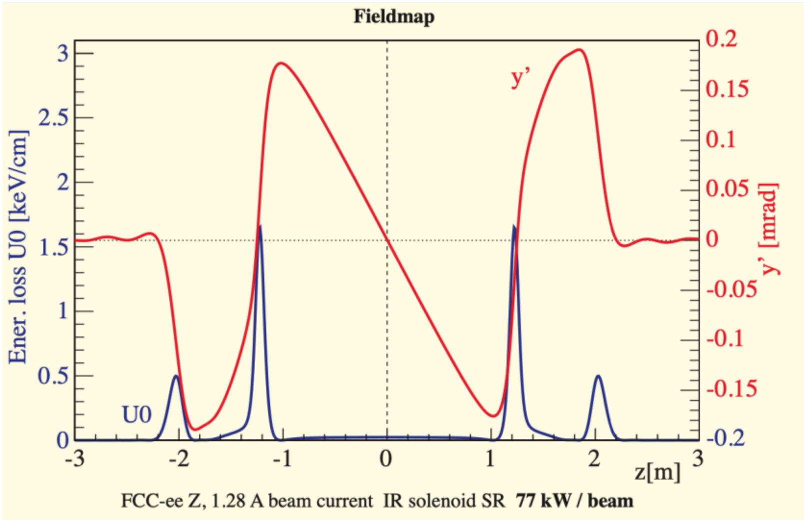
Prepared **Xsuite model of the FCC-ee ring including the experimental solenoid**

- **Synchrotron radiation** power distribution **checked against past computations**

Bz profile prepared by H. Burkhardt (map from M. Koratsinos)

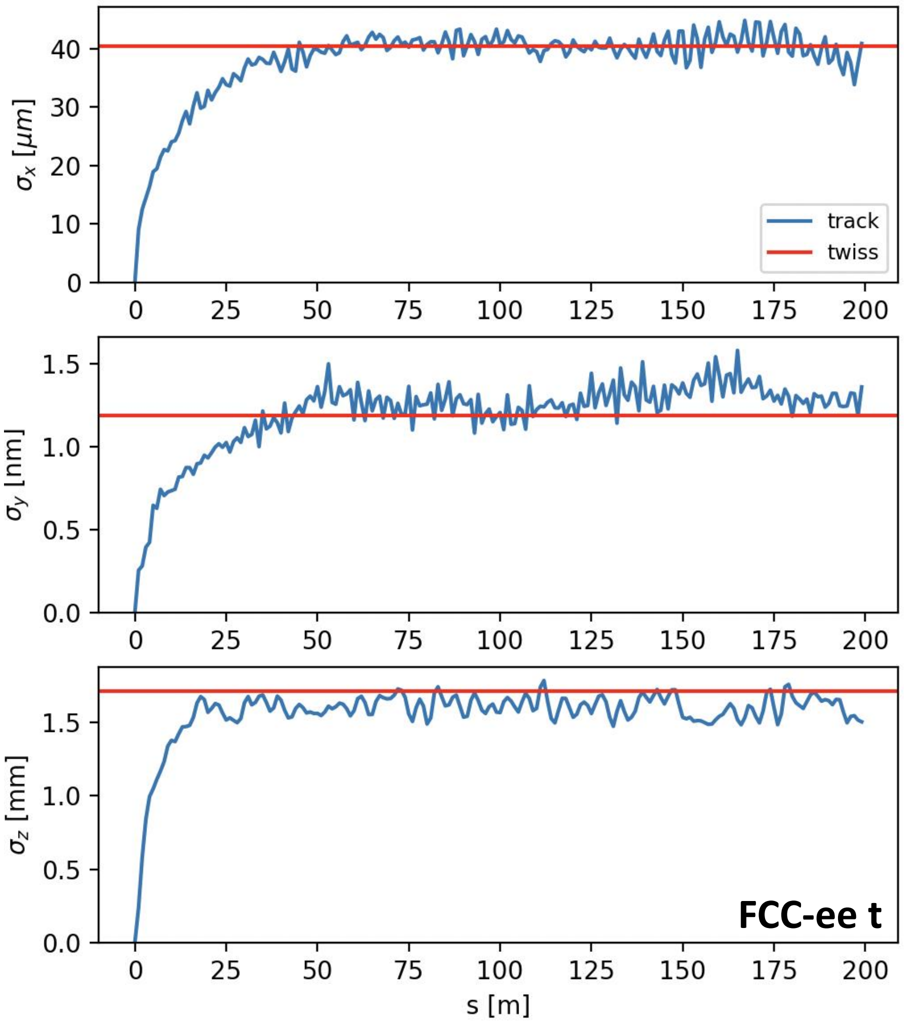


H. Burkhardt, FCC week 2022



Prepared **Xsuite model of the FCC-ee ring including the experimental solenoid**

- Checked **consistency between twiss computation and tracking** in measuring the effect of the solenoid on the equilibrium emittance.



- Xsuite solenoid elements can be used to **model experimental solenoid with arbitrary $B_s(s)$** profiles
  - The effect of **the radial field component is included** in the computation (this is automatic when tracking canonic variables)
  - **Benchmarks conducted against a time-domain tracking** through an analytic field map showed **excellent agreement**
- Effect of **synchrotron radiation included** in the computation
  - Validated against time-domain tracking and against past studies
- As the modelling does not rely on any small-angle approximation, the solenoid **tilt can be handled by suitable reference frame transformations** at entry and exit

Prepared **Xsuite model of the FCC-ee ring including the experimental solenoid**

- Xsuite match module used to implement **local corrections of orbit and optics perturbations introduced by the solenoid**
- Checked **consistency between twiss computation and tracking** in measuring the effect of the solenoid on the equilibrium emittance

Thanks for your attention