Obtaining the TMD Soft function and CS kernel from lattice QCD with the auxiliary field representation of the Wilson line

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with

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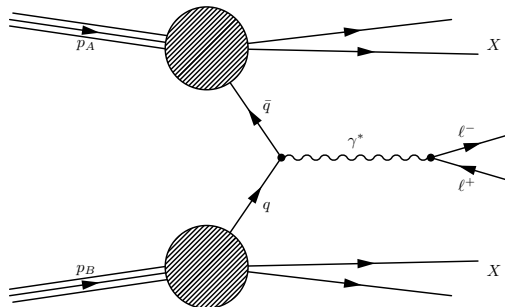**2024 Meeting on Lattice Parton Physics from Large-Momentum Effective Theory**

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Outline

1 **Motivation**

2 Our approach

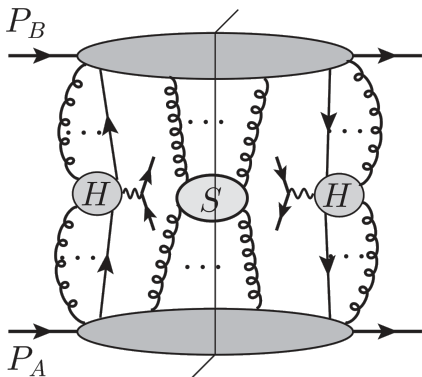**3** Computational setup

4 Results so far

Drell-Yan process

- $\Lambda_{\mathrm{QCD}} \lesssim |\vec{q}_\perp| \ll Q$

- Collins-Soper (CS) scale ζ_a, ζ_b, where $\zeta_a \zeta_b = Q^4$

- Rapidity divergences

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q \mathrm{d}Y \mathrm{d}^2 \vec{q}_\perp} = \sum_{i,j} H_{ij}\left(Q^2, \mu\right) \int \mathrm{d}^2 \vec{b}_\perp e^{i\vec{b}_\perp \cdot \vec{q}_\perp} f_i\left(x_a, \vec{b}_\perp, \mu, \zeta_a\right) f_j\left(x_b, \vec{b}_\perp, \mu, \zeta_b\right)$$
$$\times \left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$
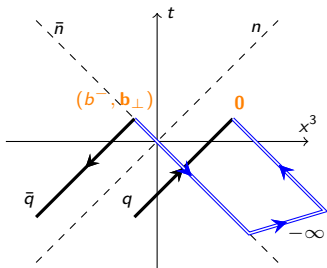
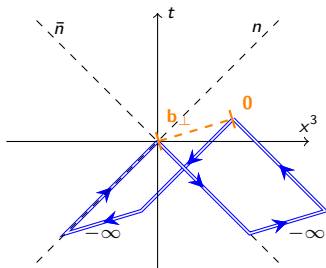# Factorization

Drell-Yan leading region [Collins, 2011]

- Leading region has contribution from soft momentum states

- Need to regulate rapidity divergences present in beam and soft functions

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}(Q,\mu) \int \mathrm{d}^2\vec{b}_\perp\, e^{i\vec{b}_\perp \cdot \vec{q}_\perp} B_i\left(x_a, \vec{b}_\perp, \mu, \frac{\zeta_a}{\nu^2}\right) B_j\left(x_b, \vec{b}_\perp, \mu, \frac{\zeta_b}{\nu^2}\right)$$
$$\times\, S_i(b_\perp, \mu, \nu)\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

Beam and soft function
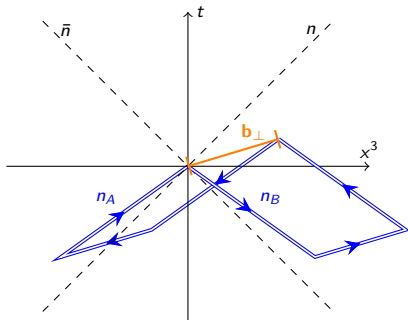

Staple gauge link for beam function

Wilson loop for soft function

$$B_i^{0(u)} \left(x, \vec{b}_\perp, \epsilon, \tau, xP^+ \right) = \int \frac{\mathrm{d}b^-}{2\pi} e^{-ib^-(xP^+)} \langle P | \, \bar{\psi}_i^0 \left(b^-, \vec{b}_\perp \right) W_{\bar{n}} \left(b^-, \vec{b}_\perp; -\infty, 0 \right)$$

$$\times \; W_\perp \left(-\infty \bar{n}; 0, b_\perp \right) W_{\bar{n}} \left(0; 0, -\infty \right) \frac{\gamma^+}{2} \psi_i^0 \left(0 \right) | P \rangle$$

$$S^0 \left(b_\perp, \epsilon, \tau \right) = \frac{1}{N_c} \langle 0 | \, W_n \left(b_\perp; 0, -\infty \right) W_{\bar{n}} \left(b_\perp; -\infty, 0 \right) W_\perp \left(-\infty \bar{n}; 0, b_\perp \right)$$

$$\times \; W_{\bar{n}} \left(0; 0, -\infty \right) W_n \left(0; -\infty, 0 \right) W_\perp \left(-\infty n; b_\perp, 0 \right) | 0 \rangle$$

Off lightcone regulator

- Collins scheme, spacelike Wilson lines:

$$n_A \equiv n - e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} - e^{+y_B}n$$

$$f_i(x_b, \vec{b}_\perp, \mu, \zeta_b) = \lim_{\epsilon \to 0} Z^i_{\mathrm{UV}}\left(\mu, \epsilon, \zeta_b\right) \lim_{\substack{y_A \to +\infty \\ y_B \to -\infty}} B_i\left(x_b, \vec{b}_\perp, \epsilon, y_B, xP^+\right)$$

$$\times \sqrt{\frac{S_i\left(b_\perp, \epsilon, y_A - y_n\right)}{S_i\left(b_\perp, \epsilon, y_A - y_B\right) S_i\left(b_\perp, \epsilon, y_n - y_B\right)}}$$

Collins-Soper kernel

- Collins-Soper (CS) kernel governs the rapidity evolution of the TMDPDF

$$K\left(b_\perp, \mu\right) = \frac{\mathrm{d}f_q\left(x, b_\perp, \mu, \zeta\right)}{\mathrm{d}\log\zeta}$$

- Can be obtained from the soft function

$$S_q\left(b_\perp, y_A, y_B, \mu\right) \overset{y_A \to +\infty}{\underset{y_B \to -\infty}{=}} S_I\left(b_\perp, \mu\right) e^{2K(b_\perp, \mu)(y_A - y_B)}$$

- By direct computation:

$$K\left(b_\perp, \mu\right) = \lim_{\substack{y_A \to +\infty \\ y_B \to -\infty}} \frac{1}{2} \frac{\partial}{\partial y_n} \log\left(\frac{S_q\left(b_\perp, y_n, y_B, \mu\right)}{S_q\left(b_\perp, y_A, y_n, \mu\right)}\right)$$

Lattice extraction of TMDPDFs

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = C_q\left(x\tilde{P}^z, \mu\right) \exp\left[\frac{1}{2}\gamma_\zeta^q\left(\mu, b_\perp\right)\log\frac{\tilde{\zeta}}{\zeta}\right] f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)$$

$$\times \left\{1 + \mathcal{O}\left(\frac{1}{(x\tilde{P}^z b_\perp)^2}, \frac{\Lambda_{\text{QCD}}^2}{(x\tilde{P}^z)^2}\right)\right\}$$

[Ebert, *et. al.*, 2019], [Ebert, *et. al*, 2022]

- C_q is a perturbatively calculable matching kernel

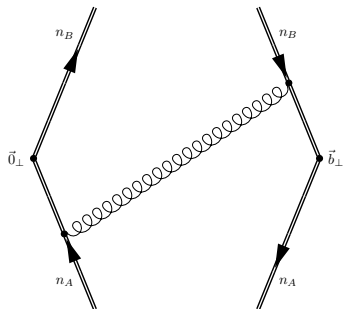- $\tilde{\zeta} = x^2 m_h^2 e^{2(y_{\tilde{P}} + y_B - y_n)}$

quasi-TMDPDF

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = \tilde{f}_q^{\text{naive}}\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) \sqrt{\frac{\tilde{S}_q^{\text{naive}}\left(b_\perp, \mu\right)}{S_q\left(b_\perp, \mu, 2y_n, 2y_B\right)}}$$

- $\tilde{f}_q^{\text{naive}}$ and $\tilde{S}_q^{\text{naive}}$ are lattice calculable objects

- S_q is the Collins soft function

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2. Our approach

3. Computational setup

4. Results so far

Euclidean space calculation

Define Euclidean space Wilson line directions as:

$$\tilde{n}_A = (in_A^0, \vec{0}_\perp, n_A^3), \qquad \tilde{n}_B = (in_B^0, \vec{0}_\perp, -n_B^3)$$

$$r_a \equiv \frac{n_A^3}{n_A^0}, \qquad r_b \equiv \frac{n_B^3}{n_B^0}$$

$$S_A^E(b_\perp, \epsilon, r_a, r_b) = g^2 C_F(\tilde{n}_A \cdot \tilde{n}_B) \int_{-\infty}^0 \mathrm{d}s \int_{-\infty}^0 \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b + s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2}$$

Coordinate space

Perform integration in coordinate space:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2} = \int_0^\infty \mathrm{d}u \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-uk^2} e^{-(b+s\tilde{n}_A - t\tilde{n}_B)^2/4u}$$

$$= \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}} \frac{1}{((b+s\tilde{n}_A - t\tilde{n}_B)^2/4)^{d/2-1}}$$
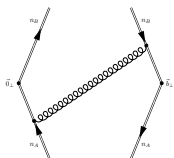
'u' integral only valid for

$$(s\tilde{n}_A - t\tilde{n}_B)^2 = s^2((n_A^3)^2 - (n_A^0)^2) + t^2((n_B^3)^2 - (n_B^0)^2) + st(n_A^3 n_B^3 + n_A^0 n_B^0) > 0$$
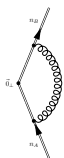
Euclidean space integral only finite when:

$$|n_A^3| > |n_A^0|, \quad |n_B^3| > |n_B^0|, \quad n_A^3 n_B^3 + n_A^0 n_B^0 > 0$$

$$\rightarrow \quad |r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$$

Soft function in Euclidean space at one loop

A   B   C   D

$$S^{(1)}(b_\perp, \epsilon, r_a, r_b)$$
$$= \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right) \right) \left\{ 2 + \log\left| \frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)} \right| \frac{r_a r_b + 1}{r_a + r_b} \right\}$$

$$r_a = \frac{1 + e^{-2y_A}}{1 - e^{-2y_A}}, \quad r_b = \frac{1 + e^{2y_B}}{1 - e^{2y_B}}$$

Finite L Wilson lines

For $L \to \infty$ and $r_a, r_b \to 1$:

$$S(b_\perp, a, r_a, r_b, L) = 1 + \frac{\alpha_s C_F}{2\pi} \left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)} \log\left(\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)} \right) \right) \log\left(\frac{b_\perp^2}{a^2} \right)$$

$$+ \frac{\alpha_s C_F}{2\pi} \left\{ -4 \log\left(\frac{b_\perp^2}{a^2} \right) + 2 \frac{\pi b_\perp}{a} + 2 \frac{\pi (|n_A| + |n_B|) L}{b_\perp} \right.$$

$$- 2 \frac{\pi (|n_A| + |n_B|) L}{a}$$

$$\left. + 2 \frac{b_\perp^2}{L^2} \left(C_1 - \frac{1}{3} \right) \right\} + \mathcal{O}\left(\frac{b_\perp^4}{L^4}, \alpha_s^2 \right)$$

$$C_1 = 1 - \frac{1}{2} \frac{1}{b_0^2 (r_b^2 - 1)} - \frac{1}{2} \frac{1}{a_0^2 (r_a^2 - 1)} \implies \frac{b_\perp^2}{L^2} \ll r_{a,b} - 1$$

- Incorrect $b_\perp$ dependence

- Linear divergence in L

- Power corrections are limited by $r_{a,b}$

Ratio

$$S_{\text{ratio}}\left(b_\perp, a, r_a, r_b, L\right)$$

$$= \frac{S\left(b_\perp, a, r_a, r_b, L\right)}{\sqrt{S\left(b_\perp, a, r_a, -r_a, L\right) S\left(b_\perp, a, -r_b, r_b, L\right)}} \qquad \text{[Ji, Liu, Liu 2020]}$$

$$= 1 + \frac{\alpha_s C_F}{2\pi}\left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)}\log\left(\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)}\right)\right)\log\left(\frac{b_\perp^2}{a^2}\right)$$

$$+ \mathcal{O}\left(\frac{b_\perp^4}{L^4}, \alpha_s^2\right)$$

- Recover correct $b_\perp$ dependence

- Linear divergence removed

- Power corrections now start are no longer limited by $r_{a,b}$

- Approaches the soft function at $L \to \infty$

Connection to Minkowski space

- At large lattice time:

$$S_{\text{ratio}}\left(b_\perp, \tilde{n}_A, \tilde{n}_B, a, \tau\right) = S_{\text{lat}}\left(b_\perp, r_a, r_b, a\right) + \mathcal{O}\left(\frac{b_\perp^4}{\tau^4}\right)$$

- Direct mapping to rapidity variables in Collins' scheme:

$$r_a = \frac{n_A^3}{n_A^0} = \frac{1 + e^{-2y_A}}{1 - e^{-2y_A}}, \quad r_b = \frac{n_B^3}{n_B^0} = \frac{1 + e^{2y_B}}{1 - e^{2y_B}}$$

$$|r_a|, |r_b| > 1, \ n_A^0 n_B^0 (r_a r_b + 1) > 0$$

- Construct matching between lattice and continuum renormalization schemes

$$S\left(b_\perp, y_A, y_B, \mu\right) = C(r_a, r_b, \mu, a) \times S_{\text{lat}}\left(b_\perp, r_a, r_b, a\right)$$

- Obtain CS kernel from:

$$S\left(b_\perp, y_A, y_B, \mu\right) = S_I\left(b_\perp, \mu\right) e^{2K(b_\perp, \mu)(y_A - y_B)}$$

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Auxiliary field definition of the Wilson line

Write Wilson line in terms of one dimensional 'fermions' that live along the path:

$$
P \exp \left\{ -ig \int_{s_i}^{s_f} \mathrm{d}s n^\mu A_\mu(y(s)) \right\}
$$

$$
= Z_\psi^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, \psi\bar{\psi} \exp \left\{ ig \int_{s_i}^{s_f} \mathrm{d}s \bar{\psi} i \partial_s \psi - \bar{\psi} n \cdot A \psi \right\}
$$

[Gervais, Nevau 1980], [Aref'eva 1980]

Auxiliary field propagator:

$$
in \cdot D H_n(y) = \delta(y) \overset{\text{Euclidean space}}{\longrightarrow} -i\tilde{n} \cdot D_E H_{\tilde{n}}(y) = \delta(y), \qquad \tilde{n} = (in_0, \vec{n})
$$

Meaningful solution only obtained with a UV cutoff due to negative energy states [Aglietti, _et. al._ 1992], [Agglietti, 1994]

Auxiliary field propagator in Euclidean space

$$K(\tau) = \left(1 - \frac{H_0|_\tau}{2n}\right)^n U_4^\dagger(\tau - 1) \left(1 - \frac{H_0|_{\tau-1}}{2n}\right)^n$$

$$H_0 \psi(x) = -iv \cdot \Delta^\pm \psi(x) = -\sum_\mu \frac{iv_\mu}{2} \left[U_\mu(x)\psi(x+\hat{\mu}) - U_{-\mu}(x)\psi(x-\hat{\mu})\right]$$

$$G(\mathbf{x}, \tau, \mathbf{x}', \tau') = K(\tau) G(\mathbf{x}, \tau - 1, \mathbf{x}', \tau')$$

For $n \to \infty$ and $a \to 0$, this will produced the same continuum equation as before.

We get correction terms of $\mathcal{O}\left(a^2, \frac{a^2}{n}\right)$

[Horgan, *et. al.*, 2009]

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Details of lattice calculation

- Using $N_f = 2 + 1$ flavor PACS-CS configurations

- non-perturbatively $\mathcal{O}(a)$-improved Wilson quark action and Iwasaki gauge action

- $32^3 \times 64$ lattice with $a = 0.0907(13)$ fm

- 400 configurations

- thyp2 smearing

- Up to 64 sources per configuration

- Using GPT/GRID

[PACS-CS '09, '11]

Numerator and denominator factors

- At large τ, we expect

$$S_{\mathrm{num}} \stackrel{\tau \to \infty}{\sim} e^{2\tau(r_a+r_b)/a}/\tau^4$$

$$S_A \stackrel{\tau \to \infty}{\sim} e^{4(\tau r_a - iz)/a}/\tau^4$$

$$S_B \stackrel{\tau \to \infty}{\sim} e^{4(\tau r_b + iz)/a}/\tau^4$$

exponential increase related to negative energy states

- We expect to see real and imaginary contributions to the denominator factors

- Combined denominator factor $\sqrt{S_A S_B}$, should be purely real

- Exponential τ dependence should cancel in ratio, giving plateau

$$S_{\mathrm{ratio}} = \frac{S_{\mathrm{num}}}{\sqrt{S_A S_B}} \stackrel{\tau \to \infty}{\sim} 1$$

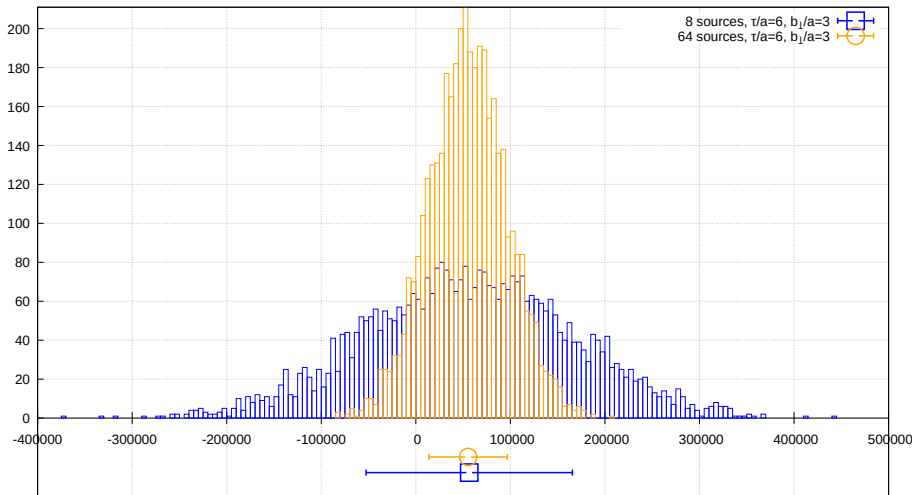# Numerator, real part
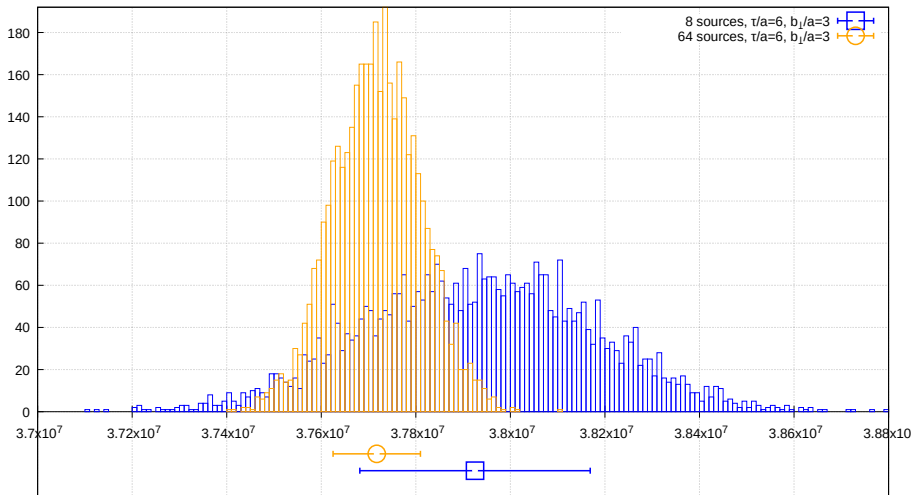
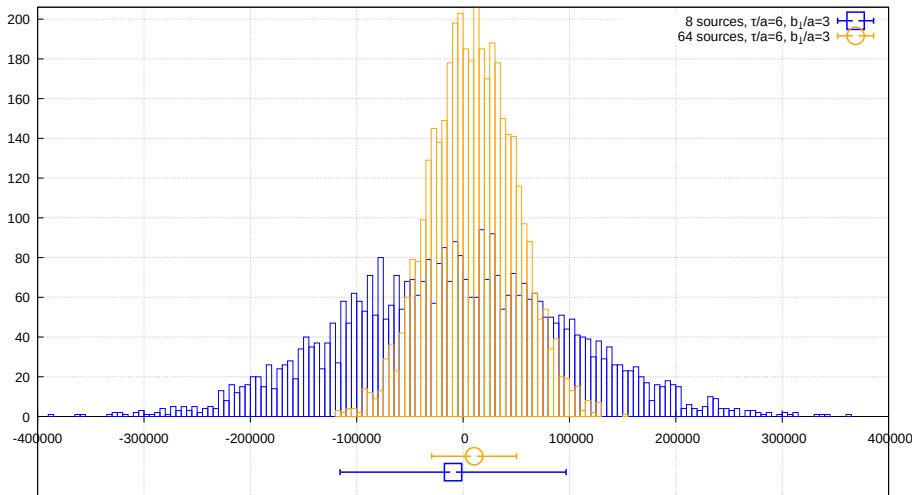# Numerator, imaginary part
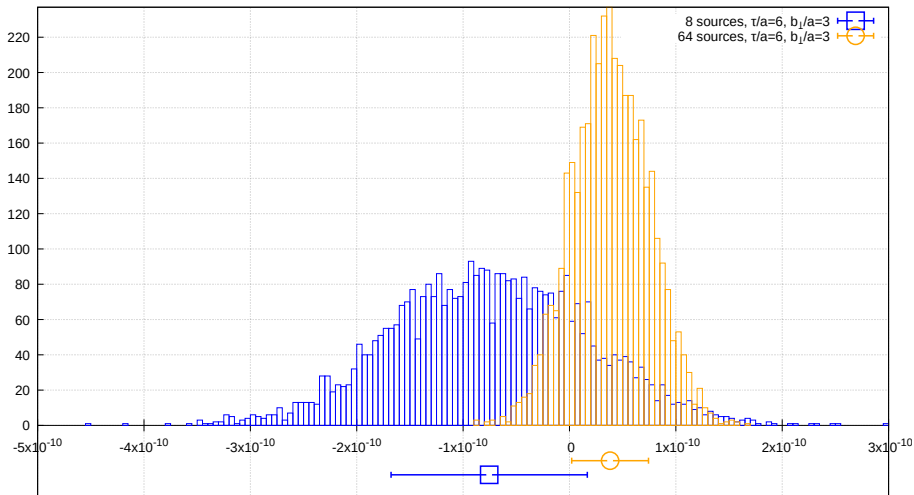
Denominator A, real part
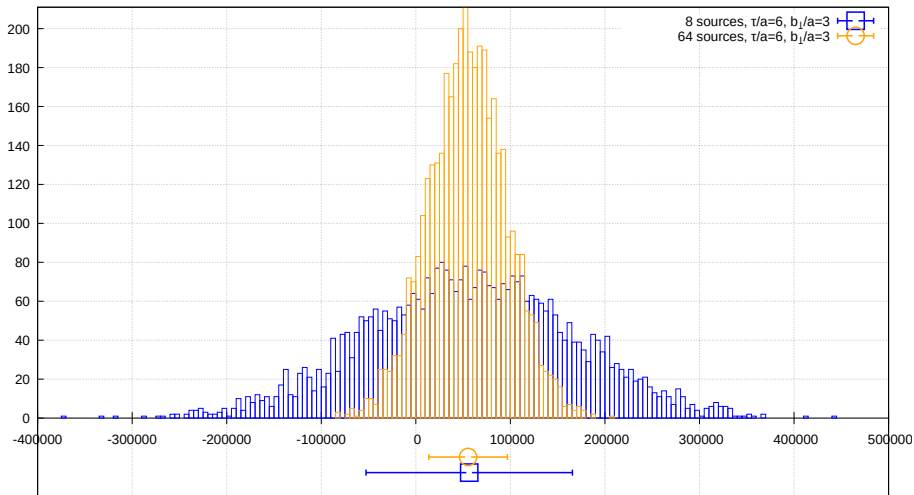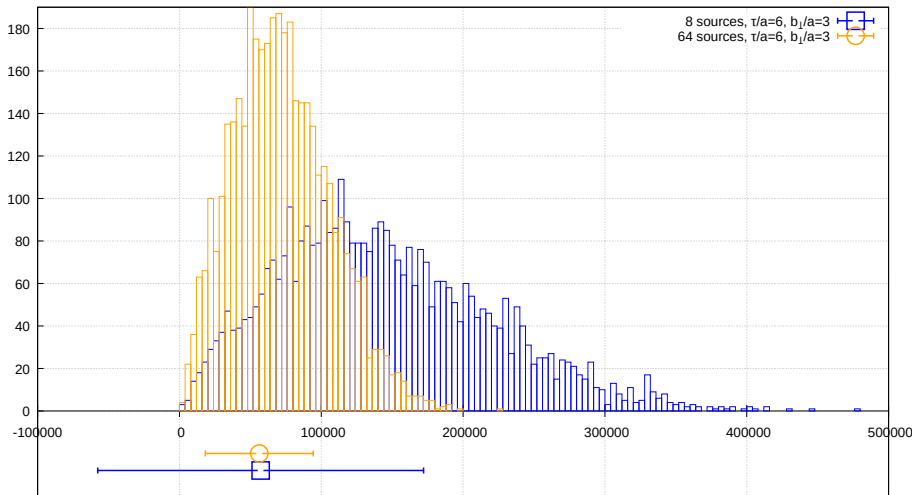
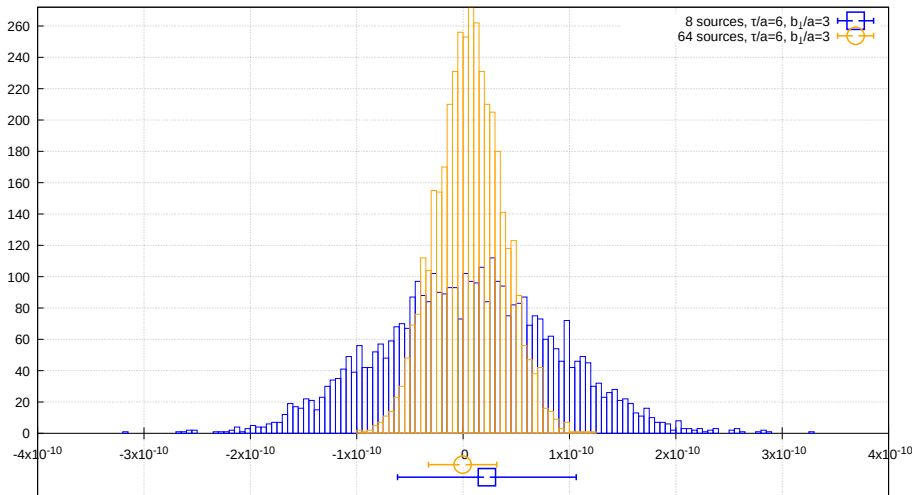# Denominator A, imaginary part
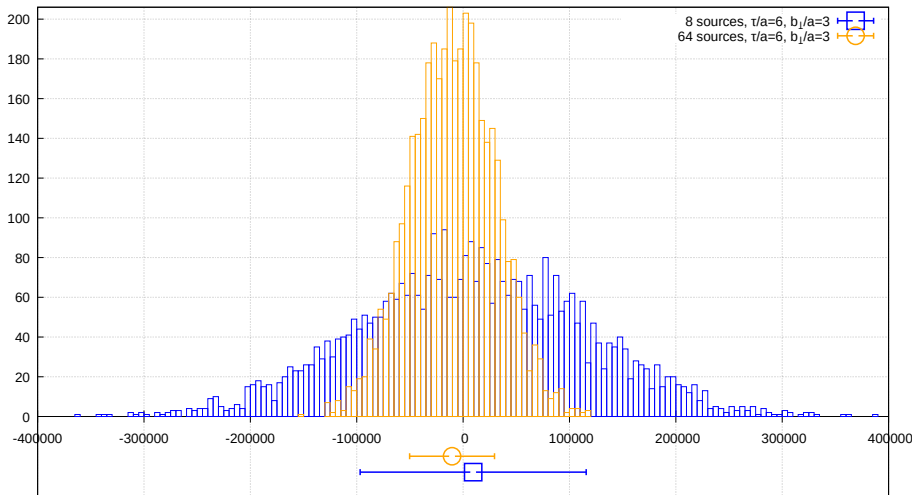
Denominator B, real part
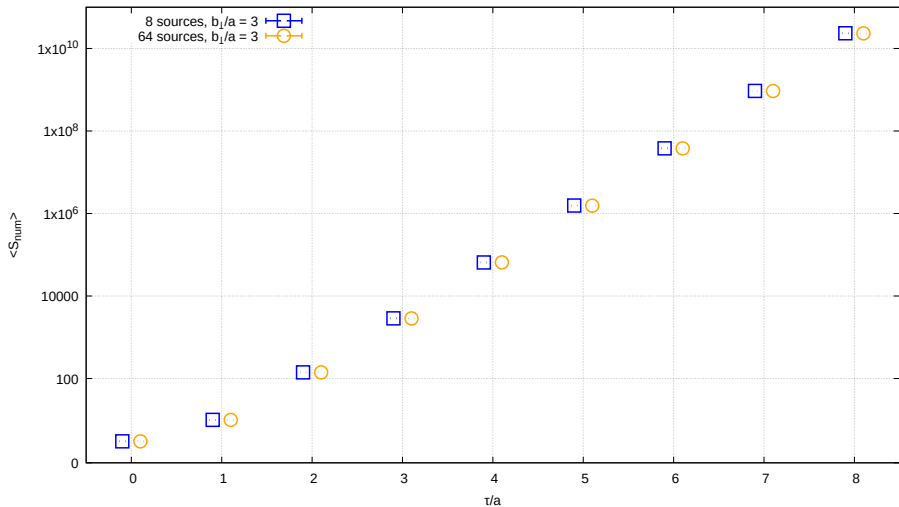
Denominator B, imaginary part
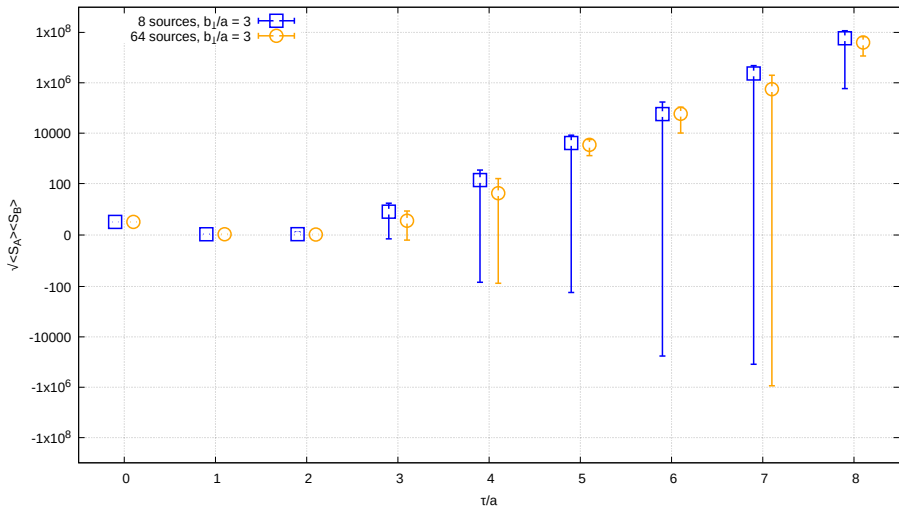
Full denominator, real part

Full denominator, imaginary part

Numerator, τ dependence

Full denominator, τ dependence

Conclusion and outlook

- Important to have multiple methods for computing the soft function

- Euclidean space calculation of soft function has a direct mapping to Minkowski space result

- Preliminary numerical investigation shows a decrease in error with increased statistics

- Expect to see a signal with ~ 1000 sources

Thank you!

Group members

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