Threshold Resummation under Large Momentum Expansion

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Theoretical papers: Xiangdong Ji, Yizhuang Liu, Yushan Su, JHEP 08 (2023) 037
Yizhuang Liu, Yushan Su, JHEP 2024 (2024) 204
Xiangdong Ji, arXiv: 2408.03378
Numerical paper: Xiangdong Ji, Yizhuang Liu, Yushan Su, Rui Zhang, in progress

Outline

1. Large Momentum Expansion
2. Threshold Resummation
3. Numerical Tests with Pion PDF

Large momentum expansion to calculate light-cone PDF $f(x,\mu)$ from quasi-PDF $\tilde{f}(y,P^z,\mu)$,

$$f(x,\mu)=\int\frac{dy}{|y|}C\left(\frac{x}{y},\frac{\mu}{|y|P^z}\right)\tilde{f}(y,P^z,\mu)+O\left[\frac{\Lambda_{\mathrm{QCD}}^2}{x^2P_z^2},\frac{\Lambda_{\mathrm{QCD}}^2}{(1-x)^2P_z^2}\right]$$

The reasoning to obtain it has been shown

Prof. Xiangdong Ji's paper in memory of Weinberg: X. Ji, arXiv: 2408.03378

- The DIS structure function can be factorized with light-cone PDF;
- The DIS structure function can be factorized with quasi-PDF;
- So the light-cone PDF and quasi-PDF has the same infrared structure. Thus an effective theory to calculate light-cone PDF from quasi-PDF can be established;
- The matching coefficient C absorbs the UV difference between $f(x,\mu)$ and $\tilde{f}(y,P^z,\mu)$, which is perturbatively calculable.

e.g. The one-loop result of C can be obtained by comparing perturbative light-cone and quasi PDFs in a massless free quark state

$$\tilde{f}^{(1)}\left(y,\frac{p^z}{\mu},\frac{1}{\epsilon_{\rm IR}}\right)=\begin{cases}\left(\frac{1+y^2}{1-y}\ln\frac{y}{y-1}+1+\frac{3}{2y}\right)_{+(1)}^{[1,\infty]}-\frac{3}{2y} & y>1\\ \left(\frac{1+y^2}{1-y}\left[-\frac{1}{\epsilon_{\rm IR}}-\ln\frac{\mu^2}{4(p^z)^2}+\ln(y(1-y))\right]-\frac{y(1+y)}{1-y}\right)_{+(1)}^{[0,1]} & 0<y<1\\ \left(-\frac{1+y^2}{1-y}\ln\frac{-y}{1-y}-1+\frac{3}{2(1-y)}\right)_{+(1)}^{[-\infty,0]}-\frac{3}{2(1-y)} & y<0\end{cases}$$

$$f^{(1)}\left(x,\frac{1}{\epsilon_{\rm IR}}\right)=\frac{(-1)}{\epsilon_{\rm IR}}\left(\frac{1+x^2}{1-x}\right)_{+(1)}^{[0,1]}\theta(x)\theta(1-x)$$

$$\tilde{C}\left(\xi,\frac{p^z}{\mu}\right)=\delta(1-\xi)+\frac{\alpha(\mu)C_F}{2\pi}\left[f^{(1)}\left(\xi,\frac{1}{\epsilon_{\rm IR}}\right)-\tilde{f}^{(1)}\left(\xi,\frac{p^z}{\mu},\frac{1}{\epsilon_{\rm IR}}\right)\right]$$

X. Xiong et al., PRD (2014)
T. Izubuchi et al., PRD (2018)
W. Wang et al., PRD (2019)
Z. Li et al., PRL (2021)
L. Chen et al., PRL (2021)
X. Ji et al., EMP (2021)

- IR divergences are cancelled, which provides 1-loop validation
- There are **threshold logs**

The effects of the threshold logs should be important in the large $x \to 1$ region

$$C\left(\xi, \frac{\mu}{p^z}\right) \sim \alpha_s^n \frac{\ln^k\left(\frac{1-\xi}{\xi}\right)}{1-\xi}$$

$$f(x,\mu) = \int \frac{dy}{|y|}\, C\left(\frac{x}{y}, \frac{\mu}{|y|P^z}\right)_+ \tilde{f}(y, P^z, \mu)$$

$$= \int_{-\infty}^{x} \frac{dy}{|y|}\, C\left(\frac{x}{y}, \frac{\mu}{|y|P^z}\right) \tilde{f}(y, P^z, \mu) - \int_1^{+\infty} d\xi'\, C\left(\xi', \frac{\mu}{|x|P^z}\right) \tilde{f}(x, P^z, \mu)$$

$$\underbrace{+\int_1^{+\infty} \frac{dy}{|y|}\, C\left(\frac{x}{y}, \frac{\mu}{|y|P^z}\right) \tilde{f}(y, P^z, \mu) + \int_x^1 \frac{dy}{|y|}\, C\left(\frac{x}{y}, \frac{\mu}{|y|P^z}\right) \tilde{f}(y, P^z, \mu)}_{\text{real}} - \underbrace{\int_{-\infty}^{1} d\xi'\, C\left(\xi', \frac{\mu}{|x|P^z}\right) \tilde{f}(x, P^z, \mu)}_{\text{virtual}}$$

- Focus on the integral region that $y > x$
- The quasi-PDF is numerically small for $y > 1$ at large momentum P^z
- The integral range $x < y < 1$ is limited for $x \to 1$
- When $x \to 1$, the real part of the integral is numerically small compared to the virtual part, where the threshold logs become important.
- A factorization is needed to resume the threshold logs

Factorization for light-cone-PDF $f(x,\mu)|_{x\to 1}$

- The IR physics is factorized into collinear and soft-collinear parts

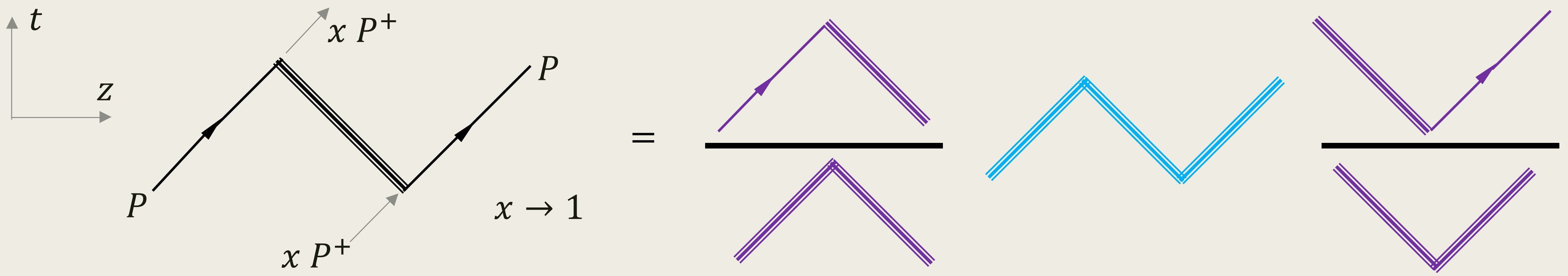
$\sim P^+(1,\lambda^2,\lambda)$, where $\lambda P^+ \sim \Lambda_{QCD}$ $\qquad$ $\sim P^+(\epsilon,\lambda^2,\sqrt{\epsilon}\lambda)$, where $\epsilon = (1-x)$ $\qquad$ Require $1 \gg \epsilon \gg \lambda$

$$f(x,\mu)\Big|_{x\to 1} = \mathcal{J}^2\left(\frac{P\cdot\bar{n}}{\mu}\right)\int\frac{d\xi}{2\pi}e^{-i(x-1)P\cdot\bar{n}\,\xi}\mathcal{S}(\xi\mu)$$

where $f(x,\mu)|_{x\to 1} = \int\frac{d\lambda}{2\pi}e^{-i\,x\,P\cdot\bar{n}\,\xi}\langle P|\bar{\psi}(\xi\bar{n})W_{\bar{n}}(\xi\bar{n},0)\gamma^+\psi(0)|P\rangle|_{x\to 1}$, $\mathcal{J}\left(\frac{P\cdot\bar{n}}{\mu}\right) = \frac{\langle\Omega|W_{\bar{n}}[-\infty\bar{n},0]\psi(0)|P\rangle}{\langle\Omega|W_{\bar{n}}[-\infty\bar{n},0]W_n[0,-\infty n]|\Omega\rangle}$

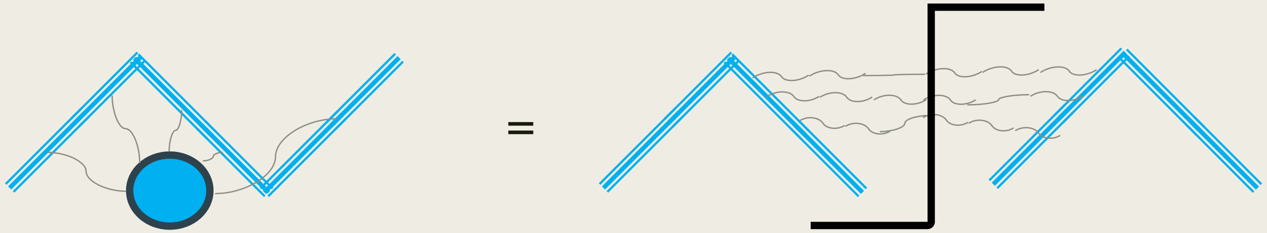
$$\int\frac{d\xi}{2\pi}e^{-i(x-1)P\cdot\bar{n}\,\xi}\mathcal{S}(\xi\mu) = \int\frac{d\xi}{2\pi}e^{-i(x-1)P\cdot\bar{n}\,\xi}\langle\Omega|W_n(+\infty,\xi\bar{n})W_{\bar{n}}(\xi\bar{n},0)W_n(0,-\infty)|\Omega\rangle$$

$n = (1,0,0,1)/\sqrt{2}$
$\bar{n} = (1,0,0,-1)/\sqrt{2}$



- Derived in SCET language in T. Becher et al., JHEP (2007)
- A general reduced diagram analysis is shown in Ji et al., PLB (2005)

Threshold soft-collinear factor



- Sum up the soft-collinear gluons with total momentum $(1-x)P^+$

$$\int \frac{d\xi}{2\pi} e^{-i(x-1)P\cdot\bar{n}\,\xi} \langle\Omega|W_n(-\infty,\xi\bar{n})W_{\bar{n}}(\xi\bar{n},-\infty)W_{\bar{n}}(-\infty,0)W_n(0,-\infty)|\Omega\rangle$$

$$= \int \frac{d\xi}{2\pi} e^{-i(x-1)P\cdot\bar{n}\,\xi} \int \frac{d^3\vec{k}}{(2\pi)^3 2E_{\vec{k}}} \sum_n \langle\Omega|W_n(-\infty,0)W_{\bar{n}}(0,-\infty)e^{-i\,\hat{P}\cdot\xi\bar{n}}|n_k\rangle\langle n_k|W_{\bar{n}}(-\infty,0)W_n(0,-\infty)|\Omega\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2E_{\vec{k}}} \boldsymbol{\delta}[\boldsymbol{k}^+ - (\mathbf{1}-\boldsymbol{x})\boldsymbol{P}^+] \sum_n |\langle\Omega|W_n(-\infty,0)W_{\bar{n}}(0,-\infty)|n_k\rangle|^2$$

Factorization for quasi-PDF $\tilde{f}(y, P^z, \mu)|_{y\to 1}$

- The IR physics is factorized into collinear and soft-collinear parts

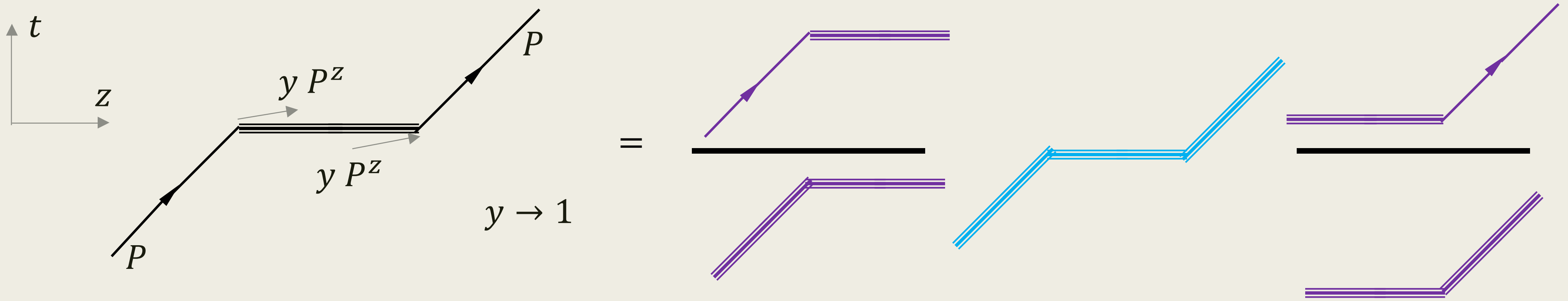
$\sim P^+(1, \lambda^2, \lambda)$, where $\lambda P^+ \sim \Lambda_{QCD}$

$\sim P^+(\epsilon, \lambda^2, \sqrt{\epsilon}\lambda)$, where $\epsilon = (1-x)$

$$\tilde{f}(y, P^z, \mu)\Big|_{y\to 1} = \int \frac{dz}{2\pi} e^{-i(y-1)P\cdot n_z z} \tilde{J}^2\left(\frac{P\cdot n_z\,\mathrm{sgn}[z]}{\mu}\right)\tilde{S}(z\mu)$$

Where $\tilde{f}(y, P^z, \mu)|_{y\to 1} = \int \frac{dz}{2\pi} e^{-i\,y\,P\cdot n_z z}\langle P|\bar{\psi}(zn_z)W_{n_z}(zn_z,0)\gamma^z\psi(0)|P\rangle|_{y\to 1}$, $\tilde{J}\left(\frac{P\cdot n_z}{\mu}\right) = \frac{\langle\Omega|W_{n_z}(+\infty n_z,0)\psi(0)|P\rangle}{\langle\Omega|W_{n_z}(+\infty n_z,0)W_n[0,-\infty n]|\Omega\rangle}$

$$\int \frac{dz}{2\pi} e^{-i(y-1)P\cdot n_z z}\tilde{S}(z) = \int \frac{dz}{2\pi} e^{-i(y-1)P\cdot n_z z}\langle\Omega|W_n(+\infty, zn_z)W_{n_z}(zn_z,0)W_n(0,-\infty)|\Omega\rangle \qquad n_z = (0,0,0,1)$$



- Verified up to one loop

- **Threshold factorization for quasi and light-cone PDFs:**

$$f(x,\mu)\Big|_{x\to 1} = \int \frac{d\lambda}{2\pi} e^{ix\lambda} e^{-i\lambda} \mathcal{J}^2\left(\frac{P\cdot\bar{n}\,\mathrm{sign}(\lambda)}{\mu}\right) \mathcal{S}\left(\frac{\lambda\,\mu}{-P^+}\right)$$

$$\tilde{f}(y,P^z,\mu)\Big|_{y\to 1} = \int \frac{d\lambda}{2\pi} e^{iy\lambda} e^{-i\lambda} \tilde{\mathcal{J}}^2\left(\frac{P\cdot n_z \mathrm{sign}(\lambda)}{\mu}\right) \tilde{\mathcal{S}}\left(\frac{\lambda\,\mu}{P^z}\right)$$

- **Matching relation:**

$$f(x,\mu)\Big|_{x\to 1} = \int \frac{dy}{|y|} \tilde{\mathcal{C}}\left(\frac{x}{y}, \frac{\mu}{|y|P^z}\right)_{\mathrm{sg}} \tilde{f}(y,P^z,\mu)\Big|_{y\to 1}$$

where

$$\mathcal{C}\left(\xi, \frac{\mu}{|y|P^z}\right)_{\mathrm{sg}} = \int \frac{d\lambda}{2\pi} e^{i\lambda(\xi-1)} \boxed{\frac{\mathcal{J}^2\left(\frac{P\cdot\bar{n}}{\mu}\right)}{\tilde{\mathcal{J}}^2\left(\frac{P\cdot n_z\,\mathrm{sign}(\lambda)}{\mu}\right)}} \boxed{\frac{\mathcal{S}\left(\frac{\lambda\,\mu}{-P^+}\right)}{\tilde{\mathcal{S}}\left(\frac{\lambda\,\mu}{P^z}\right)}}$$

$\mathcal{J}_r\left(L_{p_z}, \mathrm{sign}[\lambda], \alpha(\mu)\right)$ $\boldsymbol{\mathcal{S}_r(l_z, \alpha(\mu))}$

hard kernel **soft function**

$|x|P^z \gg |1-x|P^z \gg \Lambda_{\mathrm{QCD}}$

- The collinear and soft-collinear parts are **matched, respectively**. The RG equations of the matching coefficients are obtained separately.

Resummation

- The RG equations:

$$\frac{d}{d\ln\mu}\ln \mathcal{J}_r\left(L_{p_z}, \mathrm{sign}[\lambda], \alpha(\mu)\right) = -\Gamma_{\mathrm{cusp}}(\alpha)L_{p_z} - \tilde{\gamma}_H(\alpha) + i\ \mathrm{sign}[\lambda]\ \pi\ \Gamma_{\mathrm{cusp}}(\alpha)$$

$$\frac{d\ln \mathcal{S}_r(l_z, \alpha(\mu))}{d\ln\mu} = -\Gamma_{\mathrm{cusp}}(\alpha)l_z + \tilde{\gamma}_J(\alpha)$$

- Evolve separately: hard kernel $\mu_h \to \mu$, soft function $\mu_i \to \mu$

DGLAP logs in the threshold limit are resumed

Threshold logs are resumed

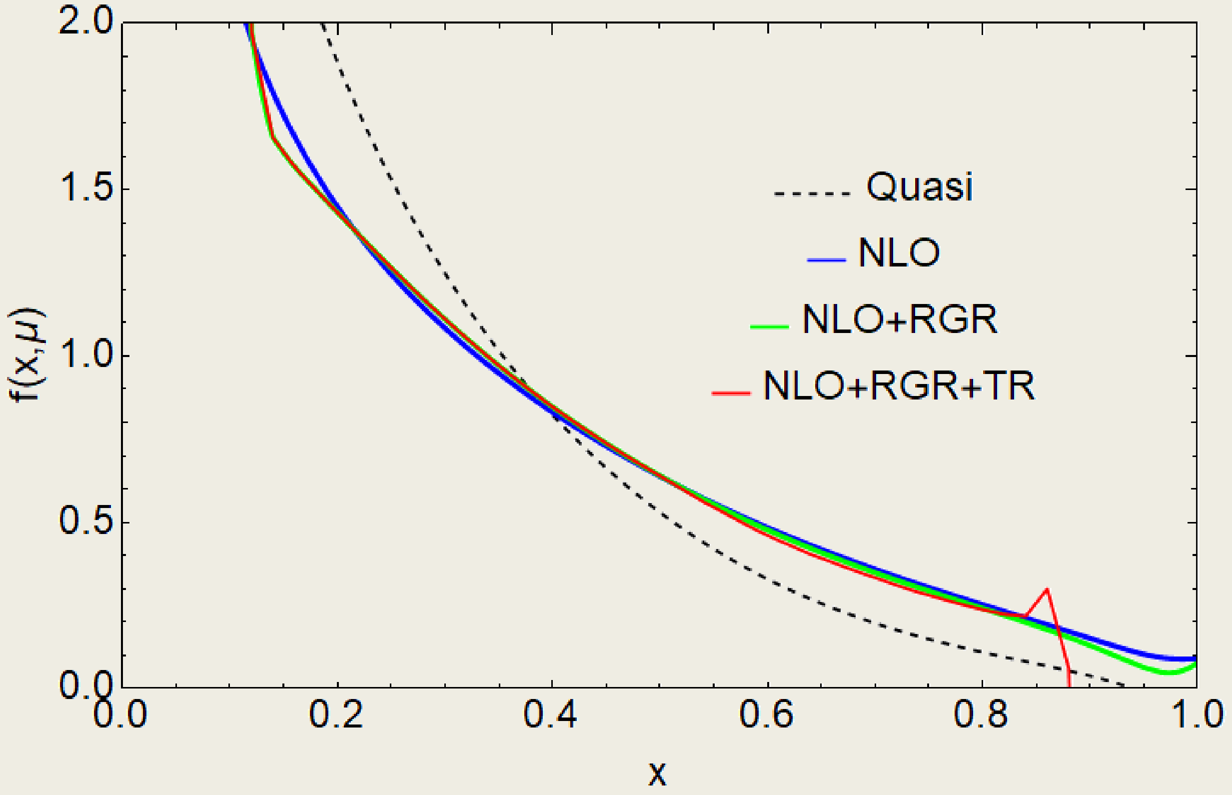
- RG resumed form

$$\tilde{\mathcal{C}}\left(\frac{x}{y}, \frac{\mu}{|y|P^z}, \mu_h, \mu_i\right)_{\mathrm{sg}} = \int \frac{dy'}{|y'|} S_r\left(\frac{x}{y'}, \frac{|y'|P^z}{\mu}, \mu_i\right) J_r\left(\frac{y'}{y}, \frac{|y|P^z}{\mu}, \mu_h\right)$$

$$S_r\left(\xi, \frac{p^z}{\mu}, \mu_i\right) = \exp[-2S(\mu_i,\mu) + a_J(\mu_i,\mu)]\mathcal{S}_r\left(l_z = -2\partial_\eta, \alpha(\mu_i)\right)\left[\frac{\sin\left(\frac{\eta\pi}{2}\right)}{|1-\xi|}\left(\frac{2|1-\xi||p^z|}{\mu_i}\right)^{\eta}\right]_* \frac{\Gamma(1-\eta)e^{-\eta\gamma_E}}{\pi}$$

$$J_r\left(\xi, \frac{p^z}{\mu}, \mu_h\right) = \exp[2S(\mu_h,\mu) - a_H(\mu_h,\mu)]\left(\frac{\mu_h}{2p^z}\right)^{2a_\Gamma(\mu_h,\mu)} |\mathcal{J}_r|\left(\ln\frac{4p_z^2}{\mu_h^2}, \alpha(\mu_h)\right)\left[\cos\left(\hat{A}(\mu_h,\mu)\right)\delta(1-\xi) + \frac{\sin\left(\hat{A}(\mu_h,\mu)\right)}{\pi(1-\xi)}\right]$$

Various perturbative matchings



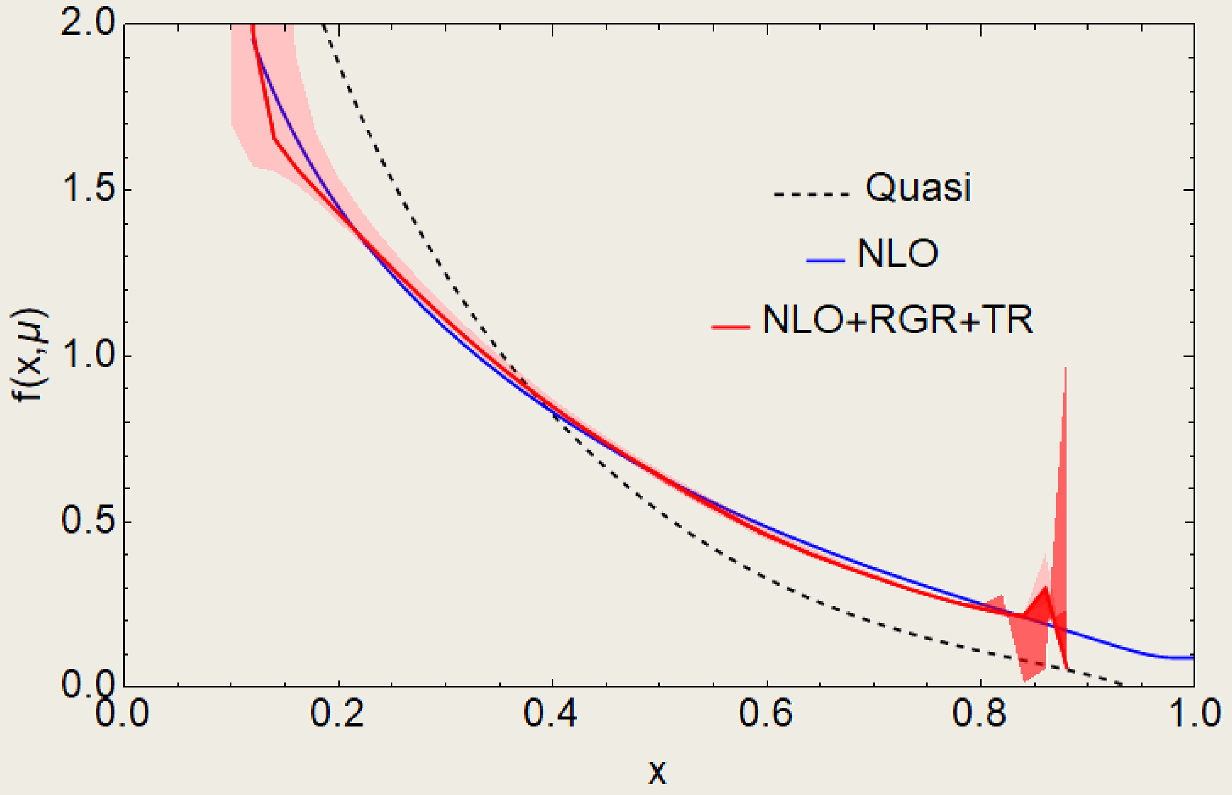
Pion valence quasi PDF is calculated based on the lattice QCD data from ANL/BNL collaboration
Gao et al., PRD (2020)
Gao et al., PRD (2021)
Gao et al., PRL (2022)

Threshold resummation influences large *x*

- NLO: fixed order matching
- NLO+RGR: DGLAP logs resummation
- NLO+RGR+TR: DGLAP and threshold logs resummation

$\mu_i = 2|1-x|P^z$, $\mu_h = 2|x|P^z$,
$P^z = 1.94\ \text{GeV}, \mu = 2\ \text{GeV}, z_s = 0.12\ \text{fm}$

Uncertainties from scale variation



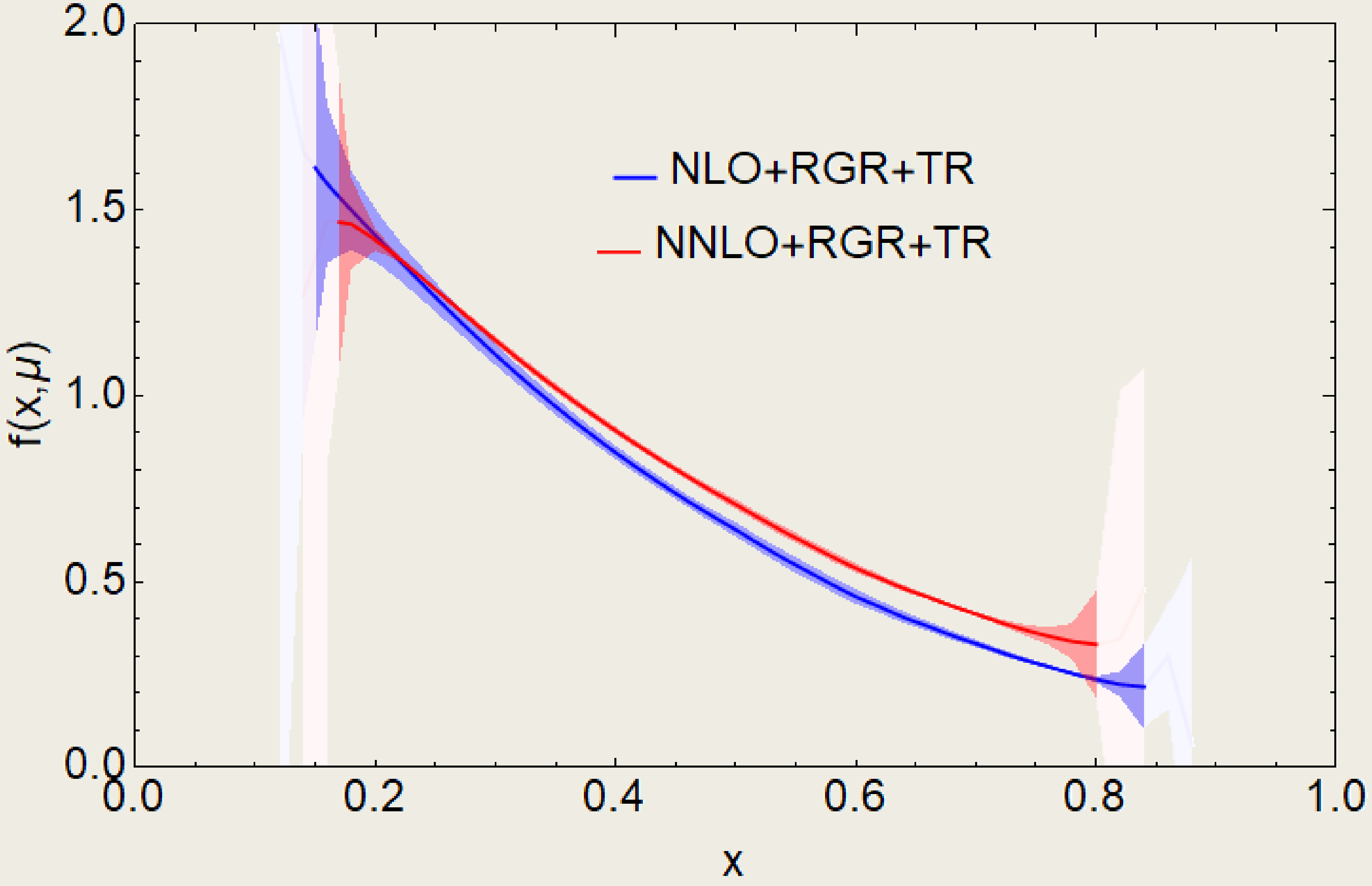
Semi-hard scale appears in the running coupling

$\sim \alpha(2|1-x|P^z)$

Uncertainties at large x are exposed

- Hard scale: (0.8~1.2)*2$|x_i|P^z$
- Semi-hard scale: (0.8~1.2)*2$|1-x_i|P^z$
- $P^z = 1.94\ \mathrm{GeV}, \mu = 2\ \mathrm{GeV}, z_s = 0.12\ \mathrm{fm}$

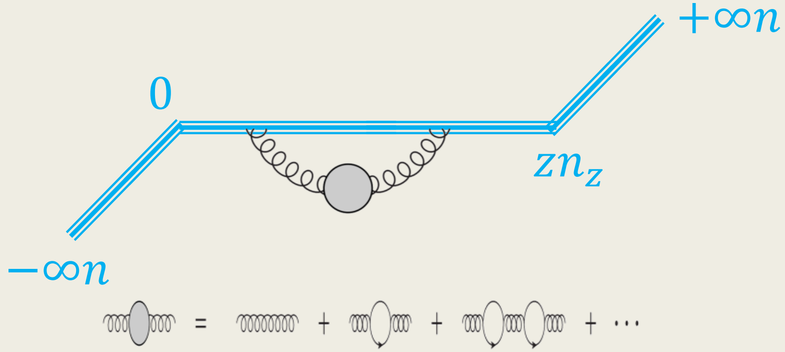
Compare NLO and NNLO wo LRR



Leading renormalon in the threshold limit

- Mass renormalon
  in the soft function

Wilson link self energy



The leading renormalon ambiguity $\sim z\,\Lambda_{\mathrm{QCD}}$ or $\sim \frac{\Lambda_{\mathrm{QCD}}}{(1-x)P^z}$

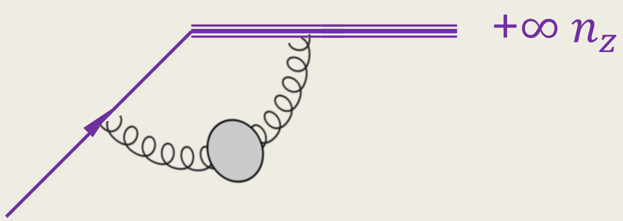
M. Beneke et al., NPB 426 (1994) 301-343
V. Braun et al., PRD 99 (2019) 1, 014013

Cancelled by the mass renormalization ambiguity

R. Zhang et al., PLB 844 (2023) 138081

- Threshold renormalon
  in the hard kernel

Light quark interacting with the coulomb field generated by the heavy quark

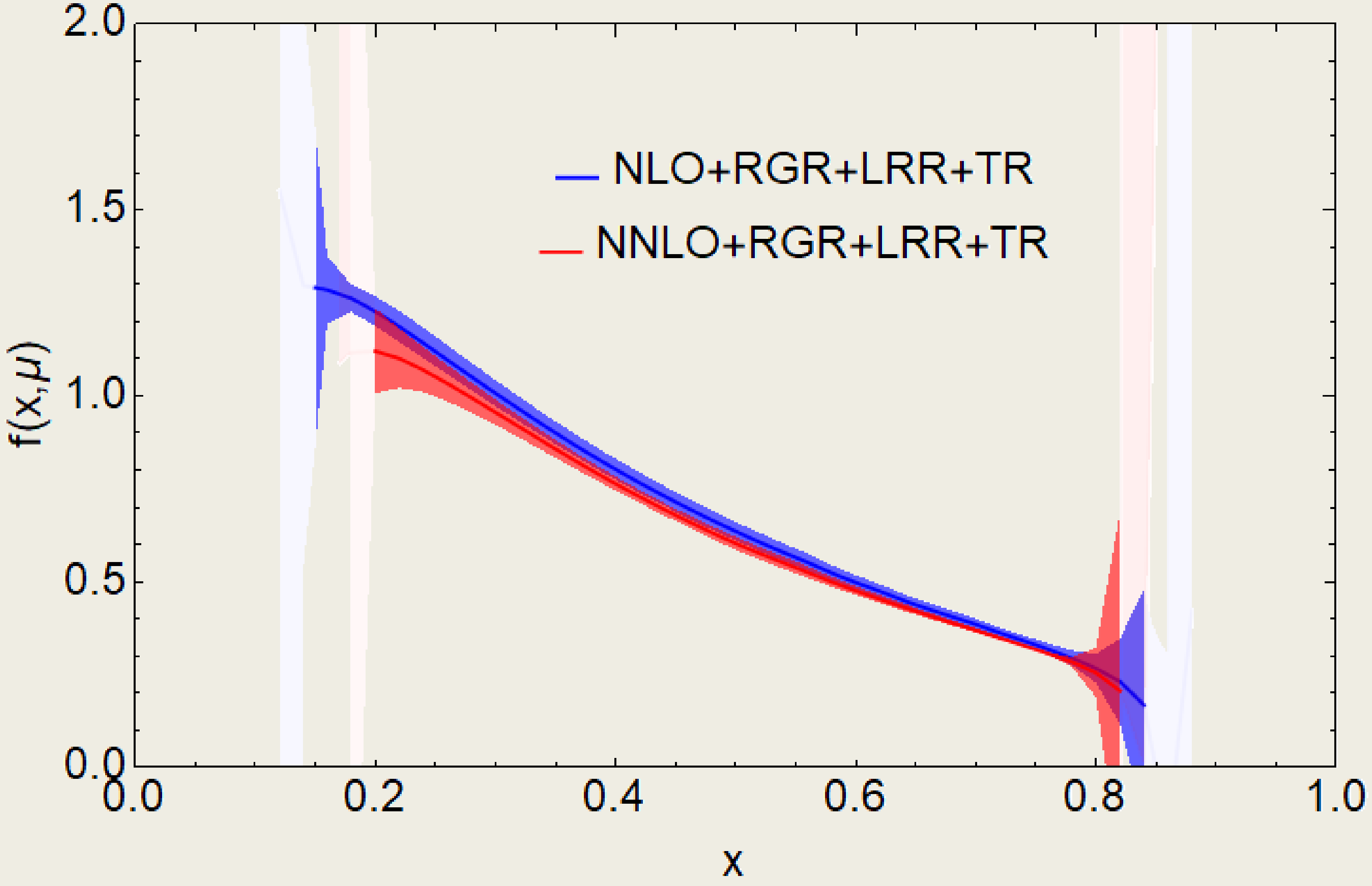


The leading renormalon ambiguity $\sim \frac{\Lambda_{\mathrm{QCD}}}{x\,P^z}$

Cancelled by the next leading threshold expansion

Yizhuang Liu et al., JHEP 2024 (2024) 204

Compare NLO and NNLO w LRR



Conclusions

- In spirit of Weinberg's EFT, the LaMET expansion can be established and used to calculate parton physics without matrix inversion

$$f(x,\mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{|y|P^z}\right) \tilde{f}(y, P^z) + O\left[\frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}, \frac{\Lambda_{\rm QCD}^2}{(1-x)^2 P_z^2}\right].$$

- To calculate large$-x$ PDF in a more reliable and controllable manner, the threshold factorization and resummation can be performed, which indicates the break down of perturbative matching at large-x.

- A good perturbative convergence between NLO and NNLO is observed with the threshold and leading renormalon resummation.

Appendix

Combination with the regular part

- The full matching contains both singular and regular parts

$$f_x(\mu) = \left[\mathcal{S}_{x,y}(P^z, \mu, \mu_h, \mu_i) + \mathcal{R}_{x,y}(P^z, \mu, \mu_h)\right] \tilde{f}_y(P^z)$$

Singular part: $\mathcal{S}(P^z, \mu, \mu_h, \mu_i)$ contains $\sim \delta(1-\xi)$, $\frac{1}{1-\xi}$

Regular part: $\mathcal{R}\,(P^z, \mu, \mu_h) = \boxed{\mathcal{M}(P^z, \mu, \mu_h)} - \boxed{\mathcal{S}(P^z, \mu, \mu_h, \mu_i = \mu_h)} \sim \delta(1-\xi) + O(1) + O(1-\xi) + \cdots$

Full kernel without threshold resummation

Singular part without threshold resummation

- Resum the threshold logs in the singular part.

  Add the regular part up to certain fixed order accuracy.

Lattice Data and Renormalization

- Pion valence PDF matrix element from BNL/ANL collaboration:

$$\tilde{H}^{\text{lat}}(z,a,P_z) = \langle \pi^+(P_z) | \bar{u}(z)\gamma^t U(z,0)u(0) - \bar{d}(z)\gamma^t U(z,0)d(0) | \pi^+(P_z) \rangle$$

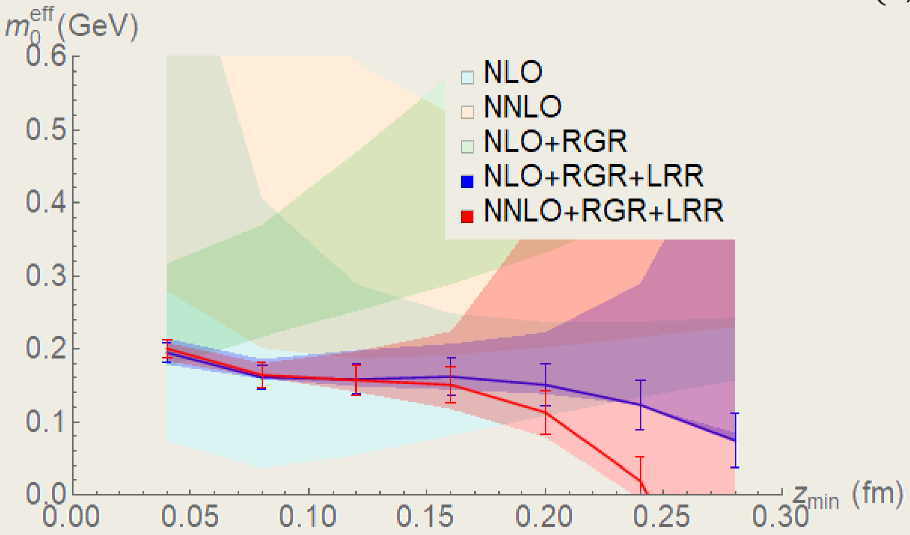
Gao et al., PRD (2020)
Gao et al., PRD (2021)
Gao et al., PRL (2022)

- Hybrid renormalized matrix element:

$$\tilde{H}(z,P_z) = \frac{\tilde{H}^{\text{lat}}(z,a,P_z)}{\tilde{H}^{\text{lat}}(z,a,0)}\theta(z_s-|z|) + \frac{\tilde{H}^{\text{lat}}(z,a,P_z)}{Z^R(z,a,\mu)\tilde{H}^{\overline{\text{MS}}}(z_s,\mu,0)}\theta(|z|-z_s)$$

where $Z^R(z,a,\mu) \sim e^{-(\delta m + m_0)z}$.

m_0 is determined through requiring $\frac{\tilde{H}^{\text{lat}}(z,a,P_z)}{Z^R(z,a,\mu)} = \tilde{H}^{\overline{\text{MS}}}(z,\mu,0)$ for $a < z \ll 1/\Lambda_{\text{QCD}}$



NLO

$$\tilde{H}^{\overline{\text{MS}}} = (1 + \alpha_s c_1^c + \alpha_s^2 c_2^c + \alpha_s^3 c_3^c + \cdots$$
$$+ \alpha_s c_1^{\text{LR}} + \alpha_s^2 c_2^{\text{LR}} + \alpha_s^3 c_3^{\text{LR}} + \cdots)\,\text{Exp}[\int d\alpha_s \frac{\gamma}{\beta}]$$

R. Zhang et al., 2305.05212

LRR: Asymptotic Form to $O[1/n^2]$
Borel sum with P.V.

RGR

Leading power accuracy

$$f_x(\mu) = [\mathcal{S}_{x,y}(P^z, \mu, \mu_h, \mu_i) + \mathcal{R}_{x,y}(P^z, \mu, \mu_h)]\tilde{f}_y(P^z)$$

Mass renormalization

Self energy of gauge link in Jet

Vertex correction of hard kernel

Next leading threshold expansion

Linear power corrections related to mass renormalon are cancelled!

R. Zhang et al., PLB 844 (2023) 138081

Leading power corrections related to threshold renormalon are cancelled!

Y. Liu et al., JHEP 2024 (2024) 204

Exponentiate the mass renormalon series

$$\tilde{J}\left(\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4}, \alpha(\mu)\right) = \tilde{J}\left(\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4}, \alpha(\mu)\right)_{\mathrm{LRR-sub}} \mathrm{Exp}\left[-|z|\mu \sum_{n=0}^{+\infty} r_n\, \alpha(\mu)^{n+1}\right]$$

Fixed order perturbation series with leading renormalon series subtracted

Mass renormalon series

Why exponentiation?

- The physical origin is related to mass
- Scale invariance of renormalon ambiguity for both $\tilde{J}$ and $\tilde{J}^{-1}$
- $\sum_{n=0}^{+\infty} r_n\, \alpha(\mu)^{n+1}$ defined with principal value is positive definite so the Fourier transformation of jet is well-defined