Impact of gauge fixing precision on the continuum limit of non-local quark-bilinear lattice operators

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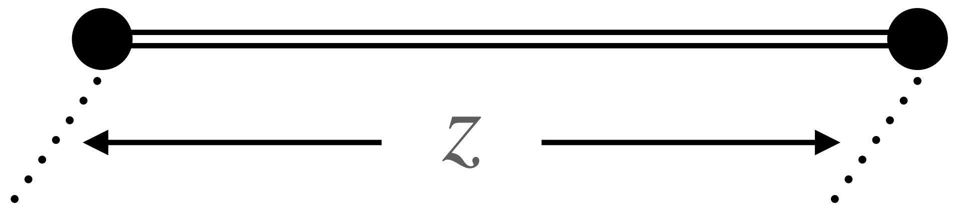
Linear divergence

Puzzles in the RI/MOM renormalization

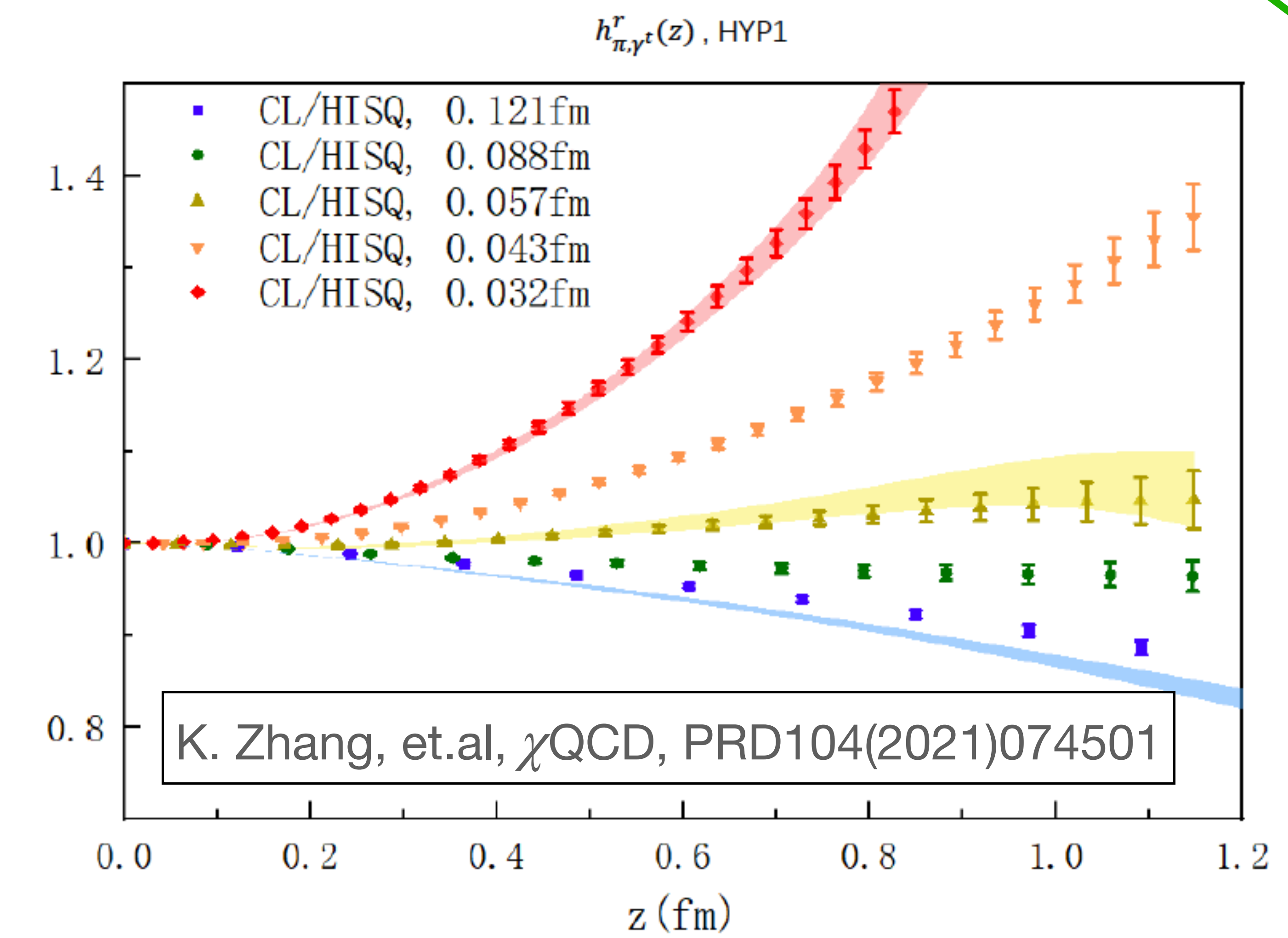
$$O_{\gamma_t}(z) = \bar{q}(0)\gamma_t U_z(0,z)q(z)$$



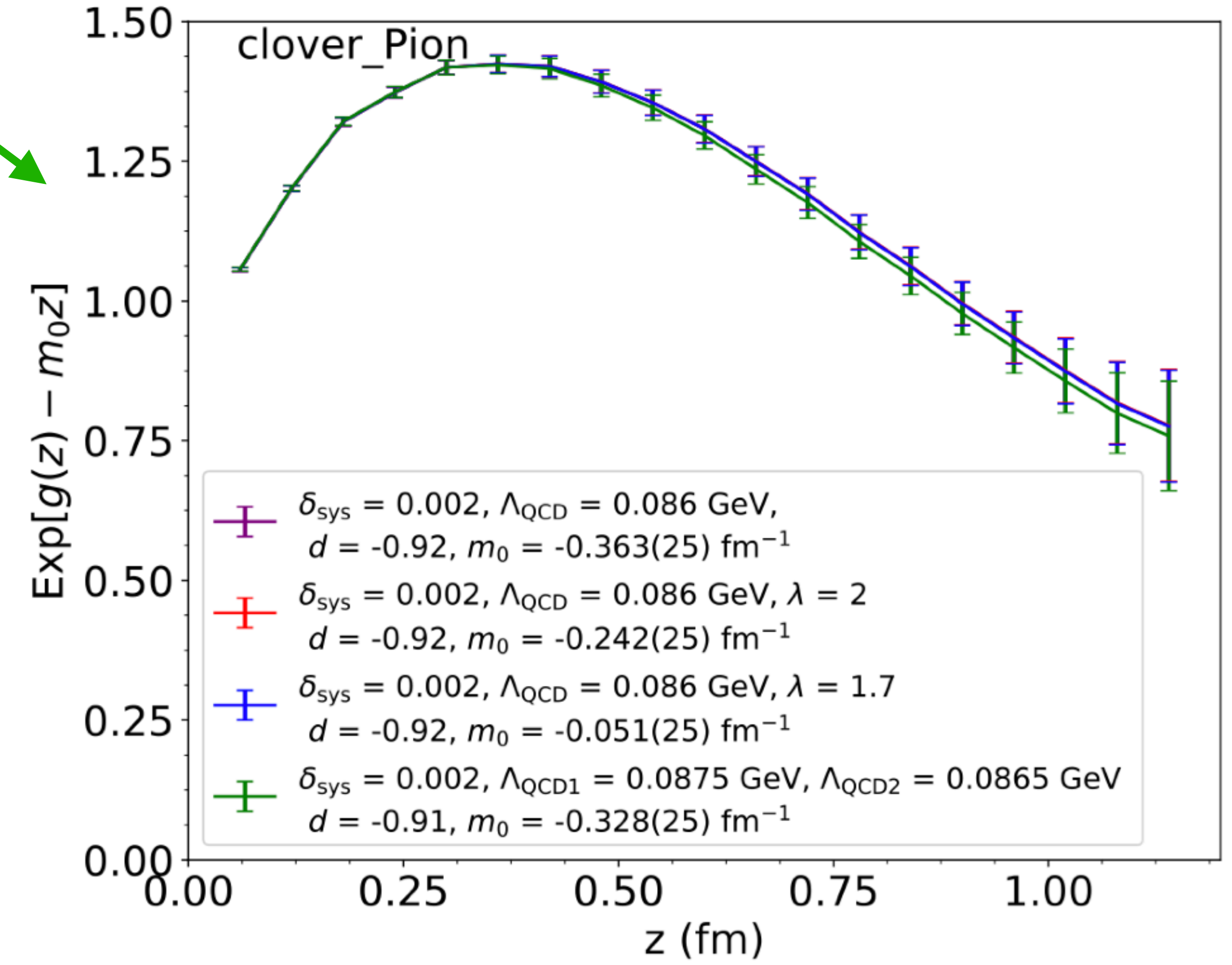
quasi-PDF operator

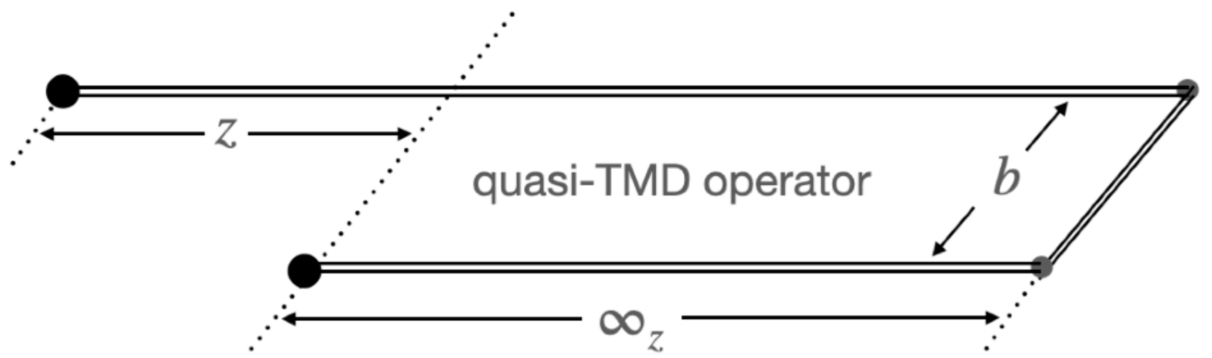
Self renormalization

RI/MOM renormalization



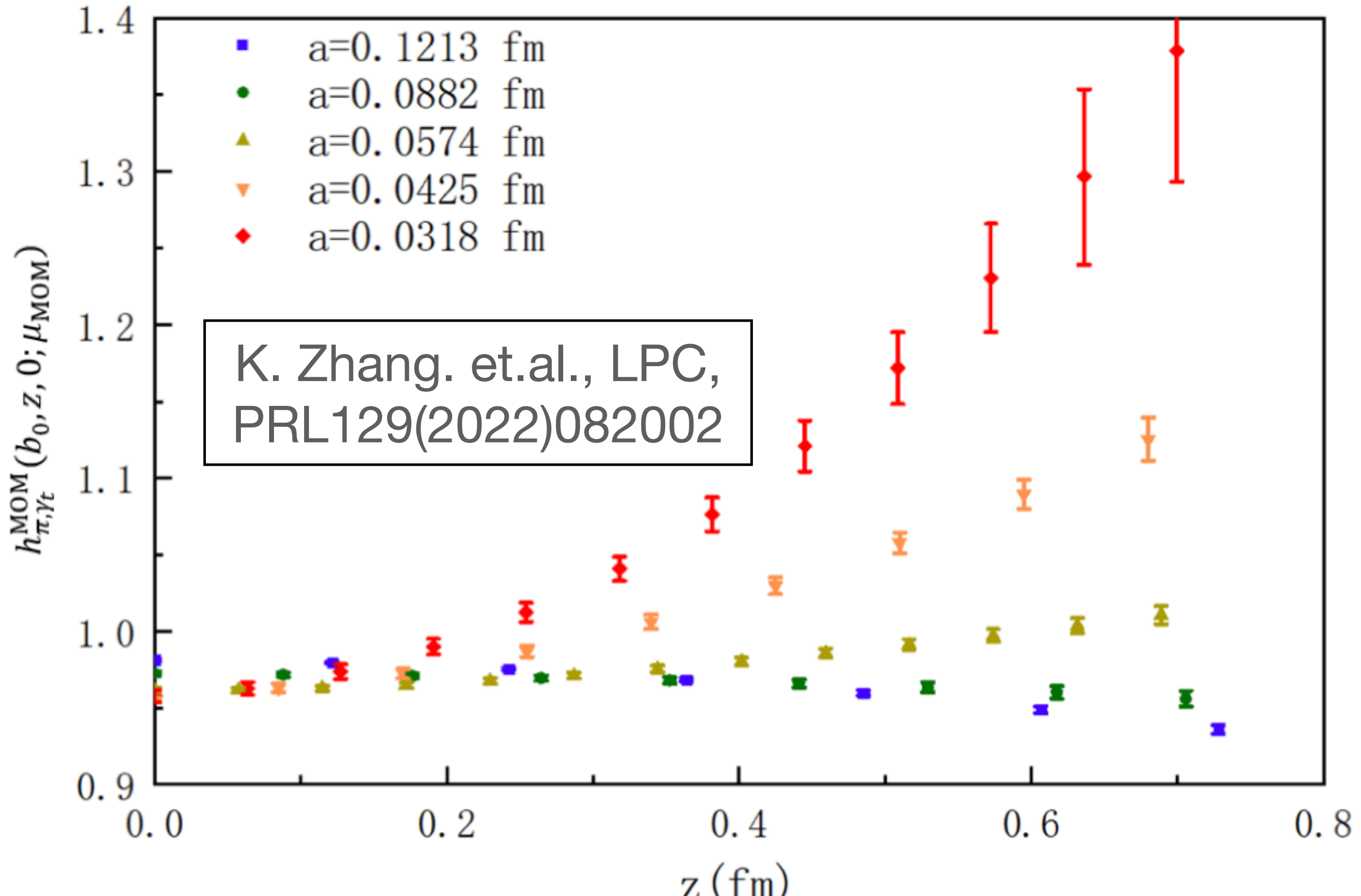
Y.-K. Huo. et.al., LPC, NPB969(2021)115443

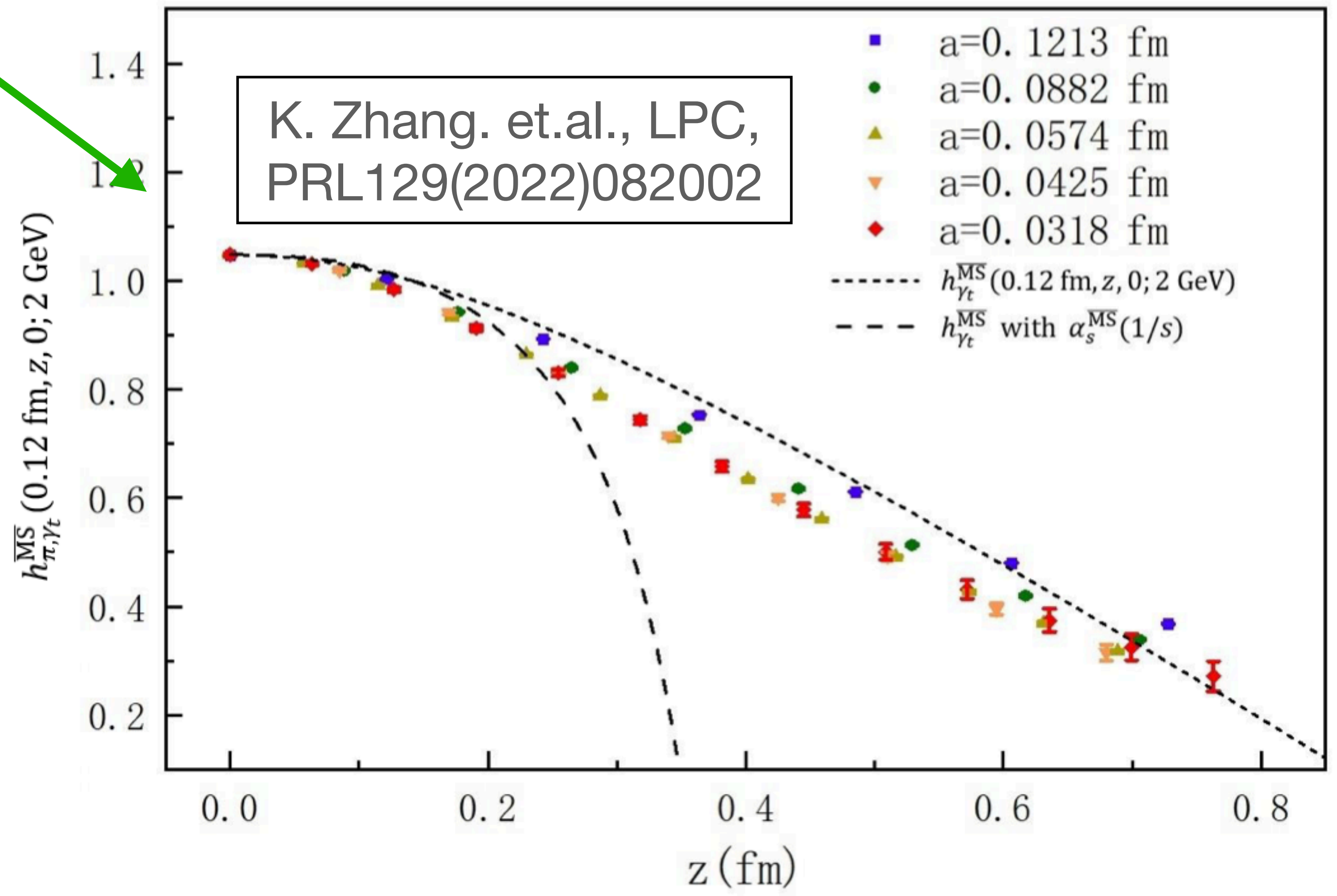




Short distance renormalization

RI/MOM renormalization





Linear divergence

Landau gauge fixing required by RI/MOM

- On the lattice, the gauge rotation is defined as $U_\mu^G(x) = G(x)U_\mu(x)G^\dagger(x + a\hat{\mu})$;
- And then the discretized gauge condition is

$$0 = \Delta^G(x) \equiv \sum_{\mu=1}^{4} \left[U_\mu^G(x) - U_\mu^G(x - a\hat{\mu}) - (U_\mu^G)^\dagger(x) + (U_\mu^G)^\dagger(x - a\hat{\mu})\right] / (2ig_0)$$

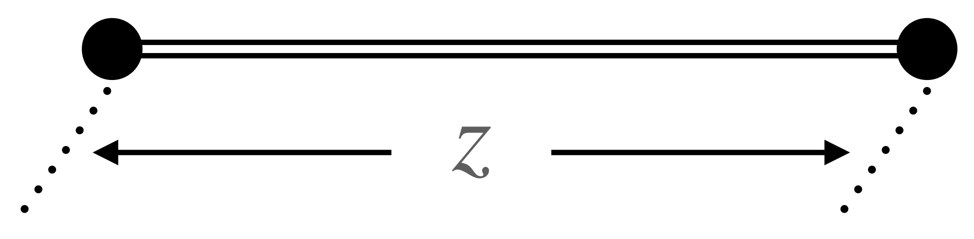
$$= a \sum_{\mu=1}^{4} \partial_\mu A_\mu^G + \mathcal{O}(a^2)\,.$$

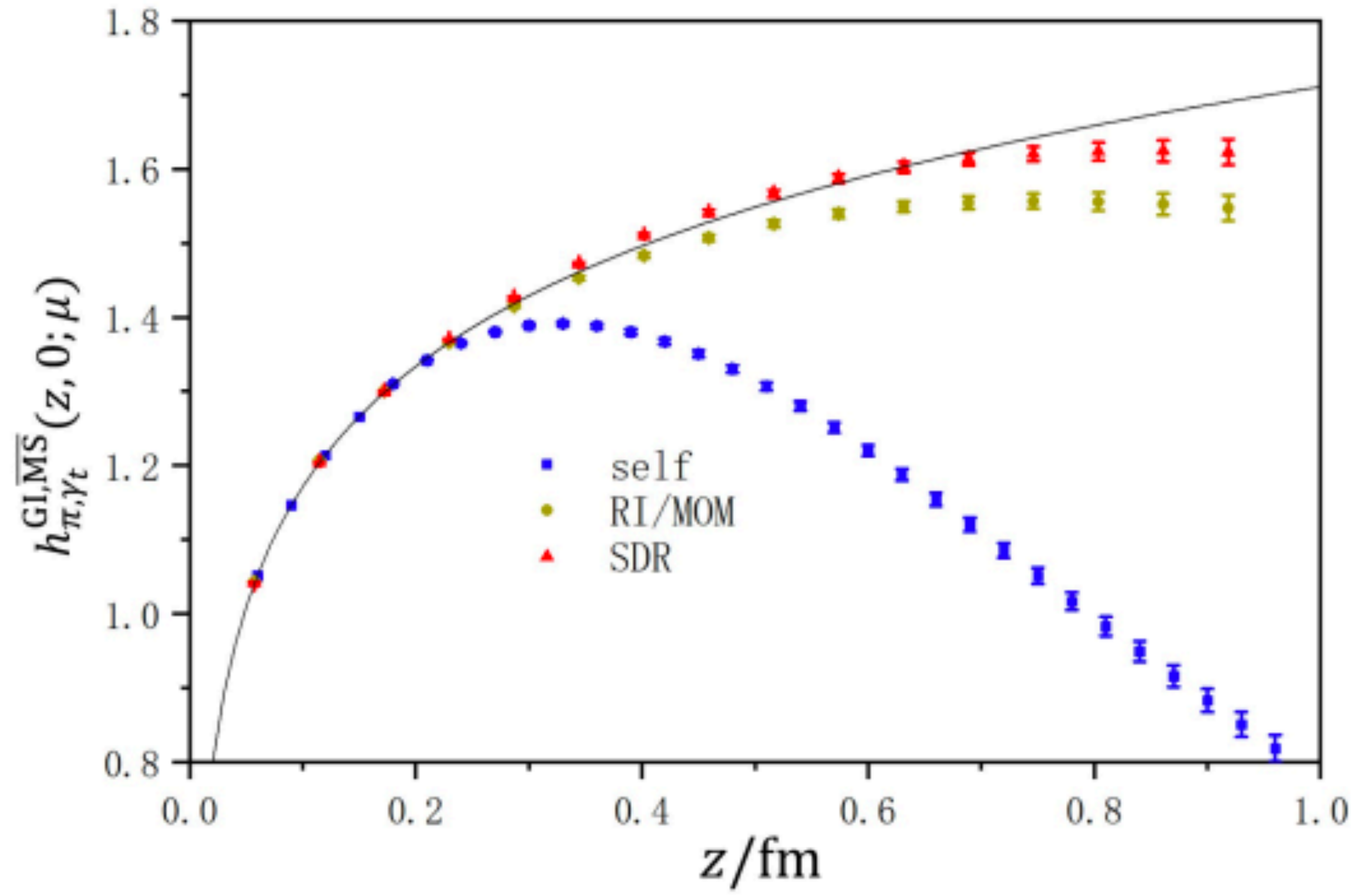
- Such a condition can only be obtained iteratively by maximum the functional $F_U[G] \equiv \frac{1}{12V} \text{Re Tr} \sum_x \sum_{\mu=1}^{4} U_\mu^G(x)$, and stop when the difference $\delta^F(n) = F_U[G(n)] - F_U[G(n-1)]$ is smaller then the required precision δ^F.

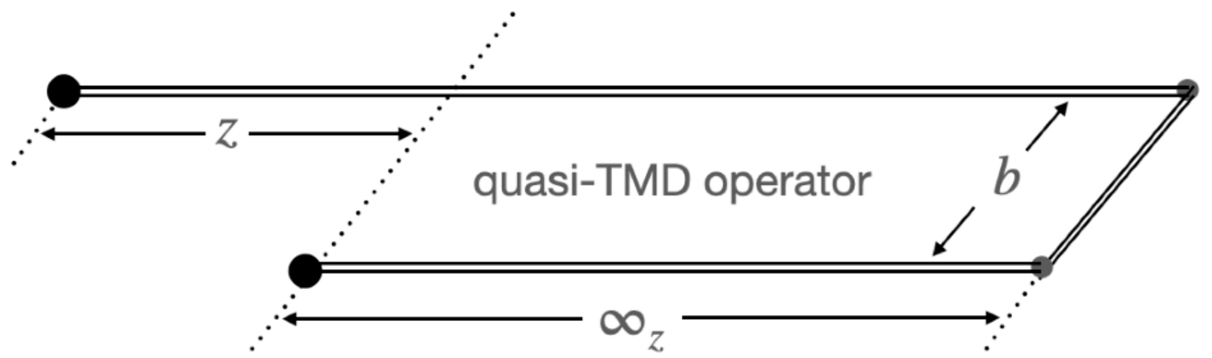
Linear divergence

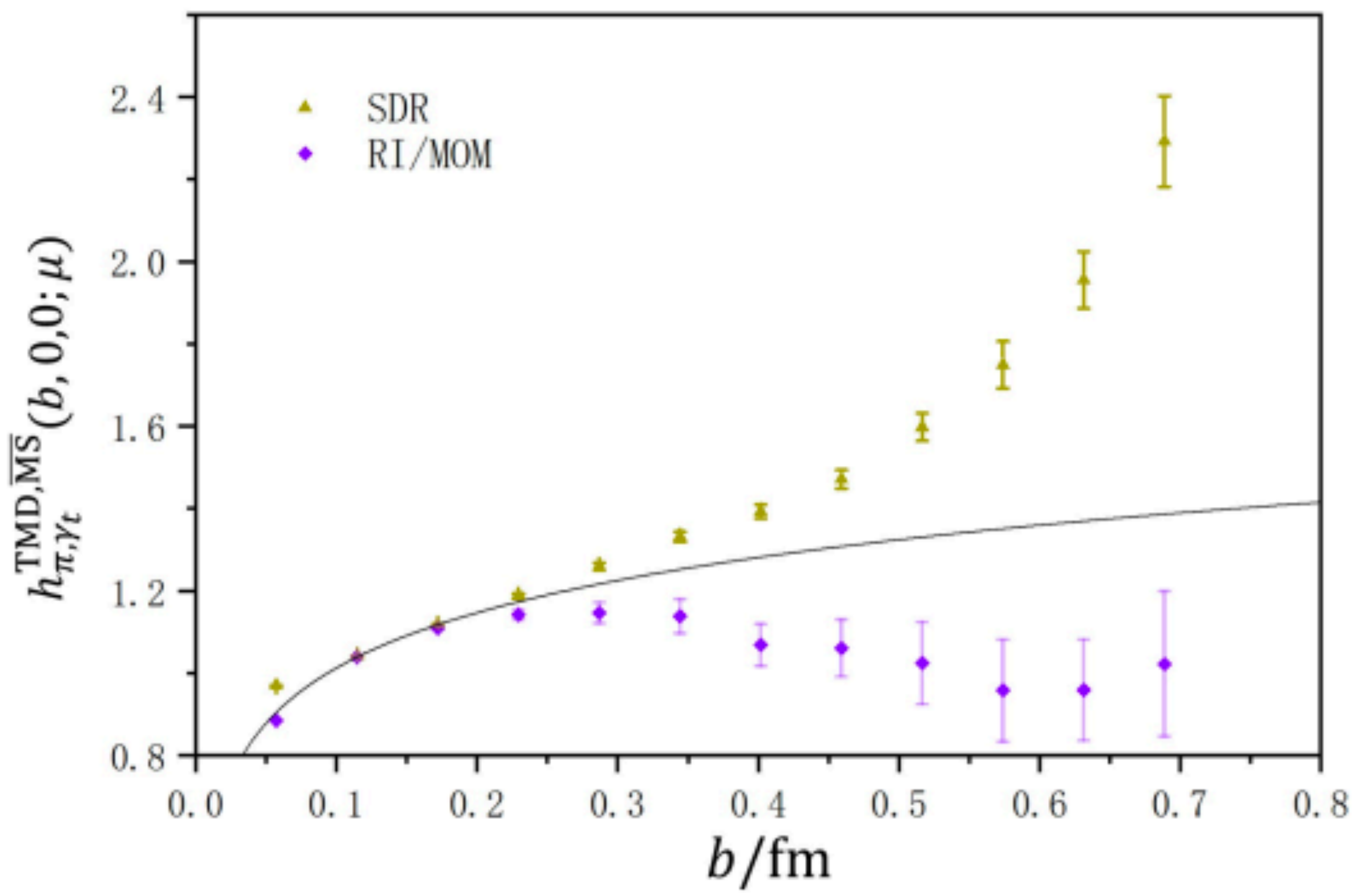
Imprecise gauge fixing

$O_{\gamma_t}(z) = \bar{q}(0)\gamma_t U_z(0,z)q(z)$









- Residual linear divergence disappears if the gauge is fixed with high enough precision.
- Different renormalization schemes agree with each other at small distance;
- But can be significantly different at large distance, due to the IR effect in the quark matrix elements used by RI/MOM.

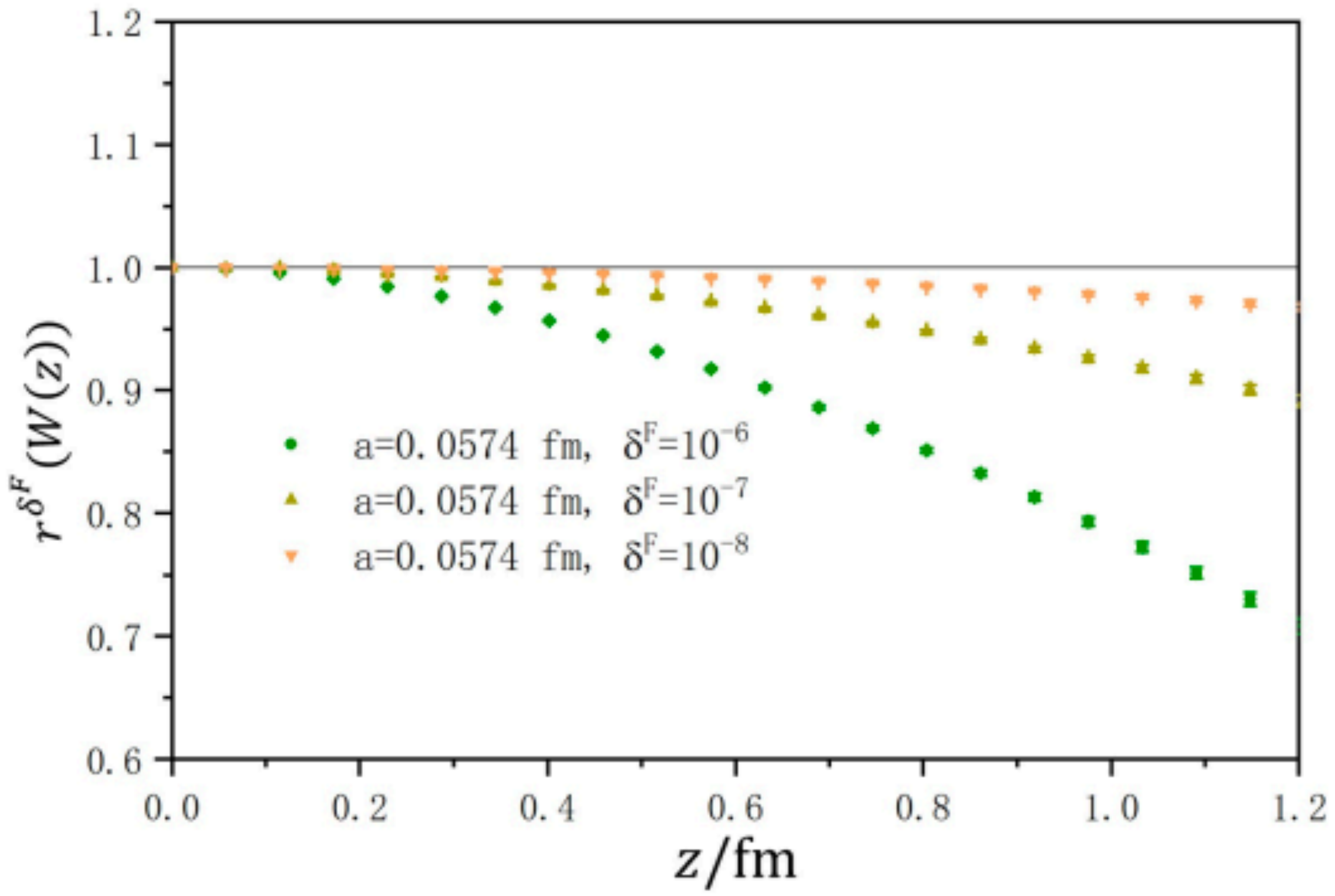
K. Zhang, et.al, LPC, 2405.14097

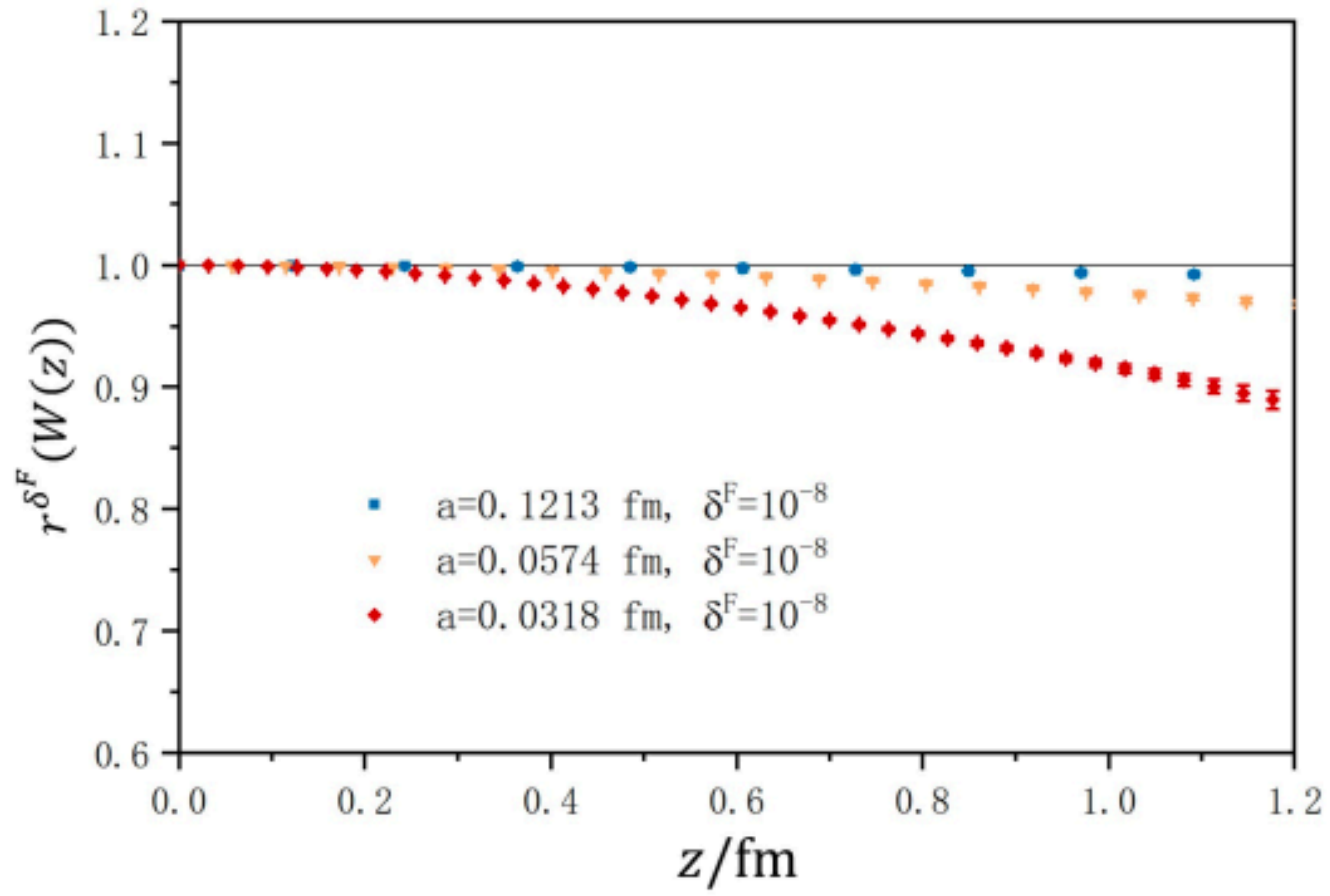
Outline

- **Wilson lines**
- Quark bilinear operator with straight Wilson line (quasi-PDF/DA)
- Quark bilinear operator with staple-shaped Wilson line (quasi-TMD PDF/DA)
- Summary

Wilson line $W(z) \equiv \frac{1}{3}\mathrm{Tr}\left[\prod_{k=0}^{n-1} U_\mu(x+ka\hat{\mu})\right]$

$$r^{\delta^F} \equiv W(z,\delta^F)/W(z,\delta_0^F)$$





Under Landau gauge with different precision

- The calculation is carried out using the 2+1+1 HISQ ensemble generated by the MILC collaboration, and uses clover valence fermion for the quark propagators.

Tag	$6/g_0^2$	L	T	a(fm)	$m_q^{\mathrm{w}}a$	c_{sw}
a12m310	3.60	24	64	0.1213(9)	-0.0695	1.0509
a09m310	3.78	32	96	0.0882(7)	-0.0514	1.0424
a06m310	4.03	48	144	0.0574(5)	-0.0398	1.0349
a045m310	4.20	64	192	0.0425(5)	-0.0365	1.0314
a03m310	4.37	96	288	0.0318(5)	-0.0333	1.0287

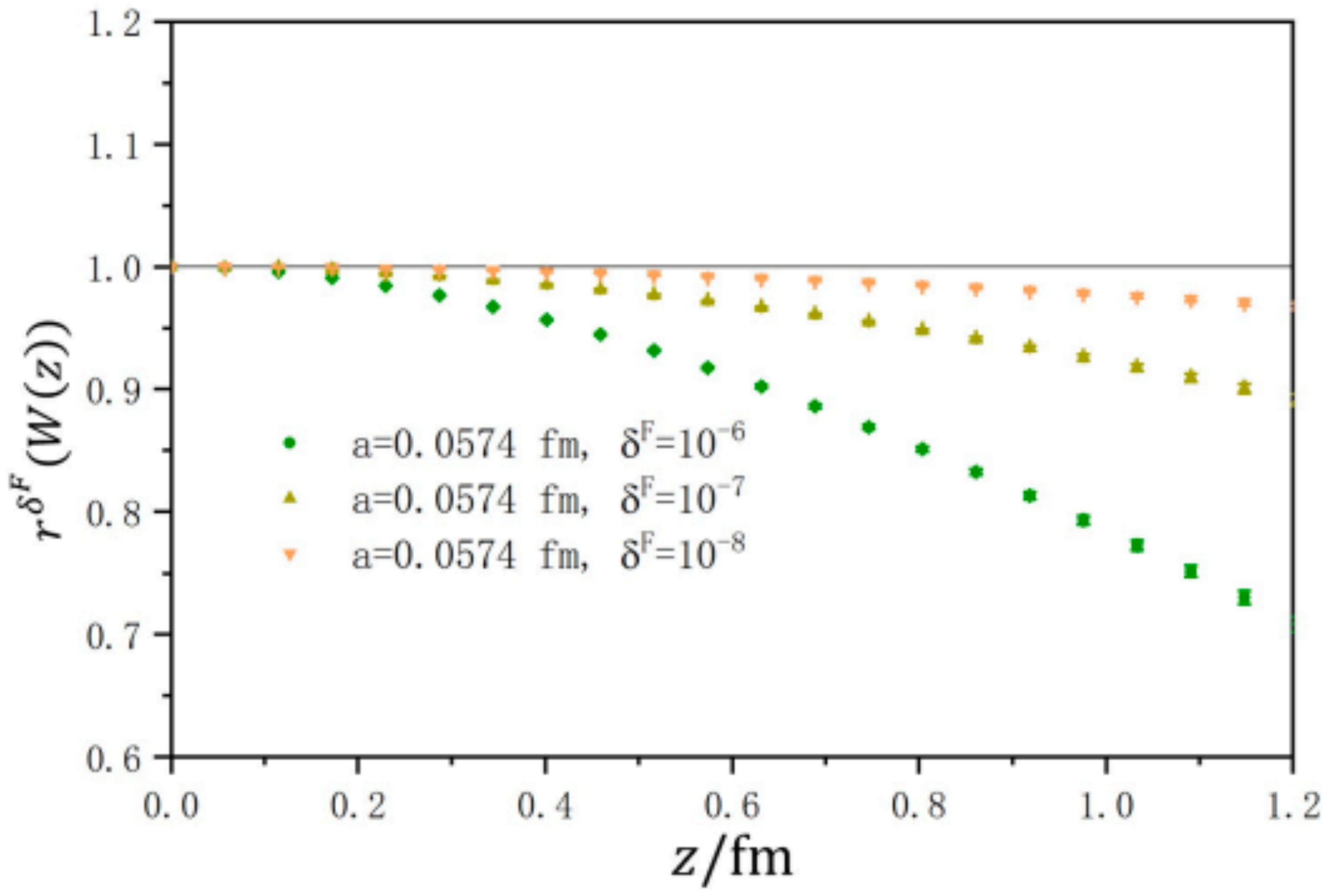
A. Bazavov, et.al, MILC, PRD 87(2013)054505

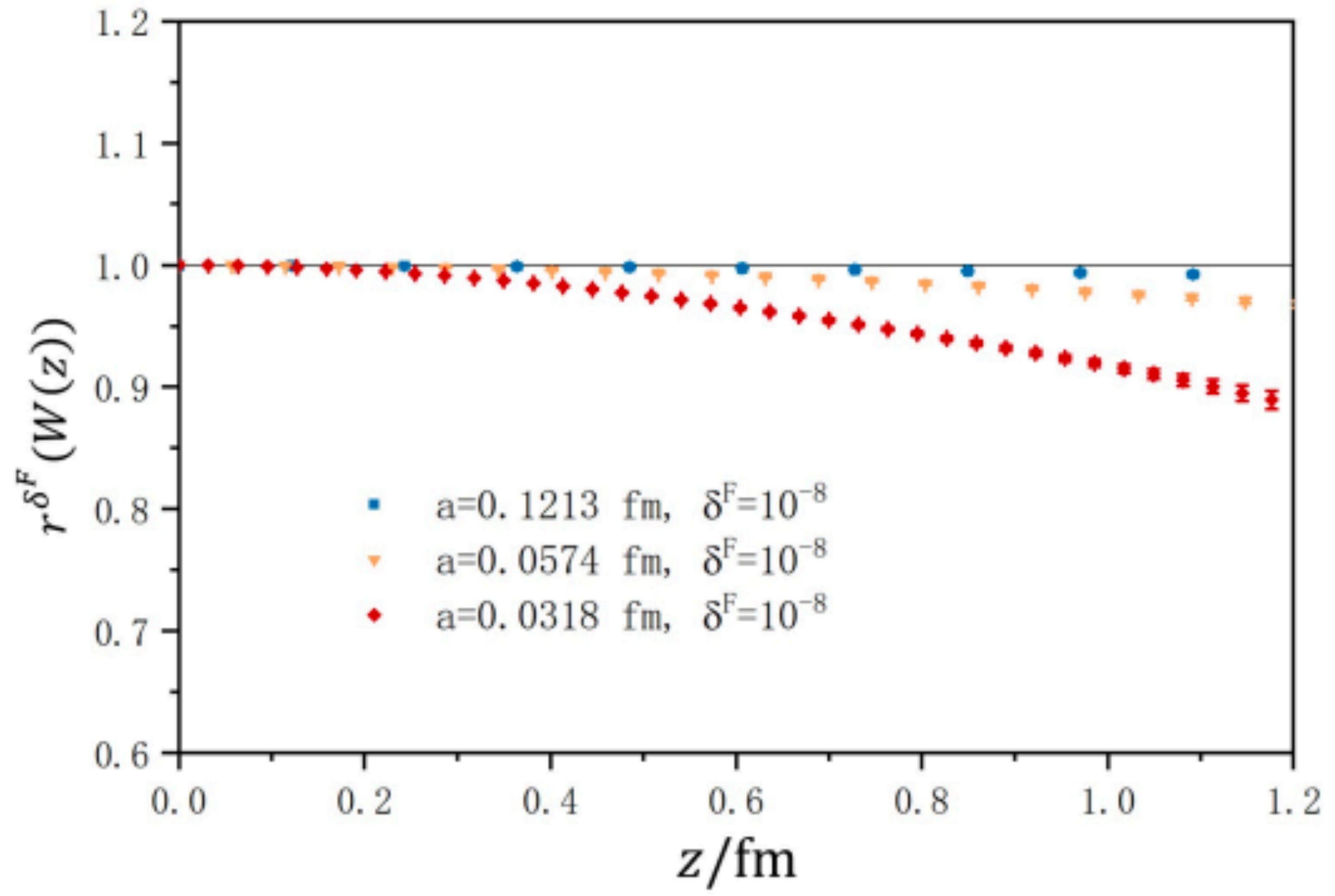
Deviation from 1 of the ratio $r^{\delta^F}(W(z))$ of the Wilson line under certain gauge fixing precision $\delta^F = 10^{-6,7,8}$ over that with $\delta_0^F = 10^{-15}$, $1 - W(z,\delta^F)/W(z,\delta_0^F)$:

- Becomes larger with fixed a but larger δ^F;
- Becomes larger with fixed δ^F but smaller a.

Wilson line $W(z) \equiv \frac{1}{3}\mathrm{Tr}\left[\prod_{k=0}^{n-1} U_\mu(x+ka\hat{\mu})\right]$

Empirical formula to fit the deviation



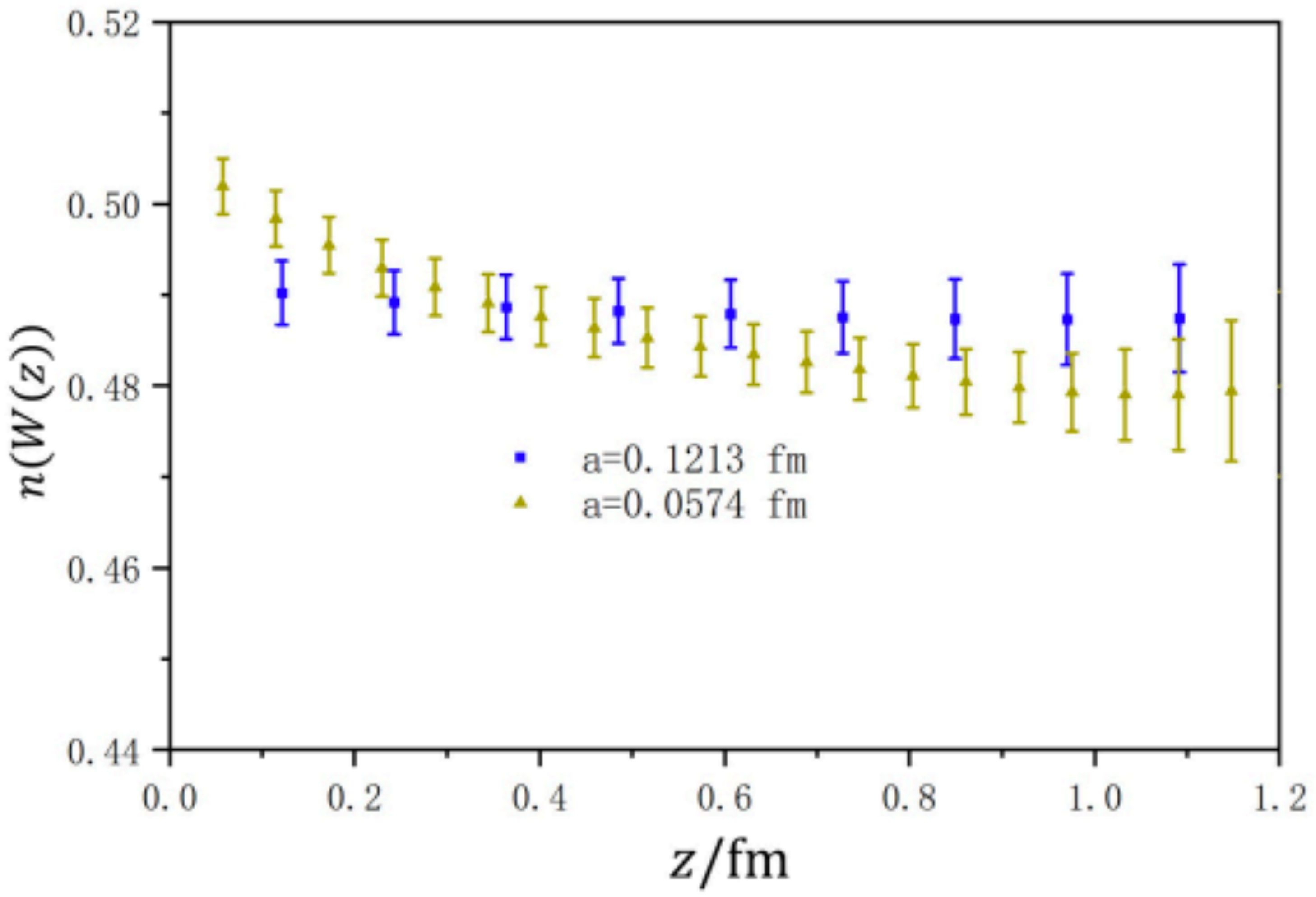


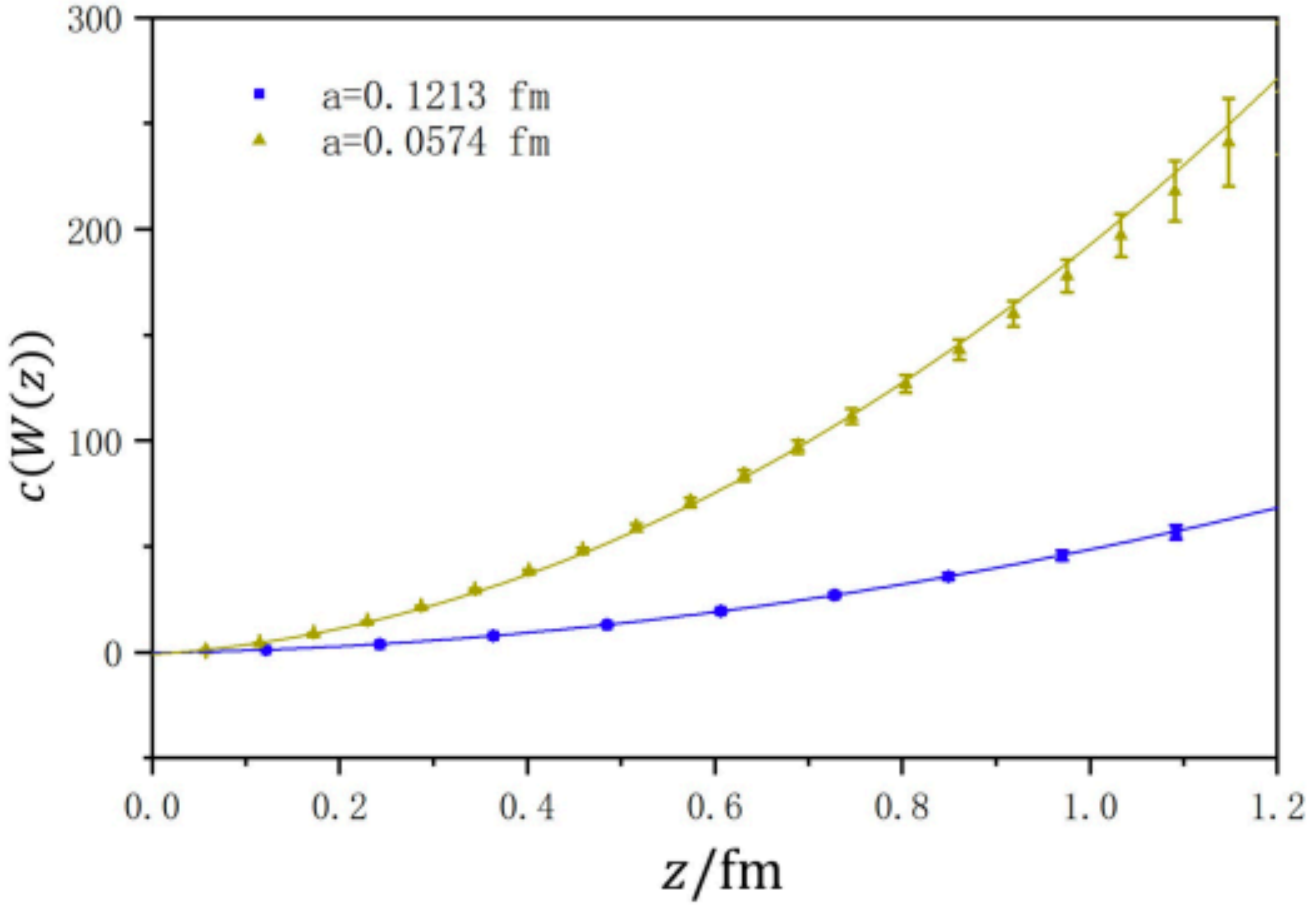
We find that this deviation can be described using the following form:

$$W(\delta^F) = W(0)e^{-c(X)(\delta^F)^{n(X)}}$$

with all the a and z we investigated:

- $n(W(z))$ is close to a constant around 0.49 and insensitive to both a and z;
- $c(W(z))$ can be further parametrized into $0.58(4)z^2/a^2 + \mathcal{O}(z/a^2)$.





Wilson line $W(z) \equiv \frac{1}{3}\mathrm{Tr}\left[\prod_{k=0}^{n-1} U_\mu(x+ka\hat{\mu})\right]$

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Why the deviation can be so large?

- $n(W(z))$ enlarged the impact of the gauge fixing residual by a huge factor $1/\sqrt{\delta^F}$;
- $c(W(z))$ introduce an UV divergence-like factor z^2/a^2 which enlarge the impact at large z.

- Using $a = 0.12$ fm, $z = 0.5$ fm, $\delta^F = 10^{-6}$: $r(W(z)) \sim$ 0.99;
- Using $a = 0.12$ fm, $z = 1.0$ fm, $\delta^F = 10^{-6}$: $r(W(z)) \sim$ 0.95;
- Using $a = 0.03$ fm, $z = 1.0$ fm, $\delta^F = 10^{-6}$: $r(W(z)) \sim$ 0.48.

Wilson line $W(z) \equiv \frac{1}{3}\mathrm{Tr}\left[\prod_{k=0}^{n-1} U_\mu(x+ka\hat{\mu})\right]$

Correlation between the gauge rotations

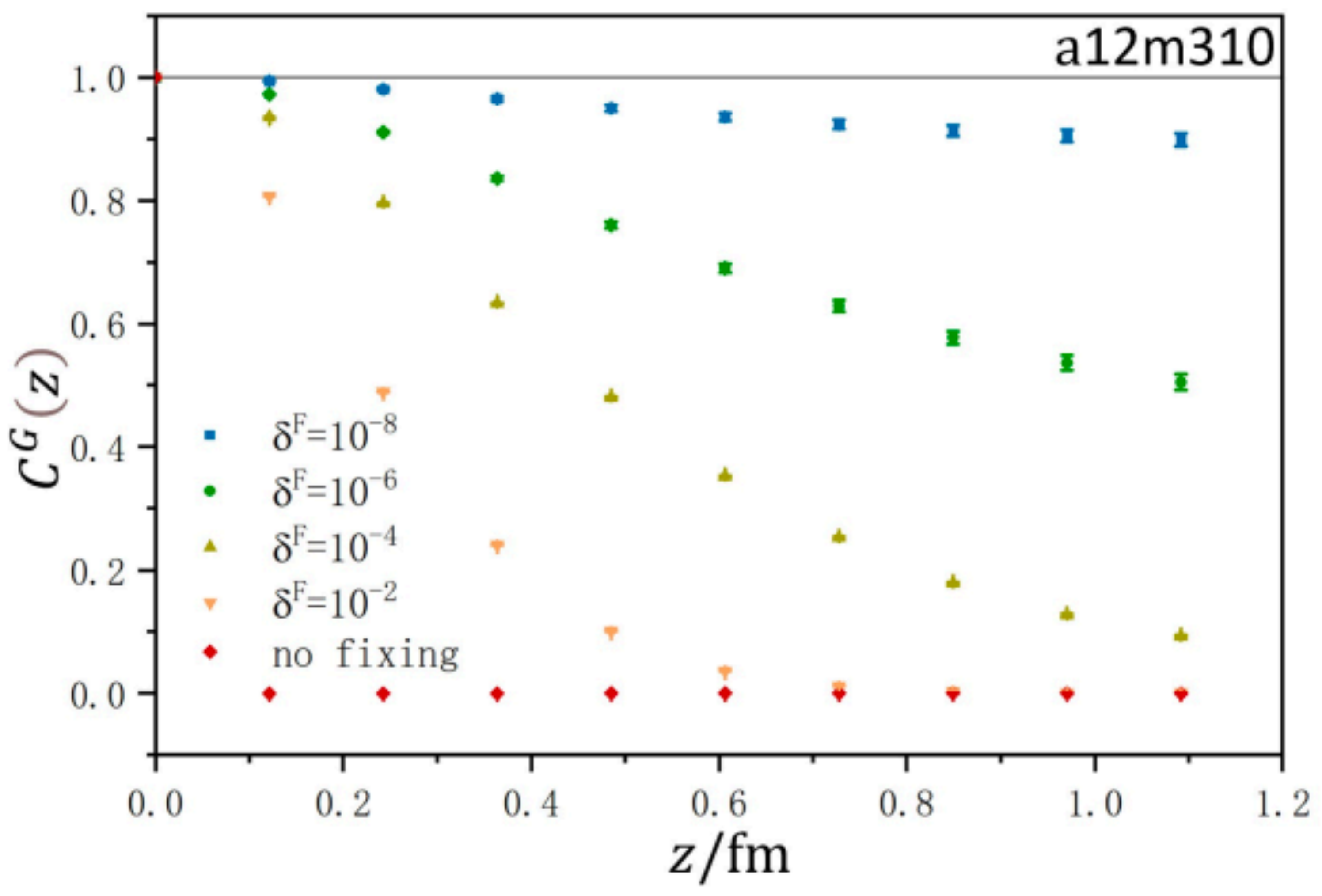
- The Wilson lines with gauge rotations $G_1(x)$ and $G_2(x)$ can be related by the following expression:

$$\begin{aligned} &U^{G_1}(x, x+z) \\ &= G_1(x)G_2^{-1}(x)U^{G_2}(x, x+z)G_2(x+z)G_1^{-1}(x+z) \\ &= [G_1(x)G_2^{-1}(x)]U^{G_2}(x, x+z)[G_1(x+z)G_2^{-1}(x+z)]^{-1}, \end{aligned}$$

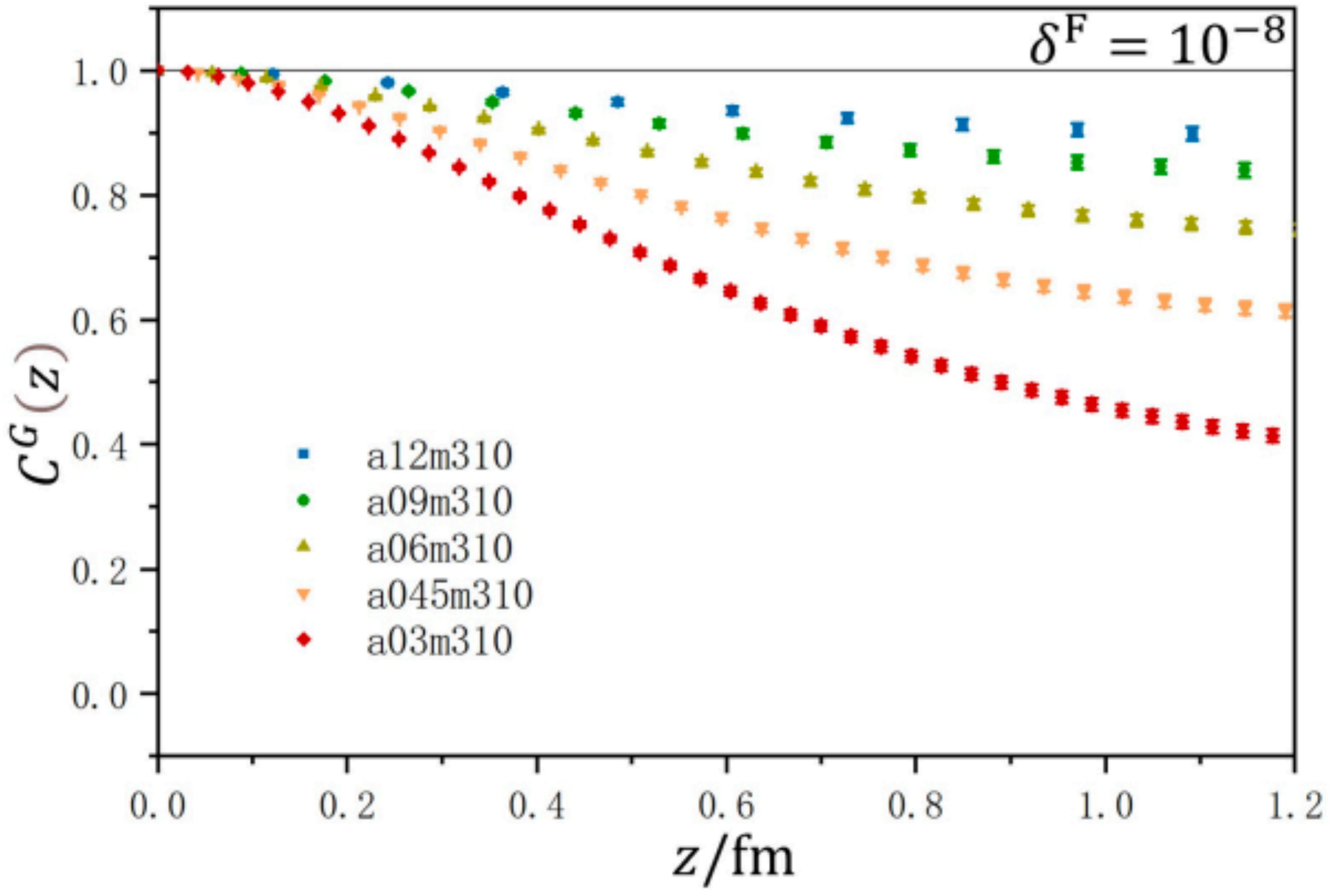
where $U^{G_{1,2}}(x, x+z) \equiv G_{1,2}(x)U(x, x+z)G^{\dagger}_{1,2}(x+z)$ just differ by the relative gauge rotations.

- Then we can define the correlation between the relative gauge rotations as:

$$C^G(z) = \frac{1}{3V}\sum_x \mathrm{Tr}\left[[G(x+z)G_0^{-1}(x+z)]^{\dagger}[G(x)G_0^{-1}(x)]\right]$$



$C^G(z, a)$ approaches 0 when $\delta^F \to \infty$, and 1 when $\delta^F \to 0$;

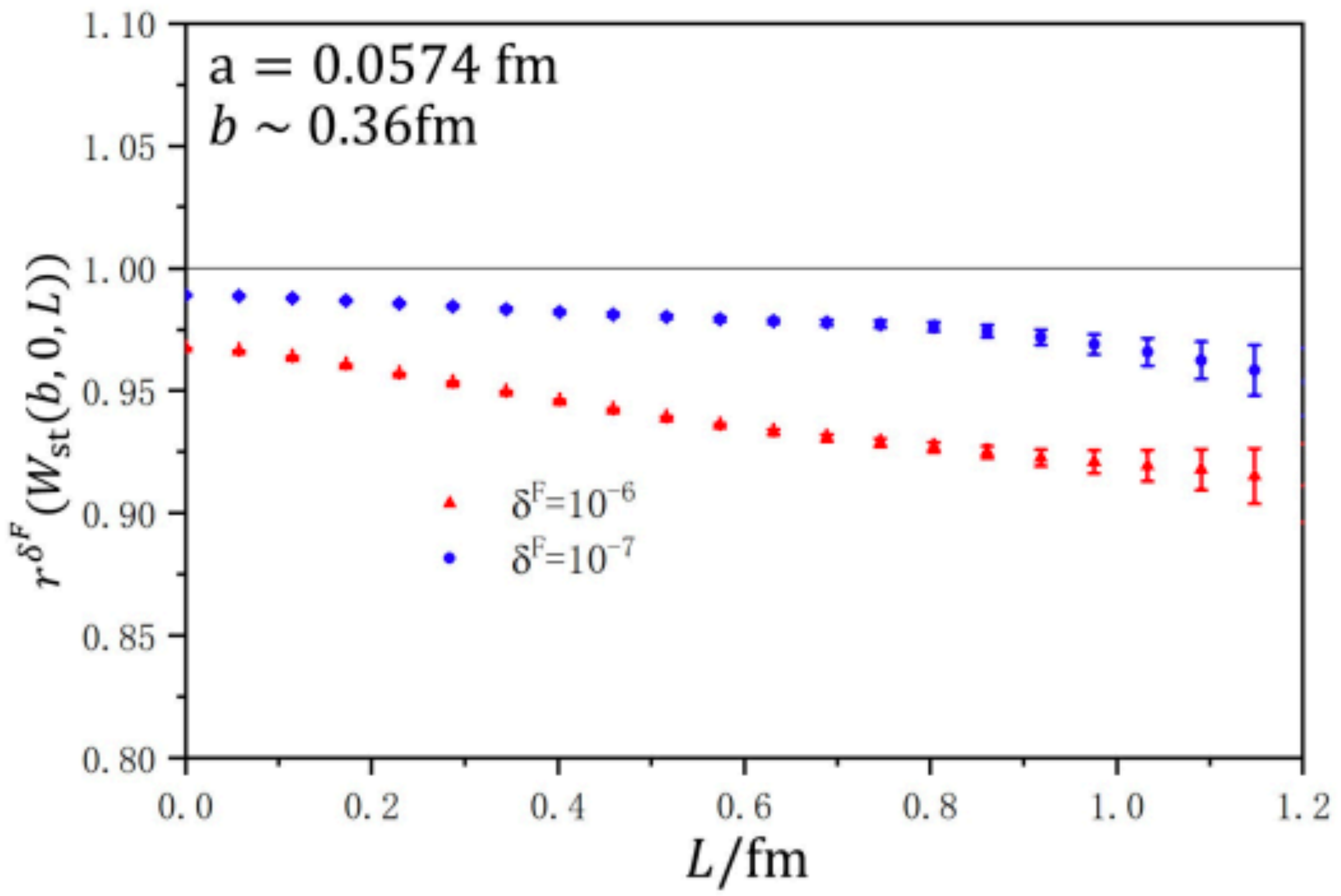


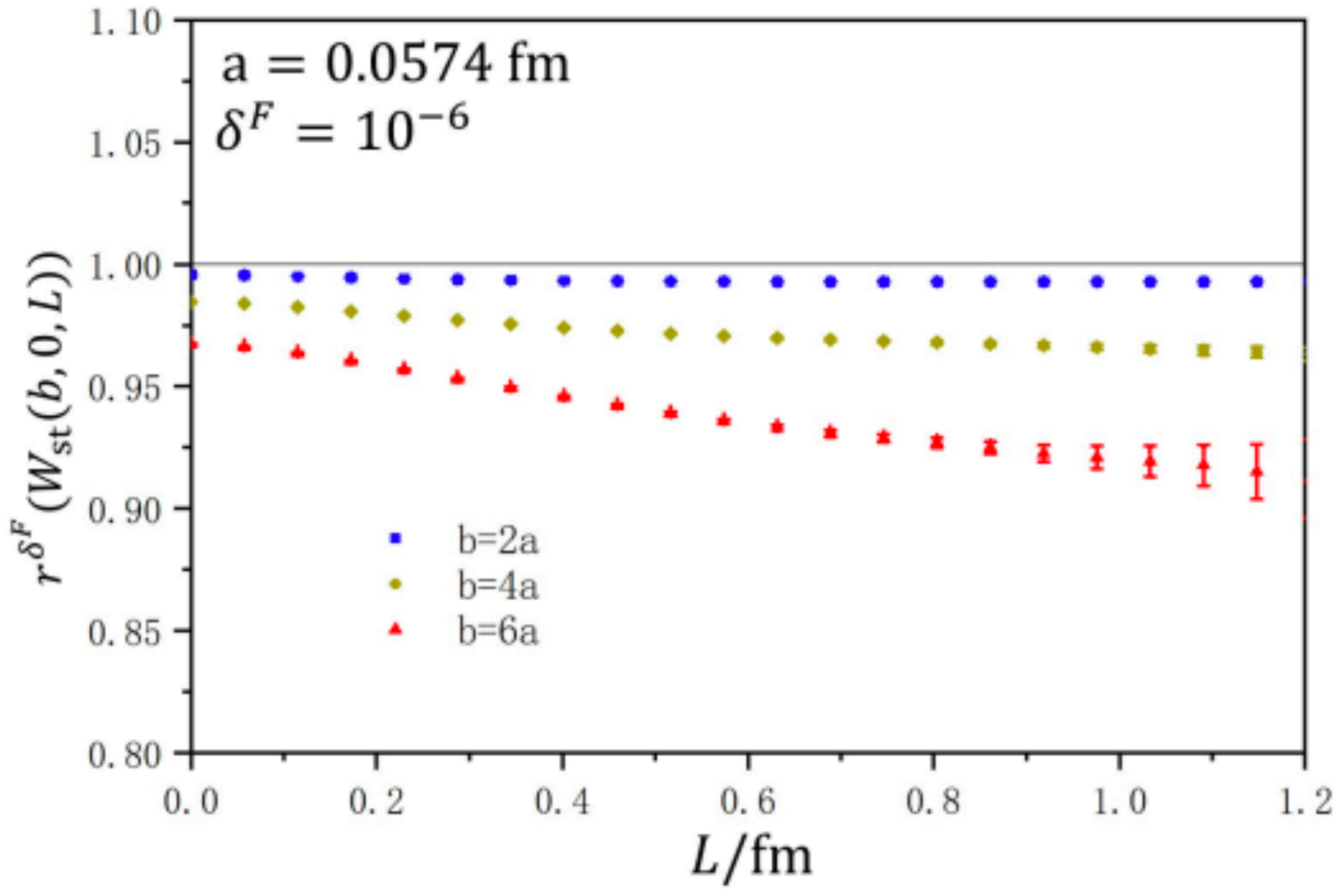
$C^G(z, a)$ would be a function of the dimension less distance z/a.

Wilson line

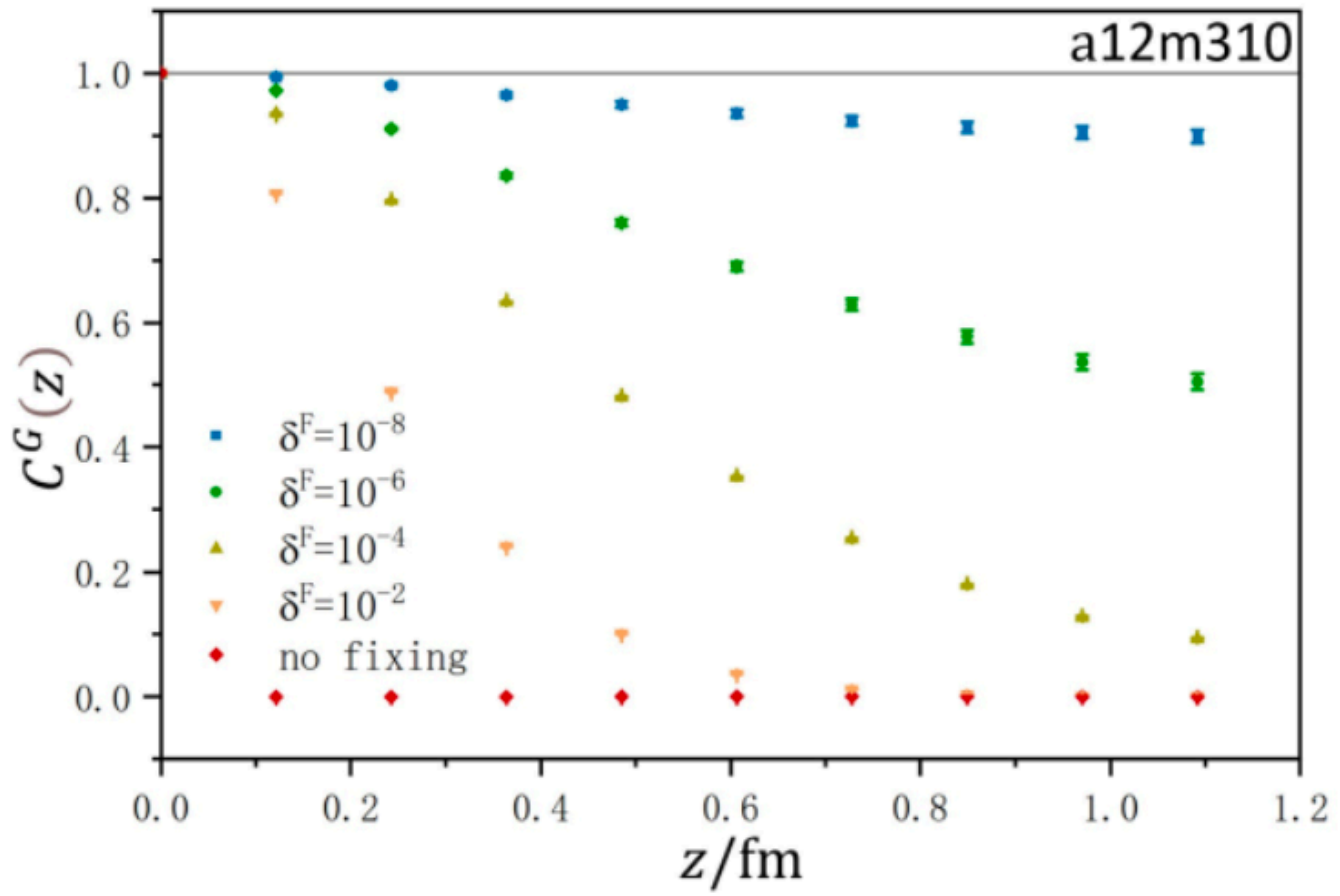
$$W_{\mathrm{st}}(b,z,L) \equiv \frac{1}{3}\mathrm{Tr}\Big\{ \mathscr{P}\exp\left[ig_0 \int_{-L}^{z} \mathrm{d}s\ \hat{n}_z \cdot A^m(b\hat{n}_\perp + s\hat{n}_z)T^m\right] \times \mathscr{P}\exp\left[ig_0 \int_{0}^{b} \mathrm{d}s\ \hat{n}_\perp \cdot A^m(s\hat{n}_\perp - L\hat{n}_z)T^m\right] \times \mathscr{P}\exp\left[ig_0 \int_{0}^{-L} \mathrm{d}s\ \hat{n}_z \cdot A^m(s\hat{n}_z)T^m\right] \Big\}$$

Staple shaped case





- The staple shaped Wilson link is needed for the quasi-TMD PDF and WF;
- Return to the straight Wilson line case with $z=0$ and $L=0$;
- Gauge fixing precision effect is not very sensitive to L, regardless the value of b.



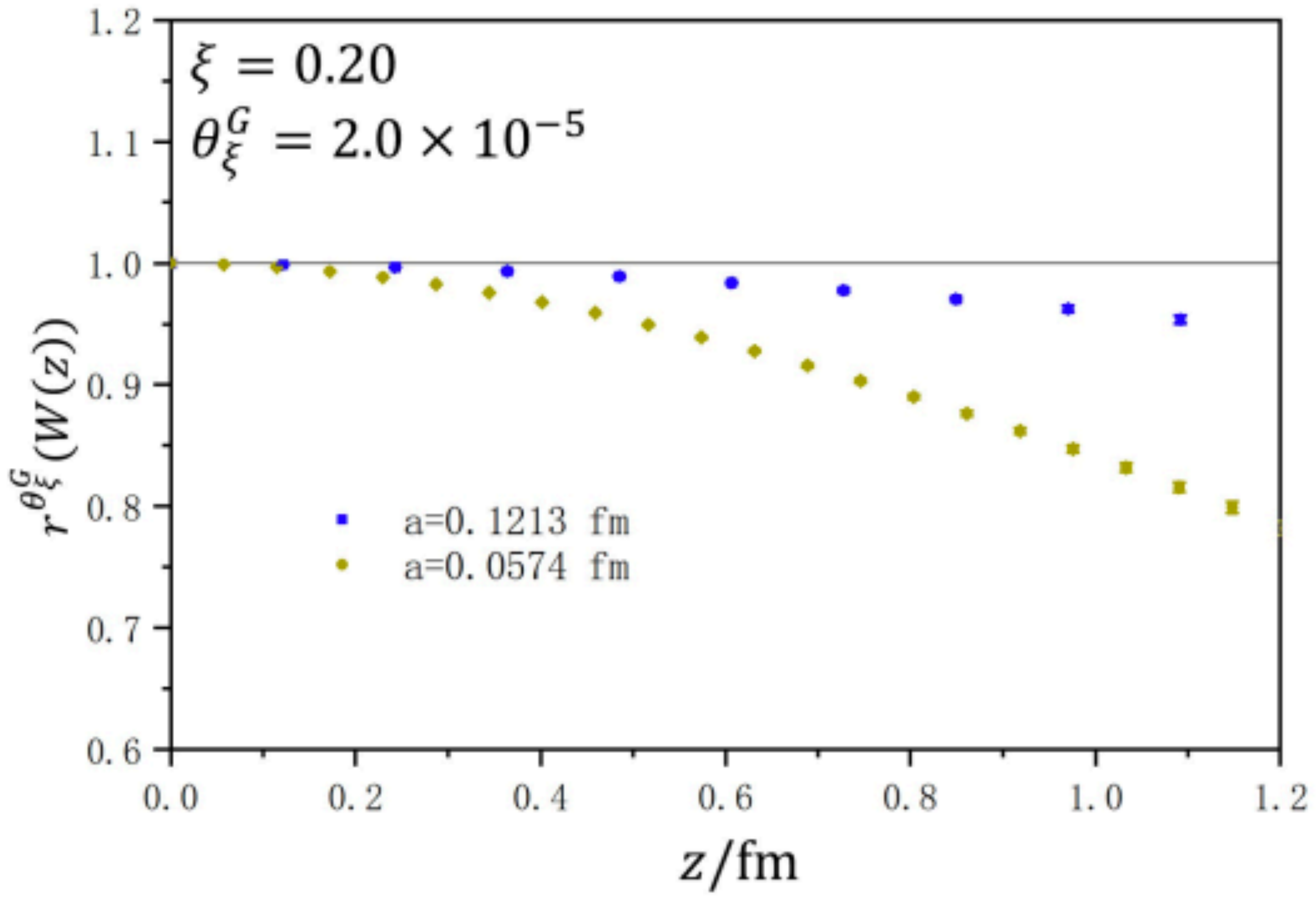
$$C^G(z) = \frac{1}{3V}\sum_x \mathrm{Tr}\left[[G(x+z)G_0^{-1}(x+z)]^\dagger [G(x)G_0^{-1}(x)]\right]$$

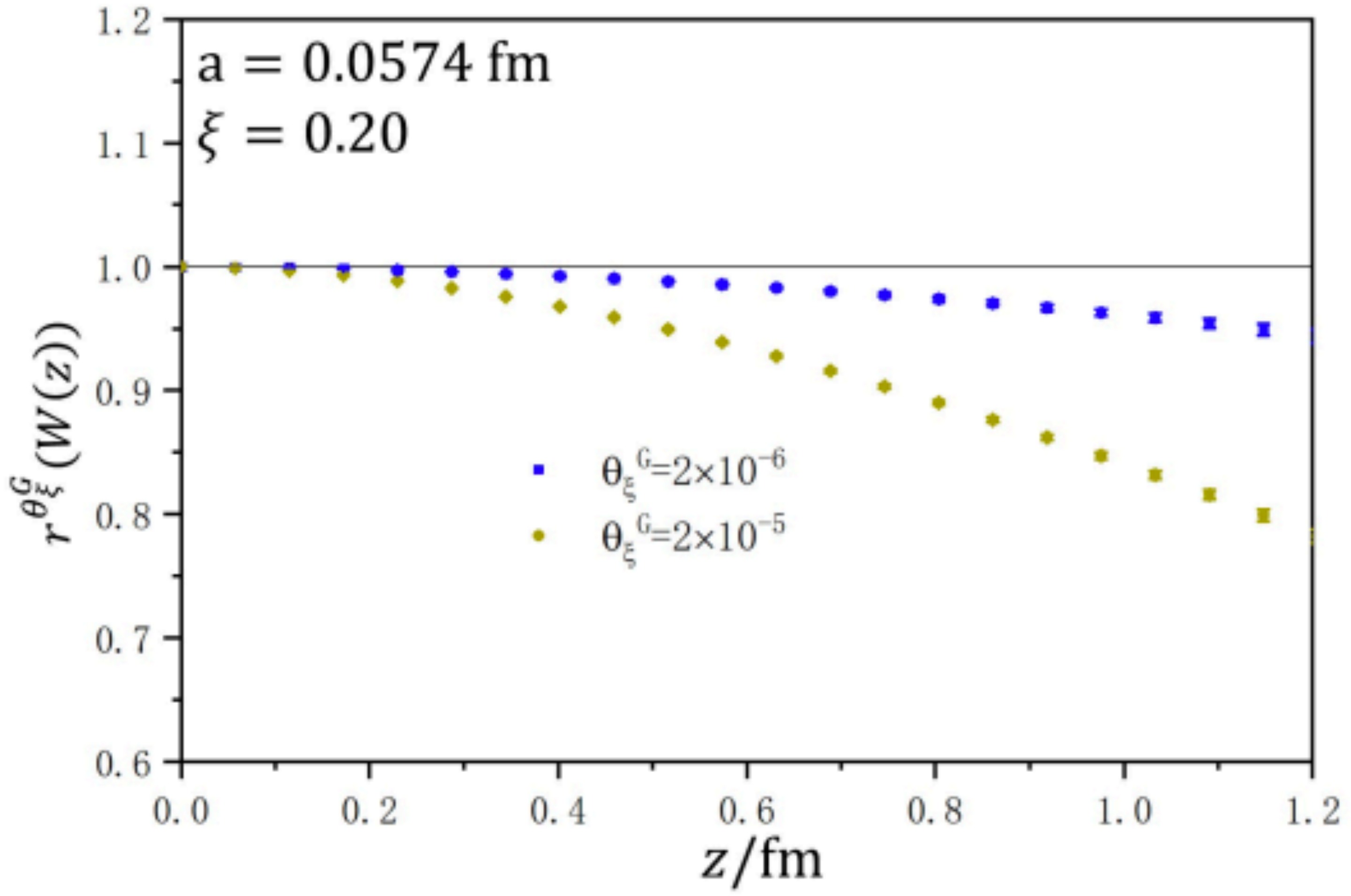
- It suggests that this effect would majorly come from the imprecise correlation $C^G(z)$ between the end points.

Wilson line $W(z) \equiv \frac{1}{3}\text{Tr}\left[\prod_{k=0}^{n-1} U_\mu(x + ka\hat{\mu})\right]$

Exploratory on the ξ gauge

$r^{\theta_\xi^G} \equiv W(z, \theta_\xi^G)/W(z, \theta_\xi^G)$





- The ξ gauge can be implemented on the lattice with limited ξ and precision $\theta_\xi^G \equiv \frac{1}{3V}\sum_x \text{Tr}[(\Delta^G(x) - \Lambda(x))^\dagger(\Delta^G(x) - \Lambda(x))]$, where $P(\Lambda(x)) \propto e^{-\frac{1}{2\xi}\text{Tr}\Lambda^2(x)}$.
- Deviation from 1 of the Wilson line ratio under ξ gauge, , $1 - W(z, \theta_\xi^G)/W(z, \theta_{\xi,0}^G = 2 \times 10^{-7})$, also shows similar gauge fixing precision dependence:
- Becomes larger with fixed a but larger δ^F;
- Becomes larger with fixed δ^F but smaller a.

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- **Quark bilinear operator with straight Wilson line (quasi-PDF/DA)**
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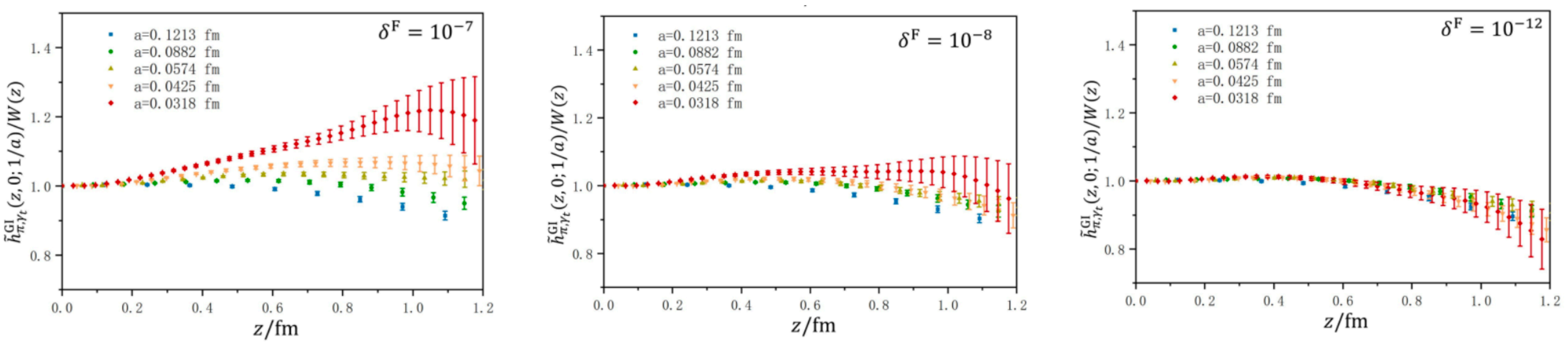
Quasi-PDF

Renormalization using the Wilson line



- Pion matrix element in the rest frame which have very tiny uncertainty:

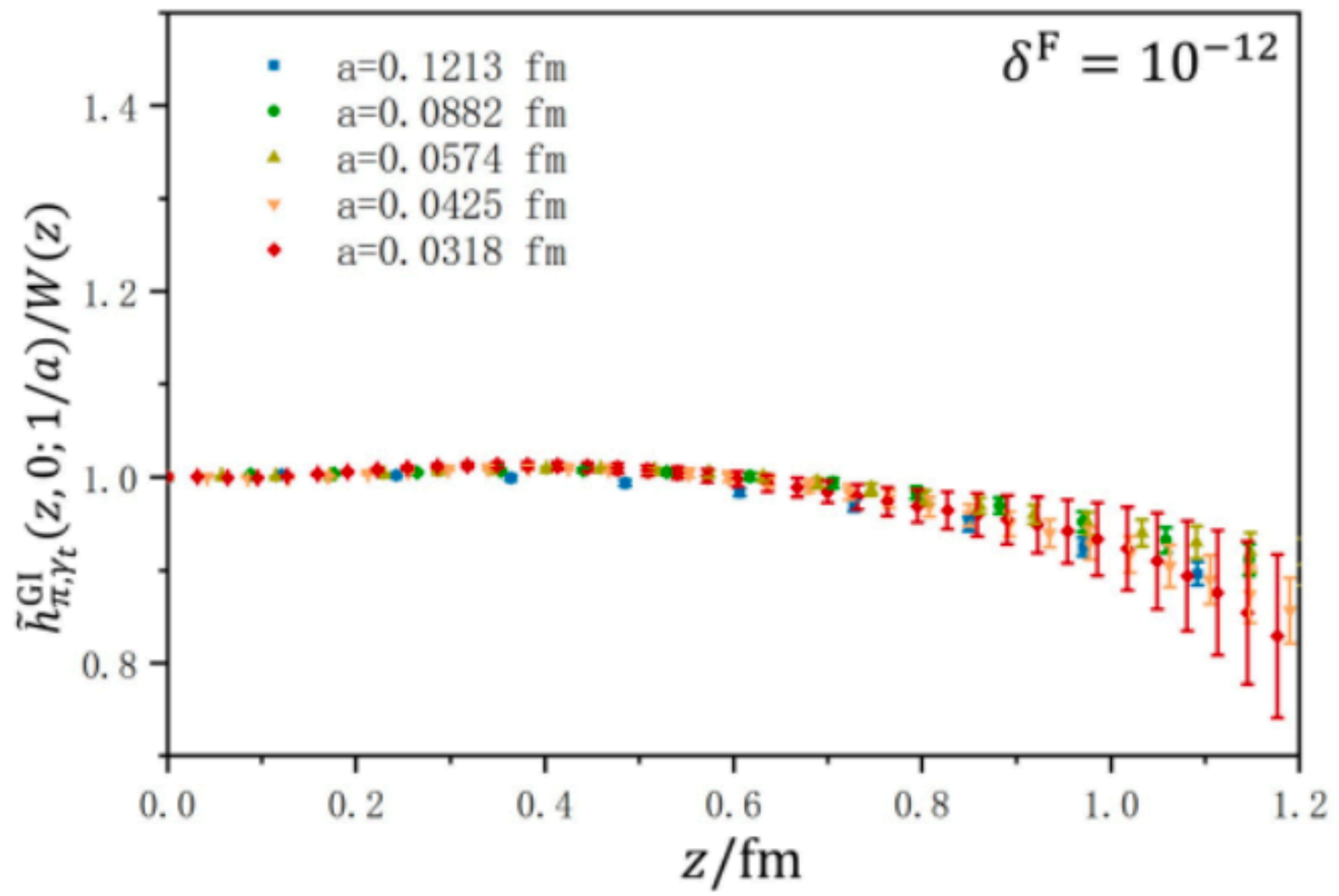
$$R_\pi(t_1, z; a, t_2) \equiv \frac{\langle O_\pi(t_2) \sum_{\vec{x}} O^{\mathrm{GI}}_{\gamma_t}(z; (\vec{x}, t_1)) O^\dagger_\pi(0) \rangle}{\langle O_\pi(t_2) O^\dagger_\pi(0) \rangle} = \langle \pi | O^{\mathrm{GI}}_{\gamma_t}(z) | \pi \rangle + \mathcal{O}(e^{-\Delta m t_1}, e^{-\Delta m (t_2 - t_1)}, e^{-\Delta m t_2});$$

- Its linear divergence is exactly the same as that of Landau gauge fixed Wilson line, when the gauge fixing is precise enough.

- Lattice spacing dependence of the ratio looks like a residual linear divergence with gauge fixing precision $\delta^F \sim 10^{-7}$.

Quasi-PDF

Renormalization using the Wilson line

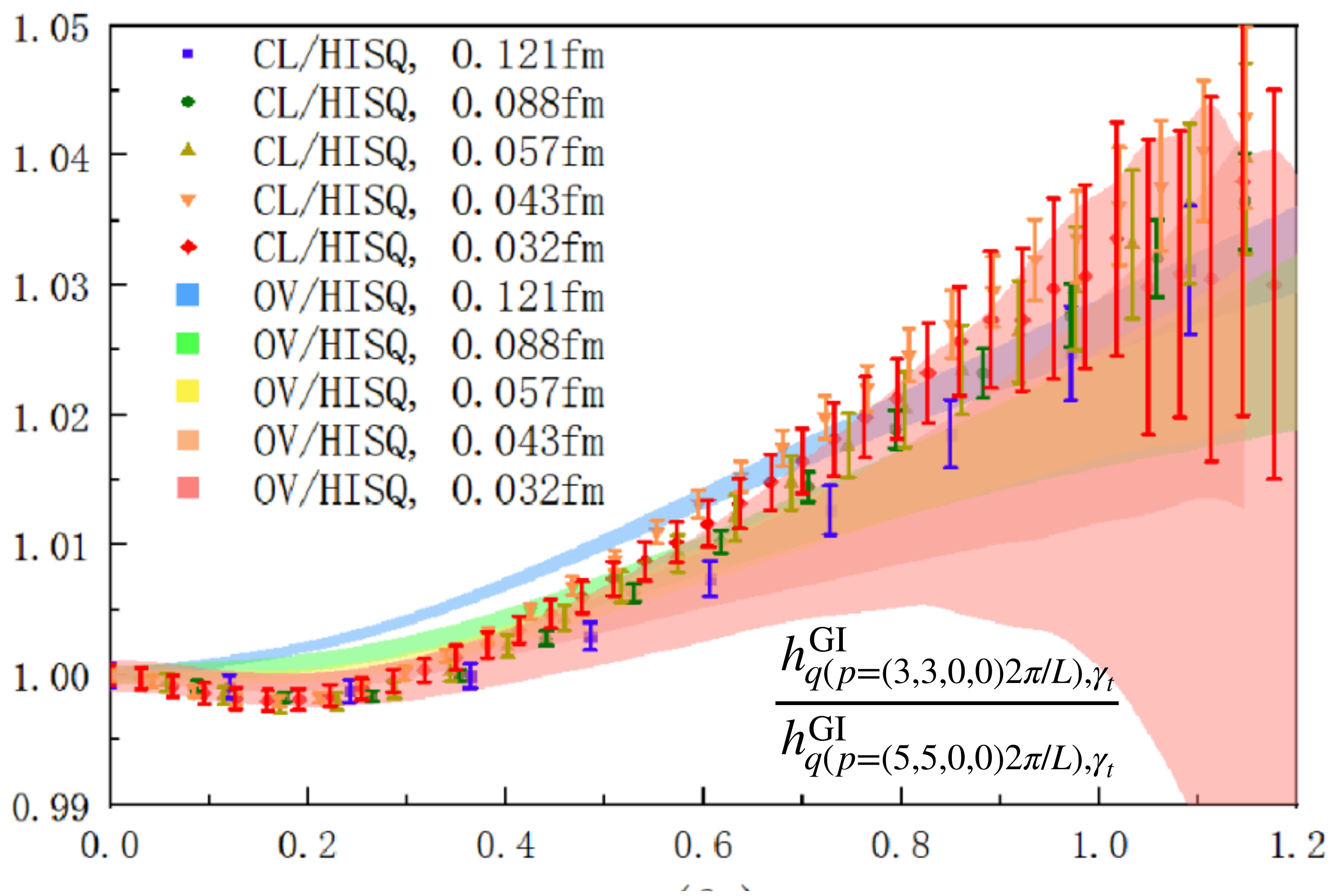


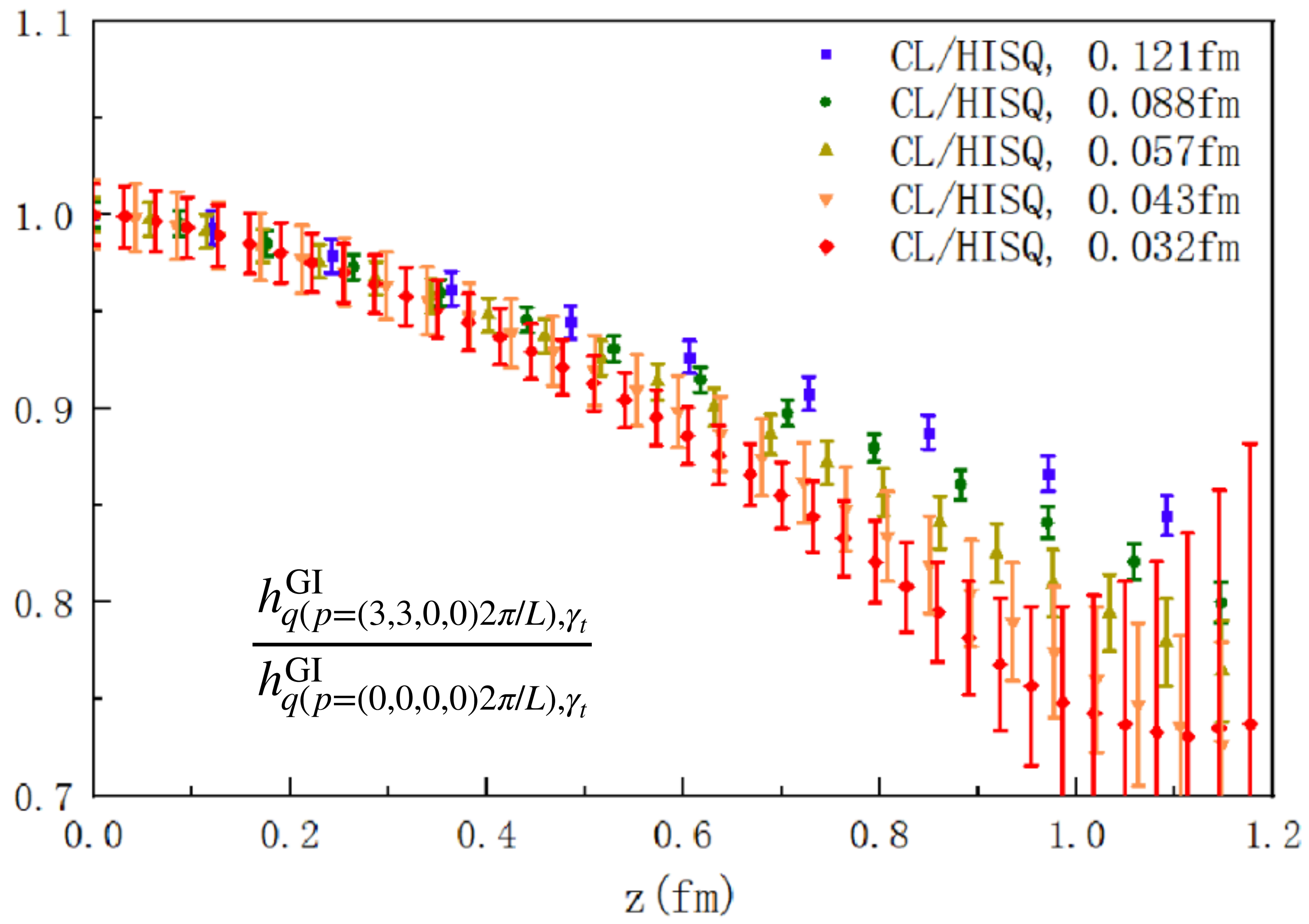
- We can further normalize the matrix element with its value at a short distance: $h^{\rm GI,SDR}_{\chi,\gamma_t}(z,P_z;1/z_0) = \dfrac{\tilde{h}^{\rm GI}_{\chi,\gamma_t}(z,P_z;1/a)/W(z)}{\tilde{h}^{\rm GI}_{\pi,\gamma_t}(z_0,0;1/a)/W(z_0)}$.
- Then we can match $h^{\rm GI,SDR}_{\chi,\gamma_t}(z,P_z;1/z_0)$ to the $\overline{\rm MS}$ scheme using the perturbative matching factor, $h^{\rm GI,\overline{MS}}_{\chi,\gamma_t}(z,P_z;\mu) = C(z,z_0,\mu)h^{\rm GI,SDR}_{\chi,\gamma_t}(z,P_z;z_0)$, where

$$
\begin{aligned}
C(z,z_0,\mu) &= \frac{\tilde{h}^{\rm GI,\overline{MS}}_{{\rm pert},\gamma_t}(z_0,0;\mu)}{W^{\overline{\rm MS}}_{\rm pert}(z_0,\mu)} W^{\overline{\rm MS}}_{\rm pert}(z,\mu) \\
&= \left\{1+\frac{\alpha_s C_F}{4\pi}\left[\xi \ln[z_0^2\mu^2]+5+\xi(2\gamma_E-\ln 4)\right]\right\} \\
&\quad \times \left\{1+\frac{\alpha_s C_F}{4\pi}(3-\xi)\left[\ln[z^2\mu^2]+2\gamma_E-\ln 4\right]\right\} + \mathcal{O}(\alpha_s^2) \\
&\xrightarrow[\xi\to 0]{} \tilde{h}^{\rm GI,\overline{MS}}_{{\rm pert},\gamma_t}(z,0;\mu) + \mathcal{O}(\alpha_s^2)\,.
\end{aligned}
$$

Quasi-PDF

Quark matrix elements





- Renormalized ME can be defined as:

$$h^{\rm GI,RI/MOM}_{\pi,\gamma_t}(z,0;\mu) \equiv Z^{\rm GI}_{\gamma_t}(z,\mu)\tilde{h}^{\rm GI}_{\pi,\gamma_t} = \frac{\tilde{h}^{\rm GI}_{\pi,\gamma_t}}{\tilde{h}^{\rm GI}_{q(p),\gamma_t}|_{\mu^2=p^2}},$$

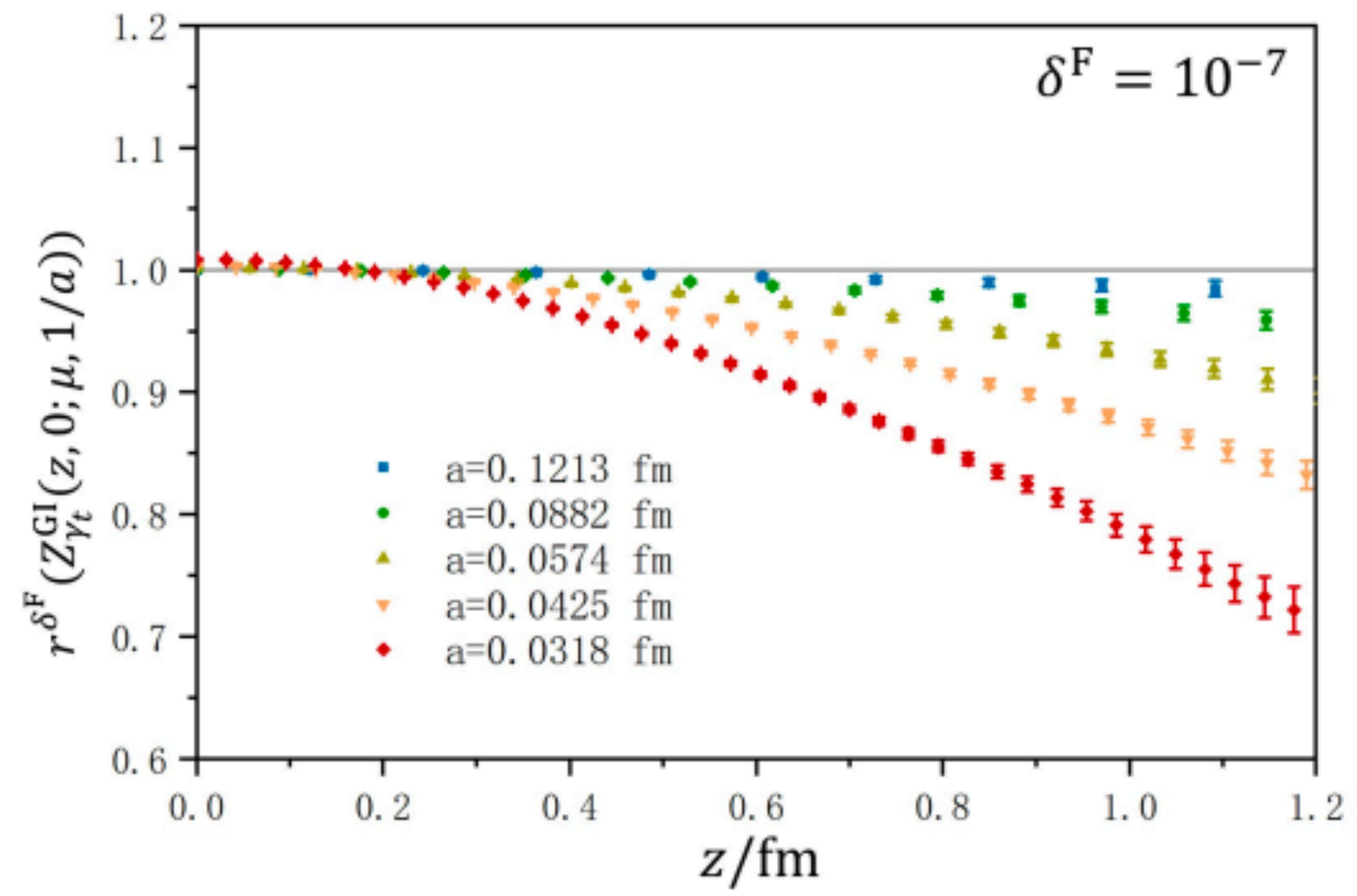
where $p = (3,3,0,0)2\pi/L$ for all the lattices with the same L but different a.

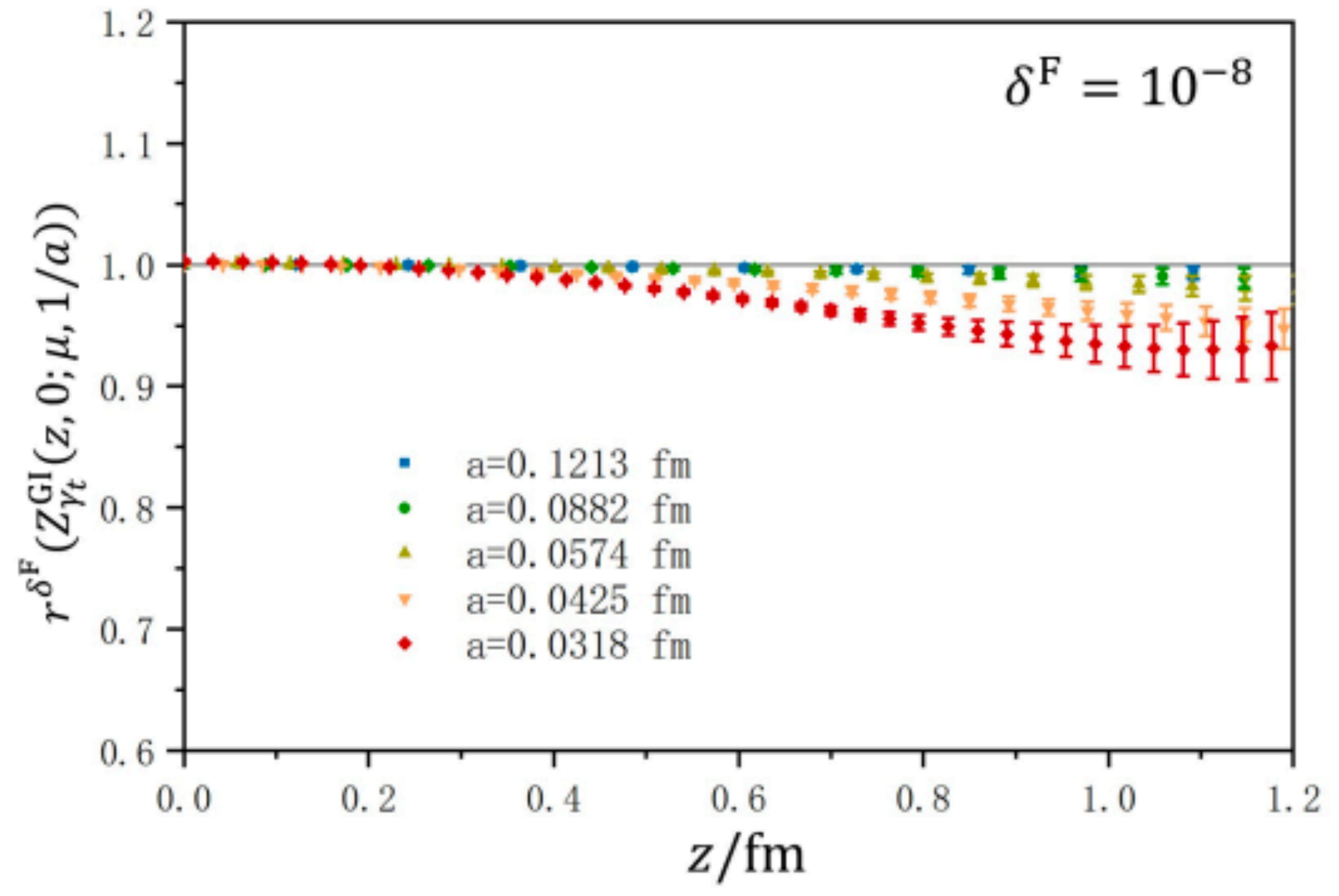
- If the UV divergence is multiplicative and then independent of the RI/MOM scale μ, the ratio $\frac{h^{\rm GI}_{q(p),\gamma_t}|_{p^2=\mu_1^2}}{h^{\rm GI}_{q(p),\gamma_t}|_{p^2=\mu_2^2}}$ should be independent of 1/a.

- It is actually true while there are some residual IR effects.

Quasi-PDF

Gauge fixing precision dependence of $Z^{\rm GI}_{\gamma_t}$



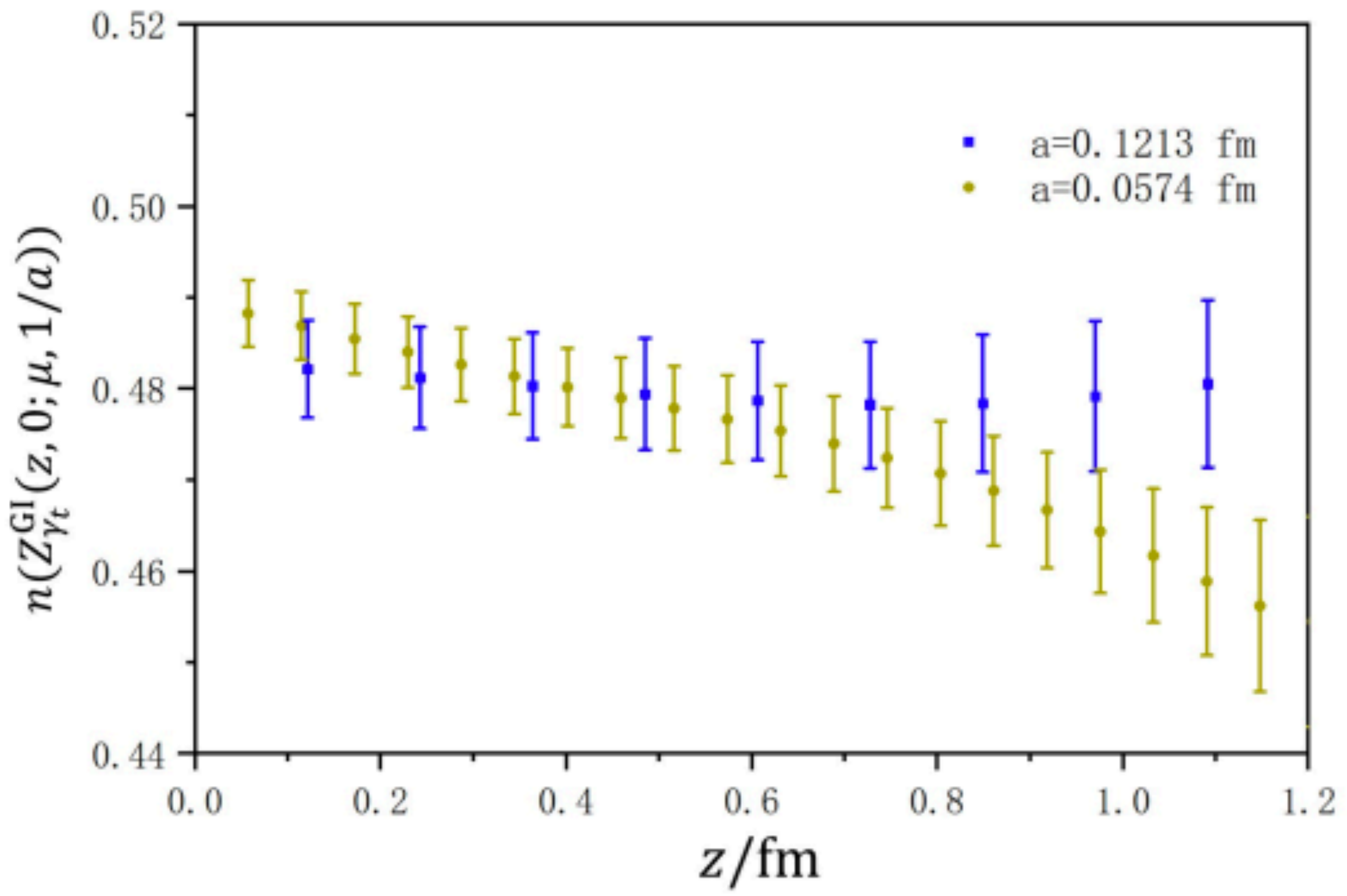


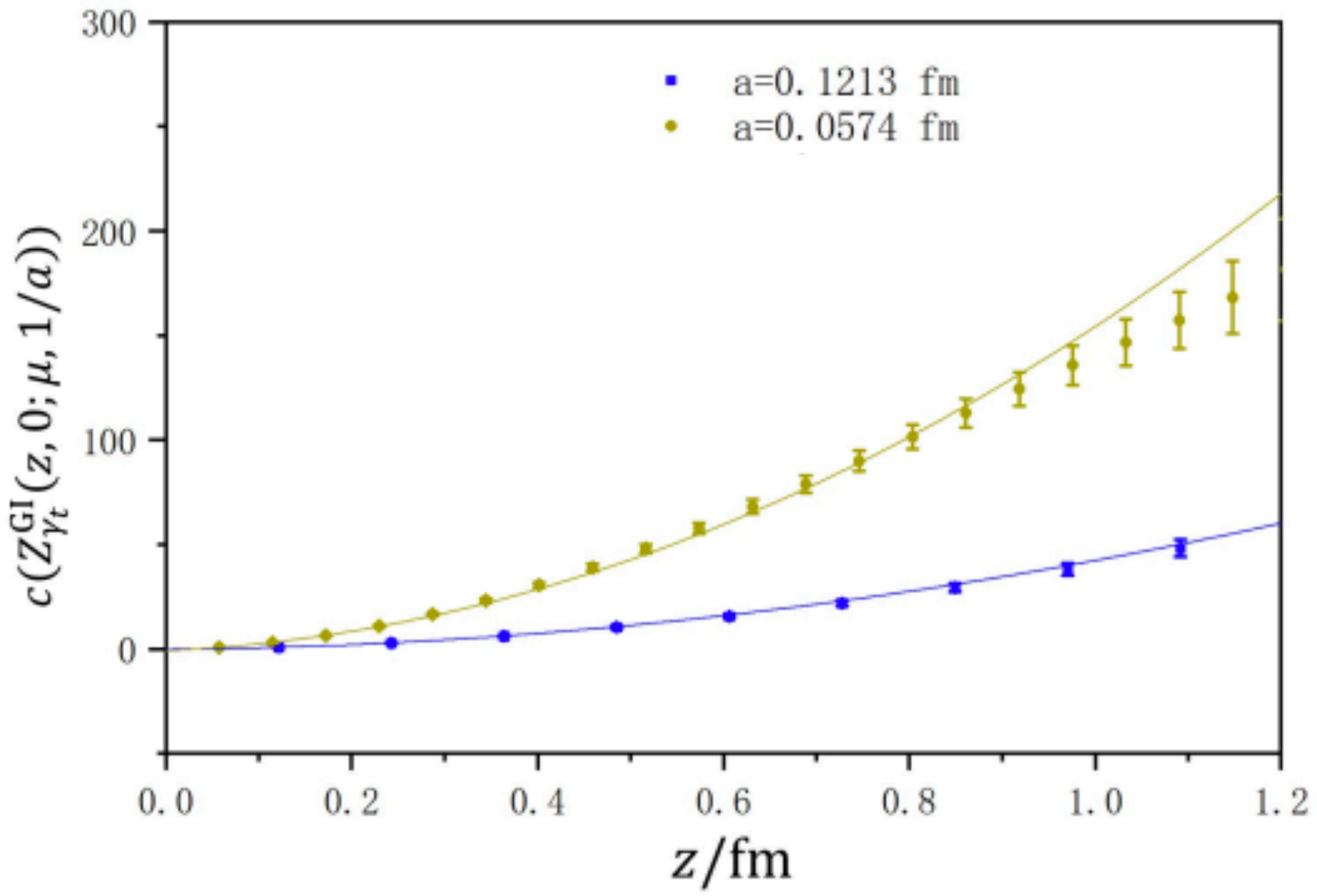
- We find similar gauge fixing precision sensitivity in $Z^{\rm GI}_{\gamma_t} \equiv 1/\tilde{h}^{\rm GI}_{q(p),\gamma_t}$;
- The deviation from 1 of the ratio with different precision can also be described using the following form:

$$W(\delta^F) = W(0)e^{-C(Z^{\rm GI}_{\gamma_t}(z))(\delta^F)^{n(Z^{\rm GI}_{\gamma_t}(z))}}$$

$$\simeq W(0)e^{-(0.55(6)z^2/a^2+\mathcal{O}(z/a^2))(\delta^F)^{0.48}}$$

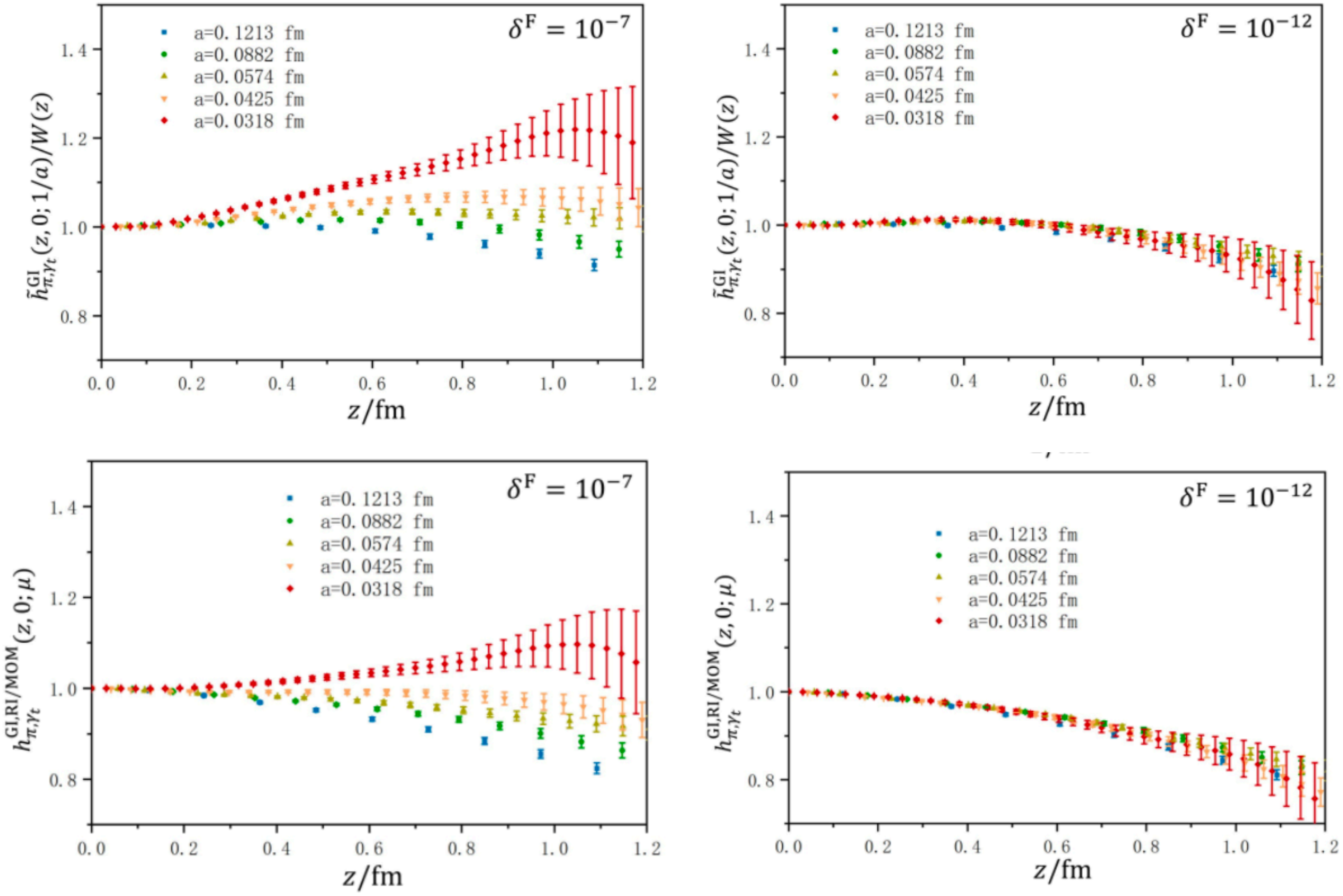
with all the a and z we investigated.





Quasi-PDF

SDR (Wilson line) v.s. RI/MOM



- Impact of the gauge fixing precision in the Wilson line and quark matrix elements are similar, while the latter case is ~10% smaller.
- The difference in the renormalized matrix element should be eliminated by the perturbative matching, at least at short distance.

Quasi-PDF

Self-renormalization

- The self-renormalization fits the hadron (e.g., pion) matrix elements in the rest frame at different lattice spacings, using the following formula:

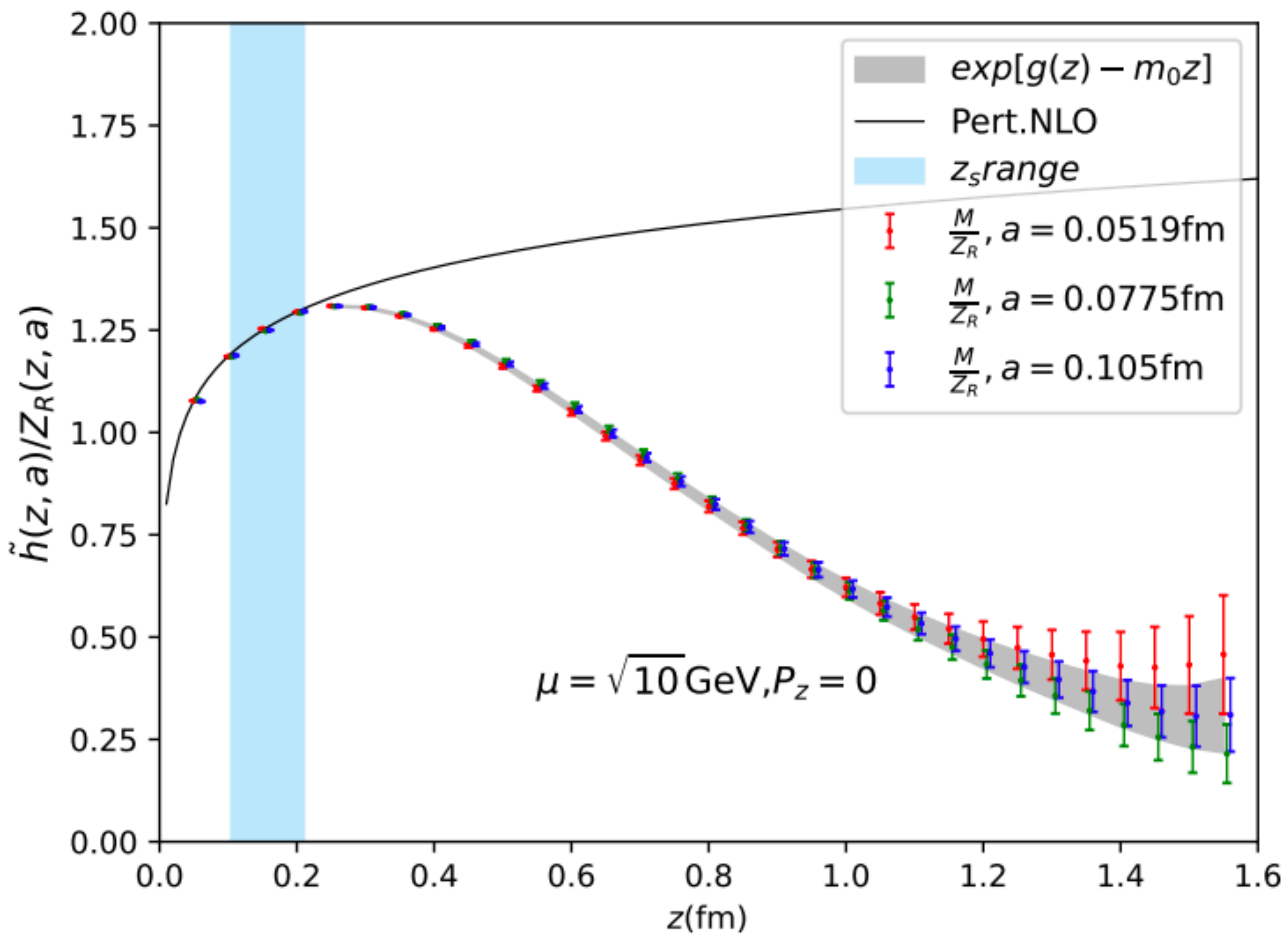
$$\ln \tilde{h}^{\rm GI}_{\pi,\gamma_t}(z,0;1/a) = \frac{kz}{a\ln[a\Lambda_{\rm QCD}]} + f(z)a^2 + m_0 z + \frac{3C_f}{b_0}\ln\left[\frac{\ln(1/a\Lambda_{\rm QCD})}{\ln(\mu/\Lambda_{\rm QCD})}\right] + \frac{1}{2}\ln\left[1+\frac{d}{\ln[a\Lambda_{\rm QCD}]}\right]^2; + \begin{cases} \ln\left[\tilde{h}^{\rm GI}_{{\rm pert},\gamma_t}(z,\mu,\Lambda_{\overline{\rm MS}})\right] & \text{if } z_0 \le z \le z_1 \\ g(z) & \text{if } z_1 < z \end{cases},$$

where $k, \Lambda_{\rm QCD}, m_0, d, g(z)$ and $f(z)$ at interpolated $z = 0.05n$ fm are free parameters.

- Then the renormalized matrix element is defined as

$$\tilde{h}^{\rm GI}_{\pi,\gamma_t}(z,0;1/a) = \begin{cases} \tilde{h}^{\rm GI}_{{\rm pert},\gamma_t}(z,\mu,\Lambda_{\overline{\rm MS}}) & \text{if } z_0 \le z \le z_1 \\ {\rm Exp}[g(z)] & \text{if } z_1 < z \end{cases}.$$

parameter	fit result
k	0.674(0.008)
$\Lambda_{\rm QCD}$(GeV)	0.16(0.03)
m_0(GeV)	0.14(0.06)
d	−0.04(0.02)
$\chi^2/d.o.f.$	1.03



Quasi-PDF

Renormalized matrix elements

- Self-renormalization:

$$Z_{\rm self}^{\overline{\rm MS}}(z,\mu;1/z_s) = e^{-\frac{kz}{a\ln[a\Lambda_{\rm QCD}]}-f(z)a^2-m_0 z-\frac{3C_f}{b_0}\ln\left[\frac{\ln(1/a\Lambda_{\rm QCD})}{\ln(\mu/\Lambda_{\rm QCD})}\right]-\frac{1}{2}\ln\left[1+\frac{d}{\ln[a\Lambda_{\rm QCD}]}\right]^2};$$

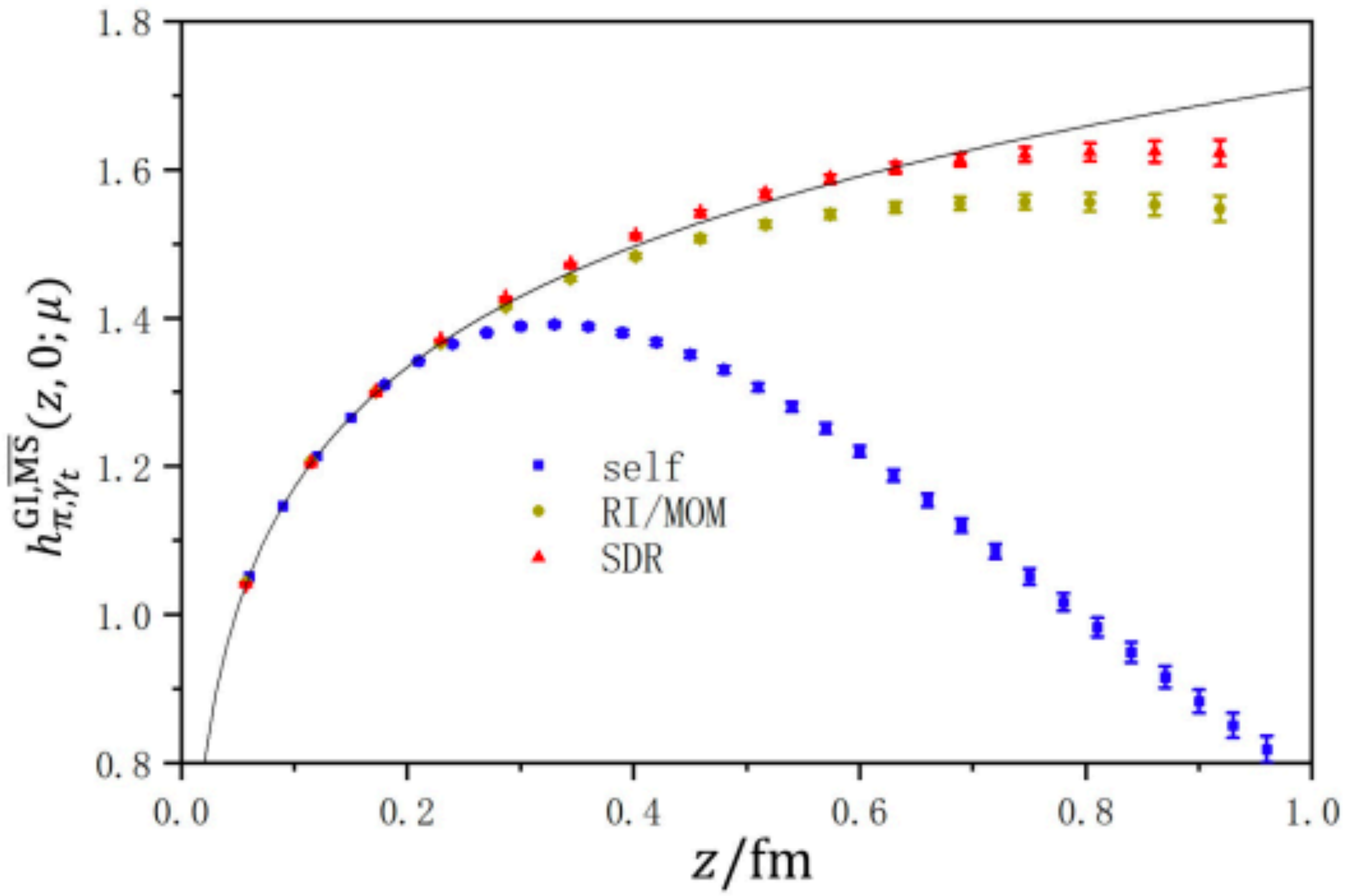
- RI/MOM renormalization:

$$Z_{\rm RI/MOM}^{\overline{\rm MS}}(z,\mu;\mu_0) = \frac{\tilde{h}^{\rm GI,\overline{MS}}_{{\rm pert},\gamma_t}(z,\mu)}{\tilde{h}^{\rm GI,RI/MOM}_{{\rm pert},\gamma_t}(z,\mu)}\frac{\tilde{h}^{\rm GI,RI/MOM}_{{\rm pert},\gamma_t}(z,\mu_0)}{\tilde{h}^{\rm GI,RI/MOM}_{q(p),\gamma_t}(z,\mu_0=\sqrt{p^2})};$$

- Short distance renormalization (SDR):

$$Z_{\rm SDR}^{\overline{\rm MS}}(z,\mu;1/z_0) = \frac{\tilde{h}^{\rm GI,\overline{MS}}_{{\rm pert},\gamma_t}(z_0,0;\mu)}{W^{\overline{\rm MS}}_{\rm pert}(z_0,\mu)}\frac{W^{\overline{\rm MS}}_{\rm pert}(z,\mu)/W(z)}{\tilde{h}^{\rm GI}_{\pi,\gamma_t}(z_0,0;1/a)/W(z_0)}.$$

- All the cases agree with each other at short distance;
- But both RI/MOM and SDR suffer from sizable renormalon effect at large z.



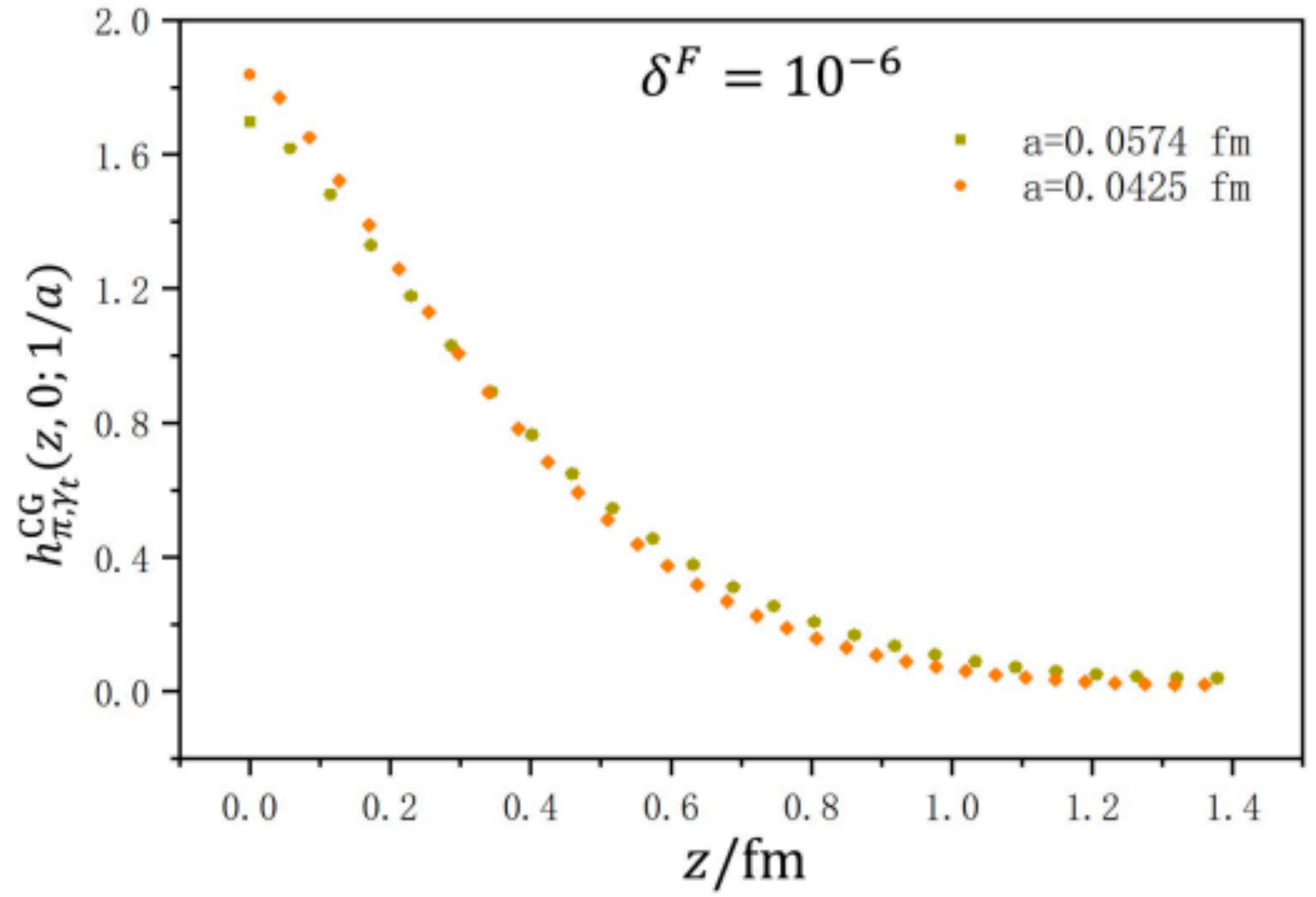
Quasi-PDF

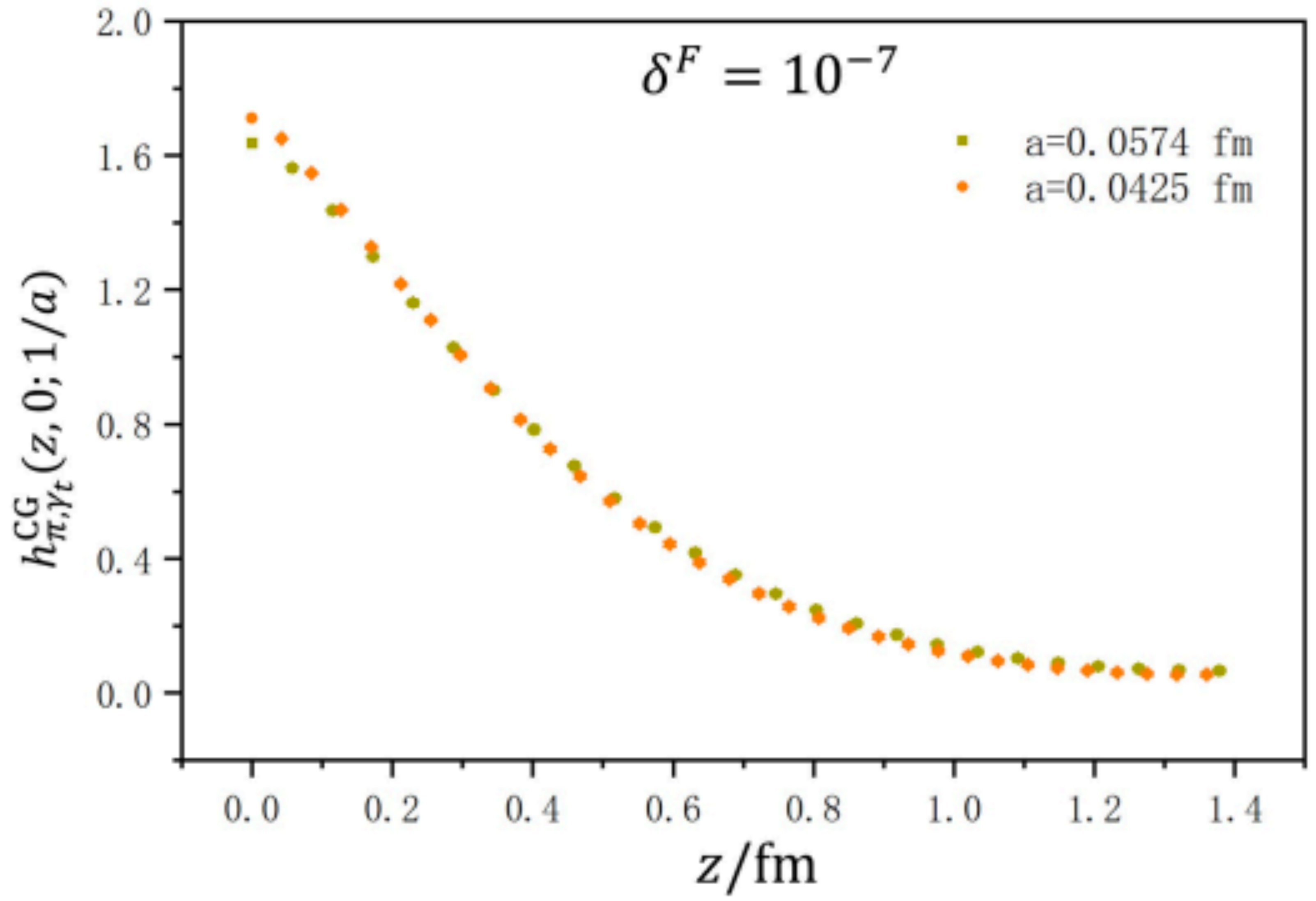
Under Coulomb gauge without Wilson line

- If we consider the quasi-PDF under the Coulomb gauge but without Wilson line,$O_{\Gamma}^{\rm CG}(z) = \bar{\psi}(0)\Gamma\psi(z)$, it should be free of the linear divergence after normalization at certain $z_0 \sim 0.2$ fm.

- Coulomb gauge fixing shall also be obtained iteratively by maximum the functional $F_U[G] \equiv \frac{1}{9V}\text{Re Tr}\sum_x \sum_{\mu=1}^{3} U_{\mu}^{G}(x)$, and stop when the difference $\delta^F(n) = F_U[G(n)] - F_U[G(n-1)]$ is smaller then the required precision δ^F.

- Practical calculation verified this prediction, when the Coulomb gauge fixing precision is high enough.





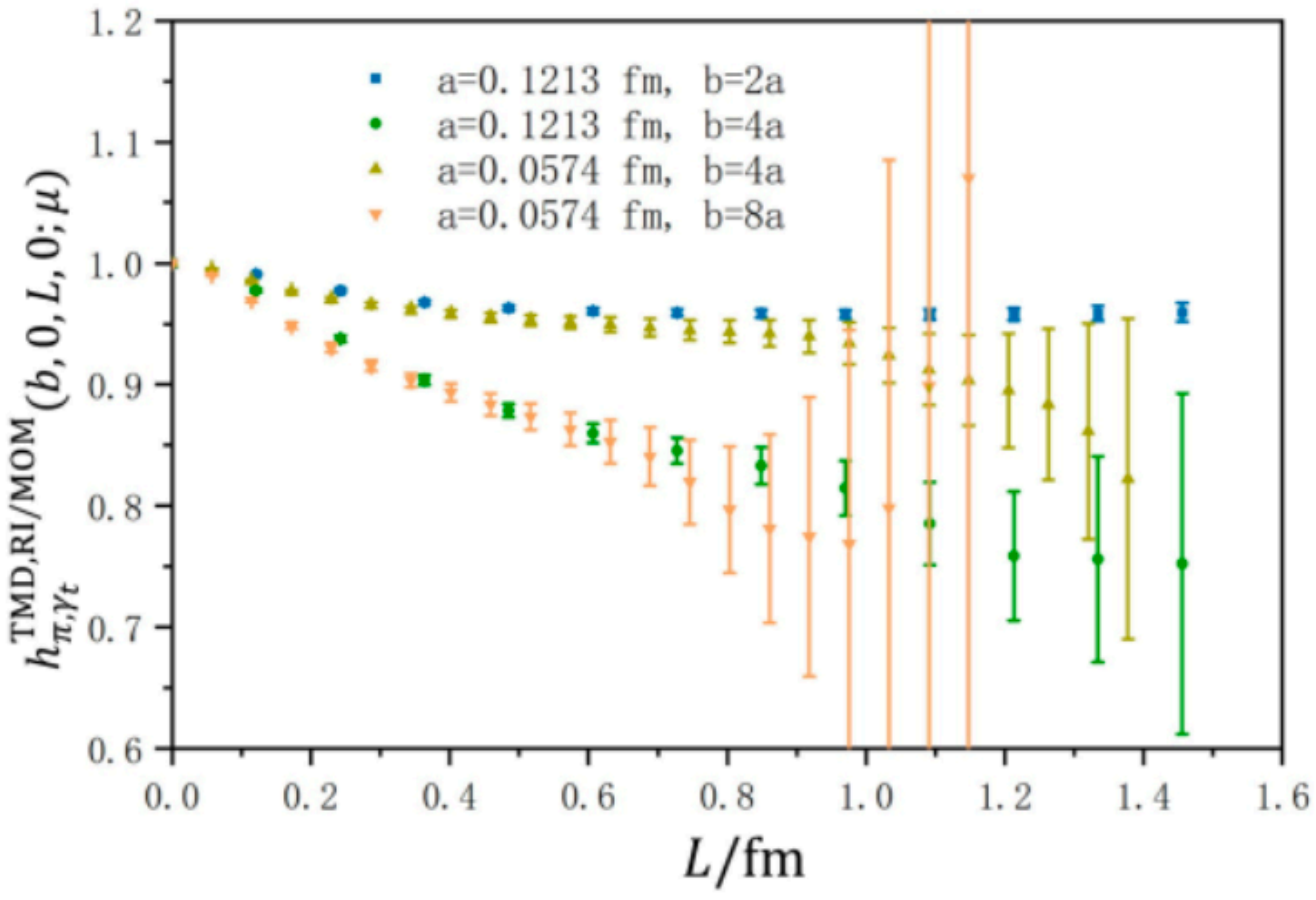
Outline

- Wilson lines
- Quark bilinear operator with straight Wilson line (quasi-PDF/DA)
- **Quark bilinear operator with staple-shaped Wilson line (quasi-TMD PDF/DA)**
- Summary

Quasi-TMD

$$O_\Gamma^{\rm TMD}(b,z,L) \equiv \bar\psi(\vec 0_\perp, 0)\Gamma \mathscr{W}(b,z,L)\psi(\vec b_\perp, z),$$

$$\mathscr{W}(b,z,L) \equiv \mathscr{P}\exp\left[ig_0\int_{-L}^{z} {\rm d}s\ \hat n_z\cdot A^m(b\hat n_\perp + s\hat n_z)T^m\right] \times \mathscr{P}\exp\left[ig_0\int_{0}^{b} {\rm d}s\ \hat n_\perp\cdot A^m(s\hat n_\perp - L\hat n_z)T^m\right] \times \mathscr{P}\exp\left[ig_0\int_{0}^{-L} {\rm d}s\ \hat n_z\cdot A^m(s\hat n_z)T^m\right].$$



L dependence of the RI/MOM renormalized ME

- To get rid of the pinch pole singularity and the linear divergence from the Wilson link self-energy, we define the "subtracted" quasi-TMD PDF ME as,

$$h^{\rm TMD}_{\chi,\gamma_t}(b,z;1/a) = \lim_{L\to\infty}\frac{\tilde h^{\rm TMD}_{\chi,\gamma_t}(b,z,L;1/a)}{\sqrt{Z_E(b,2L+z;1/a)}};$$

- The RI/MOM renormalization cancels the $\sqrt{Z_E(b,2L+z;1/a)}$ factor,

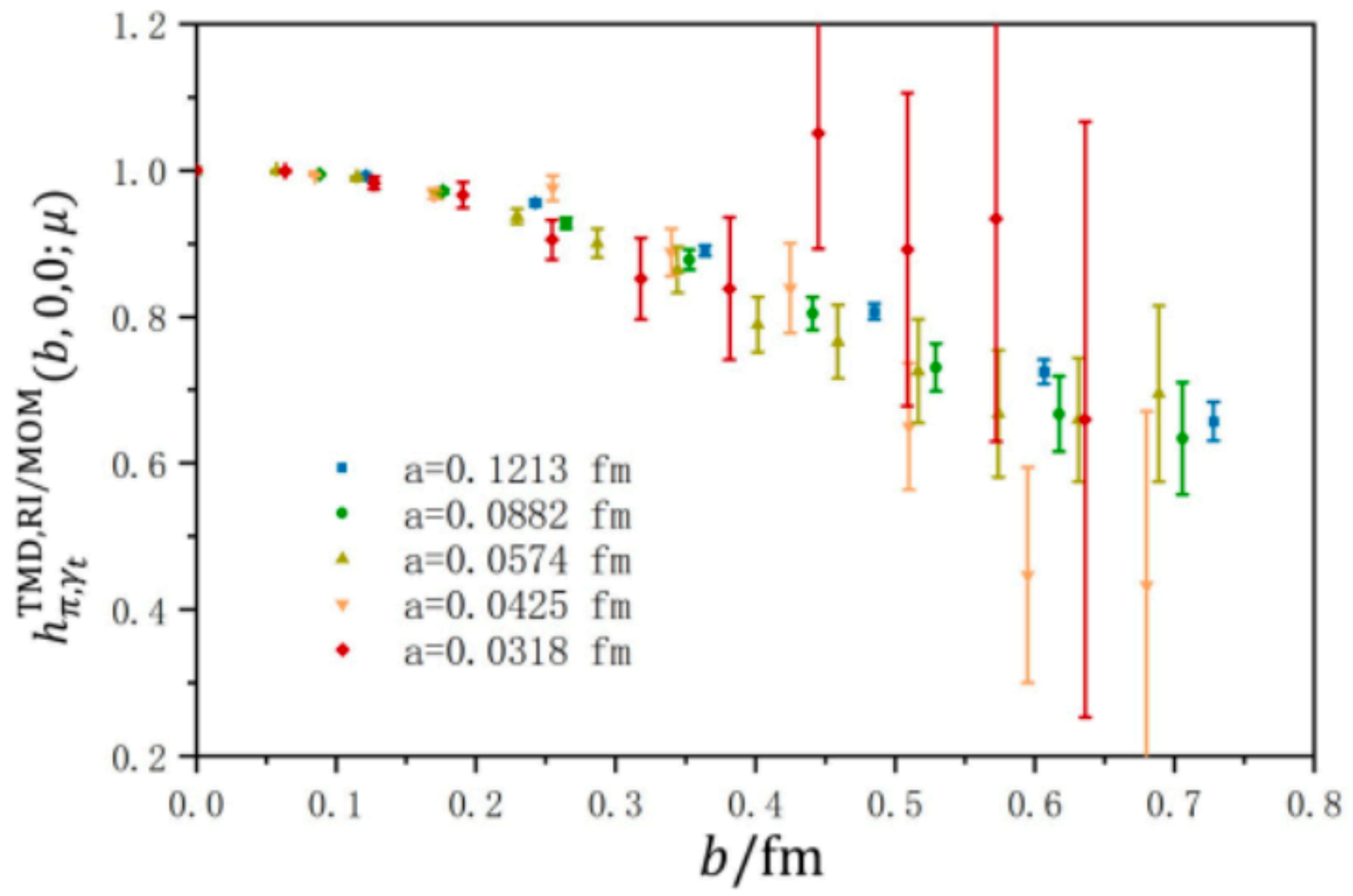
$$h^{\rm TMD,RI/MOM}_{\chi,\gamma_t}(b,z;1/a,\mu) = \lim_{L\to\infty}\frac{h^{\rm TMD}_{\chi,\gamma_t}(b,z,L;1/a)}{h^{\rm TMD}_{q,\gamma_t}(b,z,L;1/a)|_{p^2=\mu^2}} = \lim_{L\to\infty}\frac{\tilde h^{\rm TMD}_{\chi,\gamma_t}(b,z,L;1/a)}{\tilde h^{\rm TMD}_{q,\gamma_t}(b,z,L;1/a)|_{p^2=\mu^2}}.$$

- But eventually the L dependence can saturate at large enough L.

Quasi-TMD

$$O_\Gamma^{\mathrm{TMD}}(b,z,L) \equiv \bar{\psi}(\vec{0}_\perp,0)\Gamma\mathscr{W}(b,z,L)\psi(\vec{b}_\perp,z),$$

$$\mathscr{W}(b,z,L) \equiv \mathscr{P}\exp\left[ig_0\int_{-L}^{z}\mathrm{d}s\ \hat{n}_z\cdot A^m(b\hat{n}_\perp+s\hat{n}_z)T^m\right]$$
$$\times\mathscr{P}\exp\left[ig_0\int_{0}^{b}\mathrm{d}s\ \hat{n}_\perp\cdot A^m(s\hat{n}_\perp-L\hat{n}_z)T^m\right]$$
$$\times\mathscr{P}\exp\left[ig_0\int_{0}^{-L}\mathrm{d}s\ \hat{n}_z\cdot A^m(s\hat{n}_z)T^m\right].$$



RI/MOM renormalized matrix element

- With high enough gauge fixing, the RI/MOM renormalization

$$h_{\chi,\gamma_t}^{\mathrm{TMD,RI/MOM}}(b,z;1/a,\mu) = \lim_{L\to\infty}\frac{h_{\chi,\gamma_t}^{\mathrm{TMD}}(b,z,L;1/a)}{h_{q,\gamma_t}^{\mathrm{TMD}}(b,z,L;1/a)|_{p^2=\mu^2}} = \lim_{L\to\infty}\frac{\tilde{h}_{\chi,\gamma_t}^{\mathrm{TMD}}(b,z,L;1/a)}{\tilde{h}_{q,\gamma_t}^{\mathrm{TMD}}(b,z,L;1/a)|_{p^2=\mu^2}}$$

can remove all the UV divergence properly.

- Investigation of TMD under Coulomb gauge should be essential to improve the signal at large b/z.

Quasi-TMD

Renormalized matrix elements using different schemes

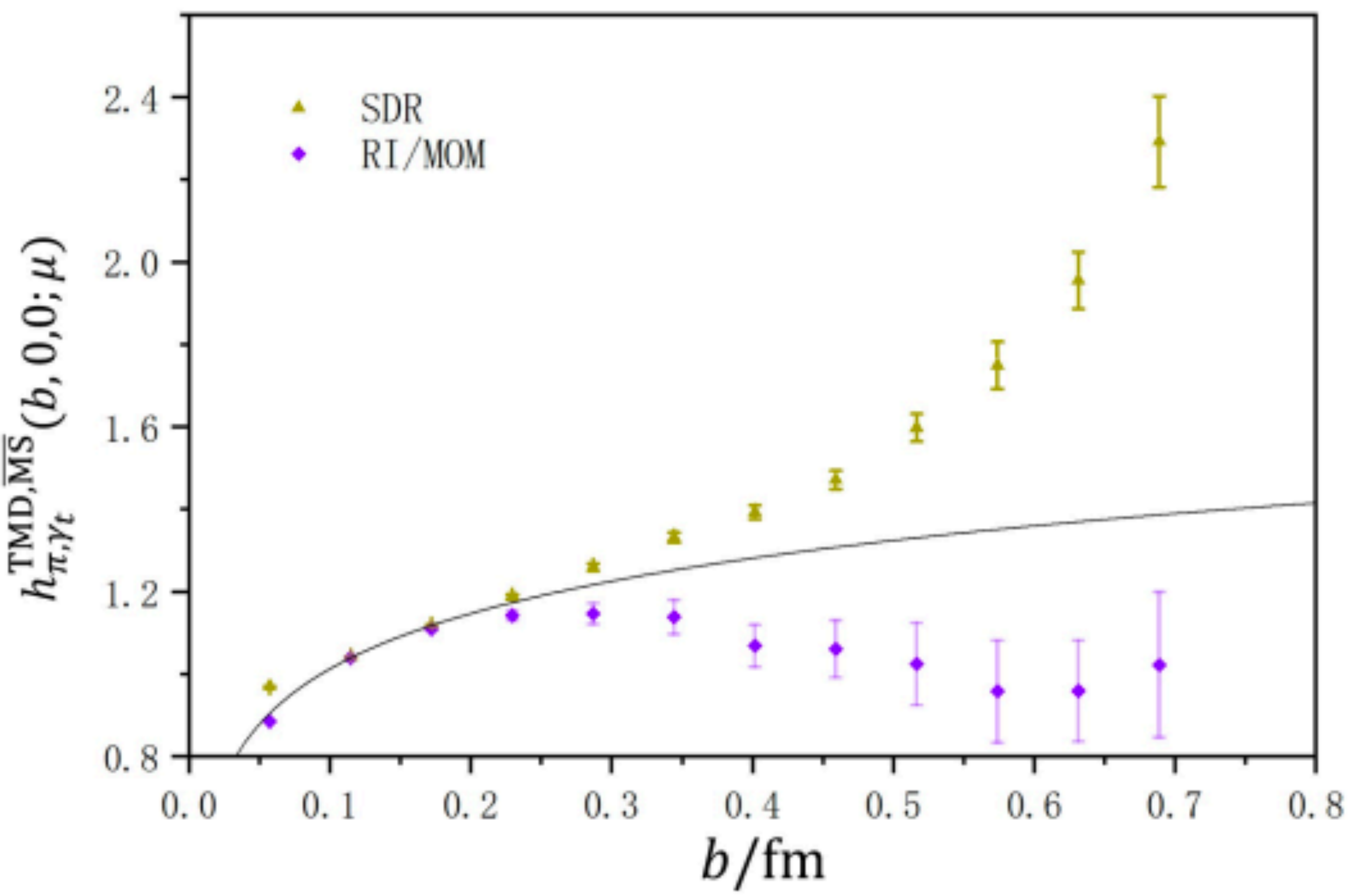
- Short distance renormalization (SDR):

$$Z_{\rm SDR}^{\overline{\rm MS}}(z,\mu;1/z_0)=\frac{h_{{\rm pert},\gamma_t}^{{\rm TMD},\overline{\rm MS}}(b_0,0,0;\mu)}{h_{\pi,\gamma_t}^{{\rm TMD},\overline{\rm MS}}(b_0,0,0;1/a)};$$

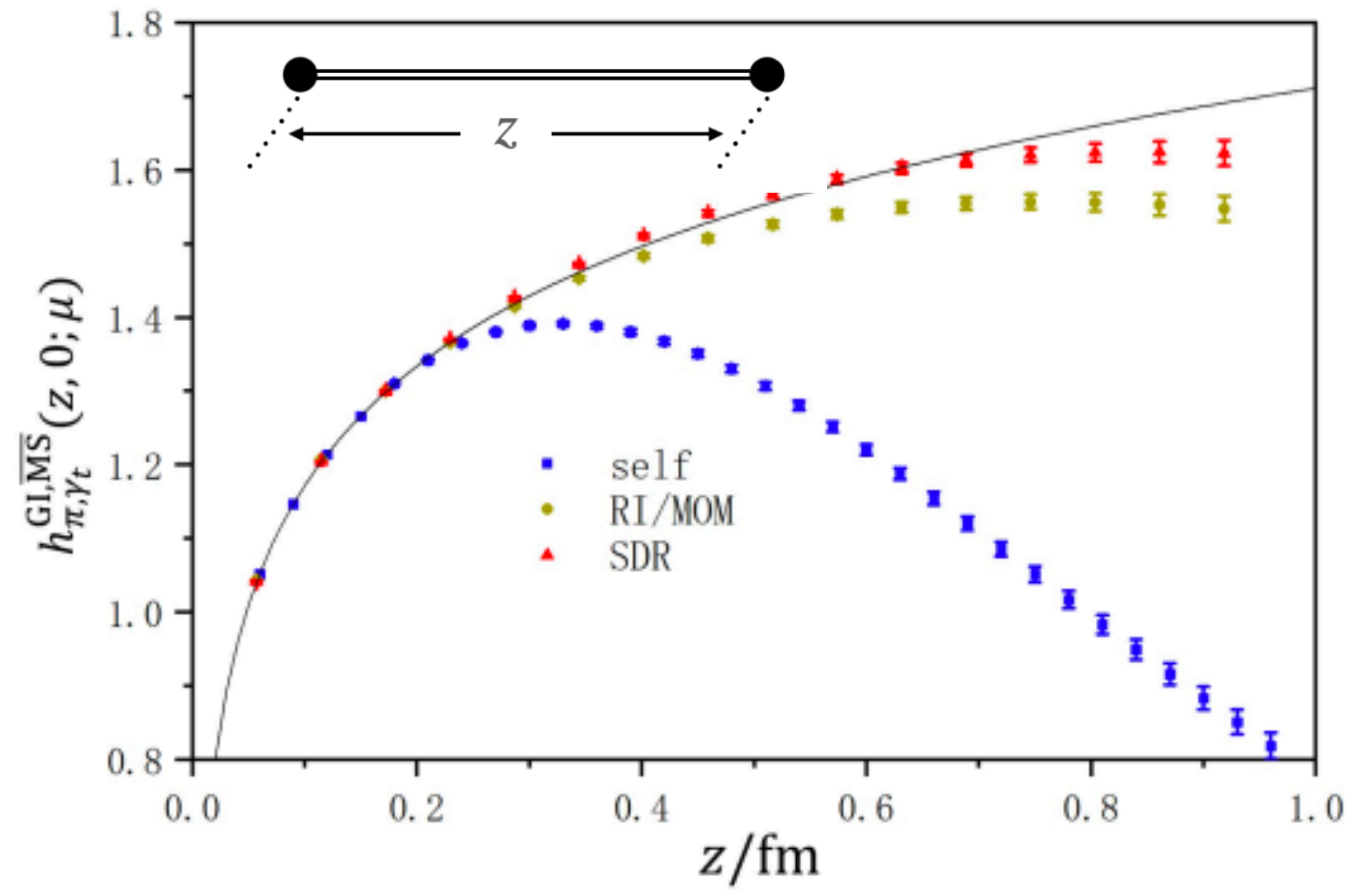
- RI/MOM renormalization:

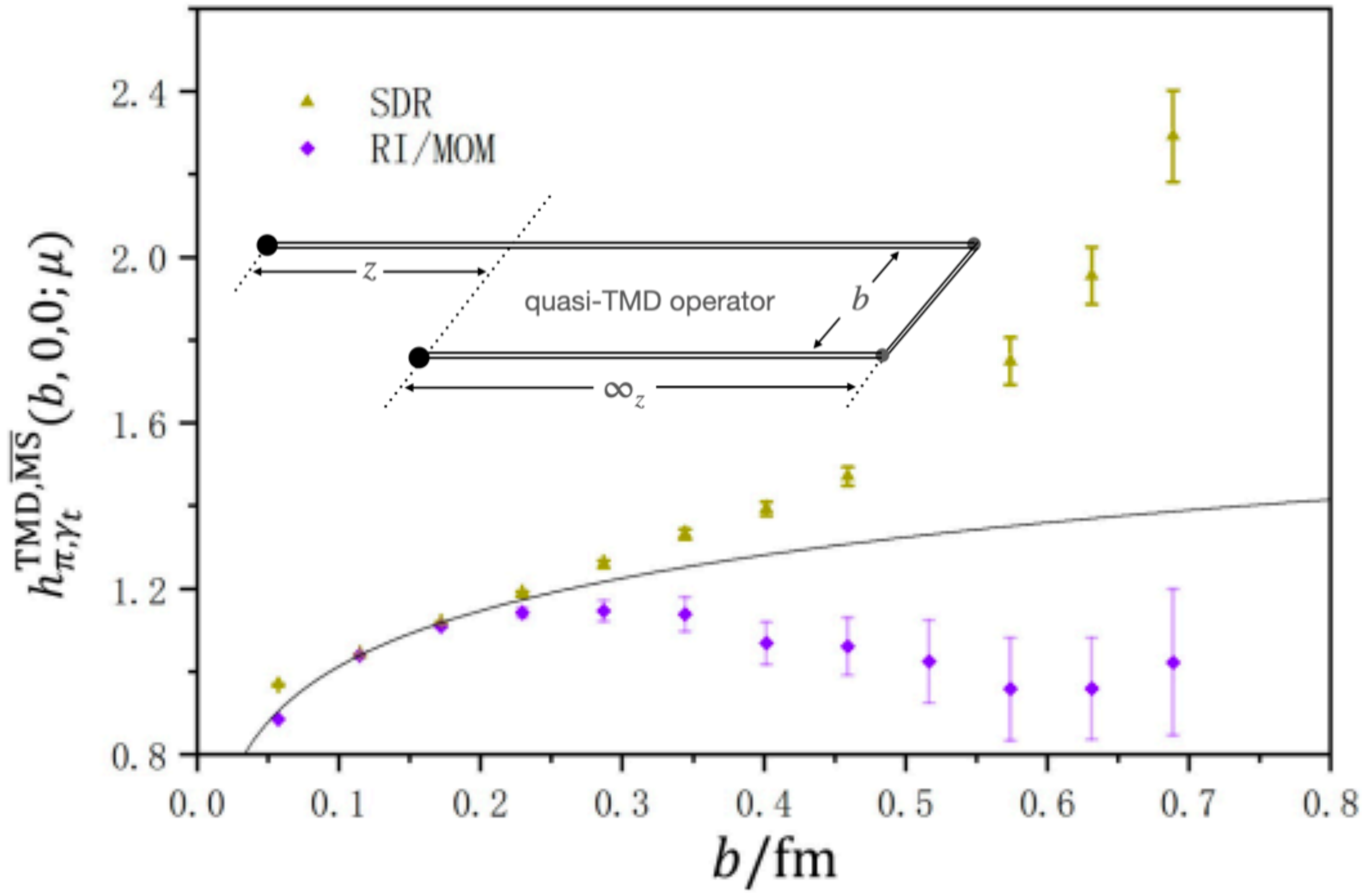
$$Z_{\rm RI/MOM}^{\overline{\rm MS}}(z,\mu;\mu_0)=\frac{\tilde{h}_{{\rm pert},\gamma_t}^{{\rm TMD},\overline{\rm MS}}(z,\mu)}{\tilde{h}_{{\rm pert},\gamma_t}^{{\rm TMD,RI/MOM}}(z,\mu)}\frac{\tilde{h}_{{\rm pert},\gamma_t}^{{\rm TMD,RI/MOM}}(z,\mu_0)}{\tilde{h}_{q(p),\gamma_t}^{{\rm TMD,RI/MOM}}(z,\mu_0=\sqrt{p^2})};$$

- All the cases agree with each other at short distance;
- But RI/MOM suffers from sizable renormalon effect at large z.



Summary





- Linear divergence can be properly removed with kinds of renormalization schemes, if the gauge fixing precision is high enough:

1. Different renormalization schemes agree with each other at small distance;

2. But they can be significantly different due to the IR effect at large distance.

- If the gauge fixing precision δ^F is finite, the residual effect can be described by an empirical form $\sim 0.6(1)z^2/a^2(\delta^F)^{0.49(1)}$ at large z.