Extracting Meson Distribution Amplitudes from Lattice QCD at Next-to-Next-to-Leading Order

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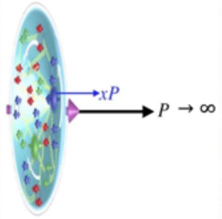
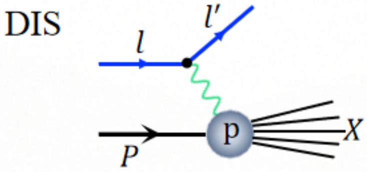
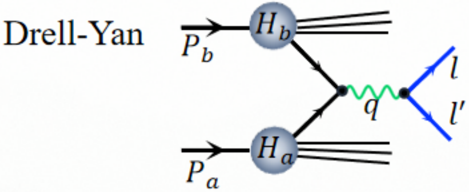
OUTLINE

Introduction

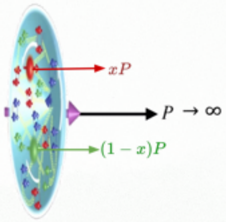
- **Collinear PDFs: the probability distribution of a single parton within a hadron (Inclusive process)**

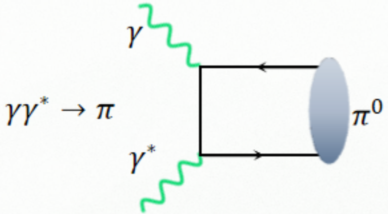


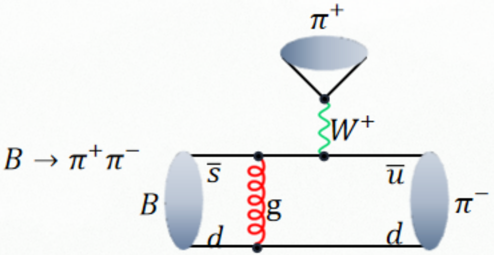




- **LCDAs: the probability distribution amplitude for partons within a hadron (Exclusive process)**





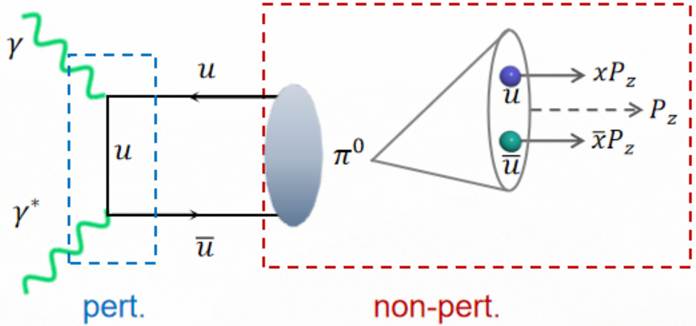


$$if_M(P^+)\phi_M(x) = \int \frac{d\xi^-}{2\pi} e^{ixp^+\xi^-} \langle 0|\bar{\psi}_1(0)\gamma^+\gamma_5 \mathcal{W}(0,\xi^-)\psi_2(\xi^-)|M(P)\rangle$$

LCDA: an essential input in exclusive processes

- **QCD factorization: a simple example**

$\gamma\gamma^* \to \pi$



$\bar{x} = 1 - x$

$$F_{\gamma\gamma^* \to \pi} \approx \textbf{hard scattering kernels} \otimes \textbf{light-cone distribution amplitudes}$$

- Exclusive processes are important to determining the fundamental parameters of SM and searching for new physics.

The research progress of LCDAs

- Asymptotic LCDAs

G. P. Lepage et.al., PRL 43 (1979)
G. P. Lepage et.al., PLB 87B(1979)

- QCD Sum rules

V. L. Chernyak et. al., NPB 201 (1982)
V. M. Braun et. al., ZPC 44 (1989)
P. Ball et. al., JHEP 08 (2007)

- Lattice calculation by OPE

G. Martinelli et. al., PLB 190 (1987)
V. M. Braun et. al., PRD 92 (2015)
G. S. Bali et. al., JHEP 08 (2019)
RQCD Collaboration, JHEP 11 (2020)

- Quark model

H. M. Choi, and C. R. Ji, PRD 75 (2007)

- Dyson−Schwinger Equation

F. Gao, L. Chang et.al.PRD 90 (2014)
Craig D. Roberts et.al., PPNP (2021)

- Lattice calculation with LaMET

J.H. Zhang et. al., PRD 95 (2017)
R. Zhang et. al., PRD 102 (2020)
J.Hua et. al. (LPC), PRL 127 (2021)
J.Hua, **F. Yao** et. al. (LPC), PRL 129 (2022)
X. Gao et. al., PRD 106 (2022)
J. Holligan et. al., NPB 993 (2023)
Ian Cloët et. al., ArXiv: 2407.00206
...

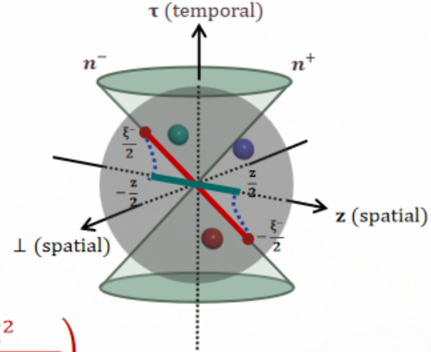
- Quantum Computing

QuNu Collaboration, PMA. 66 (2023)

- ...

Lattice calculation of LCDAs with LaMET

- **Factorization in LaMET:** X.D. Ji, PRL 110 (2013)
  X.D. Ji et. al., Rev.Mod.Phys. 93 (2021)



Quasi−LF correlation — LF correlation

$$\tilde{h}(z,\lambda,\mu) = \int_0^1 d\alpha d\beta\, C(\alpha,\beta,\mu^2 z^2)\, h(\alpha,\beta,\lambda,\mu) + \mathcal{O}\left(\Lambda_{QCD}^2 z^2, M^2 z^2\right)$$

matching coefficient — Power correction

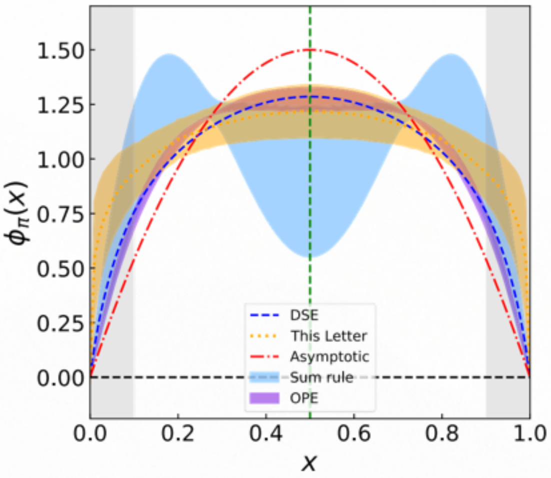
$$\tilde{\phi}_\pi(x,P_z,\mu) = \int_0^1 dy\, \mathbb{C}(x,y,P_z,\mu)\, \phi_\pi(x,\mu) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{(xP_z)^2}, \frac{\Lambda_{QCD}^2}{((1-x)P_z)^2}, \frac{M^2}{(xP_z)^2}, \frac{M^2}{((1-x)P_z)^2}\right)$$

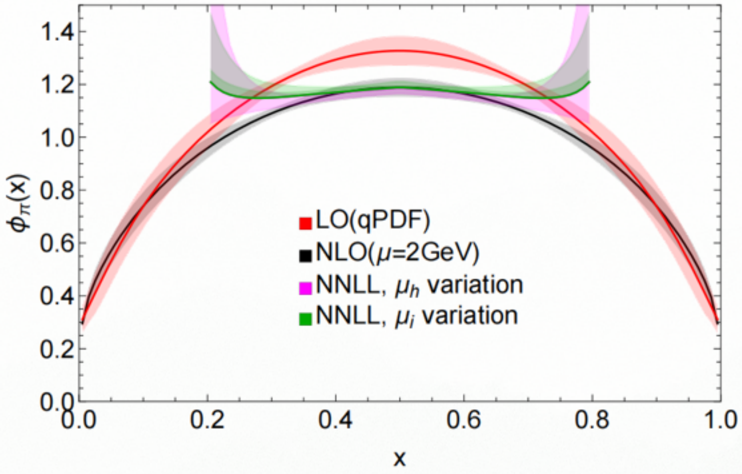
Quasi−DA — LCDA

- **Precise extractions of LCDA:** control of lattice artifacts and improvement of perturbative calculations

Quasi-LF distance: $\lambda = zP_z$

Lattice calculation of LCDAs with LaMET





- J.Hua, **F. Yao** et.al. (LPC), PRL 129 (2022)

  Continuum and infinite momentum limit

  Physical point

  Hybrid renormalization

- Ian Cloët et.al., ArXiv: 2407.00206

  a = 0.076 fm, momentum up to 1.8 GeV

  Physical point

  With RGR, LRR and threshold resummation

See Rui Zhang' s talk

Motivation

- The perturbative matching is limited **to NLO only**. The impact of higher-order correction has been much less studied.

Unpol. PDF: Z. Y. Li, Y. Q. Ma, and J. W. Qiu, PRL 126 (2021)
L. B. Chen, W. Wang, and R. L. Zhu, PRL 126 (2021)

- Improving **theoretical uncertainties** in the lattice prediction of **light meson DAs**.
  - Higher order correction
  - Renormalization group resummation (RGR) Y. S. Su, **F. Yao** et. al., NPB 991 (2023)
  - Leading renormalon resummation (LRR) R. Zhang et. al., PLB 844 (2023)
  - Threshold logarithm resummation X. D, Ji, Y. Z. Liu, and Y. S. Su, JHEP 08 (2023)

Perturbative calculation at NNLO

$$\tilde{h}^R(z,\lambda,\mu) = \int_0^1 d\alpha d\beta\, C(\alpha,\beta,z^2,\mu) h(\alpha,\beta,\lambda,\mu) + h.t.$$

1. **Correlation function in coordinate space**

   $$\langle 0|\bar{\psi}_1(0)\gamma^z\gamma_5\mathcal{W}(0,z)\psi_2(z)|\pi(P)\rangle$$

2. **Feynman diagrams at NNLO**

3. **Results in coordinate space**

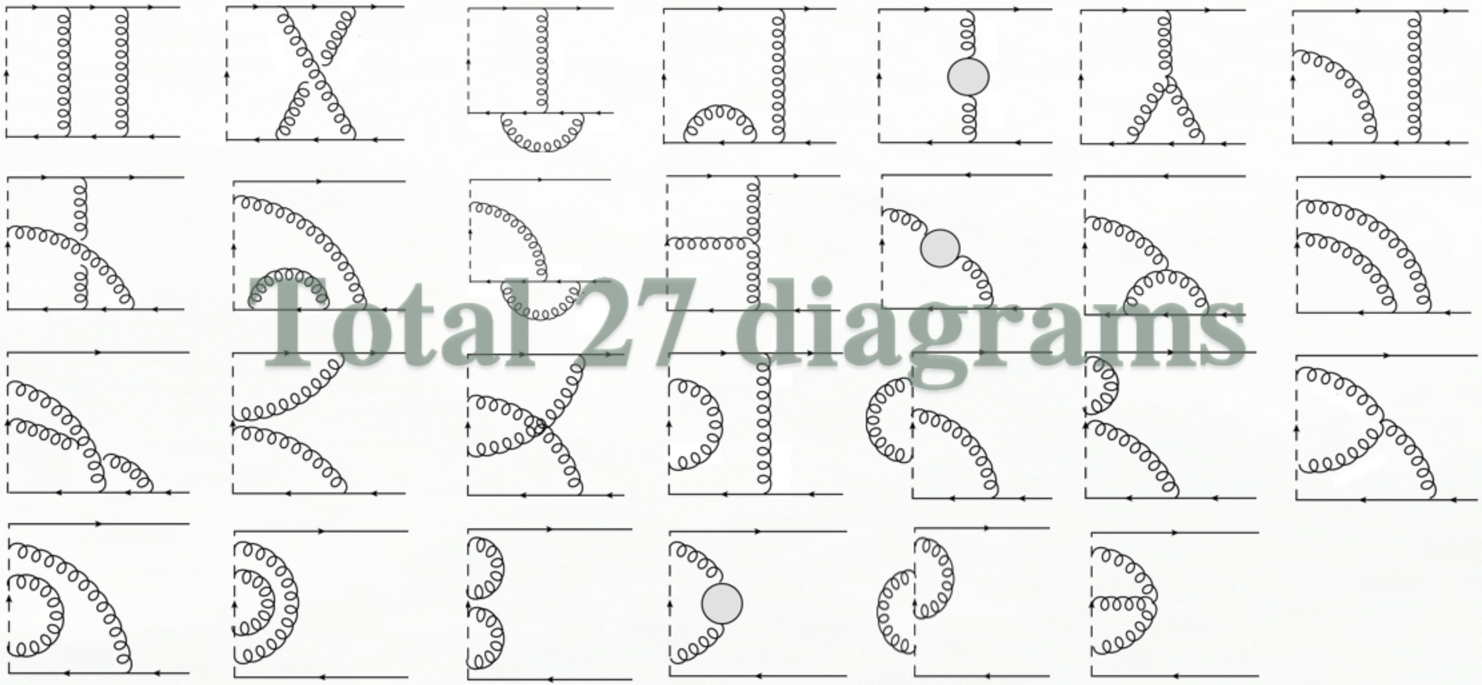
4. **Fourier transform into momentum space**

   $$\tilde{\phi}(x,\mu) = \int_0^1 dy\, \mathbb{C}(x,y,\mu)\phi(y,\mu) + h.t.$$

5. **Matching in the state-of-the-art scheme**

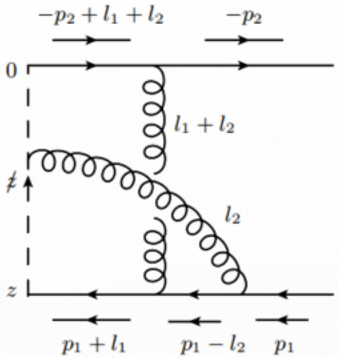
6. **Numerical test**

Feynman diagrams at NNLO



The conjugate diagrams and the **quark self-energy diagram** on external legs are not shown.

Matching coefficient calculation: an example



- **Feynman amplitude**

$$-ig^4C_F^2\left(C_F-\frac{C_A}{2}\right)\int_0^1 du\int\frac{d^dl_1d^dl_2}{(2\pi)^{2d}}\,e^{-i(p_1+l_1+ul_2)\cdot z}\gamma_\rho\frac{-\not p_2+\not l_{12}}{(-p_2+l_{12})^2}\not z\frac{\not p_1+\not l_1}{(p_1+l_1)^2}\gamma^\rho\frac{\not p_1-\not l_2}{(p_1-l_2)^2}\not z\frac{1}{l_2^2(l_1+l_2)^2}$$

- **Making a substitution:** $l_1^\alpha\to -i\partial_{z_1}^\alpha,\ l_2^\alpha\to -i\partial_{z_2}^\alpha$

$$-ig^4C_F^2\left(C_F-\frac{C_A}{2}\right)e^{-ip_1\cdot z}\gamma_\rho\gamma_\sigma\not z\gamma_\beta\gamma^\rho\gamma_\alpha\not z\int_0^1 du\Big\{(-p_2^\sigma-i\partial_{z_1}^\sigma-i\partial_{z_2}^\sigma)(p_1^\beta-i\partial_{z_1}^\beta)(p_1^\alpha+i\partial_{z_2}^\alpha)$$
$$\underbrace{\int\frac{d^dl_1d^dl_2}{(2\pi)^{2d}}\frac{e^{i(l_1\cdot z_1+l_2\cdot z_2)}}{(p_1-l_2)^2(p_1+l_1)^2(-p_2+l_{12})^2l_2^2(l_1+l_2)^2}}_{J_{12}}\Big\}_{z_1=-z,z_2=-uz},$$

typical integral ⇓

$$J_{12}\equiv\frac{\Gamma(-1-2\epsilon)}{4^{1+2\epsilon}(4\pi)^d}\int_0^1\frac{d\alpha d\beta d\gamma d\tau\,\alpha\tau^\epsilon\bar\tau}{(\alpha\bar\alpha)^{1+\epsilon}}e^{-i(\bar\alpha p_1-\alpha\beta p_2)\cdot z_1}e^{i[(\tau+\bar\tau\gamma)p_1+\beta\tau p_2]\cdot(z_2-\alpha z_1)}[-z_1^2\alpha\bar\alpha/\tau-(z_2-\alpha z_1)^2]^{1+2\epsilon}$$

Matching coefficient calculation: an example

Useful formulas for J_{12}

$$\int \frac{d^d l}{(2\pi)^d} \frac{\mathrm{e}^{-il\cdot \mathrm{z}}}{[l^2]^n} = \frac{(-1)^{-n} i}{4^{n-d/2}(4\pi)^{d/2}} \frac{\Gamma(d/2-n)}{\Gamma(n)} (-\mathrm{z}^2)^{n-d/2}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{\mathrm{e}^{-il\cdot \mathrm{z}}}{[l^2]^n} = \frac{(-1)^{-n} i\Gamma(n-d/2+1/2)}{(4\pi)^{\frac{d-1}{2}}\Gamma(n)} \int \frac{dl^z}{2\pi} \mathrm{e}^{-il^z z} \left(\frac{1}{l_z^2}\right)^{n-\frac{d-1}{2}}$$

$$\int \frac{dl^z}{2\pi} \mathrm{e}^{-il^z z} \left(\frac{1}{l_z^2}\right)^a = \frac{\Gamma(1/2-a)}{4^a \sqrt{\pi}\Gamma(a)} (z^2)^{a-1/2}$$

$$\int \frac{dl_1^z dl_2^z}{(2\pi)^2} \frac{\mathrm{e}^{-ia_1 l_1^z + ia_2 l_2^z}}{((l_1^z)^2 + (l_2^z)^2)^{n-d}} = \frac{\Gamma(-n+d+1)}{4^{n-d}\pi\Gamma(n-d)} [a_1^2 + a_2^2]^{n-d-1}$$

- Then applying the partial derivative to the typical integral J_{12}.

Matching coefficient calculation

- Appropriate change of variables and integrating over the remaining Feynman parameters:

$$\int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma \mathrm{d}\tau \mathrm{d}u\, [\ldots]\; e^{i\,[\ldots]} \;\rightarrow\; \int_0^1 d\alpha d\beta\, e^{-i(\bar{\alpha} p_1 + \beta p_2)\cdot z} \{\theta(1-\alpha-\beta)\,[\ldots] + \theta(\alpha+\beta-1)\,[\ldots]\}$$

- Conjugate diagram: $\mathrm{C}(\alpha \leftrightarrow \beta, \mathrm{z}, \boldsymbol{\epsilon})\; e^{-i(\bar{\alpha} p_1 + \beta p_2)\cdot z}$

All diagrams added:

$$\mathrm{C_B}(\alpha, \beta, \mathrm{z}, \epsilon) = \delta(\alpha)\delta(\beta) + \mathrm{a_s^B}\left(\frac{1}{\epsilon}\mathcal{C}_{11}^{(1)} + \mathcal{C}_{10}^{(1)} + \epsilon\,\mathcal{C}_{1\mathrm{E}}^{(1)}\right) + \mathrm{a_s^B}^2\left(\frac{1}{\epsilon^2}\mathcal{C}_{22}^{(2)} + \frac{1}{\epsilon}\mathcal{C}_{21}^{(2)} + \mathcal{C}_{20}^{(2)}\right)$$

$$\mathrm{C_R}(\alpha, \beta, \mathrm{z}) = \mathrm{Z}_{\alpha_s}\mathrm{Z^{UV}}\,(\,\mathrm{Z}_{\mathcal{O}} \otimes \mathrm{C_B}(\alpha, \beta, \mathrm{z}, \epsilon))$$

$$\mathrm{a}_s = \frac{\alpha_s}{4\pi}$$

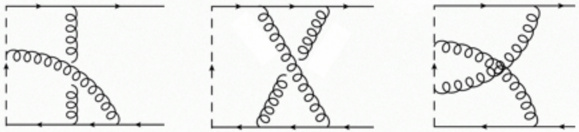
V. M. Braun et. al., JHEP 07 (2020)
V. M. Braun et. al., JHEP 03 (2016)

Matching coefficient in coordinate space

$$C_{\overline{MS}}(\alpha,\beta,\mu^2z^2) = \delta(\alpha)\delta(\beta) + a_s(\mu)\,(C_{11}L + C_{10}) + a_s^2(\mu)\,(C_{22}L^2 + C_{21}L + C_{20}) \qquad L = \ln[\mu^2 z^2]$$

F. Yao, Y. Ji and J. H. Zhang, JHEP 11 (2023)

• **PolyLog structures:** $\mathrm{Li}_{n\_}[\alpha]$, $\mathrm{Li}_{n\_}[\beta]$, $\mathrm{Li}_{n\_}[\bar{\alpha}]$, $\mathrm{Li}_{n\_}[\bar{\beta}]$, $\mathrm{Li}_{n\_}[\bar{\alpha}-\beta]$, $\mathrm{Li}_{n\_}\left[\frac{\alpha\,\beta}{\bar{\alpha}\,\bar{\beta}}\right], \ldots \quad n = 1,2,3$
• **Three color structures:** $C_F\, C_A$, $C_F{}^2$, $C_F\,\beta_0$
• **Displayed in a plus function:** $C1^{(2)}[\alpha,\beta] + C2^{(2)}[\alpha]_+\delta(\beta) + C3^{(2)}[\beta]_+\delta(\alpha) + C4^{(2)}\delta(\alpha)\delta(\beta)$
• **Two physical regions:** $\theta(1-\alpha-\beta)$, $\theta(\alpha+\beta-1)$



Matching coefficient in coordinate space

$$C_{\overline{MS}}(\alpha,\beta,\mu^2z^2) = \delta(\alpha)\delta(\beta) + a_s(\mu)\,(C_{11}L + C_{10}) + a_s^2(\mu)\,(C_{22}L^2 + C_{21}L + C_{20}) \qquad L = \ln[\mu^2z^2]$$

Cross check I	Cross check II
Recursive relation: • $C_{22} = \frac{1}{2} C_{11} \otimes (C_{11} - \beta_0)$; • $\mathcal{C}_{21}^{(2)} = C_{10} \otimes (C_{11} - \beta_0) - (Z_{21} - Z_{21}^{UV})$ **Poles can be completely cancelled which is a highly non−trivial check.**	**Reduction to helicity PDF case (forward limit):** $C_{\overline{MS}}(\alpha', \mu^2z^2) =$ $\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \delta(\alpha' - \alpha - \beta) C_{\overline{MS}}(\alpha,\beta,\mu^2z^2)$ **which is consistent with Ref.** [Z. Y. Li, Y. Q. Ma and J. W. Qiu, PRL 126 (2021)].

Ref. [Z. Y. Li, Y. Q. Ma and J. W. Qiu, PRL 126 (2021)] provides matching coefficients in unpolarized case ($\Gamma = \gamma^0$ and $\Gamma = \gamma^z$).

Matching coefficient in momentum space

Taking $p_1 = x\,P,\ p_2 = (1-x)\,P$

$$\mathbb{C}^{(2)}(\mathrm{x},\mathrm{y},\mu/P_z) = \int_0^1 d\tau \int \frac{d\lambda}{2\pi} e^{i(x-\tau)\lambda}\ \mathfrak{C}^{(2)}\left(\tau,\mathrm{y},\frac{\mu^2\lambda^2}{{P_z}^2}\right)$$

$$= \mathrm{a}_\mathrm{s}^2(\mu)\begin{cases} f_1(x,y,\mu/P_z) & x<0<y \\ f_2(x,y,\mu/P_z) & 0<x<y \\ f_2(\bar{x},\bar{y},\mu/P_z) & y<x<1 \\ f_1(\bar{x},\bar{y},\mu/P_z) & y<1<x \end{cases}$$

In progress

Quasi−LF correlation $\tilde{h}_\mathrm{R}$ $\mathrm{C}(\alpha,\beta,\mu^2\mathrm{z}^2)$

FT

Radyushkin, PRD 100 (2019)

Pseudo−DA $\mathfrak{C}(\tau,\mathrm{y},\mu^2\mathrm{z}^2)$

double FT

Quasi−DA $\mathbb{C}(\mathrm{x},\mathrm{y},\mu/P_z)$

$$\mathfrak{C}^{(2)}(\tau,\mathrm{y},\mu^2\mathrm{z}^2) = \int \frac{d\lambda}{2\pi} e^{-i\tau\lambda}\ \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, e^{i(\bar{\alpha}y+\beta\bar{y})\lambda}\ \mathrm{C}^{(2)}(\alpha,\beta,\mu^2\mathrm{z}^2)$$

$$= \mathrm{a}_\mathrm{s}^2(\mu)\begin{cases} f(\tau,y,\mu^2\mathrm{z}^2) & 0<y<\tau<1 \\ f(\bar{\tau},\bar{y},\mu^2\mathrm{z}^2) & 0<\tau<y<1 \end{cases}$$

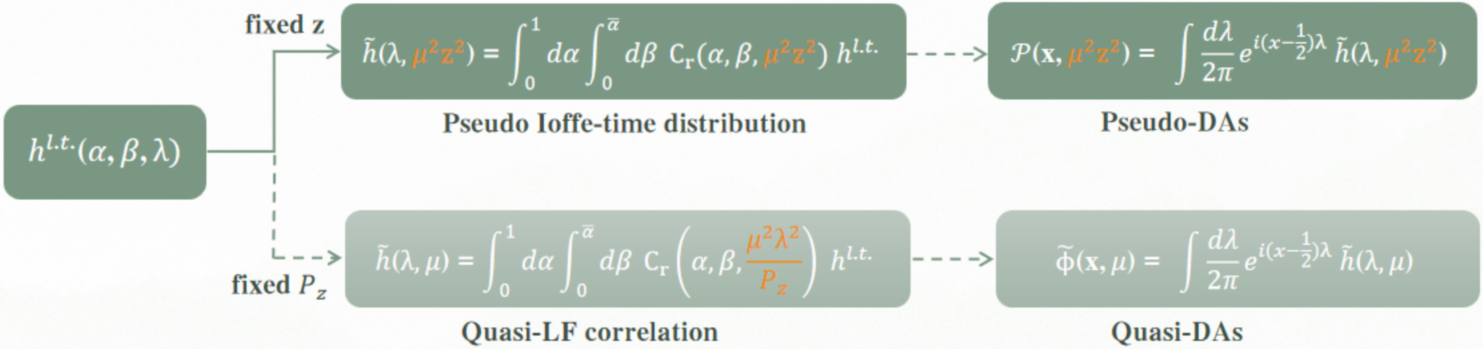
Matching coefficient in state-of-the-art scheme

	$\overline{\text{MS}}$ scheme	Ratio scheme	Ratio−hybrid scheme
Coordinate space	$C_{\overline{\text{MS}}}(\alpha,\beta,\mu^2z^2)$	$C_r(\alpha,\beta,\mu^2z^2)=\frac{C_{\overline{MS}}(\alpha,\beta,\mu^2z^2)}{C_0(\mu^2z^2)}=\delta(\alpha)\delta(\beta)+C_r^{(1)}(z)$ $+(C_{\overline{\text{MS}}}^{(2)}(z)-C_0^{(2)}(z)\delta(\alpha)\delta(\beta)-C_0^{(1)}(z)C_r^{(1)}(z))$ $C_0(z)$ represents the same operator matrix element at zero momentum.	$C_h(\alpha,\beta,\mu^2z^2)$ $=C_r(\alpha,\beta,\mu^2z^2)\left(1+\left(\frac{C_0(\mu^2z^2)}{C_0(\mu^2z_s^2)}-1\right)\theta(\lvert z\rvert-\lvert z_s\rvert)\right)$ $=C_r(\alpha,\beta,\mu^2z^2)+(\delta^{(1)}(z,z_s)+\delta^{(2)}(z,z_s))\ \theta(\lvert z\rvert-\lvert z_s\rvert)$ where z_s denotes a truncation point.
Momentum space	$\mathbb{C}_{\overline{\text{MS}}}(x,y,\mu/P_z)$	$\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\ e^{i(\bar{\alpha}y+\beta\bar{y})\lambda}\int\frac{d\lambda}{2\pi}e^{-ix\lambda}\text{Ln}[\lambda^2]\,\delta(\alpha)\delta(\beta)$ $\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\ e^{i(\bar{\alpha}y+\beta\bar{y})\lambda}\int\frac{d\lambda}{2\pi}e^{-ix\lambda}\text{Ln}^2[\lambda^2]\,\delta(\alpha)\delta(\beta)$ $C_0(z)$ and $\mathbb{C}_0(\mu/P_z)$ are consistent with Ref. [Z. Y. Li, Y. Q. Ma and J. W. Qiu, PRL 126 (2021)].	$\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\ e^{i(\bar{\alpha}y+\beta\bar{y})\lambda}\int\frac{d\lambda}{2\pi}e^{-ix\lambda}\text{Ln}[\frac{\lambda^2}{\lambda_s^2}]\,\delta(\alpha)\delta(\beta)\theta(\lvert\lambda\rvert-\lvert\lambda_s\rvert)$ $\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\ e^{i(\bar{\alpha}y+\beta\bar{y})\lambda}\int\frac{d\lambda}{2\pi}e^{-ix\lambda}\text{Ln}^2[\frac{\lambda^2}{\lambda_s^2}]\,\delta(\alpha)\delta(\beta)\theta(\lvert\lambda\rvert-\lvert\lambda_s\rvert)$ Added some additional terms on the basis of Eq.(A18) in Ref. [Y. S. Su, **F. Yao** et. al., NPB 991 (2023)].

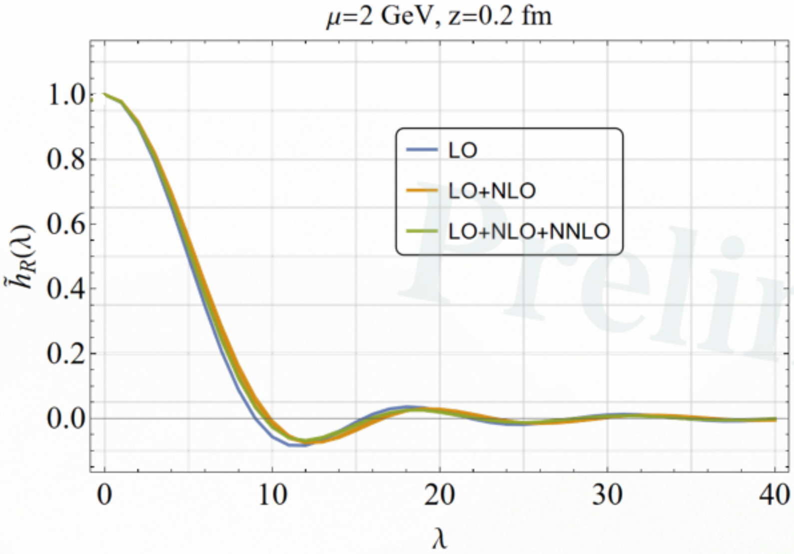
Ref. [Y. S. Su, **F. Yao** et. al., NPB 991 (2023)] provides matching coefficients in **Ratio−hybrid scheme** with $\Gamma=\gamma^0$.

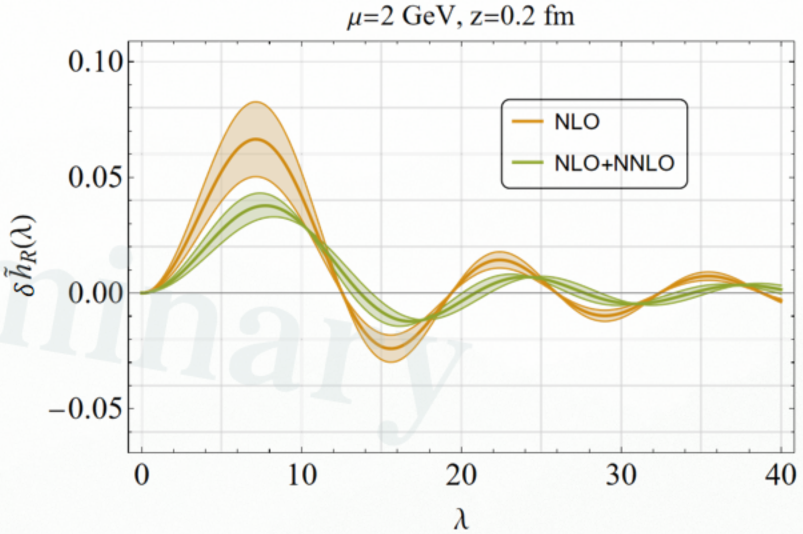
Numerical test

- **Coordinate space:** $C_\Gamma(\alpha, \beta, \mu^2 z^2) \Rightarrow$ **small-z region**
- **Asymptotic model:** $\phi(x) = 6x(1-x)$
- **Light−cone:** $h^{l.t.}(\alpha, \beta, \lambda) = \int_0^1 dx\ e^{-i(1-\alpha-\beta)x\,\lambda + i(\frac{1}{2}-\beta)\lambda}\ \phi(x)$



Numerical test: Pseudo Ioffe−time distribution (fixed z)

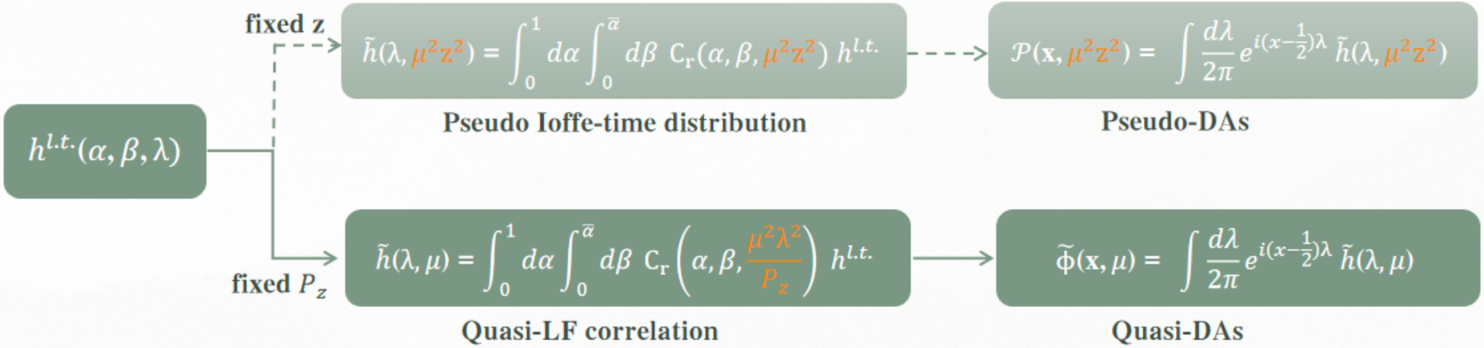




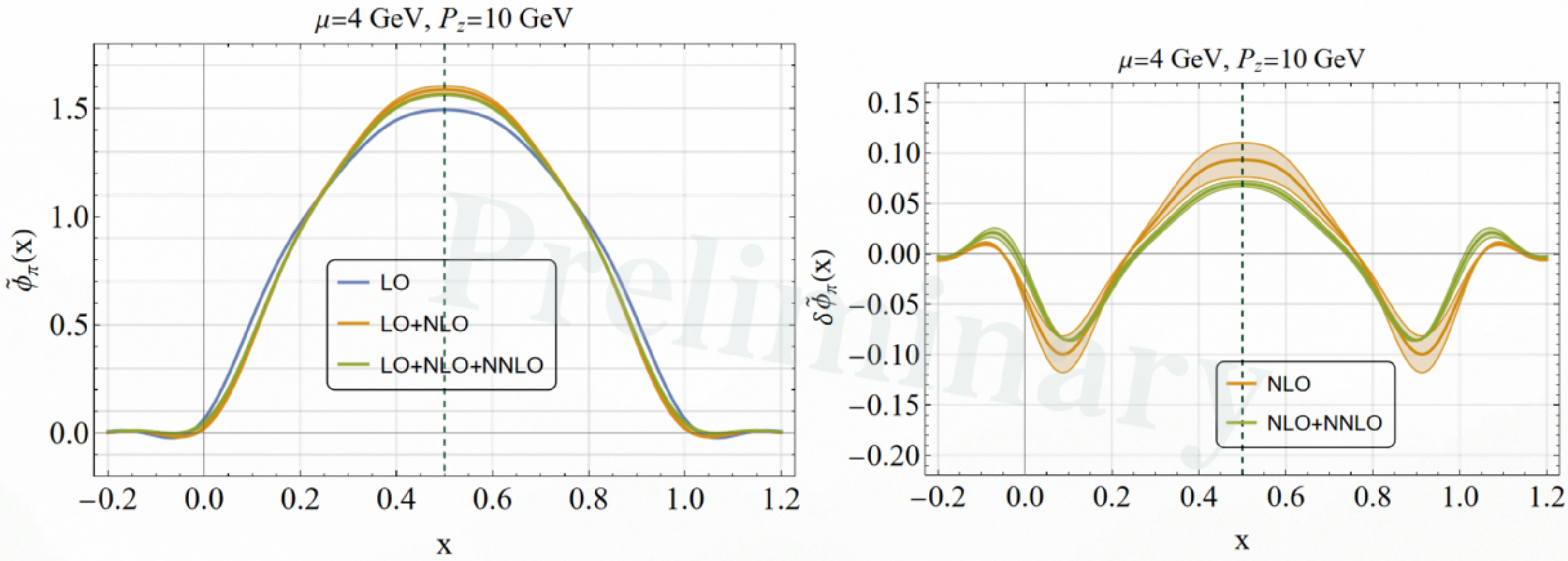
- Uncertainty: μ variation from 2 GeV to 4 GeV.
- The **scale dependence** is **reduced** when higher-order corrections are included.
- The NNLO effect amounts to ~ **30 – 40%** of the NLO correction.

Numerical test: quasi−DA (fixed large-P_z)

- **Coordinate space:** $C_r(\alpha, \beta, \mu^2 z^2)$ ⇒ **small-z region**
- **Asymptotic model:** $(x) = 6x(1 - x)$
- **Light−cone:** $h^{l.t.}(\alpha, \beta, \lambda) = \int_0^1 dx\ e^{-i(1-\alpha-\beta)x\,\lambda + i(\frac{1}{2}-\beta)\lambda}\ \phi(x)$



Numerical test: quasi−DA (fixed large-P_z)



- Uncertainty: μ variation from 4 GeV to 8 GeV.
- The **scale dependence** is **reduced** when higher-order corrections are included.
- The NNLO effect amounts to ~ **10 − 20%** of the NLO correction.

Summary and outlook

- LCDA describes the internal structure of the meson and is an essential input to the exclusive processes.
- Lattice QCD calculations can provide valuable information on LCDA.
- We present the perturbative matching process up to NNLO for extracting LCDA in LaMET:
  - In coordinate space and in momentum space
  - In state-of-the-art scheme
  - The impact of the NNLO matching tested numerically
- Follow−up:
  - The improvement of RGR, LRR and threshold resummation
  - Combined with lattice data
  - Generalized to GPDs (in progress)

Thank you for listening !

Speaker: Fei Yao (feiyao@mail.bnu.edu.cn)