Advanced Quantum Mechanics

In classical mech, to describe the motion of a particle we use the time-dependent position $\vec{x}(t)$ as the dynamical variable (Ang. mom.)

In QM, the dynamical variable is the wave f^{n} $\psi(\vec{x},t) \in \mathbb{C}$ or ket $|\psi\rangle \in \mathcal{H}$

$$\psi(\vec{x},t) \in \mathbb{C}$$

The govering eq^{n} is Schrödinger eq^{n} -

$$\hat{H}\psi(\vec{x},t) = i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x},t)$$

In mom. space $\hat{H} = \frac{\vec{p}^2}{2m} + \tilde{V}(\vec{p},t)$

$$\downarrow$$

F.T. of $V(\vec{x},t)$

Posi^{n} Mom.

$$\begin{tikzcd} \langle \vec{x}|\psi\rangle \arrow[r] \arrow[d] & \langle \vec{p}|\psi\rangle \arrow[d] \\ \underset{\in \mathbb{C}}{\psi(\vec{x},t)} \arrow[r, leftrightarrow, bend right, "\text{Fourier tr}^{s}"'] & \underset{\in \mathbb{C}}{\tilde{\psi}(\vec{p},t)} \end{tikzcd}$$

$$\tilde{\psi}(p) \equiv \int_{-\infty}^{\infty} dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\psi(x)$$

* Stationary state of energy $E \in \mathbb{R}$ is a state $\psi(\vec{x},t)$ for which

$\psi(\vec{x},t) = e^{-iEt/\hbar}\phi(\vec{x}) \rightarrow$ for such a state $\hat{H}\phi(\vec{x}) = E\phi(\vec{x})$

Q^{n} What if $E \in \mathbb{C}$?

$E = E_0 - i\Gamma$ w/ $E_0, \Gamma \in \mathbb{R}$

$$|\psi(\vec{x},t)|^2 = \psi^*(\vec{x},t)\psi(\vec{x},t) = e^{i(E^*-E)t/\hbar}|\phi(\vec{x})|^2$$

$$= e^{-2\Gamma t/\hbar}|\phi(\vec{x})|^2$$

$\Rightarrow$ Normalization can not be preserved in time

$\Rightarrow$ not a valid situation.

Now, consider the SE for a time-independent 1-D

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$$ (rewriting $\phi \rightarrow \psi$).

$$\Rightarrow \left(-\frac{d^2}{dx^2} + \frac{2m}{\hbar^2}V(x)\right)\psi(x) = \frac{2mE}{\hbar^2}\psi(x)$$

$$\Rightarrow \underbrace{\left(-\frac{d^2}{dx^2} + \tilde{V}(x)\right)}_{\text{2nd order differential operator}} \psi(x) = \tilde{E}\,\psi(x)$$

* $\psi(x)$ is square integrable i.e. $\int dx\,|\psi(x)|^2$ is finite. If 1 normalized

Factorization method

Try to write the Hamiltonian as a product of two first order differential operators a and $a^\dagger$, plus a const. ϵ

Aim: find two operators

$$a = \frac{d}{dx} + \alpha_p(x)$$

$$a^\dagger = -\frac{d}{dx} + \alpha_p(x)$$

such that $\hat{H} = -\frac{d^2}{dx^2} + \tilde{V}(x)$ can be written as

$$\hat{H} = a^\dagger a + \epsilon$$

The eqn satisfied by $\alpha_p(x)$ is a Riccati type eqn.

$$-\frac{d^2}{dx^2} + \tilde{V}(x) = a^\dagger a + \epsilon$$

$$\Rightarrow \tilde{V}(x) - \epsilon = a^\dagger a + \frac{d^2}{dx^2} = \left(-\frac{d}{dx} + \alpha_p(x)\right)\left(\frac{d}{dx} + \alpha_p(x)\right) + \frac{d^2}{dx^2}$$

$$= \cancel{-\frac{d^2}{dx^2}} - \alpha_p'(x) - \cancel{\alpha_p(x)\frac{d}{dx}} + \cancel{\alpha_p(x)\frac{d}{dx}} + \alpha_p^2(x) + \cancel{\frac{d^2}{dx^2}}$$

$$= -\alpha_p'(x) + \alpha_p^2(x)$$

$$\Rightarrow \alpha_p'(x) = \alpha_p^2(x) - \left(\tilde{V}(x) - \epsilon\right) \rightarrow$$ 1st order DE of $\alpha_p(x)$ ↓ non-linear

* Infeld-Hull approach (particular soln.)

* Mielnik approach (general soln.)

} → usually solns are transcendental fns.

Examples

(i) $\tilde{V}(x) = x^2$: Simple Harmonic Oscillator

→ Sol^{n}s are Hermite polynomials type wave-f^{n}s $\psi_n(x)$

w/ $E_n = (2n+1)$, $n = 0, 1, 2, \ldots$

The SE ⇒

$$\underbrace{\left(-\frac{d^2}{dx^2} + x^2\right)}\psi(x) = \tilde{E}\,\psi(x)$$

$$\left.\begin{array}{l} a^\dagger a + 1 \\ \text{or} \quad a a^\dagger - 1 \end{array}\right\} \Rightarrow \begin{array}{l} \epsilon = 1 \\ \alpha_p(x) = x \end{array}$$

It can also be shown

$$[a, a^\dagger] = 2$$

$$[\hat{H}, a] = -2a$$

$$[\hat{H}, a^\dagger] = 2a^\dagger$$

$$a = \frac{d}{dx} + x$$

$$a^\dagger = -\frac{d}{dx} + x$$

$$a^\dagger a = \left(-\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)$$

$$= -\frac{d^2}{dx^2} + x^2 - 1 - x\frac{d}{dx} + x\frac{d}{dx}$$

$$= -\frac{d^2}{dx^2} + x^2 - 1$$

$$\Rightarrow \quad -\frac{d^2}{dx^2} + x^2 = a^\dagger a + 1$$

N.B. The traditional forms can be obtained w/ $H \to H/2$, $a \to \frac{a}{\sqrt{2}}$, $a^\dagger \to \frac{a^\dagger}{\sqrt{2}}$.

* Suppose $\psi_0(x)$ is the ground state wave-f^{n}.

$$\Rightarrow \quad a\,\psi_0(x) = 0 \quad \text{only then} \quad \hat{H}\psi_0 = \epsilon\,\psi_0 = \psi_0$$

$$\psi_0(x) \propto e^{-\int^x \alpha_p(y)\,dy} \quad \to \text{satisfies normaliz}^{n} \text{ etc.}$$

$$\Rightarrow \quad \psi_0(x) = c_0\, e^{-x^2/2}$$

N.B. $a^\dagger \tilde{\psi}(x) = 0 \Rightarrow \tilde{\psi}(x) \propto e^{x^2/2} \to$ not a physical sol^{n}.

(ii) $$\hat{H} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2}{r} = -\frac{d^2}{dr^2} + \tilde{V}_l(r) \qquad l = 0, 1, 2, \dots$$

$$E_n \equiv E_{l,k} = -\frac{1}{(l+k)^2}, \quad k = 1, 2, 3, \dots \qquad l + k = n$$

N.B. these are lowering and raising $\hat{op}$ in l.

$$\begin{cases} a_l = \frac{d}{dr} + \frac{l}{r} - \frac{1}{l} \\ a_l^\dagger = -\frac{d}{dr} + \frac{l}{r} - \frac{1}{l} \end{cases} \qquad \alpha_{pl}(r) = \frac{l}{r} - \frac{1}{l}$$

$$a_l^\dagger a_l - \frac{1}{l^2} = \hat{H}, \quad a_l a_l^\dagger - \frac{1}{l^2} = H_{l-1}$$

Isotropic harmonic oscillator

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + \frac{1}{2}m\omega^2\vec{r}^2$$

$$= \sum_{p=x,y,z}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial p^2} + \frac{1}{2}m\omega^2 p^2\right)$$ (sum of 3, 1-D osc. w/ equal masses m and ang. freq. ω)

$$E_N = (\underbrace{n_x + n_y + n_z}_{N} + \frac{3}{2})\hbar\omega$$ Each E_N level is $\frac{1}{2}(N+1)(N+2)$ fold degenerate.

* In general degeneracy in the spectrum of a Hamiltonian can be attributed to the existence of a symmetry.

* In this case, it is rotational invariance.

$\Downarrow$ spherical co-ord can be used to solve this

ang. dep. → Y_{lm}

rad. dep. → $E_{nl} = (2n+l+\frac{3}{2})\hbar\omega$ dep. on l but not its proj.? i.e., m_l

$\Downarrow$

$(2l+1)$ - fold degenerat (indep. of m_l)

$\Downarrow$

$N = 2n+l$

→ total degeneracy $(N+1)(N+2)/2$

(including $(2l+1)$ fold degeneracy of each l - level).

Central potential

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)$$ $V(r)$ just depends on r i.e. no angular dependence, such a potential is called central potential

↓

also rotationally invariant

$$= \frac{\vec{p}^2}{2m} + V(r) \qquad \hat{\vec{p}}^2 = -\hbar^2\vec{\nabla}^2$$

In spherical polar co-ordinate

$$\vec{p}^2 = -\hbar^2 \left[\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right\} \right]$$

$$= -\hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \hbar^2 \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] \cdots ①$$

Now, note that if $\vec{a}$ and $\vec{b}$ are QM operators,

$$(\vec{a}\times\vec{b})^2 = \vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b})^2 - a_j[a_j, b_k]b_k + a_j[a_k, b_k]b_j$$

$- a_j$ [illegible] $[a_k, b_j]b_k - a_j a_k [b_k, b_j]$

$= \vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b})^2 + \gamma(\vec{a}\cdot\vec{b})$ when $[a_i, b_j] = \gamma\delta_{ij}$, $\gamma \in \mathbb{C}$

& $[a_i, a_j] = 0 = [b_i, b_j]$.

$$\Rightarrow \underbrace{\vec{L}^2 = (\vec{r}\times\vec{p})^2}_{\Downarrow} = \vec{r}^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2 + i\hbar(\vec{r}\cdot\vec{p}) \quad (\because [x_i, p_j] = i\hbar\delta_{ij})$$

$L_i = \epsilon_{ijk} x_j p_k$ → from this show that $\cdot\ [L_i, L_j] = i\epsilon_{ijk}\hbar L_k$

$\cdot\ [L_i, \vec{r}^2] = [L_i, \vec{p}^2] = [L_i, \vec{r}\cdot\vec{p}] = 0$

$\cdot\ [L_i, \vec{L}^2] = 0$

Additionally, $[L_i, x_j] = i\hbar\epsilon_{ijk}x_k$

$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k$

$\vec{L}\times\vec{L} = i\hbar\vec{L}$

Now $\vec{L}^2 = \vec{r}^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2 + i\hbar(\vec{r}\cdot\vec{p})$

$$\Rightarrow \vec{p}^2 = \frac{1}{\vec{r}^2}\left[(\vec{r}\cdot\vec{p})^2 - i\hbar\underbrace{(\vec{r}\cdot\vec{p})} + \vec{L}^2\right]$$

$$\vec{r}\cdot(-i\hbar\vec{\nabla}) = r\hat{r}\cdot(-i\hbar\vec{\nabla})$$

$$= r\hat{r}\cdot\left(-i\hbar\hat{r}\frac{\partial}{\partial r} + \cdots\hat{\theta} + \cdots\hat{\phi}\right.$$

$$= -i\hbar r\frac{\partial}{\partial r}$$

$$\therefore \left(-i\hbar r\frac{\partial}{\partial r}\right)\left(-i\hbar r\frac{\partial}{\partial r}\right)$$

$$= -\hbar^2 r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) = -\hbar^2\left[r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}\right]$$

$$\therefore \quad \vec{p}^2 = \frac{1}{r^2}\left[-\hbar^2\left\{r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}\right\} - \hbar^2 r\frac{\partial}{\partial r} + \vec{L}^2\right]$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{r^2}$$

$$= -\hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{\vec{L}^2}{r^2} \quad ---- \quad ②$$

Comparing ① and ②

$$\vec{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right).$$

∴ The Hamiltonian takes the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{2mr^2}\vec{L}^2 + V(r)$$

Thus for central force/potential Hamiltonians $[L_i, H] = 0$

$$\Rightarrow i\hbar\frac{d}{dt}\langle L_i\rangle = \langle [L_i, H]\rangle = 0$$

$\Rightarrow$ L_i $\hat{ops}$ are conserved in central potentials.

* Complete set of commuting observables:-

Full set up to O(square) → $\underbrace{H,\ x_i,\ p_i,\ L_i,\ \vec{r}^2,\ \vec{p}^2,\ \vec{r}\cdot\vec{p},\ \vec{L}^2,\cdot}$
$(i=1,2,3)$

CSCO ⇒ $\underbrace{H,\ L_z,\ \vec{L}^2}$

all eigenstates are uniquely labelled by the eval. of these $\hat{ops}$.
(Assuming no spin.
w/ spin, S_z, $\vec{S}^2$ to be added to the list)

Now, suppose we are considering an electron ~~moving~~ of the hydrogen atom. It is moving in the central force potential i.e. Coulomb potential, in this case

$$V(r) = -\frac{Ze}{r}, \quad Z = 1.$$

* In addition to the orbital angular mom, the electron in the hydrogen atom, also has an intrinsic spin

→ this comes from the observations of Stern-Gerlach experiment.

The intrinsic spin has the properties of angular momentum

$\Rightarrow$ $S_{x,y,z}$ $\qquad [S_i, S_j] = i\hbar\epsilon_{ijk} S_k$ $\qquad$ for electron it is "two-valued"

2-D repⁿ of S_i is $\quad S_i = \frac{\hbar}{2}\sigma_i \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

* N.B. Ang. mom. operators can be thought of as rotation generators.

The rotⁿ element can be given as

e.g., $\mathcal{D}(\phi) = \exp\left(-\frac{iL_z\phi}{\hbar}\right)$ $\qquad$ unitary $\hat{op}$, L_z are hermitia

In the hydrogen atom we can think of the electron to have the CSCO to be $\quad H, L_z, \vec{L}^2, S_z, \vec{S}^2$

In a general sense we can define J_i to be any kind of ang. mom. e.g. even sum of L_i and S_i $\quad (J_i = L_i + S_i)$

$[L_i, S_i] = 0$

J_i's satisfy $\quad [J_i, J_j] = i\epsilon_{ijk}\hbar J_k$

$$[J_i, \vec{J}^2] = 0$$

To get the spectrum i.e. the eigenstates we define

$$J_\pm = J_x \pm iJ_y \qquad (J_+)^\dagger = J_-$$

$$[J_+, J_-] = 2\hbar J_z$$

$$\vec{J}^2 = J_+J_- + J_z^2 - \hbar J_z = J_-J_+ + J_z^2 + \hbar J_z$$

$$[J_\pm, \vec{J}^2] = 0, \quad [J_z, J_\pm] = \pm\hbar J_\pm$$

$\therefore$ $\vec{J}^2$ and J_z are hermitian and commute, they can be simultaneously diagonalized

$$|j,m\rangle \quad w/ \quad j,m \in \mathbb{R}$$

$$\left.\begin{aligned} \vec{J}^2|j,m\rangle &= \hbar^2 j(j+1)|j,m\rangle \\ J_z|j,m\rangle &= \hbar m|j,m\rangle \end{aligned}\right\} \quad \langle j',m'|j,m\rangle = \delta_{j'j}\delta_{m'm}$$

$$* \quad \hbar^2 j(j+1) = \langle j,m|\vec{J}^2|j,m\rangle = \sum_{i=1}^{3} \| J_i |j,m\rangle \|^2 \geq 0$$

$$\Rightarrow \quad j(j+1) \geq 0 \quad \Rightarrow \quad \underbrace{j \geq 0}_{\text{we can just use this}} \quad \text{or} \quad j \leq -1$$

What is the effect of $J_\pm$ on $|j,m\rangle$?

N.B. $$\vec{J}^2 \left(J_\pm |j,m\rangle \right) = J_\pm \vec{J}^2 |j,m\rangle = \hbar j(j+1) \left(J_\pm |j,m\rangle \right)$$

$$\Rightarrow \quad \underbrace{J_\pm}_{\downarrow} |j,m\rangle \propto |j,m'\rangle \quad \text{for some } m'$$

changes the value of m

$$J_z \left(J_\pm |j,m\rangle \right) = \left([\underset{\downarrow\ \text{id.}}{J_z, J_\pm}] + J_\pm J_z \right) |j,m\rangle$$

$$= \left(\pm \hbar J_\pm + \hbar m J_\pm \right) |j,m\rangle$$

$$= \hbar (m \pm 1) \left(J_\pm |j,m\rangle \right)$$

$$\Rightarrow \quad J_\pm |j,m\rangle = C_\pm(j,m) |j, m \pm 1\rangle$$

$$\xrightarrow[\text{adjoint}]{\text{taking}} \quad \langle j,m| J_\mp = \langle j, m \pm 1| C^*_\pm(j,m)$$

$$\Rightarrow \quad \langle j,m| J_\mp J_\pm |j,m\rangle = |C_\pm(j,m)|^2$$

$$\Rightarrow \quad |C_\pm(j,m)|^2 = \langle j,m| \left(\vec{J}^2 - J_z^2 \mp \hbar J_z \right) |j,m\rangle \qquad \Big| \ \langle j,m|j,m\rangle = 1$$

$$= \hbar^2 j(j+1) - \hbar^2 m^2 \mp \hbar^2 m$$

$$= \hbar^2 \left[j(j+1) - m(m \pm 1) \right] = \| J_\pm |j,m\rangle \|^2 > 0$$

$$\Rightarrow \quad C_\pm(j,m) = \hbar \sqrt{j(j+1) - m(m \pm 1)}$$

Now given a state $|j,m\rangle$ how far we can raise or lower?

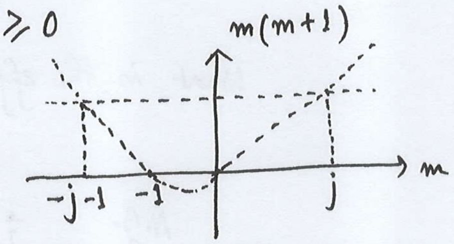
Guess $\quad |J_z| \leq |\vec{J}| \rightarrow |m| \leq \sqrt{j(j+1)}$

* For raised states to be consistent, $\| J_+ |j,m\rangle \|^2 \geq 0$

$$\Rightarrow \quad j(j+1) - m(m+1) \geq 0$$

$$\Rightarrow \quad m(m+1) \leq j(j+1)$$

$$\Rightarrow \quad -j-1 \leq m \leq j$$



(* If the e.value of $\vec{J}^2$ was not chosen as $\hbar^2 j(j+1)$ this eqn., inequality would not have had a simple soln. like this)

To maintain $m \leq j$,

[illegible] $\quad C_+(j,j) = 0$

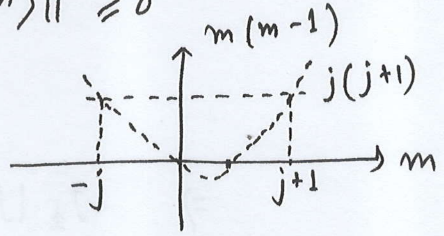
i.e. $\quad J_+|j,j\rangle = 0$

* For lowered states to be consistent, $\| J_- |j,m\rangle \|^2 \geq 0$

$$\Rightarrow \quad j(j+1) - m(m-1) \geq 0$$

$$\Rightarrow \quad m(m-1) \leq j(j+1)$$

$$\Rightarrow \quad -j \leq m \leq j+1$$



To maintain $m \geq -j$, $\quad C_-(j,-j) = 0$, i.e., $J_-|j,-j\rangle = 0$.

$\therefore$ For consistency of the multiplet of states w/ some given fixed j, m can take values from $-j$ to j w/ integer steps

$\Rightarrow$ distance $2j$ between $-j$ to j must be an integer

$$\left.\begin{array}{l} 2j \in \mathbb{Z} \quad \Rightarrow \quad j \in \mathbb{Z}/2 \\ \Rightarrow \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \end{array}\right\} \text{Quantizn. of ang. mom.}$$

Thus j can be integral or half-integral

For any j, allowed values of m are $\underbrace{-j, -j+1, \ldots, j-1, j}_{2j+1}$

E.g. for $j = 0$, $m = 0$ → singlet

$j = 1$, $m = -1, 0, 1$ → triplet

$j = 1/2$, $m = -1/2, 1/2$ → doublet

$|j,j\rangle, |j,j-1\rangle, \ldots |j,-j\rangle$

Addition of angular momenta

Examples:-

- $\vec{J} = \vec{L} + \vec{S} = \vec{L} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}$

  $\mathbb{1}$ (in $\vec{L} \otimes \mathbb{1}$) ↓ identity $\hat{op}$ in the spin space

  $\mathbb{1}$ (in $\mathbb{1} \otimes \vec{S}$) → identity $\hat{op}$ in the ∞-dim ket space spanned by the posi^n eigenkets

- Two spin-$1/2$ particles,

  $\vec{S} = \vec{S}_1 + \vec{S}_2 = \vec{S}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}_2$

  $\mathbb{1}$ (in $\vec{S}_1 \otimes \mathbb{1}$) ↓ id. in the spin space of 2

  $\mathbb{1}$ (in $\mathbb{1} \otimes \vec{S}_2$) → id. in the spin space of 1

$[S_{1x}, S_{2y}] = 0$, etc.

$\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 \; : \; s(s+1)\hbar^2$

$S_z = S_{1z} + S_{2z} \; : \; m\hbar$

$S_{1z} \; : \; m_1\hbar$

$S_{2z} \; : \; m_2\hbar$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

In such a case eigenstates can be represented as

1. $\{m_1, m_2\}$ rep^n : based on the eigenkets of S_{1z} and S_{2z}

   $|++\rangle, \; |+-\rangle, \; |-+\rangle, \; |--\rangle$

   $m_1 = 1/2 \quad m_2 = -1/2$ etc.

2. $\{s, m\}$ rep^n : based on the eigenkets of $\vec{S}^2$ and S_z

   $|1, (\pm 1, 0)\rangle \qquad |0,0\rangle$ → singlet state

   triplet state ↙ ; 1 ↓ s-val ; $(\pm 1, 0)$ ↓ m-val

1 & 2 rel^n ↓ Clebsch-Gordan co-eff^n

$$|1,1\rangle = |++\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$

$$|1,-1\rangle = |--\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \quad .//$$

Now, consider a general situation w/ two ang. mom $\hat{\text{op}}$ $\vec{J}_1$ and $\vec{J}_2$ in different subspaces V_1 and V_2.

$$[J_{\alpha i}, J_{\alpha j}] = i\hbar\, \epsilon_{ijk} J_{\alpha k} \qquad \alpha = 1, 2$$

more generally $$[J_{\alpha i}, J_{\beta j}] = i\hbar\, \epsilon_{ijk} J_{\alpha k}\, \delta_{\alpha\beta} \quad \Rightarrow \quad [J_{1i}, J_{2j}] = 0.$$

Define total ang. mom. $\vec{J} = \vec{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2$

Now, the eigenstates can be defined in two ways —

(i) Simultaneous eigenkets of $\vec{J}_1^2, \vec{J}_2^2, J_{1z}, \&\ J_{2z}$: $|j_1 j_2; m_1 m_2\rangle$

$$\vec{J}_\alpha^2 \, |j_1 j_2; m_1, m_2\rangle = j_\alpha (j_\alpha + 1)\hbar^2 \, |j_1 j_2; m_1, m_2\rangle, \quad \alpha = 1, 2$$

$$J_{\alpha z} \, |j_1 j_2; m_1, m_2\rangle = m_\alpha \hbar \, |j_1 j_2; m_1, m_2\rangle, \quad \alpha = 1, 2$$

(ii) Simultaneous eigenkets of $\vec{J}^2, \vec{J}_1^2, \vec{J}_2^2, \&\ J_z$; $|j_1 j_2; jm\rangle$

$$\vec{J}_\alpha^2 \, |j_1 j_2; jm\rangle = j_\alpha (j_\alpha + 1)\hbar^2 \, |j_1 j_2; jm\rangle, \quad \alpha = 1, 2$$

$$\vec{J}^2 \, |j_1 j_2; jm\rangle = j(j+1)\hbar^2 \, |j_1 j_2; jm\rangle$$

$$J_z \, |j_1 j_2; jm\rangle = m\hbar \, |j_1 j_2; jm\rangle$$

* N.B. $\vec{J}^2$ can not be added to the option (i) as $[\vec{J}^2, J_{1z}] \neq 0$
$[\vec{J}^2, J_{2z}] \neq 0$

$\|^{ly}$ $J_{\alpha z}$ $(\alpha = 1, 2)$ can not be added to option (ii).

* N.B. $[\vec{J}^2, J_z] = 0$, $[\vec{J}^2, \vec{J}_\alpha^2] = 0$

$$\downarrow$$

$$\sum_{\alpha=1}^{2} \vec{J}_\alpha^2 + 2 J_{1z} J_{2z} + J_{1+} J_{2-} + J_{1-} J_{2+}$$

Now, there is a thm →

Given two sets of base kets, both satisfying orthonormality & completeness, ∃ unitary op s.t. $|b\rangle = U|a\rangle$ $\qquad (U^\dagger U = \mathbb{1} = UU^\dagger)$

$$U = \sum_k |b^{(k)}\rangle\langle a^{(k)}| \Rightarrow U|a^{(\ell)}\rangle = |b^{(\ell)}\rangle \ldots$$

∴ The two base kets discussed earlier can be connected via unitary trs

$$|j_1 j_2 ; jm\rangle = \sum_{m_1, m_2} |j_1 j_2 ; m_1 m_2\rangle \underbrace{\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; jm\rangle}$$

elements of the unitary trs matrix

→ Clebsch - Gordan co-effn

- other notn $C(j_1 j_2 j ; m_1 m_2 m)$

or $C_{j_1 j_2}(jm ; m_1 m_2)$

$\sum'_{m_1, m_2} |j_1 j_2 ; m_1 m_2\rangle\langle j_1 j_2 ; m_1 m_2| = \mathbb{1}$ is used

* Properties of C-G co-effn

(i) C-G co-effn vanish unless $m = m_1 + m_2$

$$\because \; (J_z - J_{1z} - J_{2z})|j_1 j_2 ; jm\rangle = 0 \qquad (\because \vec{J} = \vec{J}_1 + \vec{J}_2)$$

$$\Rightarrow \langle j_1 j_2 ; m_1 m_2 | (J_z - J_{1z} - J_{2z}) | j_1 j_2 ; jm\rangle = 0$$

$$\Rightarrow (m - m_1 - m_2)\underbrace{\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; jm\rangle}_{\neq 0 \text{ if } m = m_1 + m_2} = 0$$

(ii) C-G co-effn vanish unless $|j_1 - j_2| \leq j \leq j_1 + j_2$

Dimensionality of the space spanned by $\{|j_1 j_2 ; m_1 m_2\rangle\}$ & $\{|j_1 j_2 ; jm\rangle\}$ are the same → $(2j_1+1)(2j_2+1)$.

(iii) C-G co-eff$^{\underline{ns}}$ form a unitary matrix

Also matrix elements are taken to be real by convention (Condon-Shortley phase convention)

$$\Downarrow$$

Inverse co-eff$^{\underline{n}}$ $\langle j_1 j_2; jm | j_1 j_2; m_1 m_2\rangle$ is the same as

$\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle$ itself

Now a real unitary matrix is orthogonal

orthonormality + reality $\Rightarrow$ $\displaystyle\sum_j{}' \sum_m \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm\rangle = \delta_{m_1 m_1'}\, \delta_{m_2 m_2'}$

Also $\displaystyle\sum_{m_1}\sum_{m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; j'm'\rangle = \delta_{jj'}\, \delta_{mm'}$

↳ special case $j' = j$, $m' = m = m_1 + m_2$

$$\Rightarrow \sum_{m_1}\sum_{m_2} |\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle|^2 = 1$$

(iv) Recurrence rel$^{\underline{n}}$

With j_1, j_2, and j fixed, co-eff$^{\underline{ns}}$ w/ different m_1 and m_2 are related to each other by recursion rel$^{\underline{n}}$.

$$\sqrt{(j \mp m)(j \pm m + 1)}\, \langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m \pm 1\rangle$$
$$= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)}\, \langle j_1 j_2; m_1 \mp 1, m_2 | j_1 j_2; jm\rangle$$
$$+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)}\, \langle j_1 j_2; m_1, m_2 \mp 1 | j_1 j_2; jm\rangle$$

Ex. Consider a particle w/ $s = 1/2$ and orb. ang. mom. l

$\Rightarrow$ $j_1 = l\ (\in \mathbb{Z})$, $m_1 = m_l$

$j_2 = s = 1/2$, $m_2 = m_s = \pm 1/2$

} Allowed j-values $j = \begin{cases} l \pm 1/2, & l > 0 \\ 1/2, & l = 0 \end{cases}$

Discuss the C-G co-eff$^{\underline{ns}}$ for this system.