Advanced Quantum Mechanics

Assignment-I

1. Consider a Hamiltonian of the form,

$$H = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2}{r}.$$

This is similar to the Hamiltonian of an electron in a hydrogen atom. Use the factorization method to extract as much information as you can. Then check if that is consistent with the exact solutions that you obtain by solving the Schrödinger equation.

2. Consider a particle with $s = 1/2$ and orbital angular momentum ℓ. We want to add these angular momenta. So, following the discussion in the class, assume $j_1 = \ell\ (\in \mathbb{Z}), j_2 = s = 1/2,\ m_1 = m_\ell,\ m_2 = m_s = \pm 1/2$. Clearly, allowed j values are,

$$j = \begin{cases} \ell \pm \frac{1}{2},\ \ell > 0 \\ \frac{1}{2},\ \ell = 0. \end{cases}$$

Discuss the Clebsch-Gordan coefficients for this system.

3. An operator $\vec{V}$ (with three components V_1, V_2 and V_3) satisfies the following commutation relation,

$$[L_i, V_j] = i\hbar\epsilon_{ijk}V_k,$$

where $\vec{L}$ represents the usual angular momentum operator. Such operators ($\vec{V}$) are called *vector operators*.
(i) Give a few examples of such operators.
(ii) Prove that the operator $e^{-i\theta L_x/\hbar}$ is actually a rotation operator corresponding to a rotation around the x-axis by an angle θ, by showing that

$$e^{-i\theta L_x/\hbar}V_i e^{i\theta L_x/\hbar} = R_{ij}(\theta)V_j,$$

where $R(\theta)$ is the corresponding rotation matrix.
(iii) Find out $e^{2i\pi^2 L_x/h}|l, m\rangle$.
(iv) Show that a rotation by π around the z-axis can also be achieved by first rotating around the x-axis by $\pi/2$, then rotating around the y-axis by π and, finally, rotating back by $(-\pi/2)$ around the x-axis.

4. Consider the vector operator

$$\vec{\mathscr{R}} = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - mk\frac{\vec{r}}{r}.$$

(i) Show that $\vec{\mathscr{R}}$ is hermitian and a vector under rotation.
(ii) Show that $[\vec{\mathscr{R}}, H] = 0$ for a Hamiltonian of the form $H = \frac{p^2}{2m} - \frac{k}{r}$.
(iii) Show that $\vec{\mathscr{R}}.\vec{L} = 0$ and $\mathscr{R}^2 = 2mH(L^2 + \hbar^2) + m^2k^2$.
(iv) Assume that the H can be properly diagonalized and its eigenvalue is $E < 0$. Then let's define the operator $\mathscr{I}^{\pm} = \frac{1}{2}\left(\vec{L} \pm \frac{\vec{\mathscr{R}}}{\sqrt{-2mH}}\right)$. It can be shown (you don't have to prove them) that $[\mathscr{I}_i^{\pm}, \mathscr{I}_j^{\pm}] = i\hbar\epsilon_{ijk}\mathscr{I}_k^{\pm}$ and $[\mathscr{I}_i^{+}, \mathscr{I}_j^{-}] = 0$. Moreover, these commutators hint that the eigenvalues of $\mathscr{I}^{\pm}$ can be written as $\hbar^2\lambda_{\pm}(\lambda_{\pm}+1)$ with $\lambda_+ = \lambda_- \equiv \lambda$ since $(\mathscr{I}^+)^2 = (\mathscr{I}^-)^2$. Using all these information you have to show that

$$H = -\frac{mk^2}{2[4(\mathscr{I}^{\pm})^2 + \hbar^2]},$$

and by identifying $(2\lambda + 1) \to n$ where $n = 1, 2, 3, \ldots$, show that the energy eigenvalue can be written as

$$E_n = -\frac{mk^2}{2\hbar^2 n^2}.$$

FYI, for $k = e^2$ it is simply the Hamiltonian for the electron in a hydrogen atom.