Condensed Matter Physics (CMP)

Introduction to Theory

CMP : Understanding the behavior of matter using the fundamentals laws of physics.

Approaches

A) Microscopic : Write Hamiltonian (quantum or classical) for the microscopic degrees of freedom + use laws of physics.

B) Phenomonological : Use symmetry, conservation laws, thermodynamics to describe matter regardless of microscopic structure.

Also: Theoretical or experimental

Examples:
* Fluids (Liquids or gases), quantum or classical
* Solids (metals, semiconductors, superconductors, insulators ... etc.)
* Plasmas, magnets ... etc.

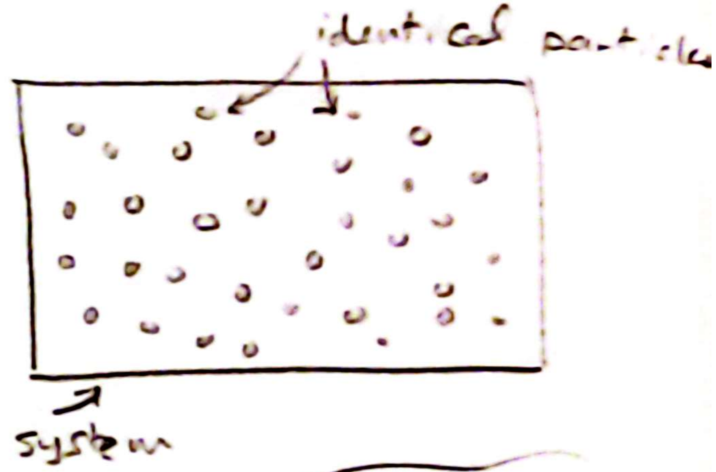
CMP $\simeq$ $\frac{2}{3}$ Physics !!

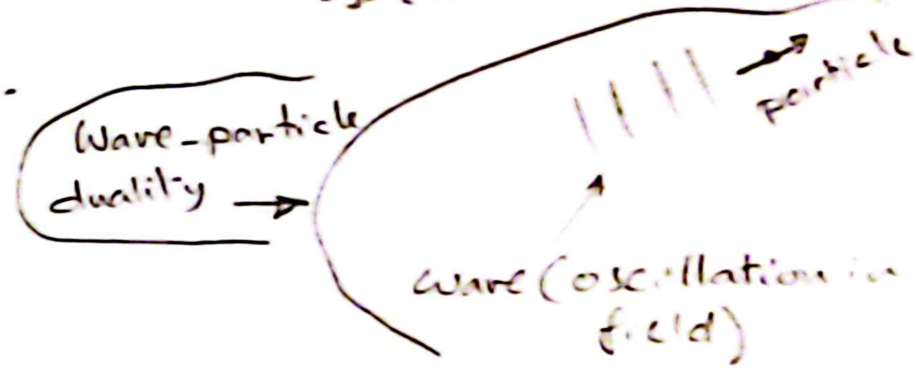
Identical Particles in Condensed Matter systems

Identical particles can be described most efficiently using second quantization

Second Quantization = Quantum Field Theory QFT

* In QFT, particles are the quanta of operator fields.
* QFT capable of describing creation and annihilation of particles.
* In ordinary wave mechanics, we have to symmetrize/antisymmetrize the wavefunction of identical bosons/Fermions. In QFT, this is already taken care of.
* Quantum processes can be visualized using simple diagrams.

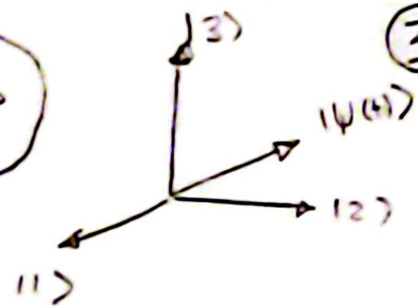




Review of basic principles

Quantum Mechanics QM

ex: 3-dimensional Hilbert space



$$|\psi(t)\rangle = \sum_n c_n(t)\,|n\rangle \qquad |n\rangle = 1, 2, \dots$$

state of system at time t.

orthonormal basis

$\langle m|n\rangle = \delta_{mn}\ ,\ \sum_n |n\rangle\langle n| = \hat{I}$ ← identity operator

$$\langle\psi(t)|\psi(t)\rangle = 1$$

* Operators act on vectors in Hilbert space:

$\hat{A}|\psi\rangle = |\chi\rangle \qquad \forall\ |\psi\rangle$ in Hilbert space.

* Schrödinger Eqn: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \qquad \hat{H}$: Hamiltonian operator

* $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle \qquad$ $|\psi_n\rangle$: energy eigenstate, E_n : ~ eigenvalue $\Big\}$ spectrum of $\hat{H}$

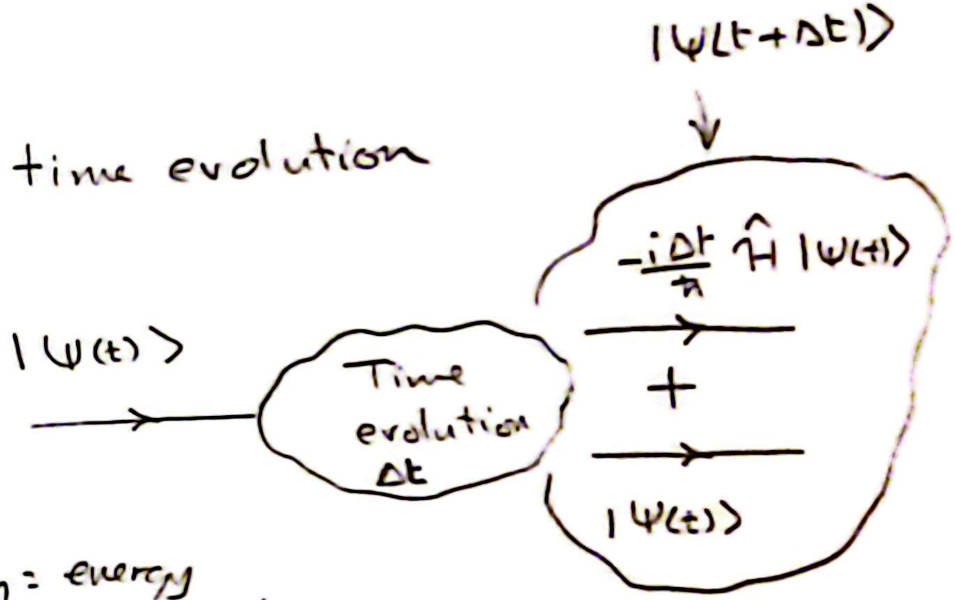
Central importance of the Hamiltonian

A) $\hat{H}$ determines time evolution

The Schrödinger Eqn can be written as:

$$|\psi(t+\Delta t)\rangle = \left[\hat{I} - \frac{i\Delta t}{\hbar}\hat{H}\right]|\psi(t)\rangle$$

$\hat{H}$ generates the non-trivial part of time evolution



B) $\hat{H}$ determines thermodynamic equilibrium

$$Z = \sum_n e^{-\beta E_n} \qquad \beta = \frac{1}{k_B T}, \quad E_n: \text{energy eigenvalue}$$

$$= e^{-\beta F} \qquad F: \text{Free energy}$$

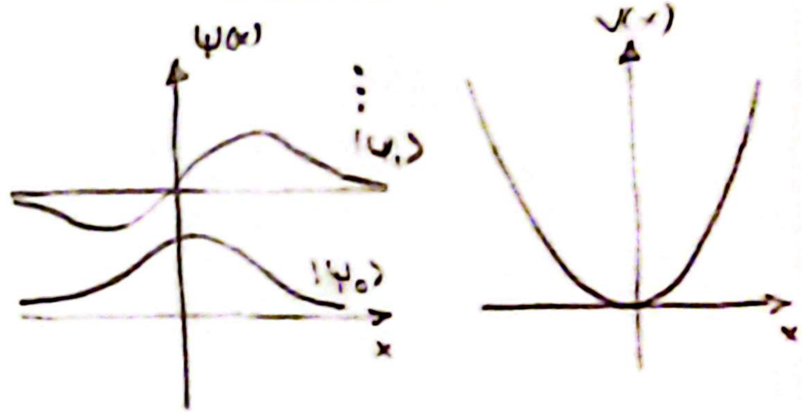
Partition function

Z, F determine all thermodynamic quantities.

Second Quantization

(Introduction)

In second quantization, we use the Hilbert space of a **single particle** to construct the Hilbert space of **multiparticles**.



- Single-particle Hilbert space
  (single-particle states)

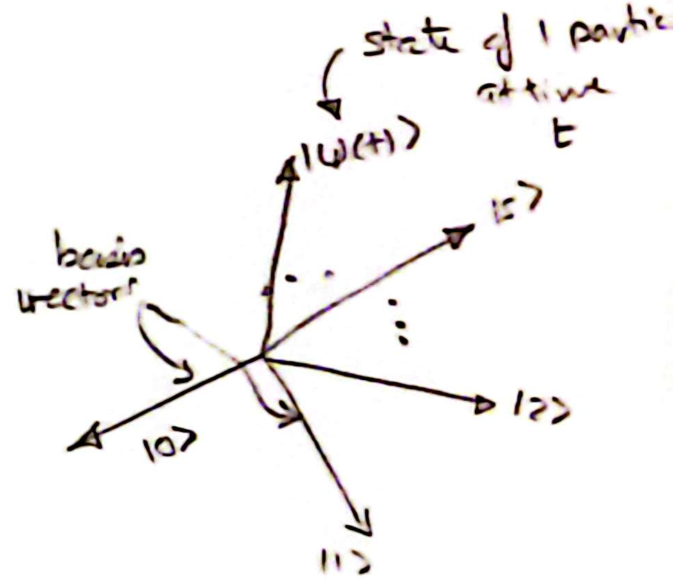
1- Start with ordinary QM problem for a single-particle (eg. Harmonic oscillator).

2- Construct an orthonormal basis for single-particle states: $|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_s\rangle \ldots$

$|0\rangle, |1\rangle, |2\rangle, \ldots |s\rangle \ldots$

$|s\rangle \equiv |\psi_s\rangle \qquad \langle s|s'\rangle = \langle \psi_s|\psi_{s'}\rangle = \delta_{ss'}$

3- The single-particle Hilbert space is the set of all vectors $|\alpha\rangle = \sum_s \alpha_s |s\rangle$, $\langle\alpha|\alpha\rangle = 1$



Hilbert space of a single particle

Note: $\langle \vec{r}\,|\psi(t)\rangle = \psi(\vec{r},t) \leftarrow$ Schrödinger wavefunction

Example:

Start with a free particle in 3-dimensions:

$$\hat{H} = \frac{\vec{p}^2}{2m} \qquad \vec{p} = -i\hbar\vec{\nabla}$$

The states $|\vec{p}\rangle$ are eigenstates of both momentum and energy. They form an orthonormal basis

$$\hat{H}|\vec{p}\rangle = \frac{p^2}{2m}|\vec{p}\rangle \quad , \quad \hat{\vec{p}}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$$

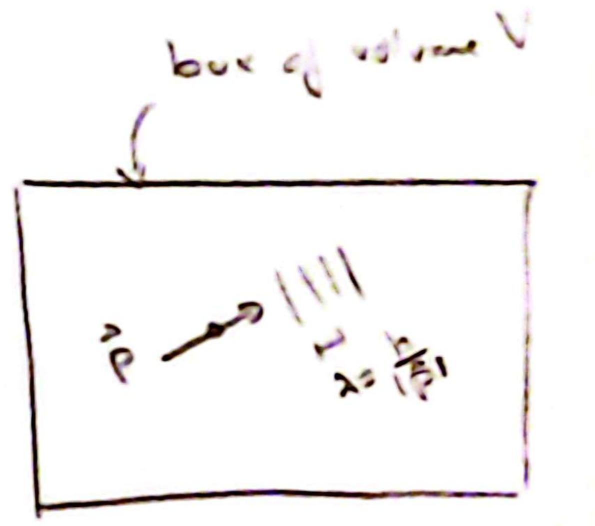
$$\langle\vec{p}|\vec{p}'\rangle = \delta_{\vec{p}\vec{p}'}$$

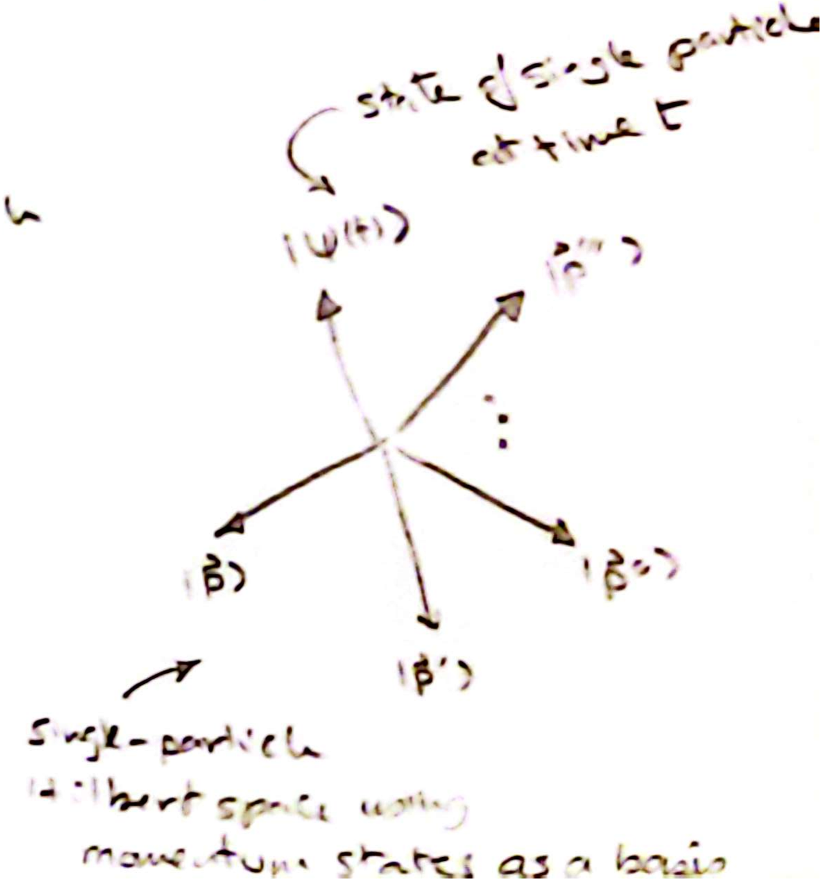
$$\langle\vec{r}|\psi_{\vec{p}}\rangle = \frac{1}{\sqrt{V}}\,e^{i\vec{p}\cdot\vec{r}/\hbar}$$

V: volume in which particle lives

$$|\psi(t)\rangle = \hat{I}\,|\psi(t)\rangle = \sum_{\vec{p}} |\vec{p}\rangle \underbrace{\langle\vec{p}|\psi(t)\rangle}_{a_{\vec{p}}(t)}$$

$$\therefore\ |\psi(t)\rangle = \sum_{\vec{p}} a_{\vec{p}}(t)\,|\vec{p}\rangle$$



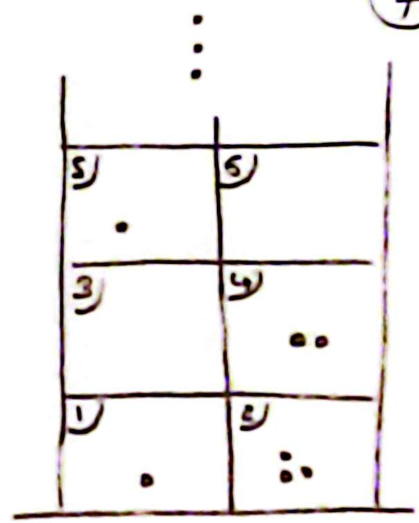


- Constructing the Multiparticle Hilbert space

1) Start with a basis for single-particle states
$|1\rangle, |2\rangle, \ldots, |s\rangle, \ldots$

2) Specify occupation numbers n_s for each state s

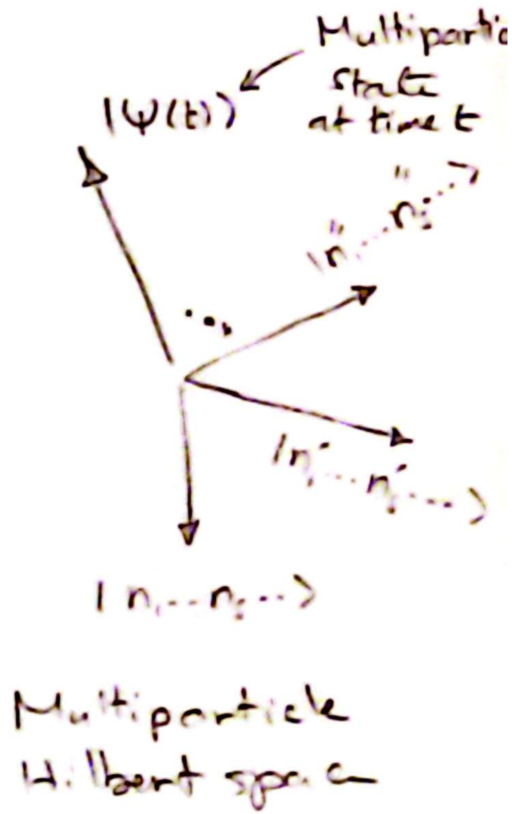
Example: In figure $n_1 = 1$, $n_2 = 3$, $n_3 = 0$, $n_4 = 2$
$n_5 = 1$, $n_6 = 0$, ... etc.



3) Construct the kets $|n_1 n_2 \ldots n_s \ldots\rangle$ as orthonormal basis vectors in the multiparticle Hilbert space.

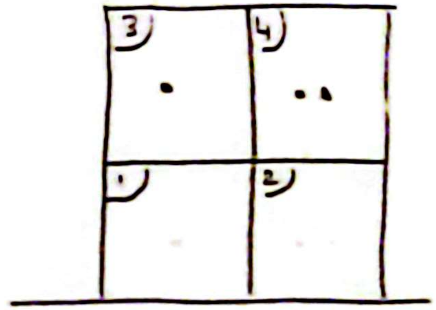
$$\langle n_1 \ldots n_s \ldots | n_1' \ldots n_s' \ldots \rangle = \delta_{n_1 n_1'} \ldots \delta_{n_s n_s'} \ldots$$

$$\hat{I} = \sum_{n_1 \ldots n_s \ldots} |n_1 \ldots n_s \ldots\rangle\langle n_1 \ldots n_s \ldots|$$



A general state in multiparticle space $|\Psi\rangle$ can be expanded in terms of the basis vectors $|n_1 \ldots n_s \ldots\rangle$:

$$|\Psi\rangle = \sum_{n_1 \ldots n_s \ldots} C_{n_1 \ldots n_s \ldots} |n_1 \ldots n_s \ldots\rangle$$



Example:

- Consider a 4-dim single-particle space with states $|1\rangle, |2\rangle, |3\rangle, |4\rangle$
- Consider 3 identical bosons occupying the 4 states above.

1) Write down all possible states with sharp n_i:

$|3000\rangle, |0300\rangle, |0030\rangle, |0003\rangle,$

$|2100\rangle, |2010\rangle, |2001\rangle, |1200\rangle, \ldots$ etc.

2) Write down the ket for the state in the figure:

$n_1 = 0 \quad n_2 = 0 \quad n_3 = 1 \quad n_4 = 2 \quad \therefore |n_1 n_2 n_3 n_4\rangle$

$\equiv |0012\rangle$

3) Consider the state $|\psi\rangle = A(|0102\rangle + 2i|3000\rangle)$

If $\langle\psi|\psi\rangle = 1$, determine A

$$\langle\psi|\psi\rangle = |A|^2 \left[\underbrace{\langle 0102|0102\rangle}_{1} + |2i|^2 \underbrace{\langle 3000|3000\rangle}_{1} \right]$$

$$= 5|A^2| = 1 \longrightarrow A = 1/\sqrt{5}$$

Bosons and Fermions

The occupation numbers n_s are represented by hermitian operators $\hat{n}_s$ with the following eigenvalues:

Bosons $n_s = 0, 1, 2, \ldots$

Fermions $n_s = 0, 1$ (Pauli exclusion)

Creation / annihilation operators

We define the operators a_s, $a_s^\dagger$ in the following way:

A) Bosons

$$a_s | n_1 \ldots n_s \ldots \rangle = \sqrt{n_s}\, | n_1 \ldots n_s - 1 \ldots \rangle$$

$$a_s^\dagger | n_1 \ldots n_s \ldots \rangle = \sqrt{n_s + 1}\, | n_1 \ldots n_s + 1 \ldots \rangle$$

Note:
$\xi_s = n_1 + \ldots + n_{s-1}$

B) Fermions

$$a_s | n_1 \ldots n_s \ldots \rangle = (-1)^{\xi_s} | n_1 \ldots n_s - 1 \ldots \rangle$$ (only when $n_s = 1$, zero otherwise)

$$a_s^\dagger | n_1 \ldots n_s \ldots \rangle = (-1)^{\xi_s} | n_1 \ldots n_s + 1 \ldots \rangle$$ (only when $n_s = 0$, zero otherwise)

The vacuum state $|0\rangle$:

We define the empty state $|0\rangle = |000\ldots0\ldots\rangle$ $(n_i = 0 \ \forall i)$

From the definition of the operators $a_s, a_s^\dagger$ we can show:

a) $a_s|0\rangle = \langle 0| a_s^\dagger = 0$ (Bosons & Fermions)

b) $[a_s, a_{s'}] = [a_s^\dagger, a_{s'}^\dagger] = 0$, $[a_s, a_{s'}^\dagger] = \delta_{ss'}$ (Bosons)

$[a_s, a_{s'}]_+ = [a_s^\dagger, a_{s'}^\dagger]_+ = 0$, $[a_s, a_{s'}^\dagger]_+ = \delta_{ss'}$ (Fermions)

where $[A,B] = AB - BA$ (Commutator)

$[A,B]_+ = AB + BA$ (anticommutator)

c) $a_s^2 = a_s^3 = a_s^4 = \ldots$ For fermions

d) $\boxed{\hat{n}_s = a_s^\dagger a_s}$ ← Bosons & Fermions