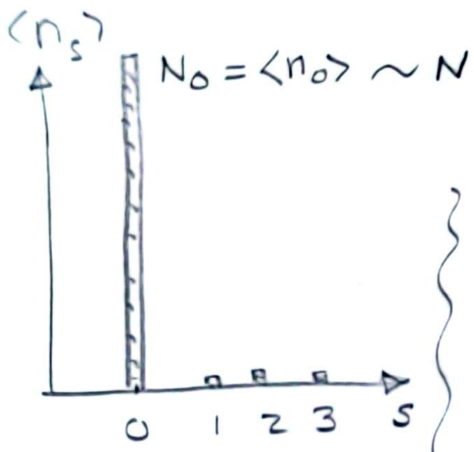
Superfluidity and Bose-Einstein Condensation (BEC)



- Macroscopic occupancy of a single-particle state

Constraint:

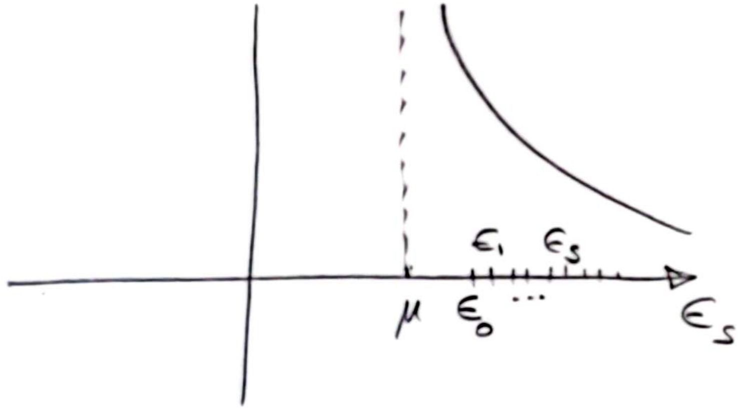
$$\sum_i n_i = N$$

$$\therefore \sum_i \langle n_i \rangle = \langle N \rangle$$

μ: Chem. pot.
vary μ till constraint is satisfied

As μ approaches ϵ_0, a macroscopic fraction N_0 condenses into the lowest state

$$\boxed{N_0/N > 0 \quad N \to \infty}$$



$$\langle n_s \rangle = \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}$$

only for non-interacting bosons

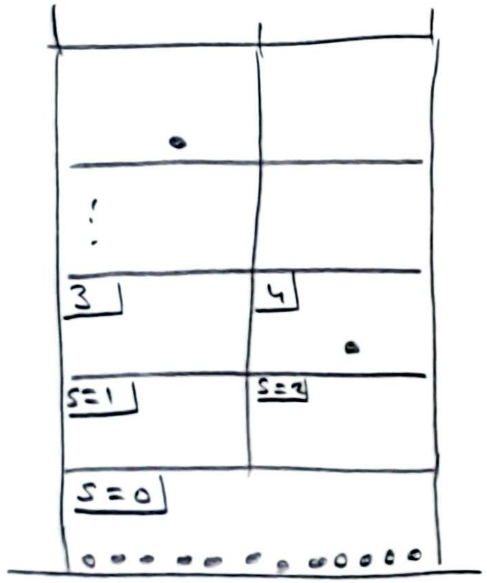
$$\hat{H} = \hat{n}_0 \epsilon_0 + \hat{n}_1 \epsilon_1 + \cdots$$

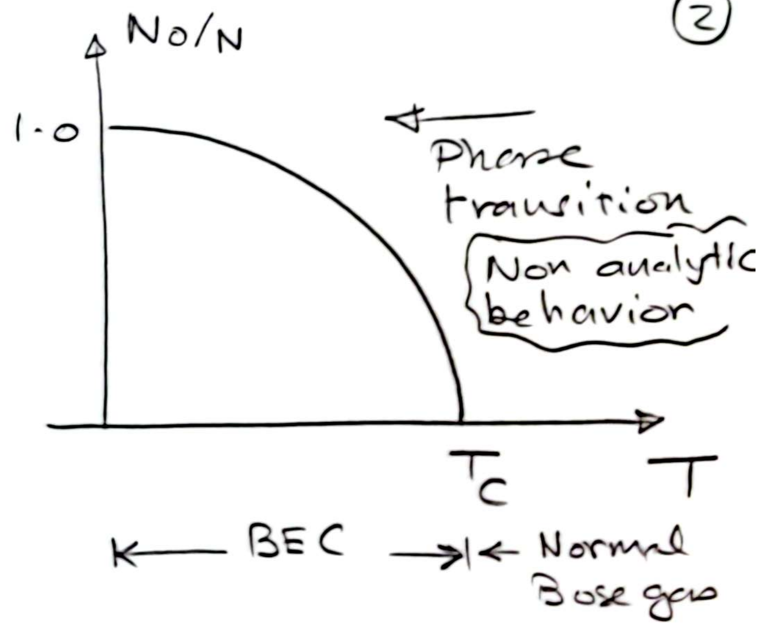
$$= \sum_i n_i \epsilon_i$$

$$= \sum_i \epsilon_i a_i^{\dagger} a_i$$

$$= \epsilon_0 a_0^{\dagger} a_0 + \cdots + \epsilon_s a_s^{\dagger} a_s + \cdots$$

$$|10^{25}\ 0\ 0\ 0\ 1\ 0\ 0\ 1 \cdots \rangle$$
$$\uparrow \quad \uparrow \quad \cdots\ n_s$$
$$n_0 \quad n_1$$

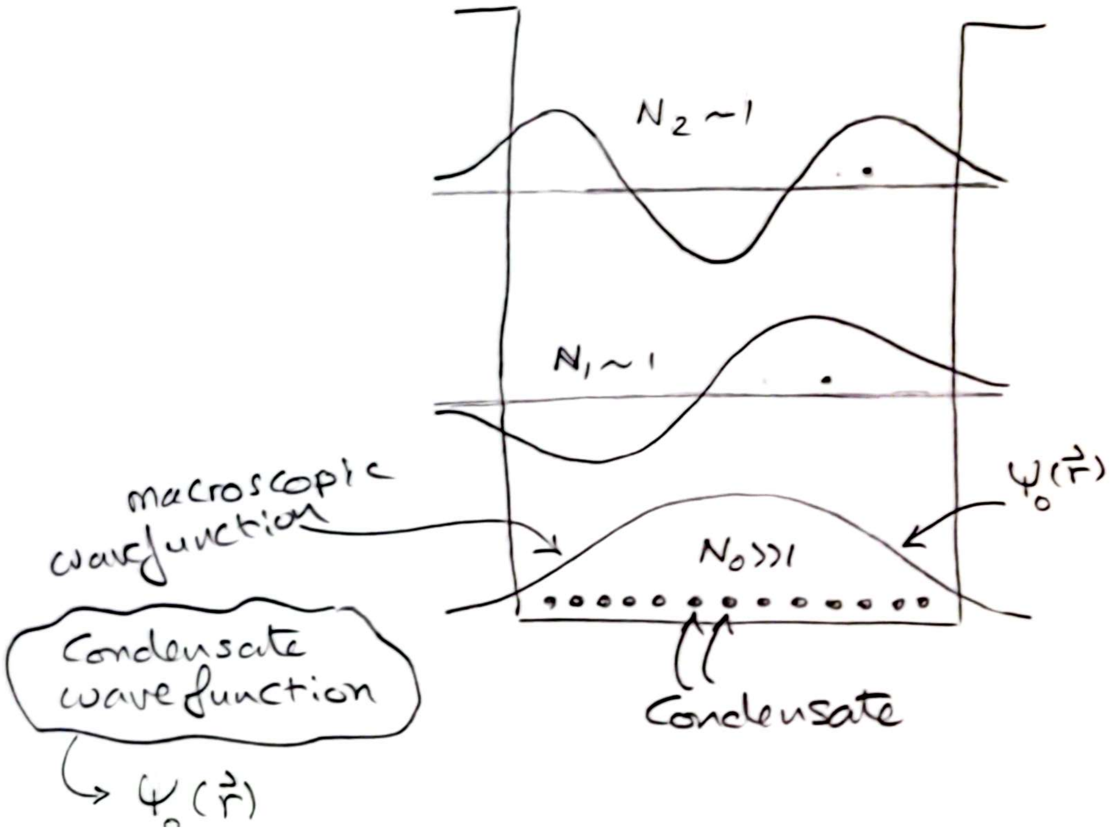




$$\hat{H} = \sum_s \epsilon_s n_s$$

$$= \sum_s \epsilon_s a_s^{\dagger} a_s$$

What about real (interacting) systems ?



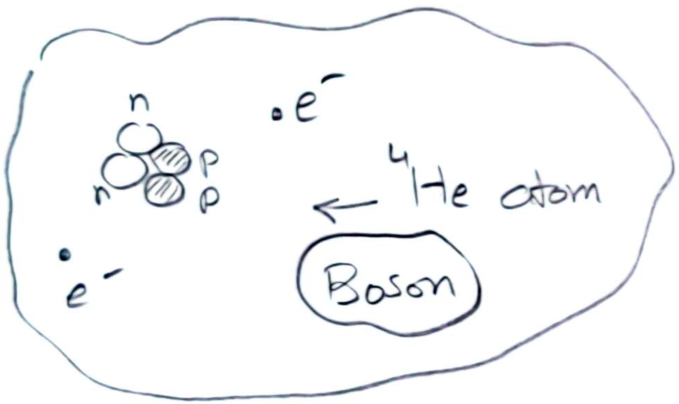
Superfluidity in real Bose systems

Generalized BEC

* Macroscopic occupation of a single-particle state, possibly varying with time, not necessarily at equilibrium.

* Kapitza, Allen and Misener (1938)

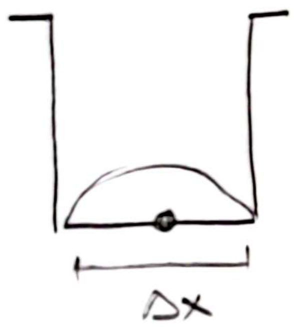
  Superfluidity found in



* Interactions are weak
* Light atoms
* Stays liquid

Localization

$$\Delta p \sim \frac{\hbar}{\Delta x}$$



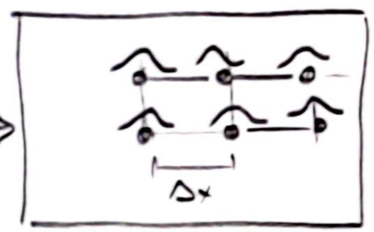
$$E = \frac{(\Delta p)^2}{2m}$$

$$E \sim \frac{\hbar^2}{2m(\Delta x)^2}$$

Localization costs kinetic energy even more when:

* Light particles
* Tight confinement

solid

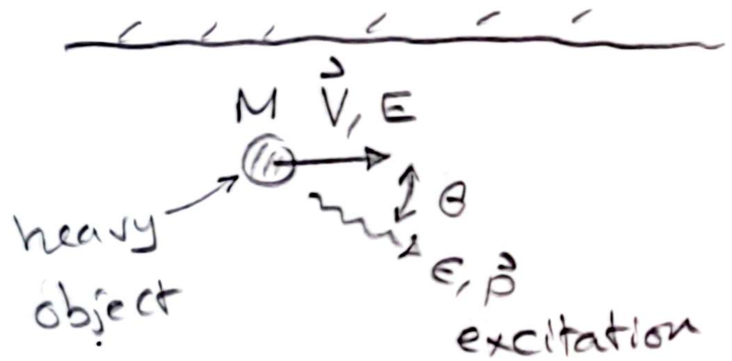


Landau's argument for superfluidity

$$\Delta E = \frac{1}{2} M (\vec{V}'^2 - \vec{V}^2) = \frac{1}{2} M (\vec{V}' + \vec{V}) \cdot (\vec{V}' - \vec{V})$$

$$\cong M \vec{V} \cdot \Delta \vec{V} \simeq M V \Delta V \cos\theta$$

$$\Delta \vec{p} = M \Delta \vec{V}$$

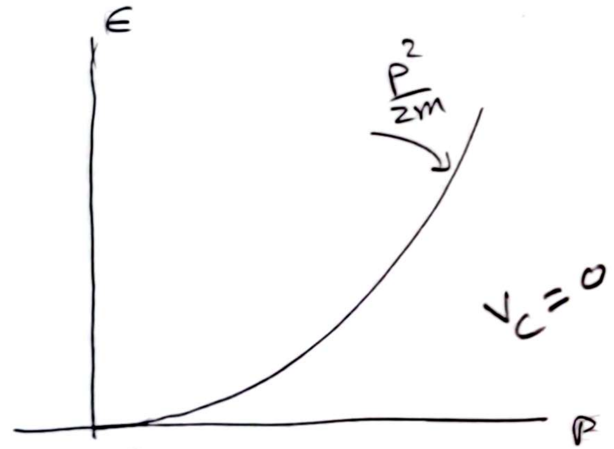


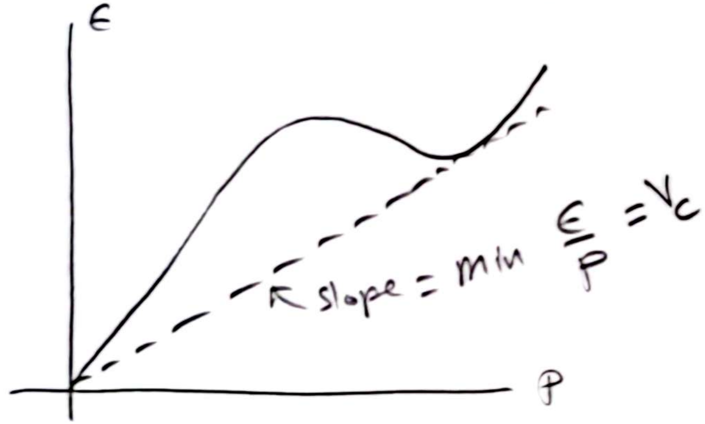
$$\Delta E = M V \Delta V \cos\theta = \epsilon$$

$$|\Delta \vec{p}| = M \Delta V \qquad = \vec{p}$$

$$\therefore \frac{|\epsilon|}{|\vec{p}|} = V |\cos\theta| \longrightarrow V = \frac{1}{|\cos\theta|} \frac{\epsilon}{|\vec{p}|}$$

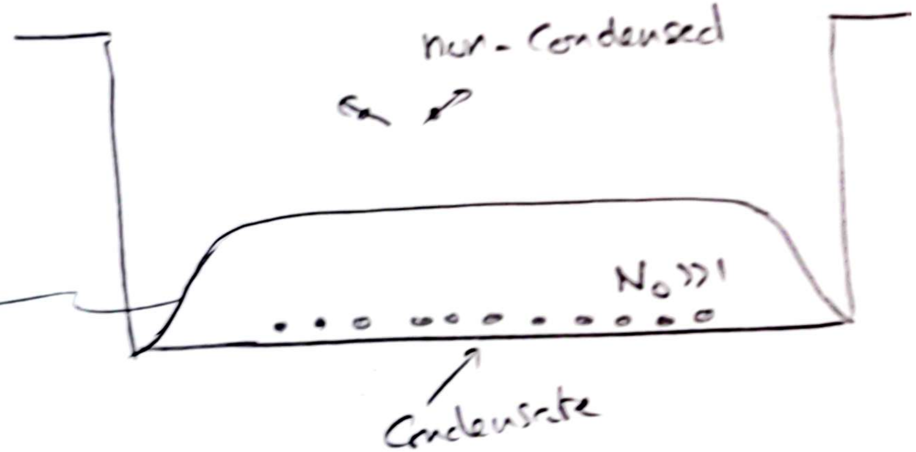
$$V_{crit.} = \left. \frac{\epsilon}{|\vec{p}|} \right|_{min}$$





Bogoliubov Theory of a weakly interacting Bose gas

N. Bogoliubov 1946

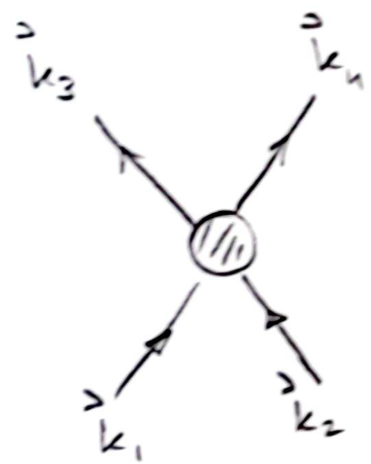


$$\mathcal{H} = \sum_{\vec k} \left(\frac{\hbar^2 k^2}{2m} - \mu\right) a_k^\dagger a_k$$

$$+ \lambda \int d\vec r \; \psi^\dagger(\vec r)\psi^\dagger(\vec r)\psi(\vec r)\psi(\vec r)$$

$$= \sum_{\vec k} \left(\frac{\hbar^2 \vec k^2}{2m} - \mu\right) a_{\vec k}^\dagger a_{\vec k}$$

$$+ \frac{\lambda}{2V} \sum_{k_1 \dots k_4} \delta_{k_1+k_2+k_3+k_4} a_{k_4}^\dagger a_{k_3}^\dagger a_{k_2} a_{k_1}$$



$$a_0 |N_0\rangle = \sqrt{N_0}\, |N_0 - 1\rangle \simeq \sqrt{N_0}\, |N_0\rangle$$

$$a_0^\dagger |N_0\rangle = \sqrt{N_0 + 1}\, |N_0 + 1\rangle \simeq \sqrt{N_0}\, |N_0\rangle$$

Secretly Coherent state

Bogoliubov's Recipe:

In all low-lying states, replace a_0, a_0^+ by $\sqrt{N_0}$ only !!!

Condensate $\vec{k}=0$

$$\mathcal{H}_{int} \cong \frac{\lambda}{2V}\left[a_0^+ a_0^+ a_0 a_0 + \sum_{\vec{k}\neq 0}\left[4\, a_k^+ a_0^+ a_k a_0 + a_k^+ a_{-k}^+ a_0 a_0 + a_0^+ a_0^+ a_k a_{-k}\right]\right]$$

$$\mathcal{H}_0 = \sum_k (\epsilon_k - \mu)\, a_k^+ a_k$$

$$= -\mu\, a_0^+ a_0 + \sum_{k\neq 0} (\epsilon_k - \mu)\, a_k^+ a_k$$

That's it !!

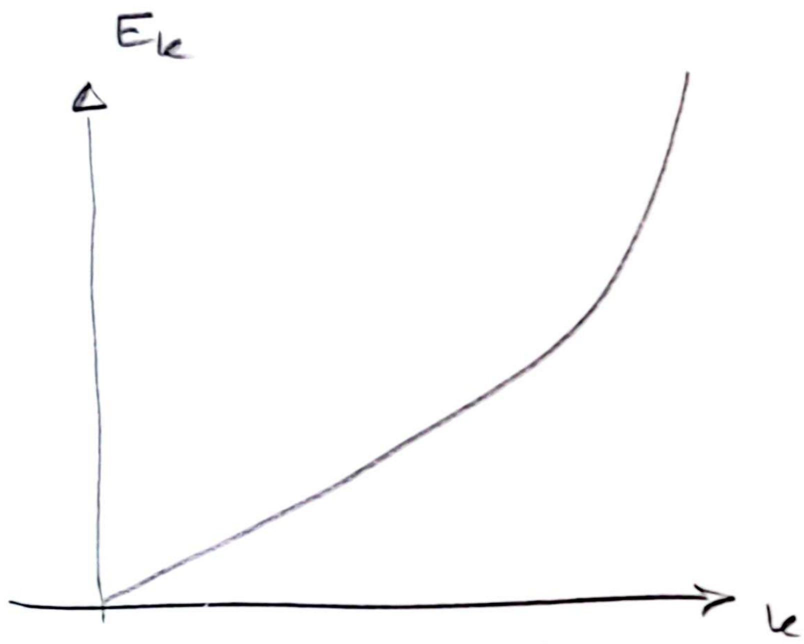
$$\mathcal{H} \cong \sum_{k\neq 0} (\epsilon_k - \mu)\, a_k^+ a_k - \mu N_0 + \frac{\lambda}{2V}\left[N_0^2 + 4N_0 \sum_{k\neq 0} a_k^+ a_k + N_0 \sum_{k\neq 0}\left(a_k^+ a_{-k}^+ + a_k a_{-k}\right)\right]$$

Bogoliubov Transformation

$$a_k = b_k \cosh\theta + b^{\dagger}_{-k} \sinh\theta$$

$$\therefore \mathcal{H} = \sum_{\vec{k}} E_k \, b^{\dagger}_k b_k$$

$$E_k = \sqrt{[\epsilon(\vec{k}) + \mu]^2 - \mu^2}$$



Dispersion for Bog. qp

Condensate + weakly interacting Bog. quasiparticles