Scattering theory

A great deal of our understanding about the structure of matter (and even properties of radiation) is gathered from various scattering processes.

· Scattering of composite particles $\rightarrow$ $p+p \rightarrow p+p+\pi^0$

$p+p \rightarrow p+n+\pi^+$

elementary particles $\rightarrow$ $e^+ + e^- \rightarrow \mu^+ + \mu^-$

} final products may not be the same as the initial particles

(These are like reactions
$a+b \rightarrow c+d+e+\cdots$)

radiation and particles $\rightarrow$ $\gamma + e^- \rightarrow \gamma + e^-$ (Compton scattering)

· Usually scattering means when the initial and final state particles are the same

$$a+b \rightarrow a+b$$

(i) Elastic $\rightarrow$ none of the particles' internal states change

(ii) Inelastic $\rightarrow$ if not elastic

Here we will consider only elastic scattering of particles w/o spin in the non-relativistic approximation.

· Also assume, the interaction potential is translationally invariant.

To discuss scattering processes, choice of frame is crucial, we can have the description in (i) lab frame or (ii) centre of mass frame (CM) } connected by well-defined relations

In CM frame, the situation reduces to a single particle scattering off a potential $V(\vec{r})$.

[Actually, if the interacⁿ between the two particles depend only on their relative separation the time-indep SE for the system

$$\left(-\frac{\hbar^2}{2m_1}\vec{\nabla}_1^2 - \frac{\hbar^2}{2m_2}\vec{\nabla}_2^2 + V(\vec{r}_1,\vec{r}_2)\right)\psi(\vec{r}_1,\vec{r}_2) = E_{tot}\,\psi(\vec{r}_1,\vec{r}_2)$$ reduces ~~this~~ to

two decoupled eigenvalue eqⁿs

(i) for the CM which moves like a free particle of mass m

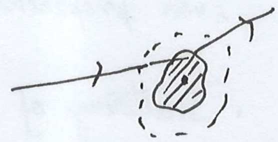
(ii) for a fictitious particle of reduced mass μ, moving in $V(|\vec{r}_1 - \vec{r}_2|)$.]

We are interested in the energy eigenstates

$$H = \frac{\vec{p}^2}{2\mu} + V(\vec{r}) \quad , \quad \psi(\vec{r},t) = \psi(\vec{r})\, e^{-iEt/\hbar}$$

(interaction is assumed to be time-indep.)

$$\left(-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r}) .$$



Potential is of finite range, or vanishes faster than $\frac{1}{r}$ as $r \to \infty$.

Say, $V(r) \neq 0$ for $r < a$

$= 0$ for $r > a$

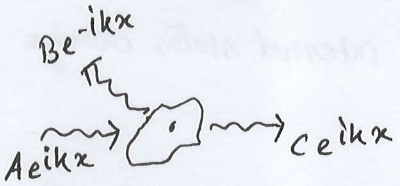
Far away from the potential $E = \frac{\hbar^2 k^2}{2\mu}$ (only +ve energy sol$^{\underline{n}}$ considered)

The SE $\Rightarrow$

$$\left(-\frac{\hbar^2}{2\mu}\left(\vec{\nabla}^2 + k^2\right) + V(\vec{r})\right)\psi(\vec{r}) = 0$$

Note that before (and long after) the collision μ (particle) behave as a free particle and can be described by a plane wave

So we have the incident, $\underbrace{\text{reflected and transmitted}}_{\text{scattered wave}}$ waves



When $V(\vec{r}) = 0$, e.g., $e^{i\vec{k}\cdot\vec{r}}$ for any $\vec{k}$, such that $\vec{k}\cdot\vec{k} = k^2$ is a sol$^{\underline{n}}$.

If the incident wave is moving towards $+\hat{z}$ direc$^{\underline{n}}$ the wave f$^{\underline{n}}$ $\phi(\vec{r}) \sim e^{ikz}$

This is a sol$^{\underline{n}}$ only if the potential vanishes

The scattered wave $\psi(\vec{r})$

$\psi(\vec{r}) \approx e^{ikr}$ (radially propagating out)

$\left(\vec{\nabla}^2 + k^2\right)e^{ikr} \neq 0 \qquad \left(= \frac{2i\,e^{ikr}k}{r} \text{ chk?}\right) \rightarrow$ fails for $r \neq 0$.

but $\left(\vec{\nabla}^2 + k^2\right)\frac{e^{ikr}}{r} = 0$ for $r \neq 0$

$\hookrightarrow$ consistent w/ the full radial sol$^{\underline{n}}$ which is of the form $u(r)/r$.

But this form $\frac{e^{ikr}}{r}$ is not general enough

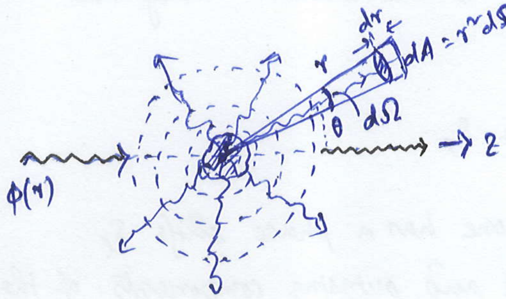
In general the scattered wave may not be spherically symmetric, and it can have some angular dependence.

So, the general form should be

$$\psi_s(\vec{r}) = f_k(\theta,\phi)\frac{e^{ikr}}{r} \quad \Big| \quad \psi_s \text{ is a sol}^{\underline{n}} \text{ for } r \gg a$$
(far away)

After the scattering has taken place, the total wave consists of a superposition of the incident plane wave and scattered wave,

$$\therefore \quad \psi(\vec{r}) = \phi(\vec{r}) + \psi_s(\vec{r})$$
$$\simeq e^{ikz} + \underbrace{f_k(\theta,\phi)}_{\text{scattering amplitude (to be determined by } V(\vec{r}))}\frac{e^{ikr}}{r}, \quad r \gg a.$$



$f_k(\theta,\phi)$ can be connected to cross section

Differential cross section $d\sigma = \dfrac{\text{\# of particles/unit time into solid angle } d\Omega}{\text{flux of incident particles}}$

flux of incident particles $\to \dfrac{\text{\# of particles}}{\text{area. time}}$

$D^{\underline{r}}$: incident flux in e^{ikz}

$$= \frac{\hbar}{\mu}\, \mathrm{Im}(\psi^* \vec{\nabla}\psi) = \frac{\hbar k}{\mu}\hat{k} \quad \text{unit vector along } z.$$

(intuitively this is

= prob. density × velocity

$= |e^{ikz}|^2 (p/\mu)$

$= 1 \times p/\mu = \frac{\hbar k}{\mu}$).

$N^{\underline{r}}$:

first note that

dn = # of particles in the infinitesimal volume of thickness dr and area $r^2 d\Omega$

$$= |\psi(\vec{r})|^2 d^3r$$

$$= \left|f_k(\theta,\phi)\frac{e^{ikr}}{r}\right|^2 r^2 d\Omega\, dr$$

$$= |f_k(\theta,\phi)|^2 d\Omega\, dr$$

with velocity $v = \frac{\hbar k}{\mu}$ all these particles will cross the area $r^2 d\Omega$ in time $dt = \frac{dr}{v}$

$\therefore$ # of particles per unit time $\Rightarrow$

$$\frac{dn}{dt} = |f_k(\theta,\phi)|^2 \frac{d\Omega\, dr}{dr/v} = \frac{\hbar k}{\mu} |f_k(\theta,\phi)|^2 d\Omega$$

$$\therefore \quad d\sigma = \frac{dn/dt}{\frac{\hbar k}{\mu}} = |f_k(\theta,\phi)|^2 d\Omega$$

Differential cross section $= \frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$
(/ per unit solid angle)

Total cross section $= \sigma = \int d\sigma = \int d\Omega\, |f_k(\theta,\phi)|^2$ //

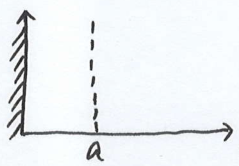
Thus we see that scattering amplitude $f_k(\theta,\phi)$ is one crucial thing that we need to find out to get the σ.

— calculated in terms of phase shift δ_l

Physical picture
Scattering takes place on each of the "partial waves", that by superposiⁿ represent the full wave-fⁿ. wave of diff. (quantized) ang. mom. l and has an incoming and outgoing component

- for each partial wave, one has a phase shift δ_l relating the incoming and outgoing components of the wave

First recall the 1-D case,



$$\phi(x) \sim \sin kx$$
$$= \frac{1}{2i}\big(e^{ikx} - \underbrace{e^{-ikx}}_{\text{incoming}}\big) \rightarrow \text{solⁿ if } V=0$$

If the potential $V(x)$ is non-zero for a finite range a, the exact solⁿ is hard to find for $x<a$, but for $x>a$, solⁿ becomes

$$\phi(x) = \frac{1}{2i}\big(e^{ikx}e^{2i\delta_k} - \underbrace{e^{-ikx}}_{\text{incoming}}\big)$$

Only possibility consistent w/ having a solⁿ of SE + conservⁿ of probability ← The outgoing wave differs by a phase factor $e^{2i\delta_k}$

δ_k : phase shift → depends on k and $V(x)$.

$\because$ We have chosen the same expression for incoming wave for $V=0$ and $V\neq 0$ solⁿ

$$\psi(x) = \phi(x) + \underbrace{\eta(x)}_{e^{ikx}e^{i\delta}\sin\delta}, \quad x>a$$

For the 3-D case, the sol^{n} valid at a large distance from the potential,

$$\psi(\vec{r}) \simeq \underbrace{e^{ikz}}_{\substack{\text{incoming} \\ \text{(same part for both} \\ V=0 \text{ and } V\neq 0)}} + \underbrace{f_k(\theta)\frac{e^{ikr}}{r}}_{\text{outgoing (scattered)}}, \quad r \gg a$$

N.B. $f_k(\theta,\phi) \to f_k(\theta)$ due to the axial sym. the system is invt. under rot^{n} about the z-axis, so no ϕ-depend^{n}.

Now, $$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} a_l P_l(\cos\theta) \underbrace{j_l(kr)}_{\text{spherical Bessel fn.}}$$ $$= \frac{1}{2i^l}\int_{-1}^{1} e^{ixu} P_l(u)\,du$$

Rayleigh's formula $\longrightarrow$ $$= \sqrt{4\pi}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^l\, Y_{l,0}(\cos\theta) \times j_l(kr)$$

Also $$Y_{l,0}(\theta) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

$$e^{ikz} = \sqrt{4\pi}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^l\, Y_{l,0}(\theta)\, j_l(kr)$$ $\to$ a plane wave is built by a linear superposi^{n} of spherical waves of all possible values of ang. mom !!!

Each l-contrib^{n} is a partial wave

Each partial wave is an exact sol^{n} when $V=0$.

But then for large r limit (i.e. $r \gg a$)

$$j_l(kr) \to \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right) = \frac{1}{2ik}\left(\underbrace{\frac{\exp\{i(kr-\frac{l\pi}{2})\}}{r}}_{\text{out-going}} - \underbrace{\frac{\exp\{-i(kr-\frac{l\pi}{2})\}}{r}}_{\text{incoming}}\right)$$

$$\therefore \psi(\vec{r}) \simeq \frac{\sqrt{4\pi}}{k}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^l\, Y_{l,0}(\theta) \times \frac{1}{2i}\left(\frac{e^{i(kr-\frac{l\pi}{2})}e^{2i\delta_l}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right), \quad r\to\infty$$

(Introducing the phase δ_l in the scattered wave)

$$= e^{ikz} + f_k(\theta)\, e^{ikr}/r$$

$$\Rightarrow \frac{\sqrt{4\pi}}{k}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^l\, Y_{l,0}(\theta)\underbrace{\frac{1}{2i}\left(e^{2i\delta_l}-1\right)}_{e^{i\delta_l}\sin\delta_l}\underbrace{\frac{e^{i(kr-l\pi/2)}}{r}}_{e^{-il\pi/2}=(-i)^l} = f_k(\theta)\, e^{ikr}/r$$

$$\Rightarrow f_k(\theta) = \frac{\sqrt{4\pi}}{k}\sum_{l=0}^{\infty}\sqrt{2l+1}\; Y_{l,0}(\theta)\, e^{i\delta_l}\sin\delta_l$$

Thus we get the scattering amplitude in terms of the phase shift.

Finally, the cross-section,

$$\sigma = \int d\Omega \, |f_k(\theta)|^2 = \int d\Omega \, f_k^*(\theta) f_k(\theta)$$

$$= \frac{4\pi}{k^2} \sum_{l,l'} \sqrt{2l+1}\,\sqrt{2l'+1}\; e^{-i\delta_l} \sin\delta_l \, e^{i\delta_{l'}} \sin\delta_{l'} \underbrace{\int d\Omega \, Y^*_{l,0}(\Omega)\, Y_{l,0}(\Omega)}_{\delta_{ll'}}$$

$$\therefore \sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l \quad // \quad \cancel{\frac{4\pi}{k^2}}$$

An important case for $f_k(\theta)$ in the forward direc^n, i.e., $\theta = 0$

$$Y_{l,0}(0) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(1) = \sqrt{\frac{2l+1}{4\pi}}$$

$$\therefore f_k(0) = \frac{\sqrt{4\pi}}{k} \sum_{l=0}^{\infty} \sqrt{2l+1}\, \sqrt{\frac{2l+1}{4\pi}}\; e^{i\delta_l} \sin\delta_l$$

$$= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l$$

$$\Rightarrow \mathrm{Im}\big(f_k(0)\big) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l = \frac{1}{k}\, \frac{k^2}{4\pi}\, \sigma$$

$$\Rightarrow \sigma = \frac{4\pi}{k}\, \mathrm{Im}\big(f_k(0)\big)$$ $\longrightarrow$ This is called optical theorem

N.B. $\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l = \sum_{l=0}^{\infty} \sigma_l$ (Partial wave expansion)

$\because \sin^2\delta_l \le 1$ for any l

$$\Rightarrow \sigma_l \le \frac{4\pi}{k^2}(2l+1)$$ $\longrightarrow$ This is called unitarity bound.

** Any spherically symmetric potential can be analyzed using phase shifts. If phase shift is known differential and total scattering cross-section can easily be calculated.

To identify the δ_l s, first consider the general sol^{n} for a free particle $(V=0)$ in spherical co-ordinates and restrict to a fixed l

$$\psi(\vec{r})\big|_l = \Big[A_l \underbrace{j_l(kr)}_{\text{spherical Bessel}} + B_l \underbrace{n_l(kr)}_{\text{spherical Neumann}}\Big] Y_{l,0}(\theta) \qquad \text{------- (1)}$$

The radial sol^{n}

(Recall the (SE) eq^{n} of the form
$$\left[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} + \left(1 - \frac{l(l+1)}{\rho^2}\right)\right]\phi(\rho) = 0$$
w/ $\rho = kr$)

$B_l \neq 0 \Rightarrow$ a non-vanishing potential because this sol^{n} becomes singular as $r \to 0$

If $V=0$ the sol^{n} should be valid everywhere and $n_l(kr)$ is singular at the origin, thus $B_l = 0$

For large kr, eq^{n} (1) $\Rightarrow$

$$\psi(\vec{r})_l \simeq \left[\frac{A_l}{kr}\sin\left(kr - \frac{l\pi}{2}\right) - \frac{B_l}{kr}\cos\left(kr - \frac{l\pi}{2}\right)\right] Y_{l,0}(\theta)$$

(using the properties of j_l and n_l)

$$= \frac{A_l}{kr}\left[\sin\left(kr - \frac{l\pi}{2}\right) - \frac{B_l}{A_l}\cos\left(kr - \frac{l\pi}{2}\right)\right] Y_{l,0}(\theta)$$

Define $\tan\Delta = -\dfrac{B_l}{A_l}$

$\Rightarrow A_l = \tilde{c}\cos\Delta$, $B_l = -\tilde{c}\sin\Delta$

$$\simeq \frac{\tilde{c}\cos\Delta}{kr}\left[\sin\left(kr - \frac{l\pi}{2}\right) - \frac{\sin\Delta}{\cos\Delta}\cos\left(kr - \frac{l\pi}{2}\right)\right] Y_{l,0}(\theta)$$

$$= \frac{\tilde{c}}{kr}\sin\left(kr - \frac{l\pi}{2} + \Delta\right) Y_{l,0}(\theta)$$

$$= \frac{\tilde{c}}{kr}\frac{1}{2i}\left[e^{i(kr - \frac{l\pi}{2} + \Delta)} - e^{-i(kr - l\pi/2 + \Delta)}\right] Y_{l,0}(\theta)$$

$$= e^{-i\Delta}\frac{\tilde{c}}{kr}\frac{1}{2i}\left[e^{i(kr - \frac{l\pi}{2} + 2\Delta)} - e^{-i(kr - \frac{l\pi}{2})}\right] Y_{l,0}(\theta)$$

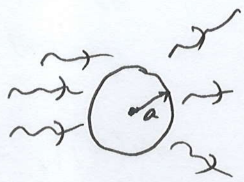
$\Rightarrow$ Δ can be easily identified w/ the required phase-shift i.e. $\Delta = \delta_l$ and it is defined as $\delta_l = \tan^{-1}\left(-\dfrac{B_l}{A_l}\right)$. //

(Recall the similar expression in p.5)

Example

Hard sphere

$$V(r) = \begin{cases} \infty , & r \le a \\ 0 , & r > a \end{cases}$$



- Write the full SE
- Sol^n can be separated into radial and angular part
  - Radial part $\rightarrow \quad R_l(r) = \frac{u}{r} = \sum_l A_l j_l(kr) + B_l n_l(kr)$
  - Angular part $\rightarrow \quad \phi(\theta) \sim \sum_{l=0}^{\infty} P_l(\cos\theta)$

$$\therefore \quad \psi(\vec{r}) = \sum_{l=0}^{\infty} \left(A_l j_l(kr) + B_l n_l(kr) \right) P_l(\cos\theta)$$

The wave-f^n must vanish at $r = a$

$$\Rightarrow \sum_{l=0}^{\infty} \left(A_l j_l(ka) + B_l n_l(ka) \right) P_l(\cos\theta) = 0$$

$$\Rightarrow \quad A_l j_l(ka) + B_l n_l(ka) = 0 \quad \hookrightarrow \forall l$$

($\because P_l(\cos\theta)$ are complete f^ns in the physical range $-1 \le \cos\theta \le 1$)

$$\text{Now,} \quad \tan\delta_l = -\frac{B_l}{A_l} = \frac{j_l(ka)}{n_l(ka)}$$

$$\therefore \quad \sin^2\delta_l = \frac{\tan^2\delta_l}{1+\tan^2\delta_l} = \frac{j_l^2(ka)}{j_l^2(ka) + n_l^2(ka)}$$

$\therefore$ Total cross-section

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \frac{j_l^2(ka)}{j_l^2(ka) + n_l^2(ka)}$$

For $ka \ll 1$

$$\sigma \simeq \frac{4\pi}{k^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)} \left(\frac{2^l \, l!}{(2l)!} \right)^4 (ka)^{4l+2}$$

At low energy the dominant contribution is from $l=0$

$$\Rightarrow \sigma \simeq \frac{4\pi}{k^2}(ka)^2 = 4\pi a^2 \quad (\text{not } \pi a^2 \text{ !!!!!})$$

→ waves w/ arbitrarily large wavelength sense the whole sphere instead of just πa^2

- It can be shown that at low energies, higher partial wave cross-sections $\sigma_{l>0}$ are much suppressed

Also, in this case

$$e^{2i\delta_l} = \frac{j_l(ka) - i n_l(ka)}{j_l(ka) + i n_l(ka)}$$

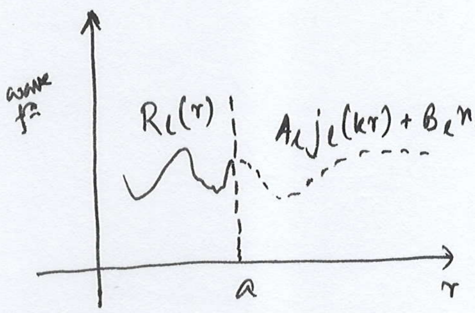
$\because e^{2i\delta} = \dfrac{1 + i\tan\delta}{1 - i\tan\delta}$.

- Phase shift for a general case

We want $l \neq 0$ phase shifts of an arbitrary potential w/ range a.

(i) Find a radial sol$^{\underline{n}}$ $R_l(r)$ for $r \leq a$.

- if the potential is not too singular at the origin $R_l(r) \sim r^l$ as $r \to 0$.



(ii) $R_l(r)$ must be matched w/ $A_l j_l(kr) + B_l n_l(kr)$ at $r=a$

↓

sol$^{\underline{n}}$ at $r > a$ where $V=0$

$\therefore$ At $r=a$ the f$^{\underline{n}}$s and their derivatives must match

$$\Rightarrow \quad R_l(a) = A_l j_l(ka) + B_l n_l(ka)$$

$$a R_l'(a) = ka\left(A_l j_l'(ka) + B_l n_l'(ka)\right)$$

(iii) Define log. derivative of radial sol$^{\underline{n}}$

$$\beta_l = \frac{a R_l'(a)}{R_l(a)} = ka\,\frac{j_l'(ka) - \tan\delta_l\, n_l'(ka)}{j_l(ka) - \tan\delta_l\, n_l(ka)} \quad \left(\text{using } \tan\delta_l = -\frac{B_l}{A_l}\right)$$

$$\Rightarrow \tan\delta_l = \frac{j_l(ka) - \frac{ka}{\beta_l} j_l'(ka)}{n_l(ka) - \frac{ka}{\beta_l} n_l'(ka)}.$$

It can also be shown that

$$e^{2i\delta_l} = -\frac{j_l - in_l}{j_l + in_l}\left(\frac{\beta_l - ka\left(\frac{j_l' - in_l'}{j_l - in_l}\right)}{\beta_l - ka\left(\frac{j_l' + in_l'}{j_l + in_l}\right)}\right)$$

- If $\beta_l \to \infty$ which happens if $R_l(a) = 0$, phase shifts of hard sphere are recovered.

- Phase shifts are particularly useful when a few of them dominate the σ, and this happens when $ka < 1$

  → at fixed k, this can happen for short-range potentials

  & → " " range, " " " low energies.

Integral eq^n for scattering

Not all potentials are spherically symmetric.

For them partial wave analysis can not be employed

We need to rewrite the SE in an integral form and use approximations (usually when the particles have high energy, or the pot. is weak)

Time-indep. SE

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r})$$

Set, $E = \frac{\hbar^2 k^2}{2m}$ (energy of a plane wave w/ wave number k)

$V(\vec{r}) = \frac{\hbar^2}{2m} U(\vec{r})$ (just a rescaling)

$\therefore$ TISE $\rightarrow$ $\left(\vec{\nabla}^2 + k^2\right)\psi(\vec{r}) = U(\vec{r})\,\psi(\vec{r})$ ------- ①

To solve this

Introduce $G(\vec{r}-\vec{r}\,')$, a Green fn. for the operator $(\vec{\nabla}^2 + k^2)$ i.e.,

$$\left(\vec{\nabla}^2 + k^2\right) G(\vec{r}-\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

* $\vec{\nabla}$ is acting on $\vec{r}$ and not on $\vec{r}\,'$

If $\psi_0(\vec{r})$ denote an arbitrary sol^n for

$$\left(\vec{\nabla}^2 + k^2\right)\psi_0(\vec{r}) = 0 \quad \text{-------- ②}$$

then any sol^n of ~~of~~ the integral eq^n

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \int d^3r'\, G(\vec{r}-\vec{r}\,')\,U(\vec{r}\,')\,\psi(\vec{r}\,')$$

is a sol^n of eq^n ①.

[This can explicitly be checked as,

$$\left(\vec{\nabla}^2 + k^2\right)\psi(\vec{r}) = \left(\vec{\nabla}^2 + k^2\right)\int d^3r'\, G(\vec{r}-\vec{r}\,')\,U(\vec{r}\,')\,\psi(\vec{r}\,')$$

(Eq^n ② already used)

$$= \int d^3r'\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,U(\vec{r}\,')\,\psi(\vec{r}\,')$$

$$= U(\vec{r})\,\psi(\vec{r})\,]$$

By noting the facts that $\left(\vec{\nabla}_{\vec{r}}^2 + k^2\right) \dfrac{e^{\pm ikr}}{r} = 0 \quad \forall\ r \neq 0 \qquad r = |\vec{r} - \vec{r}\,'|$

& $\quad \vec{\nabla}_{\vec{r}}^2 \left(-\dfrac{1}{4\pi r}\right) = \delta^{(3)}(\vec{r})$

One can guess that

$$G_\pm(\vec{r}) = -\frac{1}{4\pi} \frac{e^{\pm ikr}}{r}$$

(+) for retarded Green f^{n}; outgoing wave

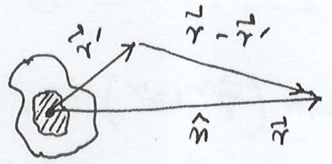
(−) for advanced Green f^{n}; ingoing wave.

For $\psi_0(\vec{r})$ we can use e^{ikz} and choose G_+

$(\vec{\nabla}^2 + k^2) e^{ikz} = 0$ anyway

$$\therefore\ \psi(\vec{r}) = e^{ikz} + \int d^3r'\, G_+(\vec{r} - \vec{r}\,')\, U(\vec{r}\,')\, \psi(\vec{r}\,')$$

where $G_+(\vec{r} - \vec{r}\,') = -\dfrac{1}{4\pi} \dfrac{e^{ik|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|}$



Far away $\quad r \gg r' \qquad r = |\vec{r}|,\quad r' = |\vec{r}\,'|$

$$|\vec{r} - \vec{r}\,'| \simeq r - \hat{n}\cdot\vec{r}\,' \qquad \hat{n} = \frac{\vec{r}}{r}$$

keep only this in the D^{r}.

keep both in N^{r}.

$$\Rightarrow\ G_+(\vec{r} - \vec{r}\,') = -\frac{1}{4\pi r} e^{ikr} e^{-ik\hat{n}\cdot\vec{r}\,'}$$

$$\therefore\ \psi(\vec{r}) = e^{ikz} + \underbrace{\left(-\frac{1}{4\pi}\int d^3r'\, e^{-ik\hat{n}\cdot\vec{r}\,'}\, U(\vec{r}\,')\, \psi(\vec{r}\,')\right)}_{f_k(\theta,\phi)} \frac{e^{ikr}}{r}$$

(still unknown as $\psi(\vec{r})$ is not determined yet)

Separating incident and scattered wave

$$\psi(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \left[-\frac{1}{4\pi}\int d^3r'\, e^{-ik_s\hat{n}\cdot\vec{r}\,'}\, U(\vec{r}\,')\, \psi(\vec{r}\,')\right] \frac{e^{ikr}}{r}$$

$k = |k_i| = |k_s|$

$\vec{k}_i = k\hat{n}_i$

$\vec{k}_s = k\hat{n}$.

Born approximation

Valid in the weak potential limit.

$\rightarrow$ somewhat accurate sol$^{\underline{n}}$s of the integral eq$^{\underline{n}}$.

$$\psi(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \int d^3r'\, G_+(\vec{r}-\vec{r}\,')\, U(\vec{r}\,')\,\psi(\vec{r}\,') \quad \text{------} \ (1)$$

Relabel $\vec{r} \to \vec{r}\,'$, $\vec{r}\,' \to \vec{r}\,''$

$$\psi(\vec{r}\,') = e^{i\vec{k}_i\cdot\vec{r}\,'} + \int d^3r''\, G_+(\vec{r}\,'-\vec{r}\,'')\, U(\vec{r}\,'')\,\psi(\vec{r}\,'') \quad \text{---} \ (2)$$

Substitute (2) in (1)

$$\psi(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \int d^3r'\, G_+(\vec{r}-\vec{r}\,')\, U(\vec{r}\,')\, e^{i\vec{k}_i\cdot\vec{r}\,'}$$
$$+ \int d^3r'\, G_+(\vec{r}-\vec{r}\,')\, U(\vec{r}\,') \int d^3r''\, G_+(\vec{r}\,'-\vec{r}\,'')\, U(\vec{r}\,'')\,\psi(\vec{r}\,'') \quad \text{--} \ (3)$$

Repeat the same procedure, after some ~~its~~ iterations we get (schematically)

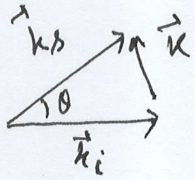
$$\psi(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \int GU e^{i\vec{k}_i\cdot\vec{r}} + \int GU \int GU e^{i\vec{k}_i\cdot\vec{r}} + \int GU \int GU \int GU e^{i\vec{k}_i\cdot\vec{r}} + \cdots$$

The approx$^{\underline{n}}$ in which we keep the first integral and set all others to zero $\longrightarrow$ first Born approx$^{\underline{n}}$.

$$\psi^{\text{Born}}(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} - \frac{1}{4\pi}\left(\int d^3r'\, e^{-i\vec{k}_s\cdot\vec{r}\,'}\, U(\vec{r}\,')\, e^{i\vec{k}_i\cdot\vec{r}\,'}\right)\frac{e^{ikr}}{r}$$

(using expression from p. 12)

$$f_k^{\text{Born}}(\theta,\phi) = -\frac{1}{4\pi}\int d^3r\, e^{-i\vec{K}\cdot\vec{r}}\, U(\vec{r})$$



Wave number transfer $\vec{K} = \vec{k}_s - \vec{k}_i$

$|\vec{k}_i| = |\vec{k}_s|$

θ : angle between $\vec{k}_i$ and $\vec{k}_s$ (it is spherical angle θ if $\vec{k}_i$ is along z-axis)

$$|\vec{K}| = 2k\sin\frac{\theta}{2}.$$

** N.B. In the Born approx$^{\underline{n}}$ $f_k(\theta,\phi)$ is simply the Fourier tr.$^{\underline{s}}$ of $U(\vec{r})$ evaluated at the mom. transfer $\vec{K}$.

* $f_k(\theta,\phi)$ captures the information of $V(\vec{r})$

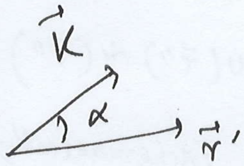
* For central potential, the expression can be simplified –
$(V(\vec{r}) = V(r))$

$$f_k^{Born}(\theta) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r\, e^{-i\vec{K}\cdot\vec{r}}\, V(r)$$

Due to spherical sym. it depends on the norm of $\vec{K}$ only

$\Rightarrow$ only θ- dependence, while $\vec{K}$ depends on both θ and ϕ

$|\vec{K}|$ only " θ.



$$f_k(\theta) = -\frac{\cancel{2}\,m}{2\pi\hbar^2}\int_0^\infty 2\pi r^2 dr \int_{-1}^{1} d(\cos\alpha)\, e^{-iKr\cos\alpha}\, V(r)$$

$$= -\frac{m}{\hbar^2}\int_0^\infty dr\, r^2 V(r)\, \frac{e^{-iKr} - e^{iKr}}{-iKr}$$

$$= -\frac{2m}{K\hbar^2}\int_0^\infty dr\, r\, V(r) \sin Kr .$$

$K = 2k\sin\theta/2$

* Born approx^n. treates the potential as a perturb^n. of the free particle waves
  – this waves thus ~~have~~ must have kinetic energies larger than the potential
  – hence the weak potential or high energy particle limit. //