Distribution function:

$$\underbrace{\Delta N}_{\#\text{ of particles}} = \frac{\overset{\text{internal dof}}{g}}{(2\pi)^3} \underbrace{f(t,\vec{x},\vec{p})}_{\substack{\text{distribution function}\\ \text{of each particle species}}} \underbrace{(\Delta\vec{x})^3(\Delta\vec{p})^3}_{\text{small vol}^m\text{ in phase space.}}$$

Why is $f(t,\vec{x},\vec{p})$ important or the right one?

$$n = \frac{g}{(2\pi)^3}\int f\, d^3p \quad \to \text{number density}$$

$$\rho = \frac{g}{(2\pi)^3}\int E f\, d^3p \quad \to \text{energy density}$$

$$P = \frac{g}{(2\pi)^3}\int \frac{p^2}{3E} f\, d^3p \quad \to \text{pressure}$$

and in general $\to$

$$\langle \varphi \rangle = \frac{1}{n}\frac{g}{(2\pi)^3}\int \varphi f\, d^3p = \underbrace{\frac{\int \varphi f d^3p}{\int f d^3p}}_{\text{Looks similar?}}$$

In thermal equilibrium $\to$ f has a simple form.

If not $\to$ $\dfrac{df_A}{dt} = \underbrace{C(f_A, f_B, \dots)}_{\text{Collision term.}}$

A, B, C, ... $\to$ Particle species

In thermal equilibrium,

$$f = \left(e^{\frac{E-\mu}{T}} \pm 1 \right)^{-1}$$

$E = \sqrt{p^2 + m^2} \rightarrow$ relativistic energy

$\mu \Rightarrow$ chemical potential

We may use the approximation $\rightarrow$

$$f \approx e^{-\frac{E-\mu}{T}} \quad \text{(Maxwell - Boltzmann)}$$

Boltzmann equation in general relativity $\rightarrow$

$$\frac{df}{d\lambda} = \underbrace{C(f)}_{\text{collision term}}$$

$\lambda \rightarrow$ affine parameter

Phase-space coordinates : (x^μ, p^μ) ; $p^\mu = \frac{dx^\mu}{d\lambda}$, $g_{\mu\nu} p^\mu p^\nu = -m^2$

$x^\mu \rightarrow$ position space-time coordinates

Also,

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0 \quad \rightarrow \text{Geodesic Eq}^n.$$

Usual 3-momentum $\rightarrow$ $p^2 = g_{ij} p^i p^j$

Direction : $\hat{p}^i \propto p^i \Rightarrow$ Normalization $\rightarrow \hat{p}^i \hat{p}^j \delta_{ij} = 1$

So, Boltzmann equation $\rightarrow$

$$\frac{df}{d\lambda} = p^0 \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{p^i}{p^0} + \frac{\partial f}{\partial E} \frac{1}{p^0} \frac{dE}{d\lambda} + \frac{\partial f}{\partial \hat{p}^i} \frac{1}{p^0} \frac{d\hat{p}^i}{d\lambda} \right]$$

Why this one in particular?

(i) If you have f, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x^i}$, $\frac{\partial f}{\partial E}$, $\frac{\partial f}{\partial \hat{p}^i}$ are easy to calculate.

(ii) $\underbrace{\frac{dE}{d\lambda}, \frac{d\hat{p}^i}{d\lambda}}$ comes from geodesic, tells us how the

→ How to calculate this?

(i) Geodesic with $\mu = 0 \Rightarrow \frac{dP^0}{d\lambda} + \Gamma^0_{\alpha\beta} p^\alpha p^\beta = 0$

Here we can use relation b/w P^0 and E

e.g., for a diagonal metric $P^0 = E(-g_{00})^{-1/2}$

(ii) Similarly, $\mu = i \Rightarrow \frac{dp^i}{d\lambda} + \Gamma^i_{\alpha\beta} P^\alpha p^\beta = 0$

For a diagonal metric, $P^i = p\hat{p}^i(-g_{ii})^{-1/2}$

(iii) Finally, $\underbrace{\frac{dA(t,x)}{d\lambda} = \frac{\partial A(t,x)}{\partial t} P^0 + \frac{\partial A(t,x)}{\partial x^i} P^i}$

→ Modified chain-rule for a total derivative.

Why will we use non-equilibrium case?

⊙ Most of the universe was in thermal equilibrium throughout, but not at all points of time.

↓

⊙ This leads to some very specific relics

Known

(1) Decoupling of photons
(2) Neutrino decoupling
(3) primordial nucleosynthesys

↓

(1) CMBR
(2) CNB
(3) Light element abundance ($D, {}^3He, {}^4He, {}^7Li$)

Speculated

(1) Inflation
(2) Baryogenesis
(3) Decoupling of Particle DM

↓

(1) Primordial Curvature perturbation, GW, ...
(2) Baryon asymmetry
(3) Relic DM

Need to know → How is our universe?
- → Homogenous
- → Isotropic

→ FLRW metric

↓

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$\boxed{\text{This is the metric we will use!}}$ → $g_{\mu\nu}$ ~~scale~~

↓

$$ds^2 = dt^2 - R^2(t)\left[\underbrace{\frac{dr^2}{1-kr^2}}_{\text{Curvature}} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]$$

↓

Scale-factor ($a(t)$)

How do particle behaves once it is decoupled?

$n \propto R^{-3} \longrightarrow$ # density

$p \propto R^{-1} \longrightarrow$ momenta

What is decoupling?

$\Gamma \rightarrow$ interaction rate of the particle (per particle)

$H = \frac{\dot{R}}{R} \rightarrow$ Hubble parameter

Coupled : $\Gamma \gtrsim H$

Decoupled : $\Gamma \lesssim H$

$\longrightarrow$ interaction is what keeps a particle in thermal equilibrium.

Evolution of the distribution function:

$$\hat{L}[f] = C[f]$$

$\longrightarrow$ Collision operator

$\longrightarrow$ Liouville operator

$$\hat{L}_{NR} = \frac{d}{dt} + \frac{d\vec{x}}{dt} \cdot \vec{\nabla}_x + \frac{d\vec{v}}{dt} \cdot \vec{\nabla}_v = \frac{d}{dt} + \vec{v} \cdot \vec{\nabla}_x + \frac{\vec{F}}{m} \cdot \vec{\nabla}_v$$

$\longrightarrow$ non-relativistic, particle of mass m subject to a force $\vec{F} = \frac{d\vec{p}}{dt}$.

$$\hat{L} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}$$

$\rightarrow$ covariant, relativistic generalization

Special case : our case → (FLRW metric)

$$f \to f(E,t)$$

$$\hat{L}[f(E,t)] = E\frac{\partial f}{\partial t} - \frac{\dot{R}}{R}|\vec{p}|^2\frac{\partial f}{\partial E}$$

Number density again →

$$n(t) = \frac{g}{(2\pi)^3}\int d^3p\, f(E,t)$$

$$\downarrow$$

$$\frac{dn}{dt} + 3\frac{\dot{R}}{R}n = \frac{g}{(2\pi)^3}\int C[f]\frac{d^3p}{E}$$

Finally it is time for the collision term →

For a process $\psi + a + b + \cdots \leftrightarrow i + j + \cdots$ the collision term,

$$\frac{g}{(2\pi)^3}\int C[f]\frac{d^3p_\psi}{E} = -\int d\Pi_\psi\, d\Pi_a\, d\Pi_b \cdots d\Pi_i\, d\Pi_j \cdots$$

$$\times (2\pi)^4\delta^4(p_\psi + p_a + p_b + \cdots - p_i - p_j - \cdots)$$

$$\times \Big[|M|^2_{\psi+a+b+\cdots \to i+j+\cdots} f_a f_b \cdots f_\psi (1\pm f_i)(1\pm f_j)\cdots$$

$$- |M|^2_{i+j+\cdots \to \psi+a+b+\cdots} f_i f_j \cdots (1\pm f_a)(1\pm f_b)\cdots(1\pm f_\psi)\Big]$$

$$d\Pi = g\frac{1}{(2\pi)^3}\frac{d^3p}{2E}$$

Two approximations →

(1) T (or CP) invariance →

$$|M|^2_{i+j+\cdots \to \psi+a+b+\cdots} = |M|^2_{\psi+a+b+\cdots \to i+j+\cdots}$$

(2) Maxwell - Boltzmann statistics →

In absence of BE Condensate or Fermi degeneracy we consider MB.

$$1 \pm f \simeq 1 \quad \text{and} \quad f_i(E_i) = \exp\left[-(E_i-\mu)/T\right]$$

Things become simpler →

$$\dot{n}_\psi + \underbrace{3Hn_\psi}_{\text{Dilution due to expansion of universe}} = \underbrace{-\int d\Pi_\psi\, d\Pi_a\, d\Pi_b \cdots d\Pi_i\, d\Pi_j \cdots (2\pi)^4 |M|^2 \times \delta^4(p_i+p_j+\cdots-p_\psi-p_a-p_b\cdots) \times \left[f_a f_b \cdots f_\psi - f_i f_j \cdots\right]}_{\text{Interactions responsible for change in } n_\psi}$$

If $|M|^2 = 0 \Rightarrow RHS = 0 \Rightarrow C = 0 \Rightarrow n_\psi \propto R^{-3}$.

How to take care of the expansion of universe?

# of particles in a comoving volume.

$\hookrightarrow$ cosmological volume with the expansion of the universe factored out.

We use entropy density $\rightarrow$ ($sR^3 = \text{const}$)

$\hookrightarrow$ conservation of entropy per comoving volume.

$$Y = \frac{n_\psi}{s}$$

$$\downarrow$$

$$\dot{n}_\psi + 3H n_\psi = s\dot{Y}$$

Time $\rightarrow$ temperature :

$$x = \frac{m}{T}$$ $\rightarrow$ usually the mass of the particle.

During radiation dominated epoch $\rightarrow$

$$t = 0.301\, g_*^{-1/2} \frac{m_{Pl}}{m^2} x^2$$

$\rightarrow$ $\underline{p = w\rho}$

$\hookrightarrow$ EOS.

The Boltzmann eq^n becomes $\rightarrow$

$$\frac{dY}{dx} = -\frac{x}{H(m)\, s} \int d\Pi_\psi\, d\Pi_a\, d\Pi_b \cdots d\Pi_i\, d\Pi_j \cdots (2\pi)^4 |M|^4$$
$$\times \delta^4(p_i + p_j + \cdots - p_a - p_b - p_\psi - \cdots)[f_a f_b f_\psi \cdots - f_i f_j \cdots]$$

$H(m) = 1.67\, g_*^{1/2}\, m^2/m_{Pl}$ $\qquad$ $H(x) = H(m)\, x^{-2}$

Application : Freeze Out

If a massive particle remained in thermal equilibrium untill the present →

abundance, $\frac{n}{s} \sim \underbrace{\left(\frac{m}{T}\right)^{3/2} \exp(-m/T)}_{\text{absolutely negligible}} \rightarrow \text{low } T$

If the interaction of the species freeze out ($\Gamma < H$) at a temp at which $\xi m/T$ is not much larger than 1, the relic abundance can be significant.

Stable species: long-lived compared to the age of the universe

Only processes that can change # →

$\psi\bar{\psi} \leftrightarrow X\bar{X}$ (annihilation and inverse annihilation)

→ all species into which ψ can annihilate

→ They are in equilibrium.

The factor $[f_\psi f_{\bar{\psi}} - f_X f_{\bar{X}}]$ →

$$f_{X,\bar{X}} = \exp(-E_{X,\bar{X}}/T)$$

Also, δ-function → $E_\psi + E_{\bar{\psi}} = E_X + E_{\bar{X}}$ so that

$$f_X f_{\bar{X}} = \exp\left[-(E_X + E_{\bar{X}})/T\right] = \exp\left[-(E_\psi + E_{\bar{\psi}})/T\right] = f_\psi^{EQ} f_{\bar{\psi}}^{EQ}$$

$E\varrho \rightarrow$ equilibrium

So, $\left[f_\psi f_{\bar\psi} - f_\chi f_{\bar\chi}\right] = \left[f_\psi f_{\bar\psi} - f_\psi^{E\varrho} f_{\bar\psi}^{E\varrho}\right]$

Boltzmann eq^n $\longrightarrow$

$$\frac{dn_\psi}{dt} + 3Hn_\psi = -\langle \sigma_{\psi\bar\psi \to \chi\bar\chi} |v| \rangle \left[n_\psi^2 - (n_\psi^{E\varrho})^2\right]$$

or,
$$\frac{dY}{dx} = \frac{-x \langle \sigma_{\psi\bar\psi \to \chi\bar\chi} |v| \rangle s}{H(m)} \left(Y^2 - Y_{E\varrho}^2\right)$$

$Y = \frac{n_\psi}{s} = \frac{n_{\bar\psi}}{s}$ $\Big] \rightarrow$ actual # of particle per comoving vol^n

$Y_{E\varrho} = \frac{n_\psi^{E\varrho}}{s} = \frac{n_{\bar\psi}^{E\varrho}}{s}$ $\Big] \rightarrow$ eq # of " " " "

$$\langle \sigma_{\psi\bar\psi \to \chi\bar\chi} |v| \rangle = (n_\psi^{E\varrho})^{-2} \int d\Pi_\psi \, d\Pi_{\bar\psi} \, d\Pi_\chi \, d\Pi_{\bar\chi} \, (2\pi)^4 \times \delta^4(P_\psi + P_{\bar\psi} - P_{\bar\chi} - P_\chi) |M|^2 \exp(-E_\psi/T) \exp(-E_{\bar\psi}/T)$$

Two regimes $\longrightarrow$

Non-relativistic $(x \gg 3)$:

$$Y_{E\varrho}(x) = 0.145 \frac{g}{g_{*s}} x^{3/2} e^{-x}$$

Ultra-relativistic $(x \ll 3)$:

$$Y_{E\varrho}(x) = 0.278 \frac{g_{eff}}{g_{*s}}$$

Finally, we can write →

$$\frac{x}{Y_{EQ}} \frac{dY}{dx} = -\frac{\Gamma_A}{H}\left[\left(\frac{Y}{Y_{EQ}}\right)^2 - 1\right]$$

$$\Gamma_A = n_{EQ} \langle \sigma_A |v| \rangle$$

$\frac{\Gamma_A}{H}$: Effectiveness of annihilations.

When $\frac{\Gamma_A}{H} < 1$, the annihilations freeze out and the # of ψ's in a comoving vol^n freeze in.

Hot Relic : ($x_f \lesssim 3$)

↳ value of x where $\Gamma_A \simeq H$

Freeze out occurs when species is still relativistic.

↳ Y_{EQ} not changing with time.

→ Asymptotic value.

$$Y(x \to \infty) = Y_\infty = Y_{EQ}(x_f) = 0.278\, g_{eff}/g_{*s}(x_f)$$

Abundance of ψ today →

$$n_{\psi_0} \simeq s_0 Y_\infty \simeq 2970\, Y_\infty \text{ cm}^{-3}$$

$$\simeq 825\left[g_{eff}/g_{*s}(x_f)\right] \text{cm}^{-3}$$

Present relic density →

$$\rho_{\psi 0} = s_0 Y_\infty m = 2.97 \times 10^3 Y_\infty (m/eV) \text{ eV cm}^{-3}$$

$$\Omega_\psi h^2 = 7.83 \times 10^{-2} \left[g_{eff}/g_{*s}(x_f) \right] \left(\frac{m}{eV} \right)$$

Now we know $\Omega_0 h^2 \lesssim 1 \rightarrow$

$$m \lesssim 12.8 \text{ eV} \left[g_{*s}(x_f)/g_{eff} \right]$$

Detour →

Universe → 4% (SM)
Universe → 26% (DM) → Hot relic? / Fuzzy relic? / Cold relic? / Non-thermal relic? / Extended structure?
Universe → 70% (DE)

If hot relic, then why not νs?

νs decouple when $T \sim$ MeV, $g_{*s} = g_* = 10.75$

$g_{eff} = 1.5$

$$\Omega_{\nu\bar{\nu}} h^2 = \frac{m_\nu}{91.5 \text{ eV}} < 0.12$$

$$\Rightarrow \quad m_\nu < 10.98 \text{ eV}$$

~~The~~ Also, TG bound $\Rightarrow$ Paulie blocking !!