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Distribution function:

$$\Delta N = \frac{g}{(2\pi)^3} f(t, \vec{x}, \vec{p}) (\Delta \vec{x})^3 (\Delta \vec{p})^3$$

→ internal dof
 → # of particles
 → distribution function of each particle species
 → small volⁿ in phase space.

Why is $f(t, \vec{x}, \vec{p})$ important or the right one?

$$n = \frac{g}{(2\pi)^3} \int f d^3p \quad \left. \vphantom{\int} \right\} \text{number density}$$

$$e = \frac{g}{(2\pi)^3} \int E f d^3p \quad \left. \vphantom{\int} \right\} \text{energy density}$$

$$P = \frac{g}{(2\pi)^3} \int \frac{p^2}{3E} f d^3p \quad \left. \vphantom{\int} \right\} \text{pressure}$$

and in general →

$$\langle \mathcal{Q} \rangle = \frac{1}{n} \frac{g}{(2\pi)^3} \int \mathcal{Q} f d^3p = \frac{\int \mathcal{Q} f d^3p}{\int f d^3p}$$

↳ Looks similar?

In thermal equilibrium → f has a simple form.

If not → $\frac{df_A}{dt} = C(f_A, f_B, \dots)$

↳ Collision term.

A, B, C, ... → Particle species

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In thermal equilibrium,

$$f = \left(e^{\frac{E-\mu}{T}} \pm 1 \right)^{-1}$$

$$E = \sqrt{p^2 + m^2} \rightarrow \text{relativistic energy}$$

 $\mu \Rightarrow$ chemical potential
We may use the approximation \rightarrow

$$f \approx e^{-\frac{E-\mu}{T}} \quad (\text{Maxwell-Boltzmann})$$

Boltzmann equation in general relativity \rightarrow

$$\frac{df}{d\lambda} = C(t)$$

$\xrightarrow{\text{collision term}}$
 $\xrightarrow{\text{affine parameter}}$

Phase-space coordinates: (x^μ, p^μ)

$\xrightarrow{p^\mu = \frac{dx^\mu}{d\lambda}, g_{\mu\nu} p^\mu p^\nu = -m^2}$
 $\xrightarrow{\text{position space-time coordinates}}$

Also, $\frac{dp^\mu}{d\lambda} + \Gamma_{\alpha\beta}^{\mu} p^\alpha p^\beta = 0$ $\xrightarrow{\text{Geodesic Eq.}}$

Usual 3-momentum $\rightarrow p^2 = g_{ij} p^i p^j$ Direction: $\hat{p}^i \propto p^i \Rightarrow$ Normalization $\rightarrow \hat{p}^i \hat{p}^j \delta_{ij} = 1$ So, Boltzmann equation \rightarrow

$$\frac{df}{d\lambda} = p^0 \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{p^i}{p^0} + \frac{\partial f}{\partial E} \frac{1}{p^0} \frac{dE}{d\lambda} + \frac{\partial f}{\partial p^i} \frac{1}{p^0} \frac{dp^i}{d\lambda} \right]$$

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Why this one in particular?

(i) If you have f , $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x^i}$, $\frac{\partial f}{\partial E}$, $\frac{\partial f}{\partial \hat{p}^i}$ are easy to calculate.

(ii) $\frac{dE}{d\lambda}$, $\frac{d\hat{p}^i}{d\lambda}$ comes from geodesic, tells us how the

↳ how to calculate this?

(i) geodesic with $\mu=0 \Rightarrow \frac{dP^0}{d\lambda} + \Gamma_{\alpha\beta}^0 P^\alpha P^\beta = 0$

Here we can use relation b/w P^0 and E

e.g., for a diagonal metric $P^0 = E(-g_{00})^{-1/2}$

(ii) Similarly, $\mu=i \Rightarrow \frac{dP^i}{d\lambda} + \Gamma_{\alpha\beta}^i P^\alpha P^\beta = 0$

For a diagonal metric, $P^i = \hat{p}^i (-g_{ii})^{-1/2}$

(iii) Finally, $\frac{dA(t,x)}{d\lambda} = \frac{\partial A(t,x)}{\partial t} P^0 + \frac{\partial A(t,x)}{\partial x^i} P^i$

↳ Modified chain-rule for a total derivative.

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Why will we use non-equilibrium case?

⊙ Most of the universe was in thermal equilibrium throughout, but not at all points of time.



⊙ This leads to some very specific relics

Known

- (1) Decoupling of photons
- (2) Neutrino decoupling
- (3) primordial nucleosynthesis



- (1) CMBR
- (2) CMB
- (3) Light element abundance (D, ³He, ⁴He, ⁷Li)

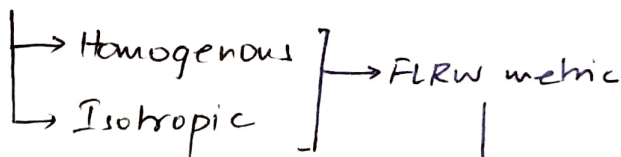
Speculated

- (1) Inflation
- (2) Baryogenesis
- (3) Decoupling of Particle DM



- (1) Primordial Curvature perturbation, Grw, ...
- (2) Baryon asymmetry
- (3) Relic DM

Need to know → how is our universe?



This is the metric we will use!

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
 $ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]$
 ↓
 Scale-factor (a(t))

↙ scale
↘ Curvature

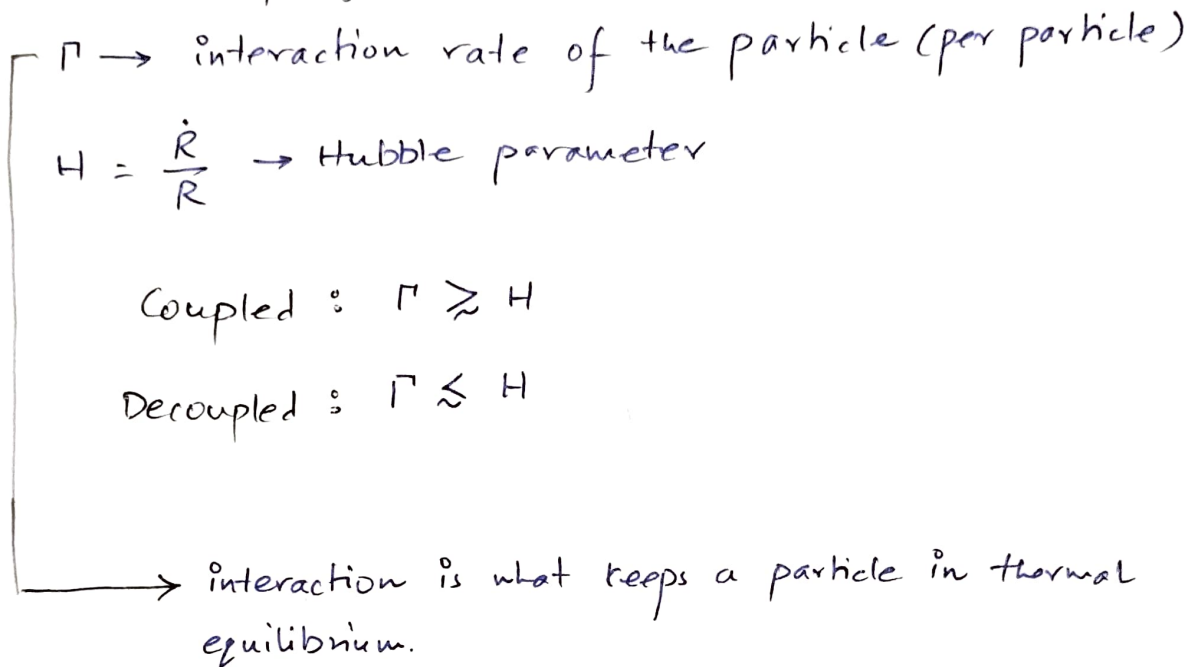
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How do particle behaves once it is decoupled?

$$n \propto R^{-3} \longrightarrow \# \text{ density}$$

$$p \propto R^{-1} \longrightarrow \text{momenta}$$

What is decoupling?



Evolution of the distribution function :

$$\hat{L}[f] = C[f]$$

\hookrightarrow Collision operator

\hookrightarrow Liouville operator

$$\hat{L}_{NR} = \frac{d}{dt} + \frac{d\vec{x}}{dt} \cdot \vec{\nabla}_x + \frac{d\vec{v}}{dt} \cdot \vec{\nabla}_v = \frac{d}{dt} + \vec{v} \cdot \vec{\nabla}_x + \frac{\vec{F}}{m} \cdot \vec{\nabla}_v$$

\hookrightarrow non-relativistic, particle of mass m subject to a force $\vec{F} = \frac{d\vec{p}}{dt}$.

$$\hat{L} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \quad \left. \vphantom{\hat{L}} \right\} \text{covariant, relativistic generalization}$$

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Special case: our case \rightarrow (FLRW metric)

$$f \rightarrow f(E, t)$$

$$\hat{L}[f(E, t)] = E \frac{\partial f}{\partial t} - \frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial f}{\partial E}$$

Number density again \rightarrow

$$n(t) = \frac{g}{(2\pi)^3} \int d^3p f(E, t)$$

$$\downarrow$$

$$\frac{dn}{dt} + 3 \frac{\dot{R}}{R} n = \frac{g}{(2\pi)^3} \int C[f] \frac{d^3p}{E}$$

Finally it is time for the collision term \rightarrow For a process $\psi + a + b + \dots \leftrightarrow i + j + \dots$ the collision term,

$$\begin{aligned} \frac{g}{(2\pi)^3} \int C[f] \frac{d^3p_\psi}{E} &= - \int d\pi_\psi d\pi_a d\pi_b \dots d\pi_i d\pi_j \dots \\ &\times (2\pi)^4 \delta^4(p_\psi + p_a + p_b + \dots - p_i - p_j - \dots) \\ &\times \left[|M|_{\psi+a+b+\dots \rightarrow i+j+\dots}^2 f_a f_b \dots f_\psi (1 \pm f_i)(1 \pm f_j) \dots \right. \\ &\left. - |M|_{i+j+\dots \rightarrow \psi+a+b+\dots}^2 f_i f_j \dots (1 \pm f_a)(1 \pm f_b) \dots (1 \pm f_\psi) \right] \end{aligned}$$

$$d\pi = g \frac{1}{(2\pi)^3} \frac{d^3p}{2E}$$

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Two approximations →

(1) T (or CP) invariance →

$$|M|_{i+j+\dots \rightarrow \psi+a+b+\dots}^2 = |M|_{\psi+a+b+\dots \rightarrow i+j+\dots}^2$$

(2) Maxwell-Boltzmann statistics →

In absence of BE condensate or Fermi degeneracy we consider MB.

$$1 \pm f \approx 1 \quad \text{and} \quad f_i(E_i) = \exp\left[-(E_i - \mu)/T\right]$$

Things become simpler →

$$\dot{n}_\psi + 3H n_\psi = - \int d\Pi_\psi d\Pi_a d\Pi_b \dots d\Pi_i d\Pi_j \dots (2\pi)^4 |M|^2$$

$$\times \delta^4(p_i + p_j + \dots - p_\psi - p_a - p_b \dots)$$

$$\times [f_a f_b \dots f_\psi - f_i f_j \dots]$$

Dilution due to expansion of universe

Interactions responsible for change in n_ψ .

$$\text{If } |M|^2 = 0 \Rightarrow \text{RHS} = 0 \Rightarrow C = 0 \Rightarrow n_\psi \propto R^{-3}$$

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How to take care of the expansion of universe?

of particles in a comoving volume.

↳ cosmological volume with the expansion of the universe factored out.

We use entropy density \rightarrow ($sR^3 = \text{const}$)

↳ conservation of entropy per comoving volume.

$$Y = \frac{n_\psi}{s}$$

↓

$$\dot{n}_\psi + 3Hn_\psi = s\dot{Y}$$

Time \rightarrow temperature:

$$x = \frac{m}{T} \rightarrow \text{usually the mass of the particle.}$$

During radiation dominated epoch \rightarrow

$$t = 0.301 g_*^{-1/2} \frac{m_{pl}}{m^2} x^2 \rightarrow \frac{p = w\rho}{\text{EOS.}}$$

The Boltzmann eqⁿ becomes \rightarrow

$$\frac{dY}{dx} = -\frac{x}{H(m)s} \int d\pi_\psi d\pi_a d\pi_b \dots d\pi_i d\pi_j \dots (2\pi)^4 |M|^4 \times \delta^4(p_i + p_j + \dots - p_a - p_b - p_\psi - \dots) [f_a f_b f_\psi \dots - f_i f_j \dots]$$

$$H(m) = 1.67 g_*^{1/2} m^2 / m_{pl}$$

$$H(x) = H(m) x^{-2}$$

(a)

Application: Freeze Out

If a massive particle remained in thermal equilibrium until the present \rightarrow

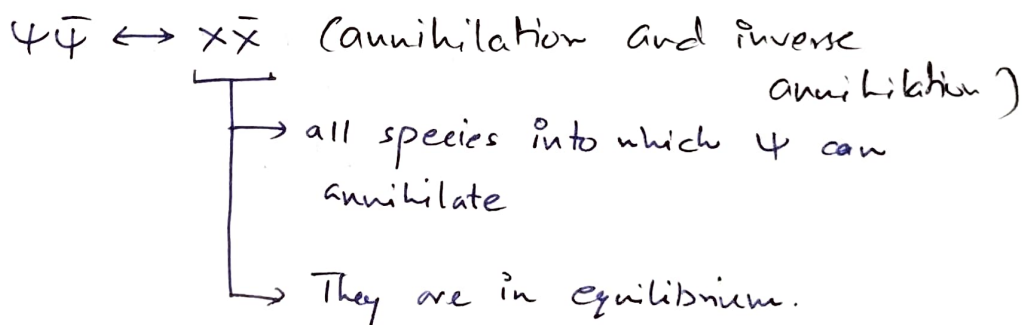
abundance, $\frac{n}{s} \sim \left(\frac{m}{T}\right)^{3/2} \exp(-m/T) \xrightarrow{\text{low } T}$

\downarrow absolutely negligible

If the interaction of the species freeze out ($\Gamma < H$) at a temp at which (m/T) is not much larger than 1, the relic abundance can be significant.

Stable species: long-lived compared to the age of the universe

Only processes that can change # \rightarrow



The factor $[f_\psi f_{\bar{\psi}} - f_X f_{\bar{X}}] \rightarrow$

$$f_{X, \bar{X}} = \exp(-E_{X, \bar{X}}/T)$$

Also, δ -function $\rightarrow E_\psi + E_{\bar{\psi}} = E_X + E_{\bar{X}}$ so that

$$f_X f_{\bar{X}} = \exp[-(E_X + E_{\bar{X}})/T] = \exp[-(E_\psi + E_{\bar{\psi}})/T] = f_\psi^{EQ} f_{\bar{\psi}}^{EQ}$$

EQ \rightarrow equilibrium

$$\text{So, } [f_{\psi} f_{\bar{\psi}} - f_{\psi}^{EQ} f_{\bar{\psi}}^{EQ}] = [f_{\psi} f_{\bar{\psi}} - f_{\psi}^{EQ} f_{\bar{\psi}}^{EQ}]$$

Boltzmann eqⁿ \rightarrow

$$\frac{dn_{\psi}}{dt} + 3Hn_{\psi} = - \langle \sigma_{\psi\bar{\psi} \rightarrow X\bar{X}} |v| \rangle [n_{\psi}^2 - (n_{\psi}^{EQ})^2]$$

or,

$$\frac{dY}{dx} = \frac{-\alpha \langle \sigma_{\psi\bar{\psi} \rightarrow X\bar{X}} |v| \rangle s}{H(m)} (Y^2 - Y_{EQ}^2)$$

$$Y = \frac{n_{\psi}}{s} = \frac{n_{\bar{\psi}}}{s} \rightarrow \text{actual \# of particle per comoving vol}^4$$

$$Y_{EQ} = \frac{n_{\psi}^{EQ}}{s} = \frac{n_{\bar{\psi}}^{EQ}}{s} \rightarrow \text{eq \# of " " " "}$$

$$\langle \sigma_{\psi\bar{\psi} \rightarrow X\bar{X}} |v| \rangle = (n_{\psi}^{EQ})^{-2} \int d\Pi_{\psi} d\Pi_{\bar{\psi}} d\Pi_X d\Pi_{\bar{X}} (2\pi)^4 \\ \times \delta^4(P_{\psi} + P_{\bar{\psi}} - P_X - P_{\bar{X}}) |M|^2 \exp(-E_{\psi}/T) \\ \exp(-E_{\bar{\psi}}/T)$$

Two regimes \rightarrow

Non-relativistic ($\alpha \gg 3$):

$$Y_{EQ}(\alpha) = 0.145 \frac{g}{g_{*s}} \alpha^{3/2} e^{-\alpha}$$

Ultra-relativistic ($\alpha \ll 3$):

$$Y_{EQ}(\alpha) = 0.278 \frac{g_{\text{eff}}}{g_{*s}}$$

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Finally, we can write \rightarrow

$$\frac{x}{Y_{\text{Eq}}} \frac{dY}{dx} = - \frac{\Gamma_A}{H} \left[\left(\frac{Y}{Y_{\text{Eq}}} \right)^2 - 1 \right]$$

$$\Gamma_A = n_{\text{Eq}} \langle \sigma_A |v| \rangle$$

$\frac{\Gamma_A}{H}$: Effectiveness of annihilations.

When $\frac{\Gamma_A}{H} < 1$, the annihilations freeze out and the # of Ψ 's in a comoving vol^m freeze in.

Hot Relic : ($x_f \lesssim 3$)

\hookrightarrow value of x where $\Gamma_A \approx H$

Freeze out occurs when species is still relativistic.

\hookrightarrow Asymptotic value.

$\hookrightarrow Y_{\text{Eq}}$ not changing with time.

$$Y(x \rightarrow \infty) = Y_\infty = Y_{\text{Eq}}(x_f) = 0.278 g_{\text{eff}}/g_{*s}(x_f)$$

Abundance of Ψ today \rightarrow

$$n_{\Psi_0} = S_0 Y_\infty = 2970 Y_\infty \text{ cm}^{-3}$$

$$= 825 \left[g_{\text{eff}}/g_{*s}(x_f) \right] \text{ cm}^{-3}$$

Present relic density \rightarrow

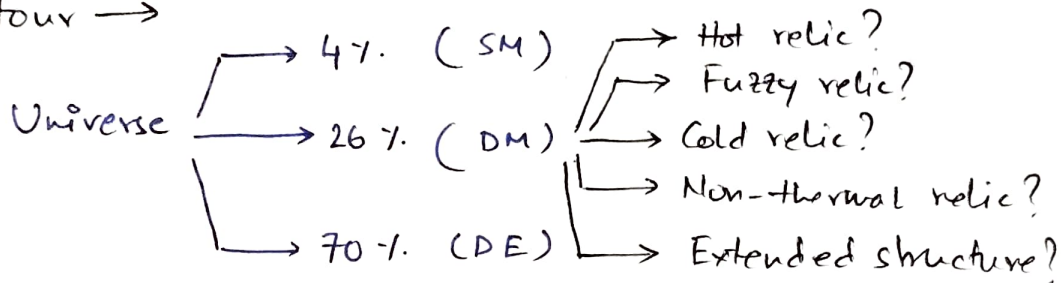
$$\rho_{\chi_0} = S_0 Y_{\infty} m = 2.97 \times 10^3 Y_{\infty} (\text{cm}^3/\text{eV}) \text{ eV cm}^{-3}$$

$$\Omega_{\chi} h^2 = 7.83 \times 10^{-2} \left[g_{\text{eff}}/g_{*s}(x_f) \right] \left(\frac{m}{\text{eV}} \right)$$

Now we know $\Omega_{\chi} h^2 \lesssim 1 \rightarrow$

$$m \lesssim 12.8 \text{ eV} \left[g_{*s}(x_f)/g_{\text{eff}} \right]$$

Detour \rightarrow



If hot relic, then why not ν_s ?

ν_s decouple when $T \sim \text{MeV}$, $g_{*s} = g_* = 10.75$

$$g_{\text{eff}} = 1.5$$

$$\Omega_{\nu} h^2 = \frac{m_{\nu}}{91.5 \text{ eV}} < 0.12$$

$$\Rightarrow m_{\nu} < 10.98 \text{ eV}$$

~~Now~~ Also, TG bound \Rightarrow Paulie blocking !!