Lecture 1

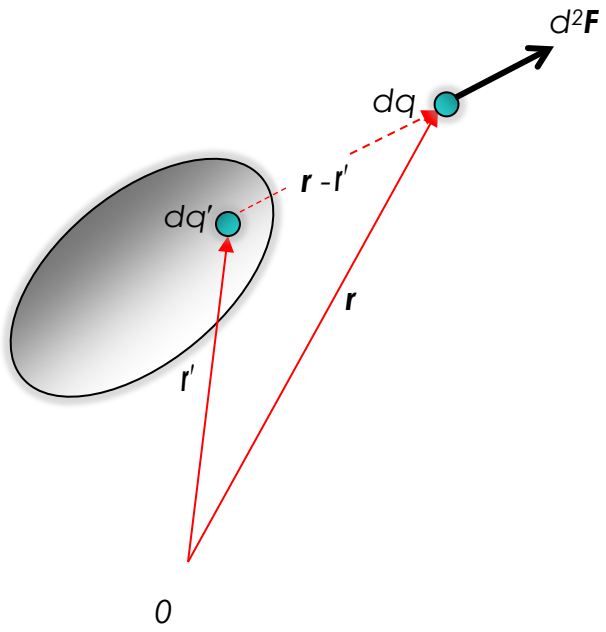
Some Selected Topics of Electromagnetism

T1 New formulation for the electric field

The most general distribution of charge is the volume distribution; i.e., any distribution of charge, whatever its size, can be considered to be distributed over some volume, even if this volume is microscopically small. So, if we have a charge distributed over a volume $\mathscr{V}$ such that the volume charge density is represented by the smooth function $\rho(\mathbf{r})$, then the charge of the element of volume dV' located at the end of the position vector $\mathbf{r}'$ is $\rho(\mathbf{r}')\,dV'$; this charge acts by a force

$$d^2\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dV'\,dq\,,$$

on the charge dq illustrated in the following figure. The force $d^2\mathbf{F}$ is written so to emphasize that it depends on the product of two infinitesimals, dV', and dq.



Thus the force acting on the charge dq by the charged body $\mathscr{V}$ is

$$d\mathbf{F}(\mathbf{r}) = \frac{dq}{4\pi\varepsilon_0} \int_{\mathscr{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dV' . \qquad (1)$$

Electrostatic field and electrostatic potential

From Eq. (1) it is clear that

$$\frac{d\mathbf{F}(\mathbf{r})}{dq} = \frac{1}{4\pi\varepsilon_0} \int_{\mathscr{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dV' .$$

Notice that the right side does not depend on the value of the charge dq, and does not change value even if dq is removed from its place, but it is rather a vector characterizing the charged volume $\mathscr{V}$. For this reason we call the vector $d\mathbf{F}(\mathbf{r})/dq$ the electrostatic field vector $\mathbf{E}$, or simply the electric field of the charged volume at $\mathbf{r}$, with

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathscr{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dV' .$$

T2 New derivation for the Electrostatic energy

When the mechanical behavior of an electrical system is to be studied, it may prove advantageous to use energy methods. In general, the energy of a system of charges, just like that of any other mechanical system, may be divided into its potential and kinetic contributions. Under static conditions, however, the entire energy of the charge system exists as potential energy, and we are particularly concerned with that potential energy arising from electrical interaction of the charges, the so-called electrostatic energy.

Electrostatic energy of a charge distribution

Charges located at an infinite distance from one another experience no force of interaction. To assemble charges into a charge density, or densities, we must do work against the Coulomb force between the charges. Any density of charge in a region of space then results in energy stored in that region. We now seek a mathematical expression for this stored energy.

The work done in bringing a point charge d^2q from a faraway position to a point $\boldsymbol{r}$ in the potential field of a finite distribution of charge is

$$d^2W = -\int_{\infty}^{\boldsymbol{r}} (d^2q)\,\boldsymbol{E}\,.\,d\boldsymbol{l} = d^2q \int_{\boldsymbol{r}}^{\infty} \boldsymbol{E}\,.\,d\boldsymbol{l} = U(\boldsymbol{r})\,d^2q\,,$$

where we considered $U(\infty)$ to be zero.

If we bring up a continuous distribution of charge of density $d\rho(\boldsymbol{r})$ to the point $\boldsymbol{r}$, where the potential is $U(\boldsymbol{r})$, we have

$$dW = \int_{all\ space} U(\boldsymbol{r})\,d\rho(\boldsymbol{r})\,dV\,. \qquad (2)$$

By Gauss's law, we have

$$d\rho = \nabla.\,d\boldsymbol{D}\,,$$

with $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ the displacement vector. Eq. (2), then, takes the form

$$dW = \int_{all\ space} U(\boldsymbol{r})\,(\nabla . d\boldsymbol{D})\,dV\,. \tag{3}$$

Using the identity

$$\begin{aligned} U\,(\nabla . d\boldsymbol{D}) &= \nabla . (U\,d\boldsymbol{D}) - \nabla U\,.\,d\boldsymbol{D} \\ &= \nabla . (U\,d\boldsymbol{D}) + \boldsymbol{E}\,.\,d\boldsymbol{D}\,, \end{aligned}$$

eq. (3) can be rewritten as

$$dW = \int_{all\ space} \left[\nabla . (U\,d\boldsymbol{D}) + \boldsymbol{E}\,.\,d\boldsymbol{D}\right] dV.$$

The first integral can be transformed to a surface integral by the divergence theorem:

$$dW = \oint_{\substack{very\ large \\ surface}} (U\,d\boldsymbol{D})\,.\,d\boldsymbol{a} + \int_{all\ space} \boldsymbol{E}\,.\,d\boldsymbol{D}\ dV.$$

The surface integral vanishes according to the assumption $U(\infty)$ leaving

$$dW = \int_{all\ space} \boldsymbol{E}\,.\,d\boldsymbol{D}\ dV. \tag{4}$$

Assuming the linearity of the used media, the vector identity

$$\boldsymbol{E}\,.\,d\boldsymbol{D} = \frac{1}{2} d\,(\boldsymbol{E}\,.\,\boldsymbol{D})\,,$$

will simplify the integral (4) to the form

$$W = \frac{1}{2} \int_{all\ space} \boldsymbol{E}\,.\,\boldsymbol{D}\ dV.$$

T3 New formulation for the Electric current

Up to this point, we have been dealing with charges at rest; now we wish to consider charges in motion. This occurs in metals, alloys, semiconductors, electrolytes, ionized gases, imperfect dielectrics, and even vacuums in the vicinity of a thermionic emitting cathode. In many conductors, the charge carriers are electrons, and in other cases, the charge may be carried by positive or negative ions.

Moving charge constitutes a current, and the process whereby charge is transported is called conduction. To be precise, the current dI through an infinitesimal area da taken in the region through which the charges move is defined as the rate of transport of the charge across da. Thus,

$$dI = \frac{d^2q}{dt}, \qquad C/s = \text{Ampere} \quad (A)$$

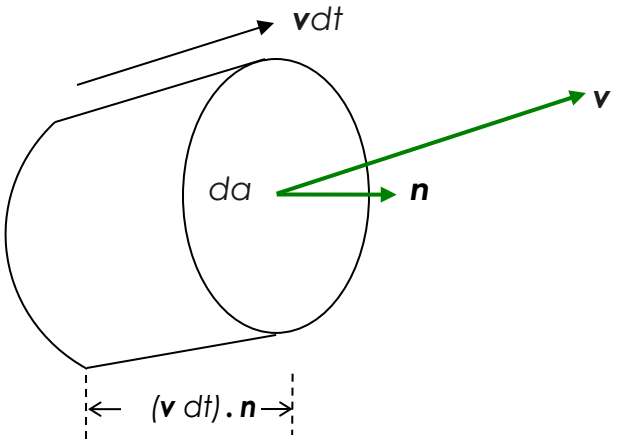
where d^2q is the charge crossing da in time dt.

By convention, the direction in which a positive carrier moves is taken as the direction, or sense, of the current. In general, an electric current arises in response to an electric field. If an electric field is imposed on a region, it will cause positive charge carriers to move in the general direction of the field and negative carriers in a direction opposite to the field. Hence, all currents produced in the process have the same direction as the field.

Current density

Consider a conducting medium that has only one type of charge carrier, of charge q. The number of these carriers per unit volume will be denoted by N. In accordance with the preceding section, we ignore their random thermal motion and assign the same drift velocity $\boldsymbol{v}$ to each carrier. We are now in a position to calculate the current through an element of area da such as is shown in the figure. During the time dt, each carrier moves a distance $\boldsymbol{v}\,dt$. From the figure it is evident that the charge d^2q that crosses da during time dt is q times

the number of all charge carriers in the volume $\boldsymbol{v} \cdot \boldsymbol{n}\, dt\, da$, where $\boldsymbol{n}$ is a unit vector normal to the area da. Then the current through da is



$$dI = \frac{\rho\, d^2V}{dt} = \frac{\rho\,(da)(\boldsymbol{v}\, dt)\,.\,\boldsymbol{n}}{dt} = \rho \boldsymbol{v}\,.\, d\boldsymbol{a}, \qquad (5)$$

as the volume charge density ρ of the charge carriers is sometimes written as Nq, with N is the carrier concentration and q is the carrier charge. If there is more than one kind of charge carrier present, there will be a contribution of the form (5) from each type of carrier. In general, the current through the area $d\boldsymbol{a} = \boldsymbol{n}\, da$ is

$$dI = (\sum_i \rho_i \boldsymbol{v}_i)\,.\, d\boldsymbol{a}\ , \qquad (6)$$

where the summation is over the different carrier types. The quantity in brackets is a vector that has dimensions of current per unit area. This quantity is called the current density and is given the symbol $\boldsymbol{J}$:

$$\boldsymbol{J} = \sum_i \rho_i \boldsymbol{v}_i\ . \qquad A/m^2$$

The current density may be defined at each point in a conducting medium and is, therefore, a vector point function. It is a useful quantity, one that enters directly into the fundamental differential equations of electromagnetic theory. Equation (6) may be written as

$$dI = \boldsymbol{J}\,.\, d\boldsymbol{a}\ ,$$

and the current through the surface $\mathscr{S}$, an arbitrarily shaped surface area of macroscopic size, is given by the integral

$$I = \int_{\mathscr{S}} \mathbf{J} . d\mathbf{a} .$$

T4 Boundary conditions

Transforming Poisson's equation into an integral equation

To handle the boundary conditions it is necessary to use the well-known Green's theorem

$$\int_{\mathscr{V}} (g\,\nabla^2 h - h\,\nabla^2 g)\,dV = \oint_{\mathscr{S}} (g\boldsymbol{\nabla} h - h\boldsymbol{\nabla} g)\cdot d\boldsymbol{a}\,, \tag{7}$$

where g and h are any two scalar fields defined in the volume $\mathscr{V}$ bounded by the closed surface $\mathscr{S}$. The Poisson differential equation for the potential can be converted into an integral equation if we choose a particular $g = U(\mathbf{r}')$, $h = 1/|\mathbf{r}-\mathbf{r}'|$, where $\mathbf{r}$ is the observation point and $\mathbf{r}'$ is the integration variable. Using the known relation

$$\nabla'^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}'),$$

and Poisson's equation $\nabla^2 U = \dfrac{\rho}{\varepsilon_0}$, we directly get

$$\int_{\mathscr{V}} [-4\pi U(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') + \frac{1}{\varepsilon_0}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}]dV' = \oint_{\mathscr{S}} [U(\mathbf{r}')\frac{\partial}{\partial n'}(\frac{1}{|\mathbf{r}-\mathbf{r}'|}) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial U(\mathbf{r}')}{\partial n'}]da',$$

where n is the normal line to the element da' of the surface $\mathscr{S}$. If $\mathbf{r}$ lies inside the volume $\mathscr{V}$ then

$$U(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathscr{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' + \frac{1}{4\pi}\oint_{\mathscr{S}} [\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial U(\mathbf{r}')}{\partial n'} - U(\mathbf{r}')\frac{\partial}{\partial n'}(\frac{1}{|\mathbf{r}-\mathbf{r}'|})]da'. \tag{8}$$

Also, if $\mathbf{r}$ lies outside the volume $\mathscr{V}$ then the left hand side of Eq. (14) vanishes. Moreover, if the surface $\mathscr{S}$ goes to infinity and the normal component of the electric field $-\partial U(\mathbf{r}')/\partial n'$ on $\mathscr{S}$ falls off faster than $1/|\mathbf{r}-\mathbf{r}'|$, then the surface integral vanishes and (8) reduces to the familiar result (5).

We are seeking boundary conditions appropriate for the Poisson (or Laplace) equation to ensure that a unique and well-behaved (i.e., physically reasonable) solution will exist inside any bounded region $\mathscr{V}$. Physical experience leads us to believe that specification of the potential on a closed surface $\mathscr{S}$ defines a unique potential problem. This is called a Dirichlet problem, or Dirichlet boundary condition. Similarly it is plausible that specification of the electric field everywhere on the surface (corresponding to a given surface-charge density) also defines a unique problem. Specification of the normal derivative is known as the Neumann boundary condition.

Formal solution of electrostatic boundary conditions using Green's functions

As a remark, for a charge-free volume, where Laplace's equation applies, the potential inside a specific volume is expressed in terms of the potential and its normal derivative only on the surface of the volume. This rather surprising result is not a solution to a boundary-value problem, but only an integral statement, since the arbitrary specification of both the potential and its derivative on the surface is overspecification of the problem.

If we had made a better choice of the function h in Eq. (13), we could have come up with a better result. Let's try again by choosing for h a function we will call Green's function:

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}'),$$

with F satisfying the Laplace equation inside the volume $\mathscr{V}$:

$$\nabla^2 F(\mathbf{r}, \mathbf{r}') = 0.$$

If we choose the scalar fields $g = U(\mathbf{r}'), h = G(\mathbf{r}, \mathbf{r}')$, and require $\nabla'^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}')$, then Green's theorem (7) will take the form

$$U(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathscr{V}} G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')dV' + \frac{1}{4\pi}\oint_{\mathscr{S}} [G(\mathbf{r}, \mathbf{r}')\frac{\partial U(\mathbf{r}')}{\partial n'} - U(\mathbf{r}')\frac{\partial}{\partial n'}G(\mathbf{r}, \mathbf{r}')]da'; \qquad (9)$$

the fields $G(\mathbf{r}, \mathbf{r}')$ are called Green's functions.

The freedom available in the definition of G means that we can make the surface integral in Eq. (9) depend only on the chosen type of boundary conditions. Thus, for Dirichlet boundary conditions we demand

$$G_D(\mathbf{r},\mathbf{r}') = 0, \qquad \mathbf{r}' \in \mathscr{S},$$

then the first term in the surface integral in (15) vanishes and the solution is

$$U(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathscr{V}} G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')dV' - \frac{1}{4\pi}\oint_{\mathscr{S}} U(\mathbf{r}')\frac{\partial}{\partial n'}G_D(\mathbf{r},\mathbf{r}')]da'.$$

For Neumann boundary conditions we will not require that $\frac{\partial}{\partial n'}G_N(\mathbf{r},\mathbf{r}') = 0$, $\mathbf{r}' \in \mathscr{S}$, since the application of Gauss's theorem to the known relation $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$ shows that

$$\int_{\mathscr{S}} \frac{\partial}{\partial n'}G_N(\mathbf{r},\mathbf{r}')\, da' = -4\pi \qquad (\neq 0).$$

Consequently the simplest allowable boundary condition on G_N is

$$\frac{\partial}{\partial n'}G_N(\mathbf{r},\mathbf{r}') = -\frac{4\pi}{A}, \quad \mathbf{r}' \in \mathscr{S},$$

where A is the total area of the boundary surface. Then the solution is

$$U(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathscr{V}} G_N(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')dV' + \frac{1}{4\pi}\oint_{\mathscr{S}} G_N(\mathbf{r},\mathbf{r}')\frac{\partial U(\mathbf{r}')}{\partial n'}\, da' + \langle U\rangle_{\mathscr{S}},$$

where $\langle U\rangle_{\mathscr{S}}$ is the average value of the potential over the whole surface $\mathscr{S}$.

T5 Earnshaw's theorem

Consider an imaginary spherical surface $\mathscr{S}$ of radius r that contains no net charge inside. The average potential over $\mathscr{S}$ is

$$\langle U \rangle_{\mathscr{S}} = \frac{1}{4\pi r^2} \oint_{\mathscr{S}} U(r,\theta,\varphi)\, da\,,$$

where da is an element of area of $\mathscr{S}$. This average can be rewritten as

$$\langle U \rangle_{\mathscr{S}} = \frac{1}{4\pi} \oint_{\mathscr{S}} U(r,\theta,\varphi)\, \frac{da}{r^2} = \frac{1}{4\pi} \oint_{\mathscr{S}} U(r,\theta,\varphi)\, d\Omega,$$

where $d\Omega$ is the solid angle the area subtends at the sphere's center. Differentiating the last form with respect to r we get

$$\frac{d}{dr}\langle U \rangle_{\mathscr{S}} = \frac{1}{4\pi} \oint_{\mathscr{S}} \frac{\partial}{\partial r} U(r,\theta,\varphi)\, d\Omega = -\frac{1}{4\pi} \oint_{\mathscr{S}} E_r(r,\theta,\varphi)\, d\Omega,$$

where $E_r(r,\theta,\varphi)$ is the normal component of the electric field at the surface $\mathscr{S}$ at the element da. On rewriting $d\Omega$ as da/r^2 we can write

$$\frac{d}{dr}\langle U \rangle_{\mathscr{S}} = -\frac{1}{4\pi r^2} \oint_{\mathscr{S}} E_r(r,\theta,\varphi)\, da$$

$$= -\frac{1}{4\pi r^2} \oint_{\mathscr{S}} \mathbf{E}(r,\theta,\varphi) \cdot d\mathbf{a}\,.$$

As $\mathscr{S}$ contains no net charge inside, and even if there are net charge outside $\mathscr{S}$, Gauss's law shows that $(d/dt)\langle U \rangle_{\mathscr{S}} = 0$, and consequently the average potential $\langle U \rangle_{\mathscr{S}}$ allover the spherical surface $\mathscr{S}$ is constant and does not depend on the radius r of $\mathscr{S}$. If we let r get smaller and smaller then $\langle U \rangle_{\mathscr{S}}$ eventually coincides with $U(0)$; i.e., $\langle U \rangle_{\mathscr{S}} = U(0)$.

Now imagine for a moment that there is a potential maximum at some point P in a region where $\rho = 0$. Then the average potential over some sphere centered on P must be lower than the potential at P, which is contrary to the above result. Thus there can never be a potential maximum in a charge-free region. For the same reason, there can never be a potential minimum either; this

is Earnshaw's theorem. Another equivalent statement of Earnshaw's theorem states that

"*A charged particle cannot be held in a stable equilibrium by electrostatic forces alone*."