Lecture 2

Conformal Mapping

A complex number z can be written in terms of two real numbers x and y according to,

$$z = x + iy,$$

where

$$i = \sqrt{-1}.$$

A single-valued function *w* of the complex variable z maps z into another complex number *w*, which can be described in terms of two real numbers *u* and *v*,

$$w(z) = u + iv.$$

Any point *(x, y)* in the Z plane yields some point (*u*, *v*) in the *W* plane. As this point moves along some curve $x = g(y)$ in the Z plane, the corresponding point in the *W* plane traces out a curve $u = h(v)$. If it should move throughout a region in the Z plane, the corresponding point would move throughout some region in the *W* plane. Thus, in general, a point in the Z plane transforms to a point in the *W* plane, a curve transforms to a curve, and a region to a region, and the function that accomplishes this is frequently spoken of as a particular transformation between the Z and *W* planes.

A function *w*(z) for which the derivative *dw*/*dz* at a point is independent of the direction of the change *dz* from the point is called an analytic function. The derivative may be written in terms of magnitude and phase:

$$\frac{dw}{dz} = Re^{i\varphi}$$

or

$$dw = Re^{i\varphi}\, dz\,.$$

By the rule for the product of complex quantities, the magnitude of *dw* is *R* times the magnitude of *dz,* and the angle of *dw* is φ plus the angle of *dz*. So the entire infinitesimal region in the vicinity of the point *w* is similar to the infinitesimal region in the vicinity of the point z. It is magnified by a scale factor *R* and rotated by an angle φ. It is then evident that, if two curves intersect at a given angle in the Z plane, their transformed curves in the *W* plane intersect at the same angle, since both are rotated through the angle φ. A transformation with these properties is called a conformal transformation.

Conjugate Functions

If the function *w* is analytic, some important relations of the functions u and v follow. We define $w' = dw/dz$. Then we can write,

$$\frac{\partial w}{\partial x} = w'\frac{\partial z}{\partial x} = w' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{1}$$

$$\frac{\partial w}{\partial y} = w'\frac{\partial z}{\partial y} = iw' = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \tag{2}$$

Eliminating w' from Eqs. (1) and (2), and equating their real and imaginary parts of the result, we get the so-called Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The functions *u* and *v* are related to each other through these equations and are called conjugate functions. An important result follows:

$$\boldsymbol{\nabla} u \,.\, \boldsymbol{\nabla} v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0, \tag{3}$$

as *u* and *v* are differentiable. Also

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = 0;$$

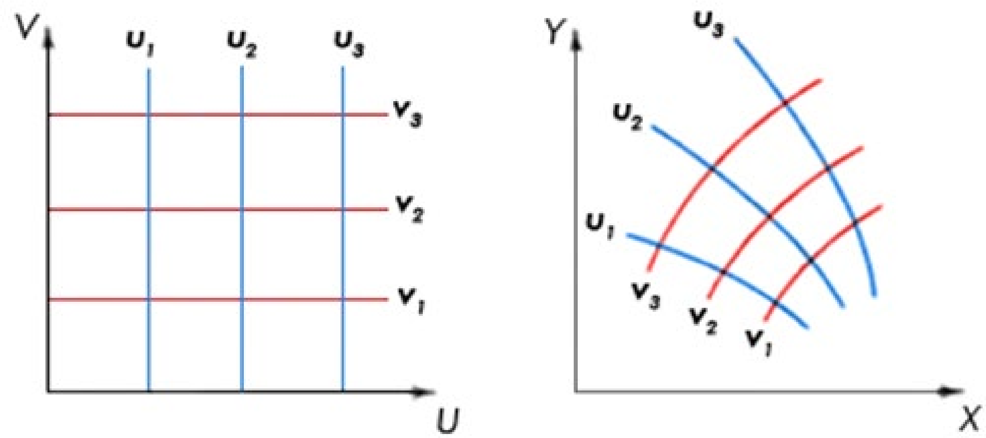
i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Likewise,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0;$$

0In particular, the lines u = constant and the lines v = constant in the W plane intersect at right angles, so their transformed curves in the Z plane must also be orthogonal (See the figure). We already know that this should be so, since the constant u lines have been shown to represent flux lines when the constant v lines



W plane   *Z* plane

A mapping of coordinate lines of the *W* plane in the *Z* plane

are equipotentials, and vice versa. From this point of view, the conformal transformation may be thought of as one that takes a uniform field in the W plane (represented by the equispaced constant u and constant v lines) and transforms it so that it fits given boundary conditions in the Z plane, always keeping the required properties of an electrostatic field.

There are few circumstances in which knowledge of the required boundary conditions will lead directly to the transformation that gives the solution. Some examples of the simpler transformations will be given to illustrate the method.

1- Consider

$$w(z) = z.$$

Here we have

$$u = x \quad \text{and} \quad v = y.$$

If we take the potential to be v, then equipotentials are surfaces of constant y, and the field lines have constant u = x. That is, the elementary problem of a uniform electric field in the x-direction is solved.

2- Consider

$$w(z) = z^2.$$

Here

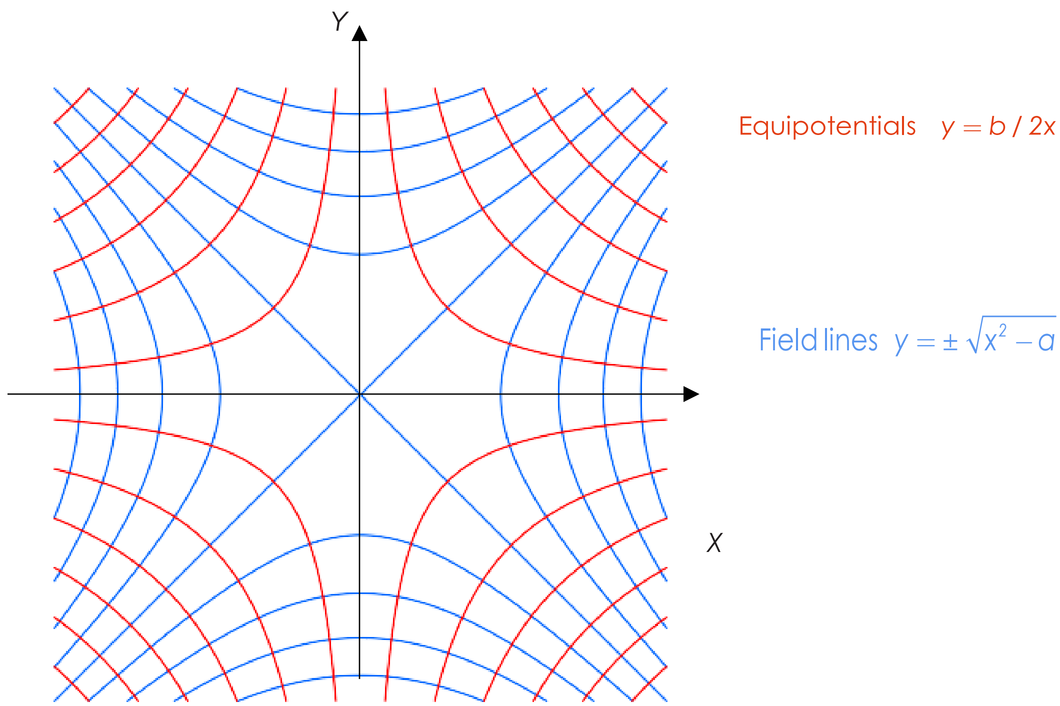
$$u = x^2 - y^2, \quad \text{and} \quad v = 2xy,$$

To see what problem is solved by the potentials u and v, we look at the surfaces of constant u and v:

$$x^2 - y^2 = a, \quad \text{and} \quad 2xy = b.$$

with a and b are constants. In this way, we have the curves

$$u: \ y = \pm\sqrt{x^2 - a}, \quad \text{and} \quad v: \ y = b/2x.$$



A configuration based on these functions is called a quadrupole, which has conductors placed at four equipotential surfaces. The electric field is given by,

$$\boldsymbol{E} = -\nabla v \equiv (-2y, -2x),$$

whose magnitude $E = 2\sqrt{x^2 + y^2}$ is proportional to the radial distance from the z-axis. Devices which exert forces that are proportional to the distance from an axis can be thought of as lenses, and so an electric quadrupole is a somewhat peculiar kind of lens.

3- Consider

$$w = \sqrt{z}\,.$$

Here

$$u^2 - v^2 = x, \quad \text{and} \quad 2uv = y\,.$$

Eliminating v from these two equations we get

$$4u^4 - 4xu^2 - y^2 = 0.$$

This equation solves to

$$u^2 = \frac{x \pm \sqrt{x^2 + y^2}}{2},$$

and hence

$$u = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}}, \quad \text{and} \quad v = \sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}.$$

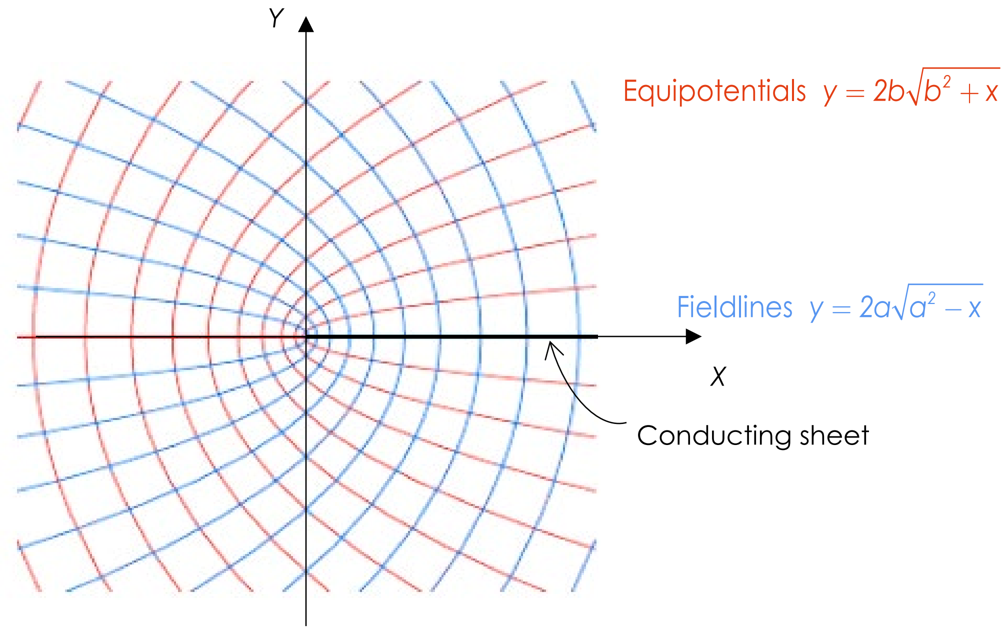
To relate the function w to electric potentials, we again consider surfaces of constant u and v:

$$\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} = a, \quad \text{and} \quad \sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} = b,$$

with a and b are constants. In this way, we have the curves

$$u:\ y = 2a\sqrt{a^2 - x}\,, \quad \text{and} \quad v:\ y = 2b\sqrt{b^2 + x}\,,$$

provided a and b are positive.



If we consider the equipotentials to be surfaces of constant v, then the electric field, shown in blue lines, is given by

$$\boldsymbol{E} = -\nabla v \equiv \frac{\sqrt{2}}{4\sqrt{x^2+y^2}}\left(\sqrt{\sqrt{x^2+y^2}-x},\ \frac{y}{\sqrt{\sqrt{x^2+y^2}-x}}\right).$$

The electric field is very strong near the origin, i.e., near the sharp edge of the conducting sheet.