Lecture 3

Green's function for the time-dependent wave equation in unbounded lossless media

The wave equations for the scalar potential U and vector potential $\boldsymbol{A}$

$$\nabla^2 U - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = -\rho(\mathbf{r},t)/\varepsilon, \quad \text{and} \quad \nabla^2 \mathbf{A} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu \mathbf{J}(\mathbf{r},t),$$

all have the basic structure

$$(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2})\,\varphi = f(\mathbf{r},t), \tag{1}$$

where $f(\mathbf{r},t)$ is a known source distribution. The factor v is the velocity of propagation in the medium, assumed here to be lossless and non-dispersive.

To solve (1) it is useful to find a Green's function for the equation. Since the time is involved, the Green's function will depend on the variables (x, x', t, t'), and will satisfy the equation,

$$(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2})\,G(\mathbf{r}-\mathbf{r}',t-t') = \delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{2}$$

Then in unbounded space the solution of (1) will be

$$\varphi(\mathbf{r},t) = \int G(\mathbf{r}-\mathbf{r}',t-t')\,f(\mathbf{r}',t')\,dV'\,dt'.$$

To find G we consider the Fourier transform of both sides of (2). The delta functions on the right have the representation,

$$\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')}\,e^{i\omega(t-t')}\,d\omega\,d^3k, \tag{3}$$

and G also has the 4dimensional Fourier transform

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{k},\omega)\,e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')}\,e^{i\omega(t-t')}\,d\omega\,d^3k. \tag{4}$$

The Fourier transform $g(\boldsymbol{k},\omega)$ is to be determined. When (3) and (4) are substituted into the defining equation (2), it turns out that $g(\boldsymbol{k},\omega)$ is

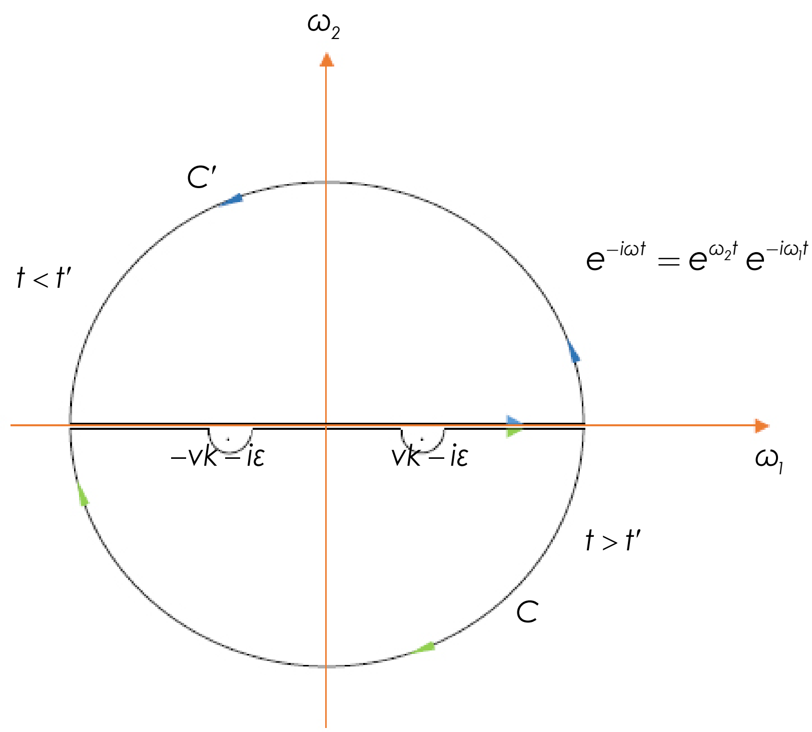
$$g(\boldsymbol{k},\omega) = \frac{1}{(2\pi)^2}\frac{1}{\frac{\omega^2}{v^2}-k^2} = \frac{v^2}{(2\pi)^2}\frac{1}{\omega^2-v^2k^2}. \tag{5}$$

When $g(\boldsymbol{k},\omega)$ is substituted into (4) and the integration over ω is begun, there appears a singularity in the integrand at $\omega = \pm kv$, and so we cannot do the integral along the real ω axis. Consequently a physically-based solution for (5) is needed. The Green's function satisfying (13) represents the wave disturbance caused by a point source at $\boldsymbol{r}'$ which is turned on only at $t' = t$. We know that such a wave disturbance propagates outwards as a spherically diverging wave with a velocity v. Hence we demand that our solution for G have the following properties:

(a) $G = 0$ everywhere for $t < t'$.

(b) G represents outgoing waves for $t > t'$.

We can do the ω integration as a Cauchy integral in the complex ω plane by adding a return contour either far above or far below the real axis. If this additional contour adds exactly zero then the closed contour integral will equal the inverse Fourier transform along that contour. The advantage of using the closed contour is that the integral can be easily done using the residues. For $t > t'$ the integral along the real axis in (4) is equivalent to the contour integral around a path C closed in the lower half-plane, since the contribution on the semicircle at infinity vanishes exponentially. On the other hand, for $t < t'$, the contour must be closed in the upper half-plane, as shown in the figure below by path C'.



Complex ω plane with contour C for $t > t'$ and contour C' for $t < t'$

In order to make G vanish for $t < t'$ we must imagine that the poles at $\omega = \pm vk$ are infinitesimally displaced below the real axis. Then the integral over C for $t > t'$ will give a nonvanishing contribution, while the integral over C for $t < t'$ will vanish. The displacement of the poles can be accomplished mathematically by writing $(\omega + i\varepsilon)$ in place of ω in (5). Then the Green's function is given by

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{v^2}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(\mathbf{k}.\mathbf{R} + \omega\tau)}}{(\omega + i\varepsilon)^2 - (vk)^2} \, d\omega \, d^3k \,,$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $\tau = t - t'$, and ε is a positive infinitesimal.

The integration over ω for $\tau > 0$ can be done with Cauchy's theorem applied around the contour C, giving

$$G(\mathbf{r},\tau)=\frac{v^2}{(2\pi)^4}\int_{-\infty}^{\infty}\oint_C \frac{e^{i\omega\tau}}{(\omega-vk)(\omega+vk)}\,d\omega\;\; e^{i\boldsymbol{k}.\boldsymbol{R}}\,d^3k$$

$$=\frac{v^2}{(2\pi)^4}\int_{-\infty}^{\infty} 2\pi i\left[\text{sum of residues of } \frac{e^{i\omega\tau}}{(\omega-vk)(\omega+vk)} \text{ about } (\pm\, vk)\right] e^{i\boldsymbol{k}.\boldsymbol{R}}\,d^3k$$

$$=\frac{v^2}{(2\pi)^3}\int_{-\infty}^{\infty} i\,\Big(\frac{e^{ivk\tau}}{2vk}-\frac{e^{-ivk\tau}}{2vk}\Big)\, e^{i\boldsymbol{k}.\boldsymbol{R}}\,d^3k$$

$$=\frac{-v}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{1}{2ik}\,(e^{ivk\tau}-e^{-ivk\tau})\, e^{i\boldsymbol{k}.\boldsymbol{R}}\,d^3k\,.$$

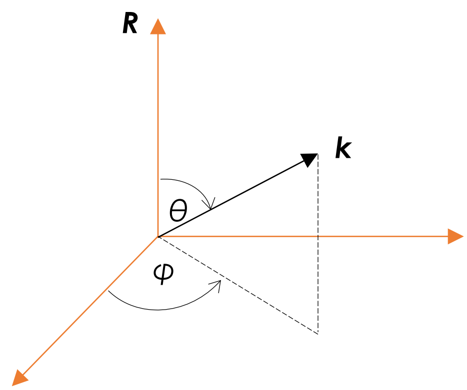
Upon using the identity $\sin x=(e^{ix}-e^{-ix})/(2i)$, the last equation simplifies to

$$G(\mathbf{r},\tau) \;=\; \frac{-v}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{\sin(v\tau k)}{k}\, e^{i\boldsymbol{k}.\boldsymbol{R}}\,d^3k\,. \tag{6}$$

Now, we still have to perform the integration over $d^3k\,(\tau>0)$. For this purpose we introduce spherical coordinates in $\boldsymbol{k}$ space, where $\boldsymbol{R}$ gives the direction of the k_z axis, as shown in the figure below. Because this axis can be chosen arbitrarily, this is always possible. We obtain

$$d^3k=k^2\sin\theta\,dk\,d\theta\,d\varphi\,,\quad \text{and} \quad \boldsymbol{k}\,.\,\boldsymbol{R}=kR\cos\theta\,.$$

When this is substituted into the expression (6), we get



$\boldsymbol{k}$ space

$$
\begin{aligned}
G(\mathbf{r},\tau) &= \frac{v}{(2\pi)^3}\int_0^\infty k\sin(v\tau k)\,(\int_0^\pi e^{ikR\cos\theta}d\cos\theta)\,(\int_0^{2\pi} d\varphi)\,dk \\
&= \frac{v}{(2\pi)^3}\int_0^\infty k\sin(v\tau k)\left|\frac{e^{ikR\cos\theta}}{ikR}\right|_0^\pi (2\pi)\,dk \\
&= \frac{v}{(2\pi)^2 R}\int_0^\infty \sin(v\tau k)\;(\frac{e^{-ikR}-e^{ikR}}{i})\,dk \\
&= \frac{-2v}{(2\pi)^2 R}\int_0^\infty \sin(v\tau k)\sin kR\,dk\,.
\end{aligned}
$$

Since the integrand is even in k, the integral can be written over the whole interval, $-\infty < k < \infty$. With a change of variable $x = vk$, the above equation can be written as

$$
\begin{aligned}
G(\mathbf{r},\tau) &= \frac{-1}{(2\pi)^2 R}\int_{-\infty}^{\infty} \sin(\tau x)\;\sin(\frac{R}{v}x)\;dx \\
&= \frac{1}{4\,(2\pi)^2 R}\int_{-\infty}^{\infty} (e^{i\tau x}-e^{-i\tau x})\,(e^{i\frac{R}{v}x}-e^{-i\frac{R}{v}x})\;dx \\
&= \frac{1}{4\,(2\pi)^2 R}\int_{-\infty}^{\infty} [e^{i(\tau+\frac{R}{v})x}-e^{i(\tau-\frac{R}{v})x}\;-e^{-i(\tau-\frac{R}{v})x}+e^{-i(\tau+\frac{R}{v})x}]\,dx \\
&= \frac{1}{8\pi R}\;[\delta(\tau+\frac{R}{v})-\delta(\tau-\frac{R}{v})\;-\delta(-\tau+\frac{R}{v})+\delta(-\tau-\frac{R}{v})]\,.
\end{aligned}
$$

The argument of the first and fourth terms never vanish (remember $\tau > 0$), hence the Green's function is

$$
G(\mathbf{r},\tau) \;=\; \frac{-1}{4\pi R}\;\delta(\tau-\frac{R}{v})\,.
$$

This Green's function is sometimes called the retarded Green's function because it exhibits the causal behavior associated with a wave disturbance. The effect observed at the point $\mathbf{r}$ at time t is due to a disturbance which originated at an earlier or retarded time $t' = t - \dfrac{|\mathbf{r}-\mathbf{r}'|}{v}$ at the point $\mathbf{r}'$.

The solution for the wave equation (1) in the absence of boundaries is so

$$\varphi(\mathbf{r},t) = \frac{-1}{4\pi}\int \frac{\delta(t' + \frac{|\mathbf{r}-\mathbf{r}'|}{v} - t)}{|\mathbf{r}-\mathbf{r}'|}\; f(\mathbf{r}',t')\, dV'\, dt' \,.$$

The integration over t' can be performed to yield the so-called retarded solution,

$$\varphi(\mathbf{r},t) = \frac{-1}{4\pi}\int \frac{[f(\mathbf{r}',t')]_{\text{ret}}}{|\mathbf{r}-\mathbf{r}'|}\; dV' \,.$$

The function $[f(\mathbf{r}',t')]_{\text{ret}}$ means that the time t' is to be evaluated at the retarded time $t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{v}$.

Turning back to electromagnetic potentials, the electrostatic potential is determined by

$$U(\mathbf{r},t) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\mathbf{r}', t - \sqrt{\varepsilon\mu}\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\; dV' \,.$$

The solution of $\boldsymbol{A}$ can be constructed in exactly the same way. The vectors $\boldsymbol{A}$ and $\boldsymbol{J}$ are first decomposed into rectangular components:

$$\nabla^2 A_i - \varepsilon\,\mu \frac{\partial^2 A_i}{\partial t^2} = -\mu\, J_i\,, \quad i = 1, 2, 3.$$

Each of these equations may be solved, giving

$$\boldsymbol{A}(\mathbf{r},t) = \frac{\mu}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}\; dV' \,,$$

which is the retarded vector potential.