General Relativity and Cosmology

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The Plan

Gravity as Geometry

Differential Geometry

Manifolds – Vectors and Tensors
Connection – Covariant Derivatives
Riemannian geometry – Curvature

General Relativity

Test particles – Einstein equations – Einstein-Hilbert action

Cosmology

The Universe around us, the red shift
Friedmann equations – Cosmic history – Inflation

Equivalence principle

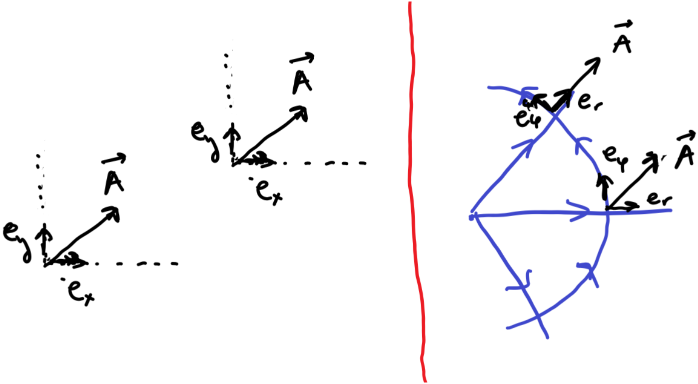
The gravitational mass
is the same
as the inertial mass.

Shouldn't we look for geometry?

Lecture 1

Basics of Differential Geometry

Parallel transport on a plane in non-Cartesian coordinates



On the right, the vector components are changed.

Let's parametrise the change of vector field components as

$$\delta A^{\nu} = -\Gamma^{\nu}_{\mu\alpha} A^{\alpha} \delta x^{\mu}$$

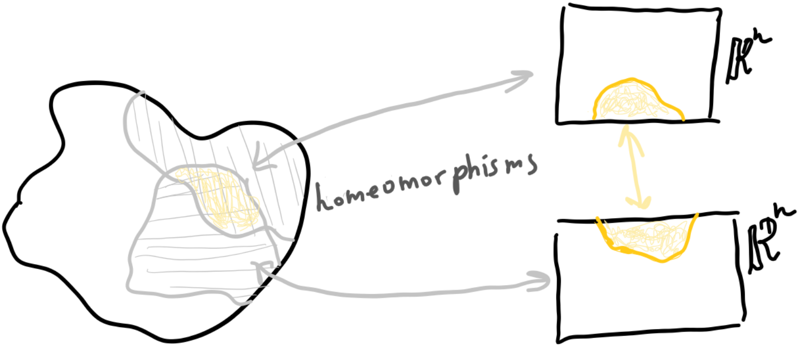
where the coefficients Γ are called the connection coefficients or Christoffel symbols.

The scalar products are now calculated as $\overrightarrow{A} \cdot \overrightarrow{B} = g_{\mu\nu} A^{\mu} B^{\nu}$, or we can define a linear functional $A_{\nu} \equiv g_{\mu\nu} A^{\mu}$.

For the functionals, $\underleftarrow{A}(\overrightarrow{B}) \equiv A_{\mu} B^{\mu}$, we need the transport law

$$\delta A_{\nu} = \Gamma^{\alpha}_{\mu\nu} A_{\alpha} \delta x^{\mu}.$$

A manifold



What is a vector?

Let's look at a germ of smooth functions at a given point.

Vectors define operators of differentiating along the vector:

$$\overrightarrow{A} \quad \longleftrightarrow \quad A^{\mu}\frac{\partial}{\partial x^{\mu}}.$$

In a given coordinate system, there is a basis of vectors $\frac{\partial}{\partial x^{\mu}}$.
Let's denote the dual basis as dx^{μ}.

Then, the linear functionals on the tangent space
(i.e. elements of cotangent space) can be represented as

$$\underleftarrow{A} \quad \longleftrightarrow \quad A_{\mu}dx^{\mu}.$$

If we perform a change of variables, $x \longrightarrow \tilde{x} = \tilde{x}(x)$,

$$A^{\mu} \frac{\partial}{\partial x^{\mu}} \longrightarrow \tilde{A}^{\mu} \frac{\partial}{\partial \tilde{x}^{\mu}},$$

the vector components must obviously be changed as

$$\tilde{A}^{\mu} = A^{\nu} \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}}.$$

Analogously, for the linear functionals

$$A_{\mu} = \tilde{A}_{\nu} \frac{\partial \tilde{x}^{\nu}}{\partial x^{\mu}},$$

i.e. the change by inverse matrix.

A metric $g_{\mu\nu}dx^\mu dx^\nu$ on the manifold
can be thought of as a linear mapping from tangent space to cotangent space $\mathfrak{g} : \ \overrightarrow{A} \longrightarrow \underleftarrow{A}$ by agreeing that $A_\mu \equiv g_{\mu\nu}A^\nu$.

It transforms as

$$g_{\mu\nu} = \tilde{g}_{\alpha\beta}\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}.$$

And so on and so forth, for tensors of arbitrary ranks.

For example, for the inverse metric $g^{\mu\nu}$:

$$\tilde{g}^{\mu\nu} = g^{\alpha\beta}\frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}.$$

Now, we can agree on the law of parallel transport as

$$\delta A^\nu = -\Gamma^\nu_{\mu\alpha} A^\alpha \delta x^\mu,$$

$$\delta A_\nu = \Gamma^\alpha_{\mu\nu} A_\alpha \delta x^\mu.$$

By introducing a connection,
we choose horizontal directions in the tangent bundle.

We then define the covariant derivative as showing the difference from the parallel transport:

$$\bigtriangledown_\mu A^\nu \equiv \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} A^\alpha,$$

$$\bigtriangledown_\mu A_\nu \equiv \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha,$$

$$\bigtriangledown_\mu g_{\alpha\beta} \equiv \partial_\mu g_{\alpha\beta} - \Gamma^\nu_{\mu\alpha} g_{\nu\beta} - \Gamma^\nu_{\mu\beta} g_{\alpha\nu},$$

$$\ldots$$

One can check that the usual laws of differentiation apply, such as

$$\bigtriangledown_\alpha (g_{\mu\nu} A^\nu) = A^\nu \bigtriangledown_\alpha g_{\mu\nu} + g_{\mu\nu} \bigtriangledown_\alpha A^\nu .$$

We also need that things like $\bigtriangledown_\mu A_\nu \equiv \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha$ are tensors, so that it is all about coordinate-independent geometry.

It implies

$$\Gamma^\kappa_{\mu\nu} = \frac{\partial x^\kappa}{\partial \tilde{x}^\rho} \left(\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{\Gamma}^\rho_{\alpha\beta} + \frac{\partial^2 \tilde{x}^\rho}{\partial x^\mu \partial x^\nu} \right)$$

as can be easily checked.

Euclidean spaces in Cartesian coordinates have $\Gamma^\alpha_{\mu\nu} = 0$.

Riemannian geometry

If to follow the example of Euclidean spaces as close as possible, we demand two properties of a connection:

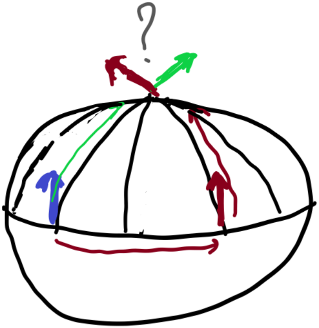
No torsion: $T^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} = 0$

No nonmetricity: $Q_{\alpha\mu\nu} \equiv \bigtriangledown_{\alpha} g_{\mu\nu} = 0$

It yields the Levi-Civita connection:

$$\Gamma^{\rho}_{\alpha\beta} = \frac{1}{2} g^{\rho\mu} \left(\partial_{\alpha} g_{\mu\beta} + \partial_{\beta} g_{\alpha\mu} - \partial_{\mu} g_{\alpha\beta} \right).$$

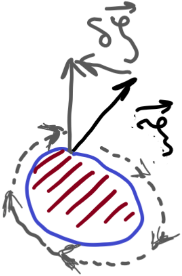
Curvature!



The result
of the parallel transport
depends on the path taken
!!!

One more property of a general connection,
on top of torsion and nonmetricity.

In particular, after a parallel transport over a closed loop, the vector might not be the same as before.



$$\delta\xi^{\mu} = -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}\, \xi^{\nu} \iint dx^{\alpha} \wedge dx^{\beta}$$

$$R^{\alpha}{}_{\beta\mu\nu} \equiv \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}$$

The Riemann tensor or the curvature tensor:

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}.$$

In a Euclidean space, this tensor is zero.
For the Levi-Civita connection, the converse is almost true.

Another possible approach to curvature is:

$$[\bigtriangledown_\mu\ ,\ \bigtriangledown_\nu]\,\xi^\alpha = \bigtriangledown_\mu \left(\partial_\nu \xi^\alpha + \Gamma^{\alpha}_{\nu\beta}\xi^\beta\right) - \bigtriangledown_\nu \left(\partial_\mu \xi^\alpha + \Gamma^{\alpha}_{\mu\beta}\xi^\beta\right) =$$

$$= \partial_\mu \left(\partial_\nu \xi^\alpha + \Gamma^{\alpha}_{\nu\beta}\xi^\beta\right) + \Gamma^{\alpha}_{\mu\rho} \left(\partial_\nu \xi^\rho + \Gamma^{\rho}_{\nu\beta}\xi^\beta\right) - \Gamma^{\rho}_{\mu\nu} \left(\partial_\rho \xi^\alpha + \Gamma^{\alpha}_{\rho\beta}\xi^\beta\right) -$$

$$-\partial_\nu \left(\partial_\mu \xi^\alpha + \Gamma^{\alpha}_{\mu\beta}\xi^\beta\right) - \Gamma^{\alpha}_{\nu\rho} \left(\partial_\mu \xi^\rho + \Gamma^{\rho}_{\mu\beta}\xi^\beta\right) + \Gamma^{\rho}_{\nu\mu} \left(\partial_\rho \xi^\alpha + \Gamma^{\alpha}_{\rho\beta}\xi^\beta\right) =$$

$$= R^{\alpha}{}_{\beta\mu\nu}\xi^\beta - T^{\rho}{}_{\mu\nu} \bigtriangledown_\rho \xi^\alpha.$$

The curvature tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}$$

is always antisymmetric:

$$R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu}.$$

For the Levi-Civita connection, more properties are valid:

$$R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\mu\nu\beta} + R^{\alpha}{}_{\nu\beta\mu} = 0,$$

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu},$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta},$$

$$\bigtriangledown_{\rho}R^{\alpha}{}_{\beta\mu\nu} + \bigtriangledown_{\mu}R^{\alpha}{}_{\beta\nu\rho} + \bigtriangledown_{\nu}R^{\alpha}{}_{\beta\rho\mu} = 0.$$