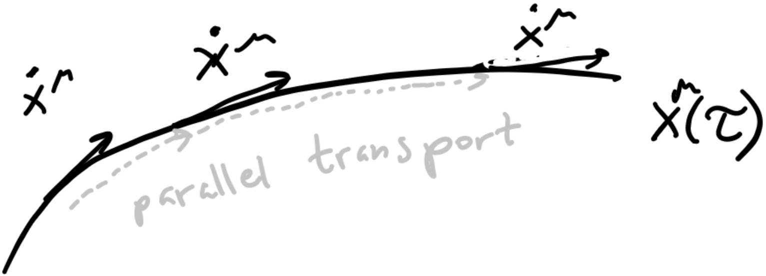
Lecture 2

General Relativity

Geodesics

We can start thinking physics from test particles.
In a flat spacetime, a free particle follows a straight line.
We need a geodesic.
The velocity of a particle is not changed. That is, the tangent vector along the trajectory gets parallely transported.



We will call a line $x^\mu(\tau)$ geodesic if its tangent vector $e^\mu \equiv \frac{dx^\mu}{d\tau}$ remains tangent when parallelly transported along the line.

Let's assume more, namely that the tangent vector is covariantly constant along the line: $\delta\frac{dx^\mu}{d\tau} = -\Gamma^\mu_{\nu\alpha}\frac{dx^\alpha}{d\tau}\delta x^\nu = -\Gamma^\mu_{\nu\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta\tau$. It gives the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

This equation is invariant under affine changes of τ. Otherwise, this is a preferred choice of parameter, the affine parameter. Under a general non-linear reparametrisation, the equation becomes more complicated.

Another familiar meaning of geodesics

Let's assume a positive definite metric (Riemannian geometry), and extremise the length of a curve,

$$\int dl = \int d\tau \sqrt{g_{\mu\nu}(x(\tau))\frac{dx^{\mu}(\tau)}{d\tau}\frac{dx^{\nu}(\tau)}{d\tau}}.$$

Assuming that $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ is constant, we get

$$\frac{1}{2}\left(g_{\mu\nu,\alpha}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} - g_{\mu\alpha,\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} - g_{\nu\alpha,\mu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\right) - g_{\mu\alpha}\frac{d^2x^{\mu}}{d\tau^2} = 0.$$

Finally, we raise the index α:

$$\frac{d^2x^{\alpha}}{d\tau^2} + \frac{1}{2}g^{\alpha\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0,$$

and get the geodesic equation with the Levi-Civita connection.

Towards Einstein equations

The next step is to find equations for gravitational interactions. We have to equate something geometric to something related to the matter such that it corresponds to the mass in the non-relativistic limit.

The natural idea would be the energy-momentum tensor $T^{\mu\nu}$.

It can be chosen symmetric $T_{\mu\nu} = T_{\nu\mu}$.

In flat space, it is conserved $\partial_\mu T^{\mu\nu} = 0$. Therefore, in curved space we expect

$$\bigtriangledown_\mu T^{\mu\nu} = 0.$$

Not precisely a conservation law, though.

On the gravity side, we have got the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$.
An obvious step is to define the Ricci tensor:

$$R_{\beta\nu} \equiv R^{\alpha}{}_{\beta\alpha\nu}.$$

With a Levi-Civita connection,
this is the only possible non-trivial contraction
and is symmetric, $R_{\mu\nu} = R_{\nu\mu}$.

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\alpha}\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\alpha}_{\rho\mu}$$

This is a rank-two symmetric tensor. Good!
But what about the "conservation laws"?

Recall the Bianchi identity:

$$\nabla_\rho R^\alpha{}_{\beta\mu\nu} + \nabla_\mu R^\alpha{}_{\beta\nu\rho} + \nabla_\nu R^\alpha{}_{\beta\rho\mu} = 0.$$

Let's do some contractions: ρ and β

$$\nabla^\beta R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\nu} - \nabla_\nu R_{\alpha\mu} = 0$$

and α and μ

$$2 \nabla^\mu R_{\mu\nu} - \nabla_\nu R = 0$$

with the scalar curvature $R \equiv g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu$.

Or we can say

$$\nabla^\mu (2R_{\mu\nu} - g_{\mu\nu} R) = 0.$$

Thus we have found a tensor (the Einstein tensor)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

which satisfies the "covariant conservation" equation

$$\bigtriangledown_\mu G^{\mu\nu} = 0.$$

And we can try the equations of motion in the form

$$G_{\mu\nu} \propto T_{\mu\nu}$$

or, equivalently (in four dimensions),

$$R_{\mu\nu} \propto \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

with $T \equiv T_\mu^\mu$.

Let's try to guess the coefficient in a very rough manner.

At the usual energies, the main effect comes from the curvature of time, because the spatial curvature does not show up in the every-day life. So, we write a metric for a static body in the following form:

$$ds^2 \approx c^2 \left(1 + \frac{2\Phi(x)}{c^2}\right) dt^2 - g_{ik}(x) dx^i dx^k$$

with g_{ik} close to the Euclidean metric.
In this case we get $\Gamma^i_{00} \approx \partial_i \Phi$, and the geodesic equation for a falling non-relativistic $\left(\frac{dx^\mu}{d\tau} \approx \delta^\mu_0\right)$ body:

$$\frac{d^2 x^i}{d\tau^2} \approx -\Gamma^i_{00} \approx -\partial_i \Phi,$$

so that the function $\Phi(x)$ plays the role of Newtonian potential.

The energy momentum tensor of the source body is naturally taken to be $T^{\mu}_{\nu} = \rho(x)\delta^{\mu}_{0}\delta^{0}_{\nu}$ where ρ is the energy density (i.e. the mass density multiplied by c^2).

We get approximately $R_{00} \propto \frac{1}{2} T_{00}$.
Now, we can calculate the curvature in our approximation,

$$R_{00} \approx \Gamma^{i}_{00,i} \approx \triangle\Phi.$$

However, we know from the Newtonian gravity that $\triangle\Phi = 4\pi\mathcal{G}\frac{\rho}{c^2}$.

And therefore, the Einstein equation finally reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi\mathcal{G}}{c^4}T_{\mu\nu}.$$

Nota bene! From now on, I will put $c = 1$.

Perturbation theory around a given background

When $g_{\mu\nu} \longrightarrow g_{\mu\nu} + h_{\mu\nu}$, from

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\rho\mu} \left(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta}\right)$$

we get

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\left(\bigtriangledown_\alpha h^{\mu}_{\beta} + \bigtriangledown_\beta h^{\mu}_{\alpha} - \bigtriangledown^{\mu} h_{\alpha\beta}\right) + \mathcal{O}(h^2),$$

and from

$$R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\alpha}\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\alpha}_{\rho\mu}$$

we find

$$\delta R_{\mu\nu} = \bigtriangledown_\rho \delta\Gamma^{\rho}_{\mu\nu} - \bigtriangledown_\nu \delta\Gamma^{\rho}_{\rho\mu} + \mathcal{O}((\delta\Gamma)^2).$$

The action principle

One can derive that, if g is the determinant of the matrix $g_{\mu\nu}$, then $\frac{\partial}{\partial g_{\mu\nu}} g = g^{\mu\nu} g$.
Also, we have seen that the variation of the Ricci tensor is a total covariant derivative (note second derivatives).

Therefore, variation of the (Einstein-Hilbert) action

$$S = \int d^4x \sqrt{-g} R = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}$$

yields the vacuum gravity equation of

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0.$$

The Bianchi identity $\bigtriangledown_\mu G^{\mu\nu} \equiv 0$ is a consequence of the diffeomorphism invariance of the action.

We minimally couple physical matter to gravity by putting arbitrary metric into its action.

And then the action of

$$S = -\frac{1}{16\pi\mathcal{G}} \int d^4x\sqrt{-g}R + S_{\mathrm{matter}}$$

produces the equation of motion

$$G_{\mu\nu} = 8\pi\mathcal{G}\,T_{\mu\nu}$$

with the energy-momentum tensor defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}S_{\mathrm{matter}}.$$

GR is very well confirmed by observations.
(Mercury perihelion, gravitational lensing, gravitational waves...)

But there are also problems...

– Intricacy of quantum physics

– Singularities

– The Dark Sector of the Universe

– Recently, more tensions in cosmology

Hence, very active research in modifications of GR.
One more motivation is that we love making troubles.