Quantum Field Theory - Part 2

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Classical Mechanics

Quantum Mechanics

Special Relativity

Classical Field Theory

QFT

Field Theory

Since the relativity introduces the annihilation–creation processes, the particle numbers in a system is not fixed any more, and thus even in a single particle system it can turn to be a multi-particle system. In this context, dealing with the individual particles and the relevant parameters associated with them does not provide a comprehensive approach. The fields instead of wave functions of the particles are more convenient to consider the systems, especially when the system is quantized.
In the field theory, the variables are fields, and the coordinates are considered as to be continuous parameters. They are not explicit variables, but the fields happen to be functions of them. The Lagrangian and Hamiltonian are written in terms of these fields and their derivatives:

Lagrangian Field Theory

Even though we deal with the systems through their Lagrangian and Hamiltonian, the most fundamental quantity is the action integral, which is simply the time integral of the Lagrangian. In the local field theory, one can express the Lagrangian as the spatial integral of Lagrangian Density ($\mathcal{L}$) as

$$\mathcal{S} = \int dtL \equiv \int d^4x\mathcal{L}(\phi, \partial_\mu\phi)$$

$$\delta\mathcal{S} = 0 \Longrightarrow \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} \right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

Lagrangian usually involves fields more than one, in these case there is one such equation as given above for each field.

Hamiltonian Field Theory

The conjugate momentum of a field can be obtained from the Lagrangian as:

$$
\begin{aligned}
p(x) &= \frac{\partial L}{\partial \dot{\phi}(x)} \\
&= \frac{\partial}{\partial \dot{\phi}(x)} \int d^3 y \mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int d^3 x \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \\
&= \int d^3 x \pi(x) d^3 x
\end{aligned}
$$

where $\phi(x)$ is the momentum density as in $px = \int \pi(x) d^3 x$ and

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$$

and the Hamiltonian is

$$
\begin{aligned}
H &= \sum_x p(x)\dot{\phi}(x) - L \\
&= \int d^3x \left[\pi(x)\dot{\phi}(x) - \mathcal{L}\right] \\
&\equiv \int d^3x \mathcal{H}
\end{aligned}
$$

$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L} \longrightarrow$ Hamiltonian density.

In the field point of view, the densities seem more convenient to use. Thus, hereafter, we will drop saying density. In the rest of these notes, Lagrangian, Hamiltonian, momentum etc mean their densities.

Kelin-Gordon (Scalar) Field

Using the relativistic relation, one can derive an equation for a relativistic particle by following Schrodinger's approach: i.e. replacing the parameters with the operators: $E \longrightarrow i\partial/\partial t$ and $\vec{p} \longrightarrow i\vec{\nabla}$ and multiply it with a wave function:

$$E^2 - \vec{p}^2 - m^2 = 0 \quad \longrightarrow \quad \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 - m^2\right)\phi(x) = 0$$

$$\left(\partial_\mu \partial^\mu - m^2\right)\phi(x) = 0$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$: Lagrangian for free Klein-Gordon field

Kelin-Gordon (Scalar) Field

The conjugate momentum of the Klein-Gordon field:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) \\
&= \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - m^2\phi^2 \\
&= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - m^2\phi^2
\end{aligned}
\qquad \Bigg| \qquad
\begin{aligned}
\pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
&= \dot{\phi}
\end{aligned}
$$

$\Rightarrow \pi(x) = \dot{\phi}(x)$: Conjugate momentum associate with $\phi(x)$.

Kelin-Gordon (Scalar) Field

Expanding the Klein-Gordon field in Fourier space as

$$\begin{aligned}
\phi(x) &= \int \frac{d^4p}{(2\pi)^2} e^{ip^\mu x_\mu} \Phi(p) \\
&= \int \frac{dE}{(2\pi)^{1/2}} \frac{d^3p}{(2\pi)^{3/2}} e^{i(E.t - \vec{p}\vec{x})} \Phi(E, \vec{p})
\end{aligned}$$

where,

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\phi(x) &= \int \frac{dE}{(2\pi)^{1/2}} \frac{d^3p}{(2\pi)^{3/2}} e^{i(E.t - \vec{p}\vec{x})} (-E^2) \Phi(E, \vec{p}) \\
\vec{\nabla}^2\phi(x) &= \int \frac{dE}{(2\pi)^{1/2}} \frac{d^3p}{(2\pi)^{3/2}} e^{i(E.t - \vec{p}\vec{x})} (-\vec{p}^2) \Phi(E, \vec{p})
\end{aligned}$$

Kelin-Gordon (Scalar) Field

putting all together in the Klein-Gordon equation:

$$\begin{aligned}
\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 - m^2\right)\phi(x) &= \int \frac{dE}{(2\pi)^{1/2}}\frac{d^3p}{(2\pi)^{3/2}}\left[e^{i(E.t-\vec{p}\vec{x})}\right. \\
&\quad \left.(-E^2 - \vec{p}^2 - m^2)\Phi(E,\vec{p})\right] \\
&= 0 \\
[E^2 + (\vec{p}^2 + m^2)]\Phi(E,\vec{p}) &= 0
\end{aligned}$$

If we define $\omega_p = \sqrt{\vec{p}^2 + m^2}$, then the equation of motion obtained above turns to be one very similar to that of the simple harmonic oscillator.

$$\mathcal{H} = \frac{1}{2}\pi^2(x) + \frac{1}{2}\omega_p^2\phi^2(x)$$

Kelin-Gordon (Scalar) Field

and using the ladder operators;

$$\begin{aligned} a &= \frac{1}{\sqrt{2\omega}}(\omega\phi + i\pi) \\ a^\dagger &= \frac{1}{\sqrt{2\omega}}(\omega\phi - i\phi) \end{aligned} \quad \xrightarrow{inverting} \begin{cases} \phi &= \frac{1}{\sqrt{2\omega_p}}(a + a^\dagger) \\ \pi &= -i\sqrt{\frac{\omega_p}{2}}(a - a^\dagger) \end{cases}$$

and expanding them in Fourier space

$$\begin{aligned} \phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{i(\omega_p t - \vec{p}\cdot\vec{x})} + a_p^\dagger e^{-i(\omega_p t - \vec{p}\cdot\vec{x})} \right) \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{\omega_p}{2}} \left(a_p e^{i(\omega_p t - \vec{p}\cdot\vec{x})} - a_p^\dagger e^{-i(\omega_p t - \vec{p}\cdot\vec{x})} \right) \end{aligned}$$

Klein-Gordon (Scalar) Field - Hamiltonian

$$\begin{aligned} H &= \int d^3x \mathcal{H} \\ &= \int d^3x \left(\frac{1}{2}\pi^2(x) + \frac{1}{2}\omega_p^2 \phi^2(x) \right) \\ &= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \left(a_p^\dagger a_p + \varepsilon_0 \right); \quad \varepsilon = \frac{1}{2}(2\pi)^{-3}\omega_p \end{aligned}$$

$$\mathcal{E} = \frac{1}{2}(2\pi)^{-3} \int d^3k \omega_p$$

The last integral, in principal, gives infinity if it is taken over the whole range of the momentum. If one assumes that this theory is valid up to some high energy ($\Lambda \gg m$), then $\mathcal{E} \propto \Lambda^4$, where Λ is called "ultraviolet cut-off scale".

Klein-Gordon (Scalar) Field - Conserved Current

The Klein-Gordon equation, in contrast to the other fundamental equations of physics, includes second-order time-derivative, which leads some problems. It can be seen best in the conserved probability current. Let us write the Klein-Gordon equation for the scalar field ϕ and its complex conjugate by splitting the time-derivative terms to those with the spatial derivative:

$$\begin{aligned} -\frac{\partial^2 \phi}{\partial t^2} &= -\nabla^2 \phi + m^2 \phi, \\ -\frac{\partial^2 \phi^*}{\partial t^2} &= -\nabla^2 \phi^* + m^2 \phi^*. \end{aligned}$$

If we multiply the first equation with ϕ^* and the second equation with ϕ from the left:

$$\begin{aligned} -\phi^* \frac{\partial^2 \phi}{\partial t^2} &= -\phi^* \nabla^2 \phi + m^2 \phi^* \phi, \\ -\phi \frac{\partial^2 \phi^*}{\partial t^2} &= -\phi \nabla^2 \phi^* + m^2 \phi^* \phi. \end{aligned}$$

Klein-Gordon (Scalar) Field - Conserved Current

If the second equation is subtracted to the first one, it drives to the following equation:

$$\phi\frac{\partial^2\phi^*}{\partial t^2} - \phi^*\frac{\partial^2\phi}{\partial t^2} = \phi\nabla^2\phi^* - \phi^*\nabla^2\phi$$

where,

$$\begin{aligned}
\phi\frac{\partial^2\phi^*}{\partial t^2} - \phi^*\frac{\partial^2\phi}{\partial t^2} &= \frac{\partial}{\partial t}\left[\phi\frac{\partial\phi^*}{\partial t} - \phi^*\frac{\partial\phi}{\partial t}\right] - \left[\frac{\partial\phi}{\partial t}\frac{\partial\phi^*}{\partial t} - \frac{\partial\phi^*}{\partial t}\frac{\partial\phi}{\partial t}\right] \\
\phi\nabla^2\phi^* - \phi^*\nabla^2\phi &= \nabla\left[\phi\nabla\phi^* - \phi^*\nabla\phi\right] - \left[\nabla\phi\nabla\phi^* - \nabla\phi^*\nabla\phi\right]
\end{aligned}$$

The second terms in both equations are clearly equal to zero. Thus, the first terms will be equal to each other. One can form them as a continuity equation which is satisfied by the conserved currents as

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

Klein-Gordon (Scalar) Field - Conserved Current

If we write down the density and the current with the solutions of free Klein-Gordon field;

$$
\begin{aligned}
\phi &= Ne^{i(E.t-\vec{p}\cdot\vec{r})} \\
\phi^* &= Ne^{-i(E.t-\vec{p}\cdot\vec{r})} \\
\rho &= \phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t} \\
\vec{J} &= \phi\vec{\nabla}\phi^* - \phi^*\vec{\nabla}\phi
\end{aligned}
\implies
\begin{cases}
\rho &= 2iEN^2 \\
E &= \pm\sqrt{\vec{p}^2 + m^2}
\end{cases}
$$

Since the energy can have negative values, the probability density can also be negative.

Klein-Gordon (Scalar) Field - Antiparticles

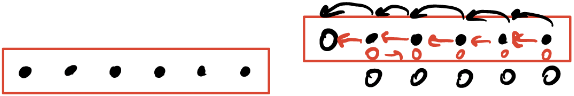
As it is shown the probability density for Klein-Gordon field can be negative, which is not acceptable for our understanding of the probability. In such an ill case, we can reconsider the states with negative energies. A full solution to the free Klein-Gordon equation can be written and manipulated as follows:

$$\begin{aligned}\phi(\vec{x},t) &= N_1 e^{-i(E.t-\vec{p}\cdot\vec{r})} + N_2 e^{i(E.t-\vec{p}\cdot\vec{r})} \\ &= N_1 e^{-i(E.t-\vec{p}\cdot\vec{r})} + N_2 e^{-i(E.(-t)-\vec{p}\cdot\vec{r})} \\ &= N_1 e^{-i(E.t-\vec{p}\cdot\vec{r})} + N_2 e^{-i(E.t'-\vec{p}\cdot\vec{r})} \;, t' = -t\end{aligned}$$

A change of time variable as $t' = -t$ i the second term can be interpreted as a state which moves backward in time. In this context, this state corresponds to the same particle represented by the first term but with a difference that it moves back in time. Such states are called "Antiparticles", which are, indeed, regular particles but moving back in time.

Klein-Gordon (Scalar) Field - Antiparticles

Feynman-Stuckelberg Interpretation



When a valance electron from atoms of a metal (generically shown in left above) is removed, it leaves a hole. Then, the closest valance electron jumps into this hole to fill it. This is continued by the other valance electrons, and as a result the hole moves in the opposite direction of the electrons' motion. One observes it as a positive charged particle moves in the direction of the hole. Feynman-Stuckelberg interpretation of antiparticles is based on this observation. When a regular particle moves back in time, it leaves a hole in our universe, and this hole moves forward in time as we do. We observe this hole as a particle with opposite electric charge. In this context, for instance, a positron (the antiparticle of electron) is also an electron moving back in time and leaving a hole in our universe.

Homework Assignments

HW 5: Using the commutation relation of the ladder operators:
$[a_p, a^\dagger_{p'}] = (2\pi)^{3/2}\delta^{(3)}(x - y)$
$[a_p, a_{p'}] = [a^\dagger_p, a^\dagger_{p'}] = 0$
calculate the commutation relation between $\phi(x)$ and its conjugate momentum $\pi(y)$: $[\phi(x), \pi(y)] \stackrel{?}{=} i\delta^{(3)}(x - y)$