Lecture 3

Elements of Cosmology

The Universe around us

On average, the geometry seems to be very simple:

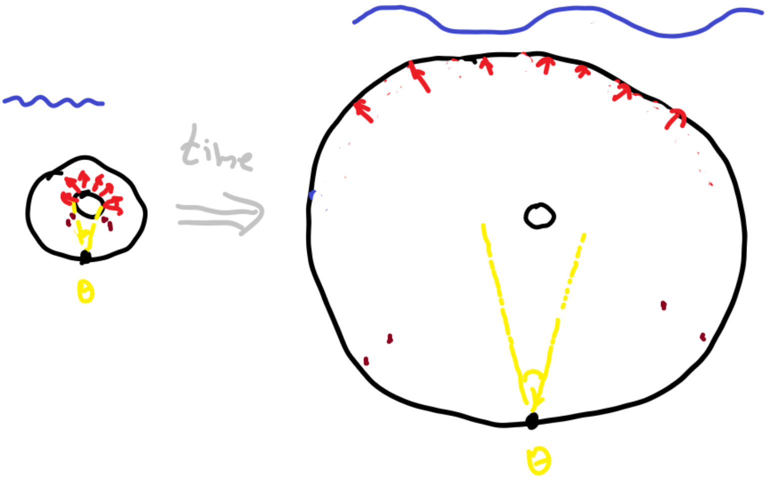
$$ds^2 = dt^2 - a^2(t) \cdot \left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

with the scale factor $a(\tau)$,
and the so-called Hubble "constant" $H = \frac{\dot a}{a}$ for the expansion rate.

Between emission (e) and absorption (a),
the wavelengths get larger. Let's define the red shift

$$z \equiv \frac{\Delta\lambda}{\lambda_e} = \frac{\lambda_a - \lambda_e}{\lambda_e} = \frac{a(t_a)}{a(t_e)} - 1.$$

$\lambda \propto a(\tau)$ $\nu \propto \frac{1}{a(\tau)}$



$$d_A(1+z) = d = \frac{d_L}{1+z}$$

How do they measure cosmological distances?
It's a hard job. There is a ladder of distances.
The main rungs: Parallax – Cepheids – Supernovae Ia

Roughly, the contents of the Universe are:
5% ordinary matter
25% Dark Matter
70% Dark Energy
in terms of the energy density.

Dark Matter should be cold, i.e. no pressure (CDM).
Dark Energy looks pretty much like a cosmological constant (Λ).
Hence, the ΛCDM model.

Friedmann equations

The only non-vanishing connection coefficients are

$$\Gamma^i_{0k} = \Gamma^i_{k0} = \frac{\dot{a}}{a}\delta_{ik} \equiv H\delta_{ik}, \qquad \Gamma^0_{ik} = a\dot{a}\delta_{ik} = a^2 H\delta_{ik}.$$

The calculation of the Ricci tensor yields

$$R_{00} = -3\frac{\ddot{a}}{a} = -3\dot{H} - 3H^2, \qquad R_{0i} = 0,$$

$$R_{ik} = \left(a\ddot{a} + 2\dot{a}^2\right)\delta_{ik} = a^2\left(\dot{H} + 3H^2\right)\delta_{ik}.$$

It then gives the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = -6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = -6\left(\dot{H} + 2H^2\right).$$

We must choose the matter distribution compatible with the symmetries of the model.
And therefore, we take $T^0_0 = \rho(t)$, the energy density, and $T^i_k = -p(t)\delta_{ik}$ where p is the pressure.
This is an homogeneous ideal fluid at rest.

Finally, the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi\mathcal{G} T_{\mu\nu}$$

for G^0_0 and G^i_k reduce to

$$3H^2 = 8\pi\mathcal{G}\rho$$

and

$$2\dot{H} + 3H^2 = -8\pi\mathcal{G}p$$

respectively. These are the Friedmann equations.

$$3H^2 = 8\pi\mathcal{G}\rho, \qquad 2\dot{H} + 3H^2 = -8\pi\mathcal{G}p, \qquad H \equiv \frac{\dot{a}}{a}$$

If $p = -\rho$, we get $\dot{H} = 0$, hence $a(t) \propto e^{Ht}$.
This is the de Sitter space, with a cosmological constant.

Otherwise, let's take $p = w\rho$, with a constant $w > -1$.
Then we have $2\dot{H} + 3H^2 = -3wH^2$ or

$$\frac{d}{dt}\left(\frac{1}{H}\right) = \frac{3}{2}(w+1)$$

and therefore

$$H = \frac{1}{\frac{3}{2}(w+1)t + \mathrm{const}}, \quad \text{hence} \quad a(t) = A \cdot (t - t_0)^{\frac{2}{3(w+1)}}.$$

$$a(t) =\propto t^{\frac{2}{3(w+1)}}, \qquad H = \frac{1}{\frac{3}{2}(w+1)t}$$

Deceleration for $w > -\frac{1}{3}$

Expansion regimes

Radiation domination, $w = \frac{1}{3}$.

$$a(t) \propto \sqrt{t}, \qquad \rho \propto H^2 \propto \frac{1}{t^2} \propto \frac{1}{a^4}$$

Matter domination, $w = 0$.

$$a(t) \propto t^{\frac{2}{3}}, \qquad \rho \propto H^2 \propto \frac{1}{t^2} \propto \frac{1}{a^3}$$

Dark Energy, tends to $w \approx -1$ and constant ρ.

The puzzle of cosmic history

After the initial singularity of hot Big Bang,
the Universe expands and cools down.

At about $z = 1100$, recombination occured
leaving us with cosmic microwave background (CMB) radiation
which propagates then almost freely.

Dark Energy comes to domination only now,
most of the time the expansion used to be decelerated.
The emitters separated by a bit more than an angular degree
could not have exchanged any signal before emission!

However, the spectrum of CMB is pretty much Planck,
with almost the same temperature from all directions.
The dipole anisotropy $\frac{\delta T}{T} \sim 10^{-3}$ is due to our motion,
while the intrinsic perturbations $\frac{\delta T}{T} \sim 10^{-5}$. Why???

Inflation

The idea is that, in the initial stage, the Universe had undergone an accelerated, almost exponential expansion of at least e^{60} times, then occurs a "graceful exit": transition to the decelerated regime of initial temperature at around the Grand Unified scale.

It would be enough (with some wishful thinking)
to explain the thermal equilibrium of different emitters in the sky.

For that to happen, we need a large negative pressure, with the equation of state parameter just slightly more than $w = -1$.

Can we do that?

Let's go for a canonical scalar field

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \right).$$

The energy-momentum tensor is obviously

$$T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - g_{\mu\nu} \cdot \left(\frac{1}{2} (\partial_\alpha \phi)(\partial^\alpha \phi) - V(\phi) \right).$$

In the cosmological regime, we need $\phi = \phi(t)$ which yields

$$\rho = \frac{1}{2}\dot{\phi}^2 + V \equiv \mathcal{K} + \Pi, \qquad p = \frac{1}{2}\dot{\phi}^2 - V \equiv \mathcal{K} - \Pi$$

in terms of kinetic and potential energies.

Having got

$$w = \frac{\mathcal{K} - \Pi}{\mathcal{K} + \Pi},$$

can we make the potential energy much larger than the kinetic one?

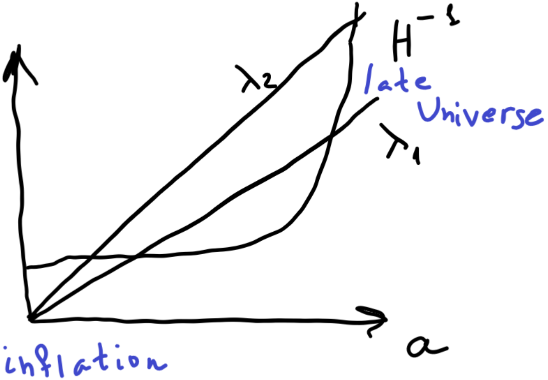
Yes! The equation of motion is

$$\Box\varphi + V'(\varphi) = 0 \qquad \text{with} \qquad \Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right)$$

In the cosmological regime $\phi = \phi(t)$ we get the so-called Hubble friction:

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0.$$

The problems are solved,
and as a bonus, we get possible quantum origin of fluctuations.



The picture is very nice and beautiful.
But... troubles such as "Hubble tension" appear in the last years...

Conclusions

Differential geometry is a very interesting and exciting field of mathematics.

Gravity is described not like a gauge field in a fixed spacetime, but rather as differential geometry of the spacetime itself.

General Relativity is a very exciting and beautiful theory, but not without its own problems and issues.

GR has made cosmology possible as a physical science.

In cosmology, one can find relation to almost every field of physics.

Nowadays, there are many new interesting things to come!