Quantum Field Theory - Part 3

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Classical Mechanics → QFT

Quantum Mechanics → QFT

Special Relativity → QFT

Classical Field Theory → QFT

Dirac (Fermionic) Field

The problem with the negative probability density can be handled with the antiparticle interpretation, and furthermore it may solve the causality problem which yields non-zero propagation when $x^2 > t^2$. The probability of a particle propagation forward in time is equal to propagation backward in time (antiparticle propagation).
All these problems somehow also related to the negative energy states and the second order time derivative terms in Klein-Gordon equation. In this case, one can ask if it is possible to derive an equation for the relativistic quantum field theory which involves first order time derivatives, but a naive approach as the following

$$E = \pm\sqrt{p^2 + m^2} \xrightarrow{?} \begin{cases} \dfrac{-i\partial}{\partial t}\psi & = \pm(\sqrt{-\vec{\nabla}^2 + m^2})\psi \\ & \simeq \pm m \left[1 - \dfrac{-\vec{\nabla}^2}{m^2} + \dfrac{1}{2!}\left(\dfrac{-\vec{\nabla}}{m^2}\right)^2 + \cdots\right]\psi \end{cases}$$

ends up with infinite series!

Dirac (Fermionic) Field

Instead of taking the square root directly, let us consider the following ansatz:

$$E^2 - \vec{p}^2 - m^2 \longrightarrow P^\mu P_\mu - m^2 \equiv (\alpha^\mu P_\mu + \beta m)(\alpha^\nu P_\nu + \beta m)$$

where α^μ is a 4-vector and β is a scalar parameter which are not known at the moment. These parameters can be found by simply taking the square of the expression in right hand side; but please note that, since we are dealing with the operators, we do not assume these parameters commute each other; thus, the order in their multiplication does matter.

$$\begin{aligned} \alpha^\mu \alpha^\nu &\neq \alpha^\nu \alpha^\mu, \quad \text{if } \mu \neq \nu \\ \alpha^\mu \beta &\neq \beta \alpha^\mu \end{aligned}$$

Dirac (Fermionic) Field

$$
\begin{aligned}
P^\mu P_\mu - m^2 &\equiv (\alpha^\mu P_\mu + \beta m)(\alpha^\nu P_\nu + \beta m) \\
&= \alpha^\mu \alpha^\nu P_\mu P_\nu + (\alpha^\mu \beta P_\mu + \beta \alpha^\nu P_\nu) m + \beta^2 m^2
\end{aligned}
$$

Here it is clear that $\beta^2 = -1$ (Do not jump on the conclusion that $\beta = i$). Let us consider the other terms one by one. The first term by Einstein convention

$$
\begin{aligned}
\alpha^\mu \alpha^\nu P_\mu P_\nu &= \sum_{\mu=0}^{3} \alpha^\mu P_\mu \sum_{\nu=0}^{3} \alpha^\nu P_\nu \\
&= \sum_{\mu=\nu=0}^{3} (\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu) P_\mu P_\nu \\
&+ \sum_{\mu \neq \nu=0}^{3} (\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu) P_\mu P_\nu \\
&= P_\mu P^\mu
\end{aligned}
$$

$$\{\alpha^\mu, \alpha^\nu\} = (\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu)$$

$$P_\mu P^\mu = P^\mu P_\mu = \eta^{\mu\nu} P_\mu P_\nu$$

$$\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

Dirac (Fermionic) Field

Comparing the terms with those given in the right hand side:

$$\{\alpha^\mu, \alpha^\nu\} = \begin{cases} 2, & \text{if } \mu = \nu = 0 \\ -2, & \text{if } \mu = \nu = 1, 2, 3 \\ 0, & \text{if } \mu \neq \nu \end{cases}$$

Combining all together leads the following form for the anti-commutation among the components of α^μ:

$$\{\alpha^\mu, \alpha^\nu\} = 2\eta^{\mu\nu}$$

Dirac (Fermionic) Field

Similarly the second term can be manipulated as follows:

$$\beta\alpha^{\nu}P_{\nu} = \beta\alpha^{\mu}P_{\mu} \Rightarrow (\alpha^{\mu}\beta P_{\mu} + \beta\alpha^{\nu}P_{\nu})m = m(\alpha^{\mu}\beta + \beta\alpha^{\mu})P_{\mu} = 0$$

$$\{\alpha^{\mu}, \beta\} = 0$$

$$\begin{aligned} P^{\mu}P_{\mu} - m^2 \quad &= (\alpha^{\mu}P_{\mu} + \beta m)(\alpha^{\nu}P_{\nu} + \beta m) \quad = 0 \\ &\Rightarrow \alpha^{\mu}P_{\mu} + \beta m = 0 \\ &\beta\alpha^{\mu}P_{\mu} + \beta^2 m = 0 \\ &\gamma^{\mu}P_{\mu} - m^2 = 0 \\ &\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \end{aligned}$$

Dirac (Fermionic) Field

Replacing the momentum with its correspondence operator as $P_\mu \longrightarrow i\partial_\mu$ and multiply the whole equation with a field Ψ yield the Dirac Equation:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0$$

What are these γs?

Clearly they are not regular numbers, since they do not commute each other. The first alternative to the regular numbers are matrices, since we already know the matrices do not have to commute under multiplication. If they are matrices, then what are their size? If we check 2×2 matrices, we can write only three matrices which are totally anti-commute each other (**What are they?**), but we need four of them. The next step is to try 4×4 matrices.

Dirac (Fermionic) Field

Dirac Picture:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 : \quad \gamma^0 = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \sigma^i & 0_{2\times 2} \\ 0_{2\times 2} & \sigma^i \end{pmatrix}$$

Weyl Picture:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 : \quad \gamma^0 = \begin{pmatrix} 0_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0_{2\times 2} & \sigma^i \\ -\sigma^i & 0_{2\times 2} \end{pmatrix}$$

In conclusion, Dirac equation is a matrix equation, thus its solutions are not simply scalar functions, but they are "*SPINORS*"

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \equiv \begin{pmatrix} \psi_L \\ \\ \psi_R \end{pmatrix} \quad ; \quad \psi_R = i\sigma_2 \psi_L^*$$

Dirac (Fermionic) Field - Solution for Free Field

Recall that the Dirac equation refers to a set of equations (or to a matrix equation), and since we have been dealing with the free Dirac field, the solutions are expected to have a plane wave form:

$$\Psi(p) = e^{-P^{\mu}x_{\mu}}u(p) \longrightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = e^{-P^{\mu}x_{\mu}}u(p) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

which yields 4 independent solutions for $u(p)$.

$$(i\gamma^{\mu}\partial_{\mu} - m)u(x) = 0 \xrightarrow{\text{Fourier Transformation}} (\gamma^{\mu}p_{\mu} - m)u(p) = 0$$

Dirac (Fermionic) Field - Solution for Free Field
Rest Frame ($\vec{\mathbf{p}} = 0$)

$$p^\mu x_\mu = p^0 x^0 - \vec{p} \cdot \vec{x} = E.t \Rightarrow \Psi(p) = e^{-iEt} u(p) \; ; \quad P^\mu = (E, 0)$$

$$\begin{aligned} (\gamma^\mu P_\mu - m) u(p) &= (\gamma^0 E - m) u(p) = 0 \\ \gamma^0 E u(p) &= m u(p) \end{aligned}$$

Considering $E^2 = m^2$ in the rest frame, the solutions can be obtained straightforwardly as follows:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Dirac (Fermionic) Field - Solution for Free Field Lab Frame ($\vec{\mathbf{p}} \neq 0$)

$$\tilde{u}(p) \equiv \begin{pmatrix} \tilde{u}_A \\ \tilde{u}_B \end{pmatrix}$$

$$(\gamma^\mu P_\mu - \mathcal{I}_{4\times 4} m)u(p) = \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix} \begin{pmatrix} \tilde{u}_A \\ \tilde{u}_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} (E-m)\tilde{u}_A - \vec{\sigma}\cdot\vec{p}\tilde{u}_B &= 0 \\ \vec{\sigma}\cdot\vec{p}\tilde{u}_A - (E-m)\tilde{u}_B &= 0 \end{aligned}$$

where,

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}$$

Dirac (Fermionic) Field - Solution for Free Field
Lab Frame ($\vec{\mathbf{p}} \neq 0$)

$$\tilde{u}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_1 + ip_2}{E+m} \end{pmatrix}, \quad \tilde{u}_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_1 - ip_2}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix}, \quad \tilde{u}_3 = \begin{pmatrix} \frac{p_3}{E+m} \\ \frac{p_1 + ip_2}{E-m} \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{u}_4 = \begin{pmatrix} \frac{p_1 - ip_2}{E+m} \\ \frac{-p_3}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Dirac (Fermionic) Field - Solution for Free Field Lab Frame ($\vec{\mathbf{p}} \neq 0$)

All these solutions obey $E^2 = \vec{p}^2 + m^2$; and thus one can expect two of them lead to the cases of positive energy, while the other two to those with negative energy. If these solutions are substituted into the Dirac equation, it can be seen that $\tilde{u}_1$ and $\tilde{u}_2$ correspond to the positive energy cases with $E = \sqrt{\vec{p}^2 + m^2}$, while $\tilde{u}_3$ and $\tilde{u}_4$ to the negative energy cases as $E = -\sqrt{\vec{p}^2 + m^2}$.
In the case of Dirac field, the following solutions

$$\begin{aligned} v_1(E, \vec{p}) &= \tilde{u}_4(-E, -\vec{p}) \\ v_2(E, \vec{p}) &= \tilde{u}_3(-E, -\vec{p}) \end{aligned}$$

represent the anti-particles with positive energies.

Dirac (Fermionic) Field - Spin and Helicity

We now find two solutions for each of particle and antiparticle. These two solutions for each case, indeed, also represent the spin states of the fermions. In our notation, for a fermion, whose momentum is along the $z-$axis, $\psi = u^1(p) = u^1(p)e^{-ipx}$ represents a spin-up fermion, while $\psi = u^1(p)e^{-ipx}$ describes a spin-down fermion. Similarly, $\psi = v^1(p)e^{-ipx}$ corresponds to a spin-up anti-fermion, while $\psi = v^2(p)e^{-ipx}$ means the spin-down anti-fermion. These are just the spin eigenstates of the spin-operator S_z. In application to high energy physics, the most fundamental quantity is helicity, which is defined as

$$\hat{h}\psi = \frac{\vec{S} \cdot \vec{p}}{|\vec{S}||\vec{p}|}\psi = \pm\psi$$

The spinors can correspond to two eigenvalues of $\hat{h}$ shown as ± 1, depending on the direction of their momenta. The states with $h = 1$ are called **right-handed** and $h = -1$ corresponds to the **left-handed** fermions. The helicity becomes crucial especially when one deals with the weak interactions, neutrinos etc.

Dirac (Fermionic) Field - Homework Assignment

HW 6: Find the solutions of the free Dirax field by applying a generalized boost to the rest-frame solutions. The boost generator is given as

$$\text{Boosts}: \quad S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

and under a boost,

$$\Psi \longrightarrow \Psi' = \mathcal{S}\Psi = \begin{pmatrix} e^{\frac{\sigma^i}{2}\phi} & 0 \\ 0 & e^{\frac{-\sigma^i}{2}\phi} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

HW 7: Calculate the probabilty density for $(i\gamma^\mu \partial_\mu - m)\Psi = 0$ and show that it is always positive (Hint: Use the Dirac conjugation $\bar{\psi}$, please see the next slide).

Dirac (Fermionic) Field - Lagrangian

To obtain a convenient Lagrangian for the Dirac field, which yields Dirac equation, one must form possible scalar combinations of the Dirac spinors. In the four dimentional spacetime, the jenerators for the boosts and rotations of the spinors are given as

$$\text{Boosts}: \quad S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$\text{Rotations}: \quad S^{0i} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$\bar{\psi}\psi$:	Scalar	$\bar{\psi} = \psi^\dagger\gamma^0$: Dirac conjugation
$\bar{\psi}\gamma^\mu\psi$:	Vector	
$\bar{\psi}\sigma^{\mu\nu}\psi$:	2^{nd} rank tensor	$\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$
$\bar{\psi}\gamma^\mu\gamma^5\psi$:	Axial Vector	
$\bar{\psi}\gamma^5\psi$:	Pseudo Scalar	

Dirac (Fermionic) Field - Lagrangian and Hamiltonian

Since the Lagrangian is a scalar in the relativistic regime, all the terms in it should also be Lagrangian. Thus, one needs to pick up a scalar bi-linear form from the list given above:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

The conjugate momentum of ψ is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \gamma^0 \equiv i\bar{\psi}$$

and the Hamiltonian is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi$$

The solutions with $\tilde{u}^{1,2}(p)$ happen to be eigenstates of this hamiltonian with positive energy E_p, and those with $v^{1,2}(p)$ represent the eigenstates with the negative energy $-E_p$.

Dirac (Fermionic) Field - Lagrangian and Hamiltonian

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_{s=1}^{2} \left[a_p^s u^s(p) e^{-ipx} + b_p^s v^s(p) e^{ipx} \right]$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_{s=1}^{2} \left[a_p^{s\dagger} \bar{u}^s(p) e^{ipx} + b_p^{s\dagger} \bar{v}^s(p) e^{-ipx} \right]$$

Substituting the Fouriers transforms of the spinor into the Hamiltonian leads to

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s \left(E_p a_p^{s\dagger} a_p^s - E_p b^{s\dagger} b_s \right)$$

Now we can interpret $a^\dagger$ and a to be the creation and annihilation operators for the fermions, respectively, while $b^\dagger$ and b are the creation and annihilation operators for the anti-fermions.

Dirac (Fermionic) Field - Lagrangian and Hamiltonian

The annihilation operators give zero when they are applied to the vacuum state as is happen in the other cases (scalar fiels, harmonic oscillator, zero-particle states etc.). On the other hand, fermions exhibits a different attitude. They follow the Fermi-Dirac statistics which is based on the fact that two fermions cannot occupy the same state. It can be expressed in terms of the creation/annihilation operators as

$$\begin{aligned} a_p^\dagger a_p^\dagger |0> &= 0 \\ b_p^\dagger b_p^\dagger |0> &= 0 \end{aligned}$$

On the other hand, if one imposes commutation relations in quantizing the Dirac field, as in the case of scalar field, fermions with the same quantum states can be created more than ones, which clearly contradicts with Fermi-Dirac statistics. In this case, a proper quantization for the Dirac fields arises from the anti-commutation relations given as:

$$\begin{aligned} \{a_p^s, a_q^{r\dagger}\} &= \{b_p^s, b_q^{r\dagger}\} \qquad = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\delta^{rs} \\ \{\psi_a(x), \bar{\psi}_b(y)\} &= \delta^{(3)}(\vec{x} - \vec{y})\delta_{ab} \end{aligned}$$