Alexandria Quantum Computing Group (AleQCG)

مركز التميز في الحاسبات الكمية

CENTER OF EXCELLENCE FOR QUANTUM COMPUTERS

Basics of Quantum Computing
Day 3

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2nd ArPS summer School on Advanced Physics

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Zewail City of Science and Technology

shaaban Khalil

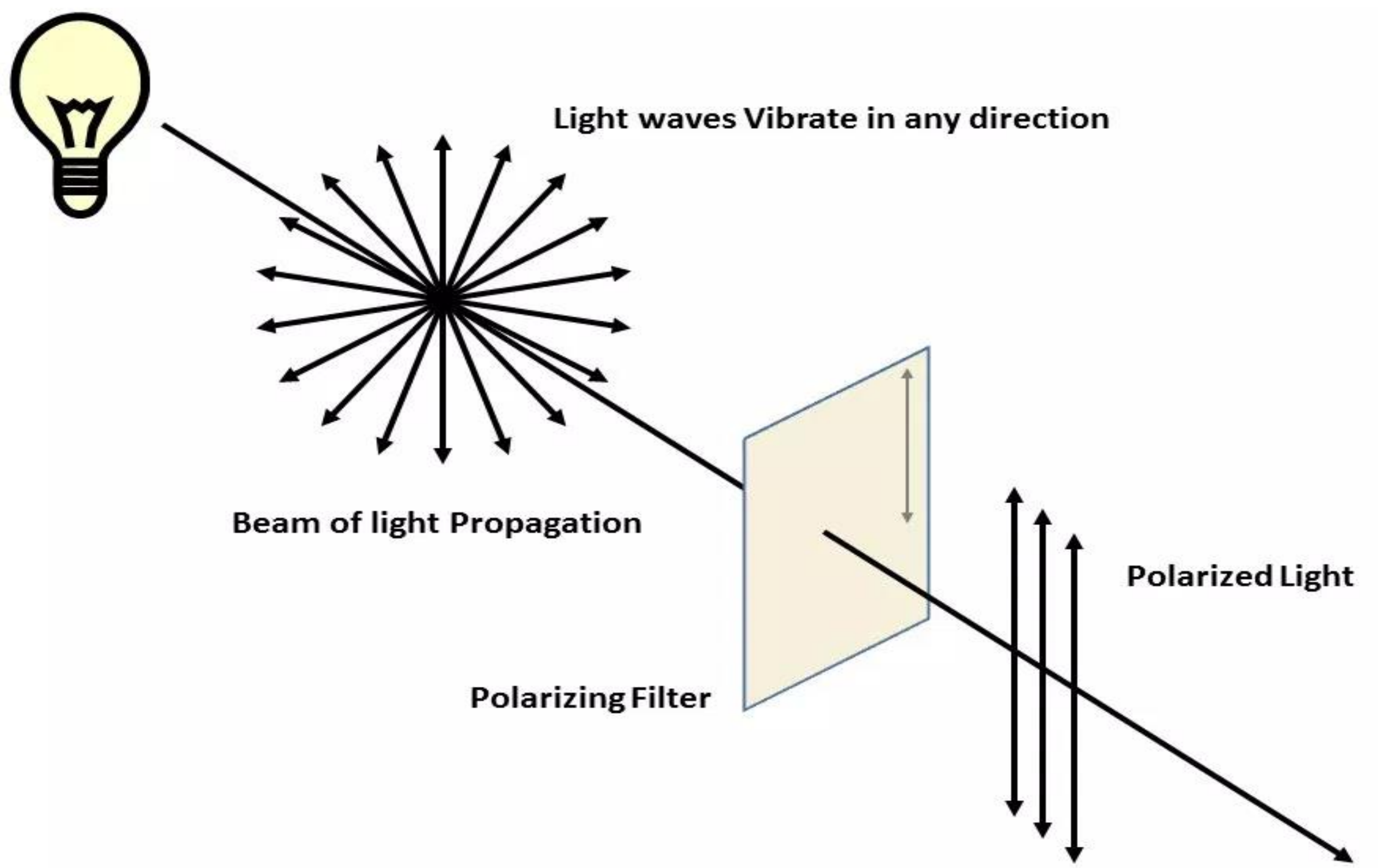
Description

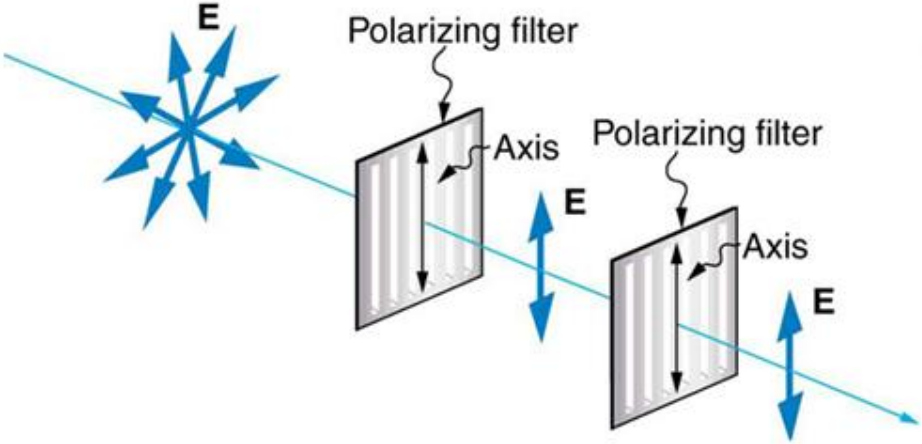
Quantum Communication Protocols

Outline

- **Light Polarization**
- **Quantum Dense Coding**
- **Quantum Teleportation**

Light Polarization

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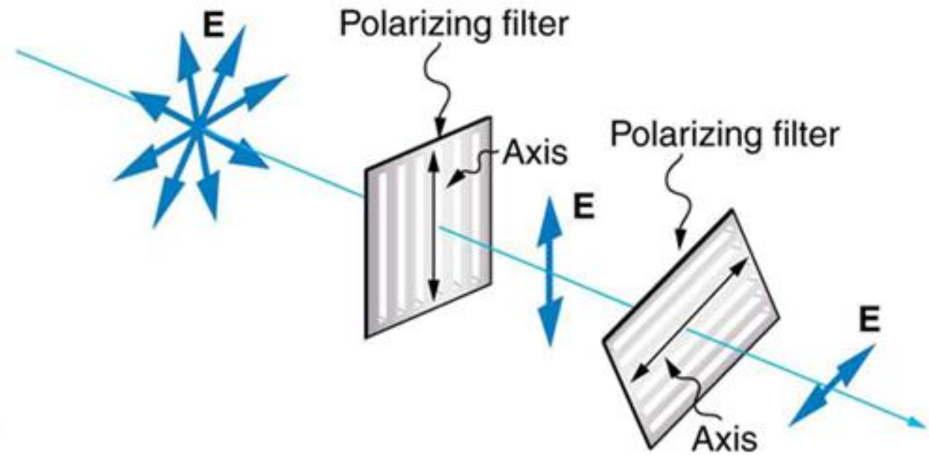
(a)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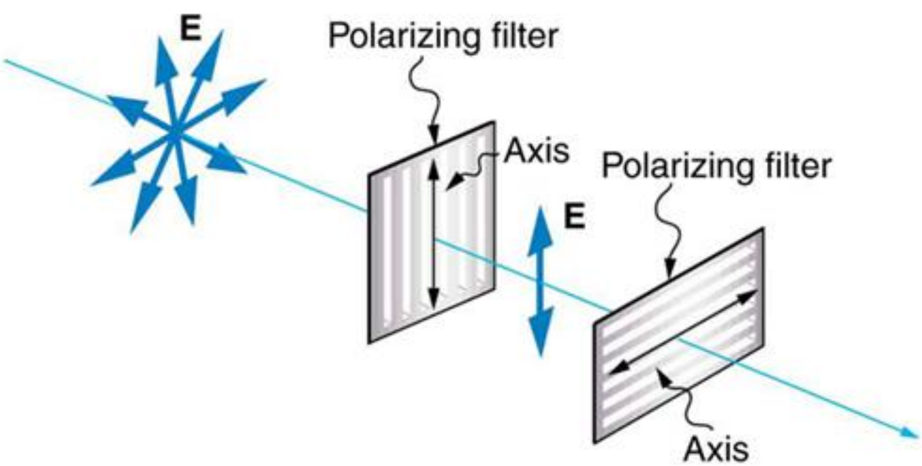
(b)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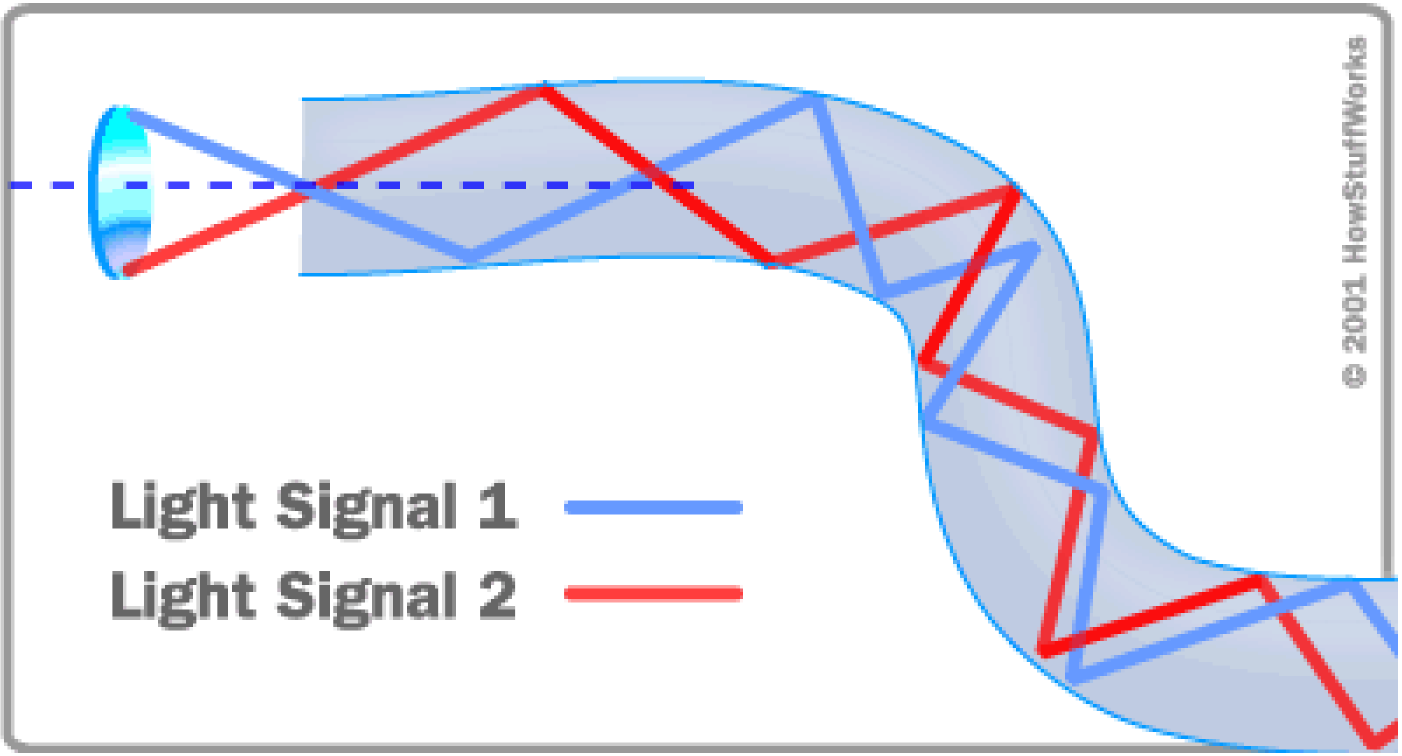
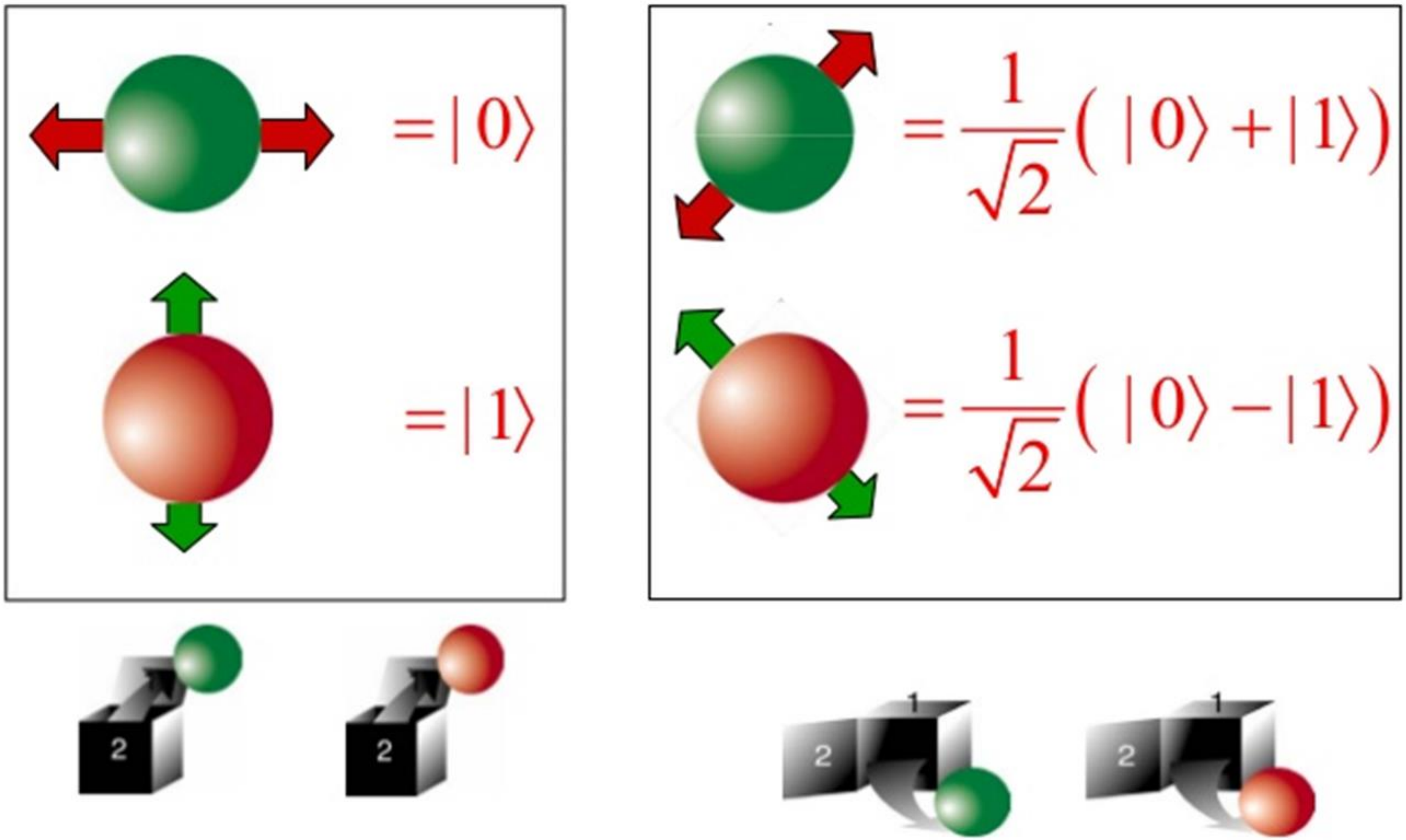
(c)

(d)

Optical Fiber

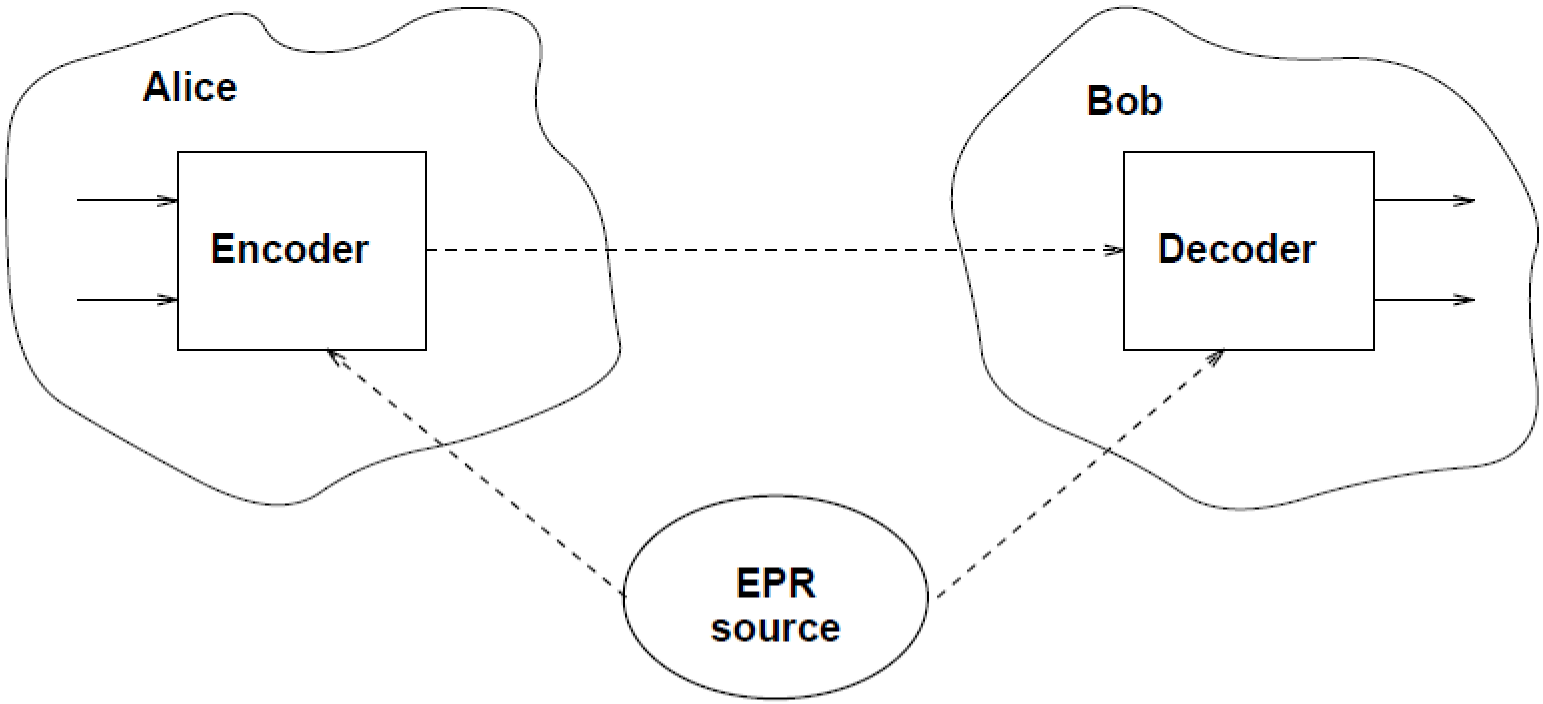


Photon polarization as a qubit

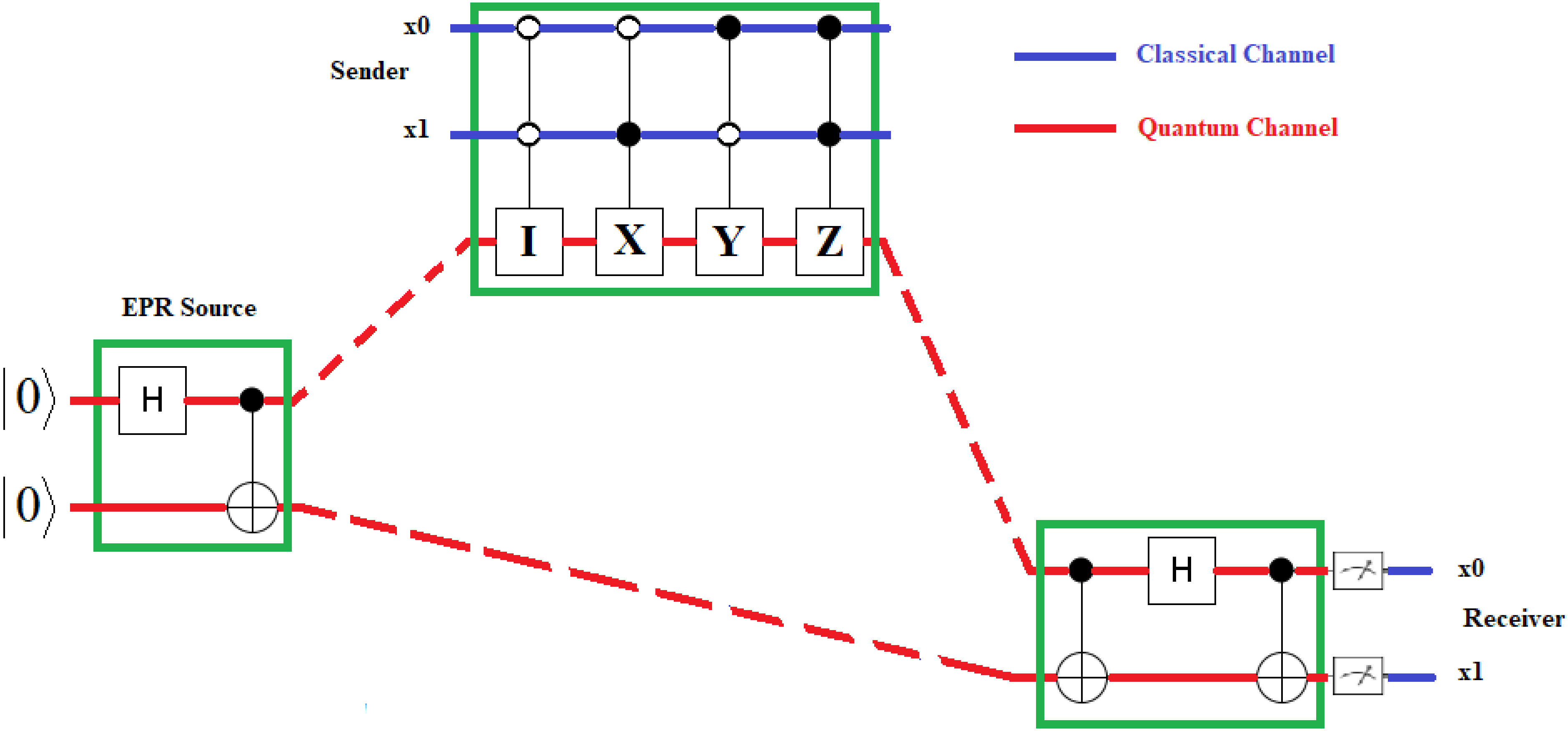


Quantum Dense Coding -1

Is used to send two classical bits using a single quantum channel.



Quantum Dense Coding - 2



Sender

Alice. Alice receives two classical bits, encoding the numbers 0 through 3. Depending on this number Alice performs one of the transformations $\{I, X, Y, Z\}$ on her qubit of the entangled pair ψ_0. Transforming just one bit of an entangled pair means performing the identity transformation on the other bit. The resulting state is shown in the table.

Value	Transformation	New state
0	$\psi_0 = (I \otimes I)\psi_0$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
1	$\psi_1 = (X \otimes I)\psi_0$	$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle)$
2	$\psi_2 = (Y \otimes I)\psi_0$	$\frac{1}{\sqrt{2}}(-\|10\rangle + \|01\rangle)$
3	$\psi_3 = (Z \otimes I)\psi_0$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$

Alice then sends her qubit to Bob.

Receiver

Bob. Bob applies a controlled-NOT to the two qubits of the entangled pair.

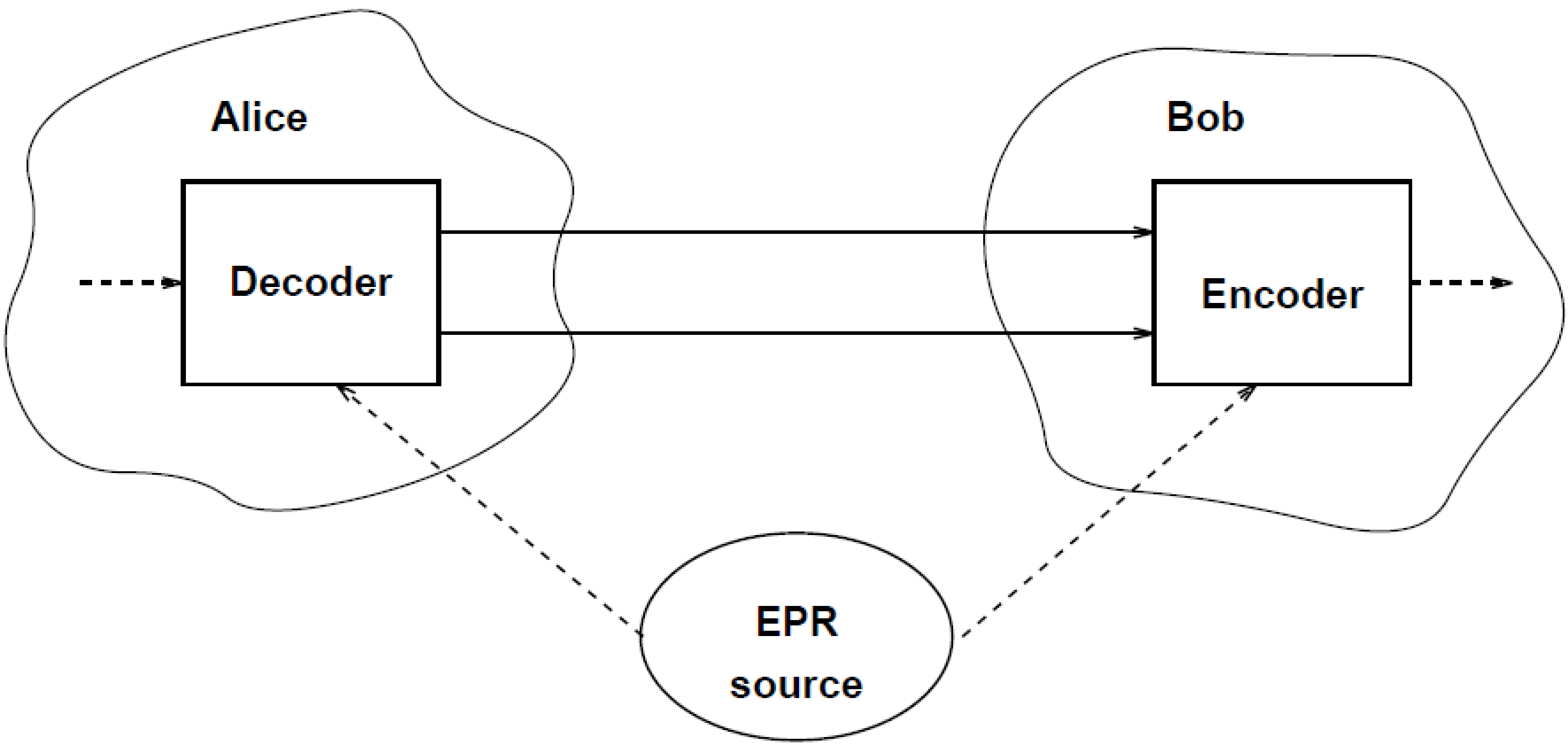
Initial state	Controlled-NOT	First bit	Second bit
$\psi_0 = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 10\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$	$\lvert 0\rangle$
$\psi_1 = \frac{1}{\sqrt{2}}(\lvert 10\rangle + \lvert 01\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 11\rangle + \lvert 01\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 0\rangle)$	$\lvert 1\rangle$
$\psi_2 = \frac{1}{\sqrt{2}}(-\lvert 10\rangle + \lvert 01\rangle)$	$\frac{1}{\sqrt{2}}(-\lvert 11\rangle + \lvert 01\rangle)$	$\frac{1}{\sqrt{2}}(-\lvert 1\rangle + \lvert 0\rangle)$	$\lvert 1\rangle$
$\psi_3 = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 10\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$	$\lvert 0\rangle$

Bob now applies H to the first bit:

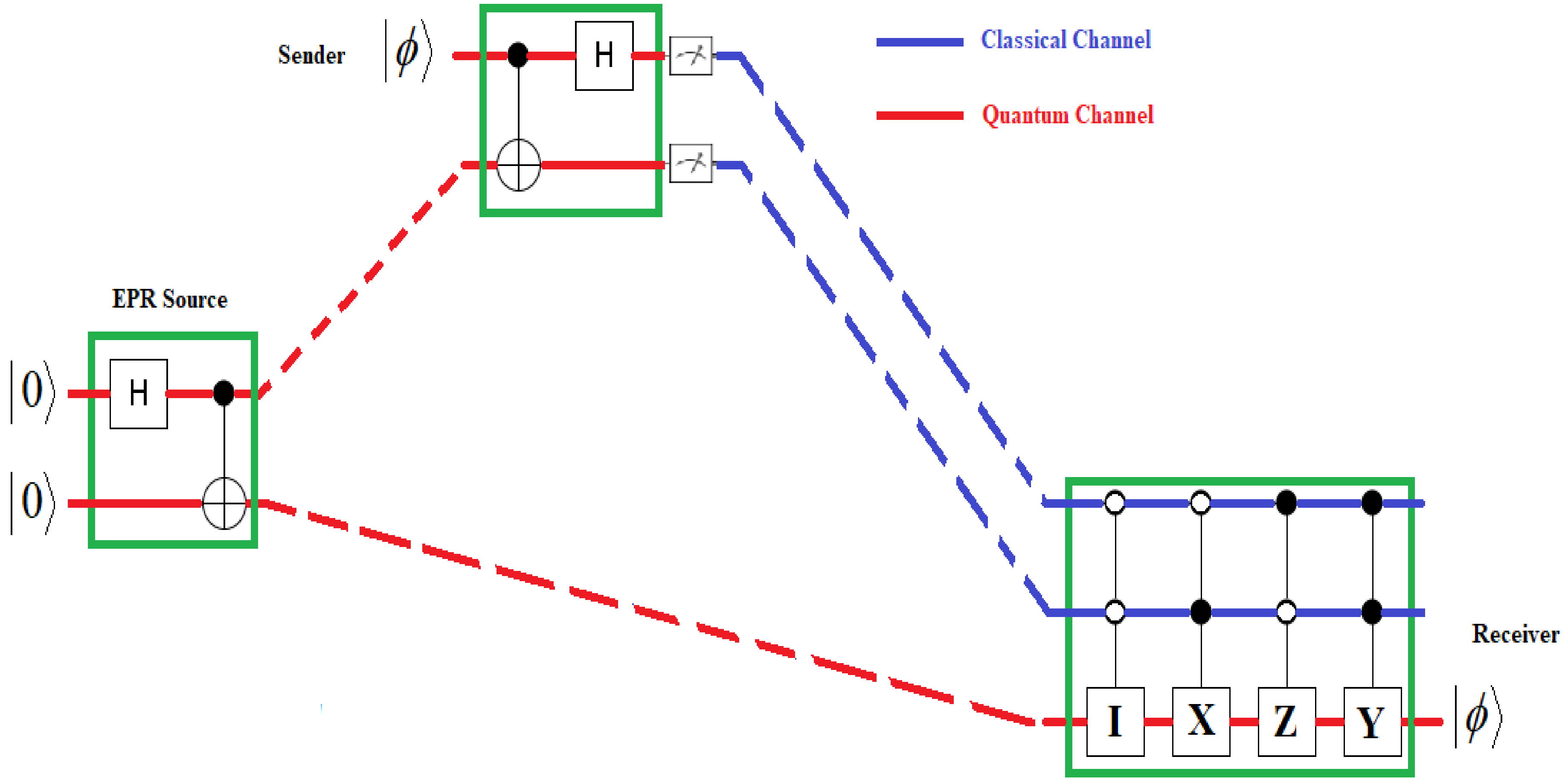
Initial state	First bit	H(First bit)
ψ_0	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$	$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle) + \frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)\right) = \lvert 0\rangle$
ψ_1	$\frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 0\rangle)$	$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle) + \frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)\right) = \lvert 0\rangle$
ψ_2	$\frac{1}{\sqrt{2}}(-\lvert 1\rangle + \lvert 0\rangle)$	$\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle) + \frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)\right) = \lvert 1\rangle$
ψ_3	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$	$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle) - \frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)\right) = \lvert 1\rangle$

Quantum Teleportation - 1

- Is used to teleport an unknown quantum state using two classical channels.



Quantum Teleportation -2



Quantum Teleportation - 3

Alice. Alice has a qubit whose state she doesn't know. She wants to send the state of ths qubit

$$\phi = a|0\rangle + b|1\rangle$$

to Bob through classical channels. As with dense coding, Alice and Bob each possess one qubit of an entangled pair

$$\psi_0 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Alice applies the decoding step of dense coding to the qubit ϕ to be transmitted and her half of the entangled pair. The starting state is quantum state

$$\begin{aligned}\phi \otimes \psi_0 &= \frac{1}{\sqrt{2}}(a|0\rangle \otimes (|00\rangle + |11\rangle) + b|1\rangle \otimes (|00\rangle + |11\rangle)) \\ &= \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle),\end{aligned}$$

of which Alice controls the first two bits and Bob controls the last one. Alice now applies $C_{not} \otimes I$ and $H \otimes I \otimes I$ to this state:

$$\begin{aligned}&(H \otimes I \otimes I)(C_{not} \otimes I)(\phi \otimes \psi_0) \\ &= (H \otimes I \otimes I)(C_{not} \otimes I)\frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle) \\ &= (H \otimes I \otimes I)\frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle) \\ &= \frac{1}{2}(a(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + b(|010\rangle + |001\rangle - |110\rangle - |101\rangle)) \\ &= \frac{1}{2}(|00\rangle(a|0\rangle + b|1\rangle) + |01\rangle(a|1\rangle + b|0\rangle) + |10\rangle(a|0\rangle - b|1\rangle) + |11\rangle(a|1\rangle - b|0\rangle))\end{aligned}$$

Alice measures the first two qubits to get one of $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$ with equal probability. Depending on the result of the measurement, the quantum state of Bob's qubit is projected to $a|0\rangle + b|1\rangle$, $a|1\rangle + b|0\rangle$, $a|0\rangle - b|1\rangle$, or $a|1\rangle - b|0\rangle$ respectively. Alice sends the result of her measurement as two classical bits to Bob.

Note that when she measured it, Alice irretrievably altered the state of her original qubit ϕ, whose state she is in the process of sending to Bob. This loss of the original state is the reason teleportation does not violate the no cloning principle.

Bob. When Bob receives the two classical bits from Alice he knows how the state of his half of the entangled pair compares to the original state of Alice's qubit.

bits received	state	decoding
00	$a\|0\rangle + b\|1\rangle$	I
01	$a\|1\rangle + b\|0\rangle$	X
10	$a\|0\rangle - b\|1\rangle$	Z
11	$a\|1\rangle - b\|0\rangle$	Y

Quantum Algorithms

Outline

- Quantum Parallelism
- Superposition Preparation
- Parallel Evaluation of a function
- Marking Solutions
- Grover’s Quantum Search Algorithm

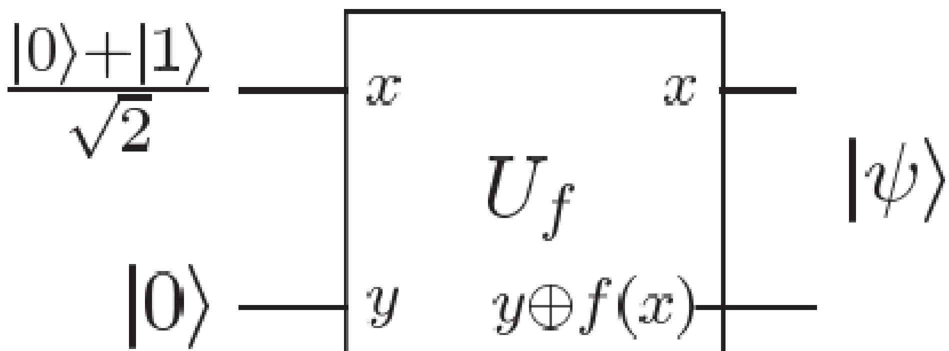
Quantum Algorithms

- What class of computations can be performed using quantum circuits?
- How does that class compare with the computations which can be performed using classical logical circuits?
- Can we find a task which a quantum computer may perform better than a classical computer?

Quantum Parallelism

- *Quantum parallelism* is a fundamental feature of many quantum algorithms.

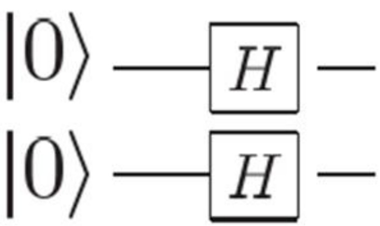
- Allows quantum computers to evaluate a function *f*(*x*) for many *different* values of *x* simultaneously.



$$\frac{|0, f(0)\rangle + |1, f(1)\rangle}{\sqrt{2}}.$$

Superposition Preparation

- This procedure can easily be generalized to functions on an arbitrary number of bits, by using a general operation known as the *Hadamard transform*, or sometimes the *Walsh– Hadamard transform*.
- *For n* = 2, $H^{\otimes 2}|00\rangle$

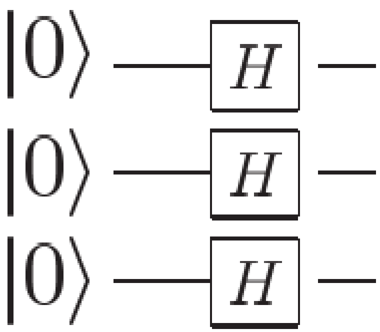


$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Superposition Preparation

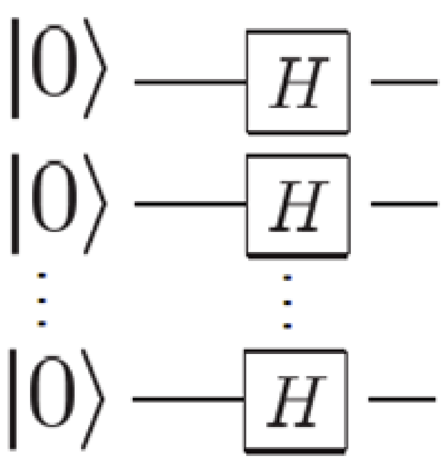
- *For n* = 3, $H^{\otimes 3}|000\rangle$



$$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$$

$$=\frac{|000\rangle+|001\rangle+|010\rangle+|011\rangle+|100\rangle+|101\rangle+|110\rangle+|111\rangle}{2\sqrt{2}}$$

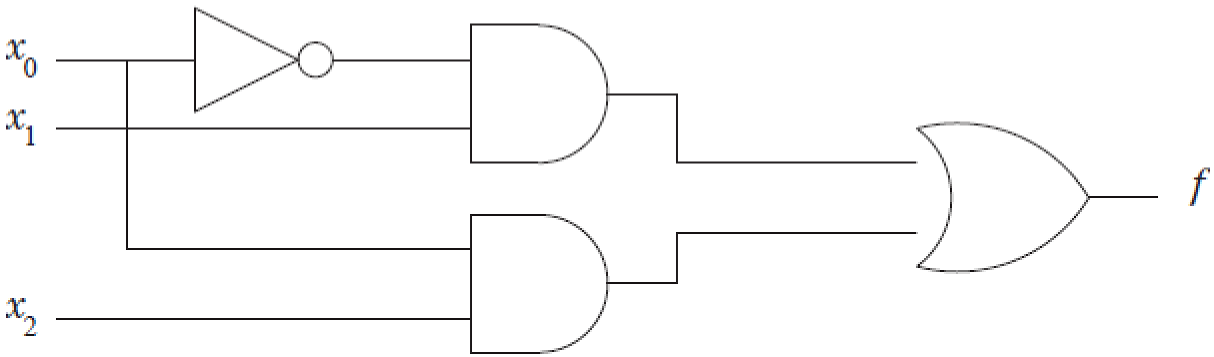
- For arbitrary n>0



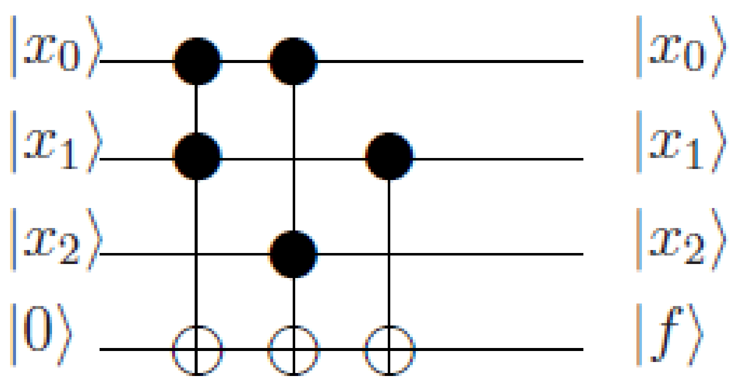
$$H^{\otimes n}|0\rangle^{\otimes n}=\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$$

Boolean Quantum Circuits

$$f = \overline{x_0}x_1 + x_0x_2$$



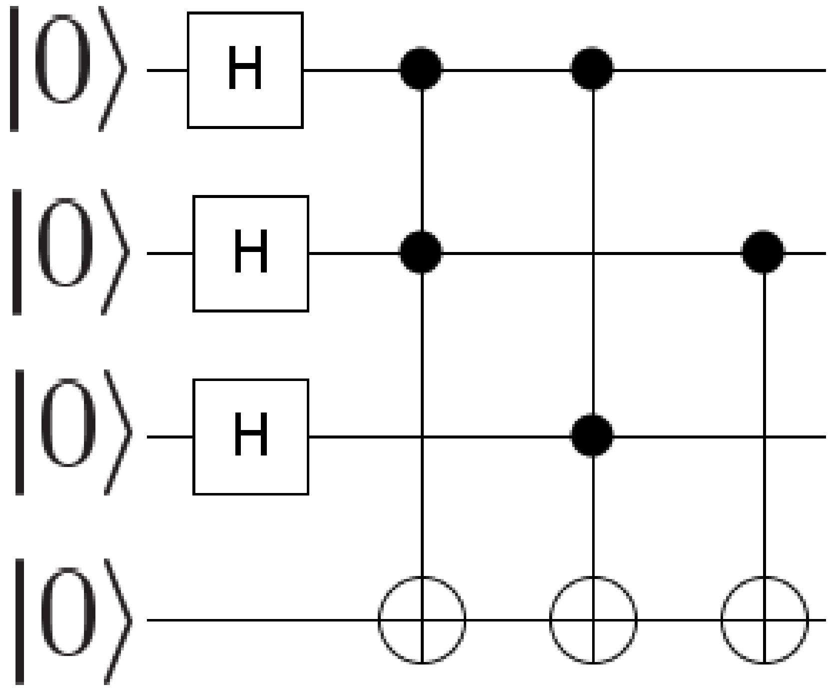
Digital circuit



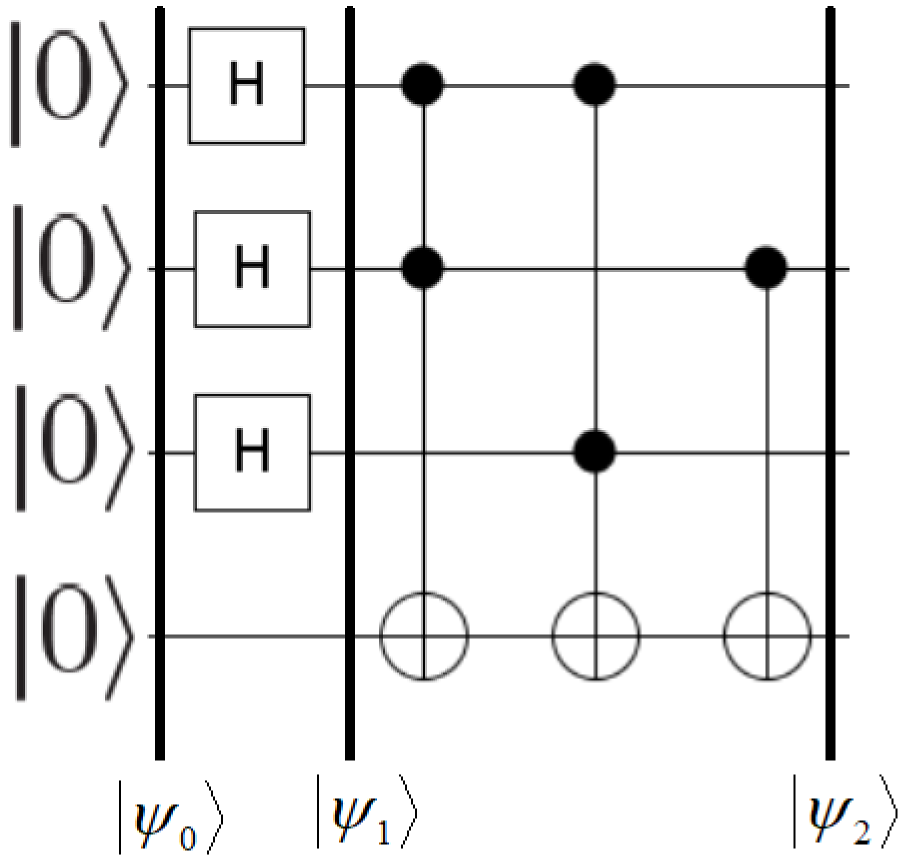
Quantum circuit

x_0	x_1	x_2	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Parallel Evaluation of f



Tracing the Circuit



$$|\psi_0\rangle = |000\rangle \otimes |0\rangle$$

$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle$$

$$= \left(\frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2\sqrt{2}}\right) \otimes |0\rangle$$

$$= \left(\frac{|000,0\rangle + |001,0\rangle + |010,0\rangle + |011,0\rangle + |100,0\rangle + |101,0\rangle + |110,0\rangle + |111,0\rangle}{2\sqrt{2}}\right)$$

$$|\psi_2\rangle = \left(\frac{|000,0\rangle + |001,0\rangle + |010,1\rangle + |011,1\rangle + |100,0\rangle + |101,1\rangle + |110,0\rangle + |111,1\rangle}{2\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}}\big(|000\rangle + |001\rangle + |100\rangle + |110\rangle\big) \otimes |0\rangle + \frac{1}{2\sqrt{2}}\big(|010\rangle + |011\rangle + |101\rangle + |111\rangle\big) \otimes |1\rangle$$

Effect of Entanglement

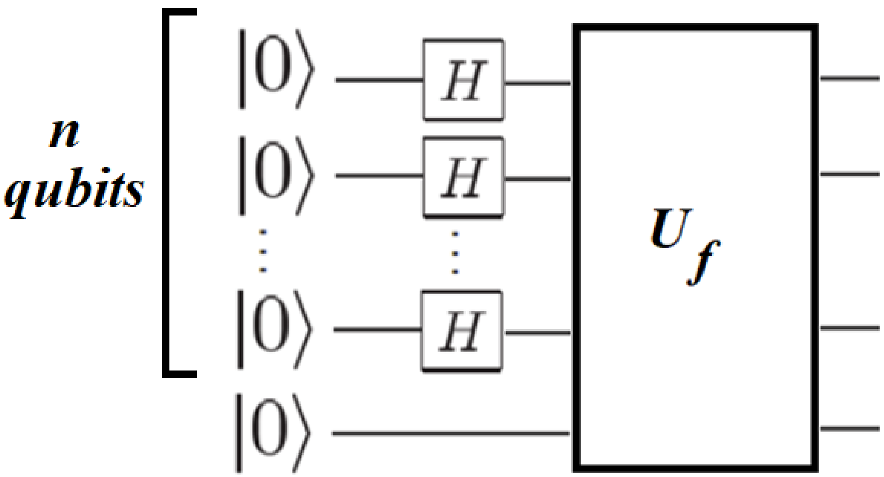
- If we measure the extra qubit and finds |0>, then the system collapses to

$$\frac{1}{2}\left(|000\rangle + |001\rangle + |100\rangle + |110\rangle\right) \otimes |0\rangle$$

- If we measure the extra qubit and finds |1>, then the system collapses to

$$\frac{1}{2}\left(|010\rangle + |011\rangle + |101\rangle + |111\rangle\right) \otimes |1\rangle$$

General Form (Marking by Entanglement)



$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |0\rangle$$

$$|\psi_1\rangle = \left(H^{\otimes n} \otimes I\right)|\psi_0\rangle$$

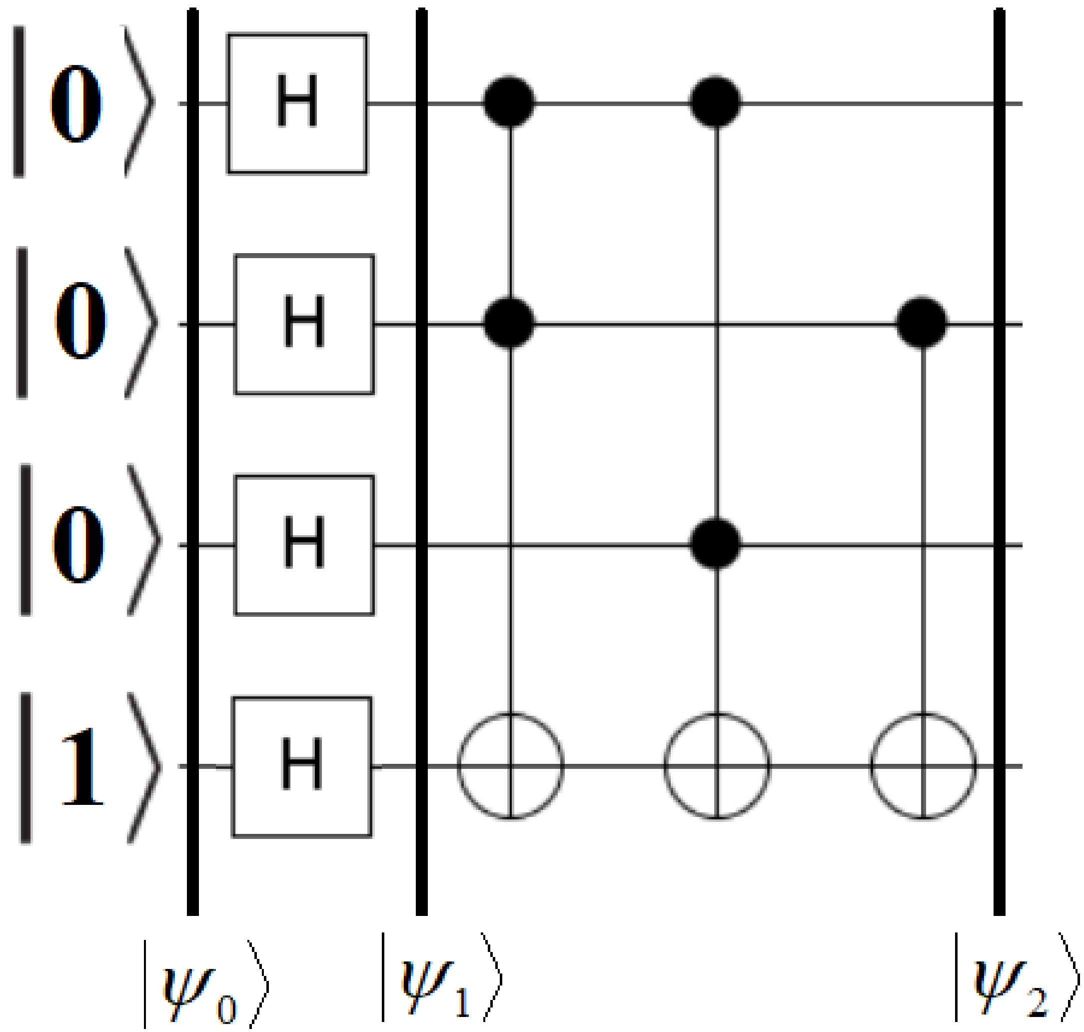
$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes |0\rangle$$

$$|\psi_2\rangle = U_f |\psi_1\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \left(|x\rangle \otimes |f(x)\rangle\right)$$

This method is called Marking the solutions by Entanglement

Marking the Solutions by Phase Shift



$$|\psi_0\rangle = |000\rangle \otimes |1\rangle$$

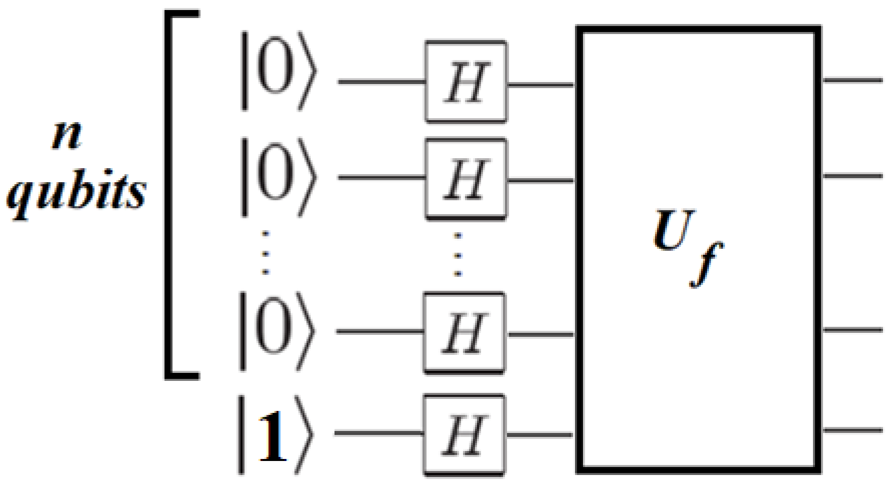
$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \left(\frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_2\rangle = \left(\frac{|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle}{2\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |100\rangle + |110\rangle\right) - \frac{1}{2\sqrt{2}}\left(|010\rangle + |011\rangle + |101\rangle + |111\rangle\right)\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

General Form (Marking by Phase Shift)



$$|\psi_0\rangle = |0\rangle^{\otimes n}$$

$$|\psi_1\rangle = H^{\otimes n}|\psi_0\rangle$$

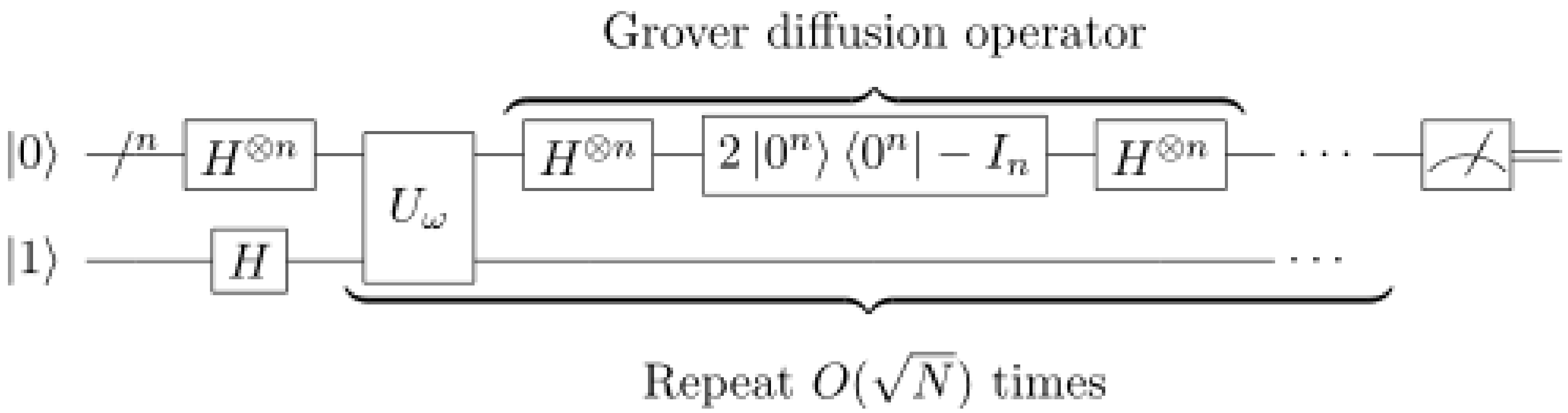
$$= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$$

$$|\psi_2\rangle = U_f|\psi_1\rangle$$

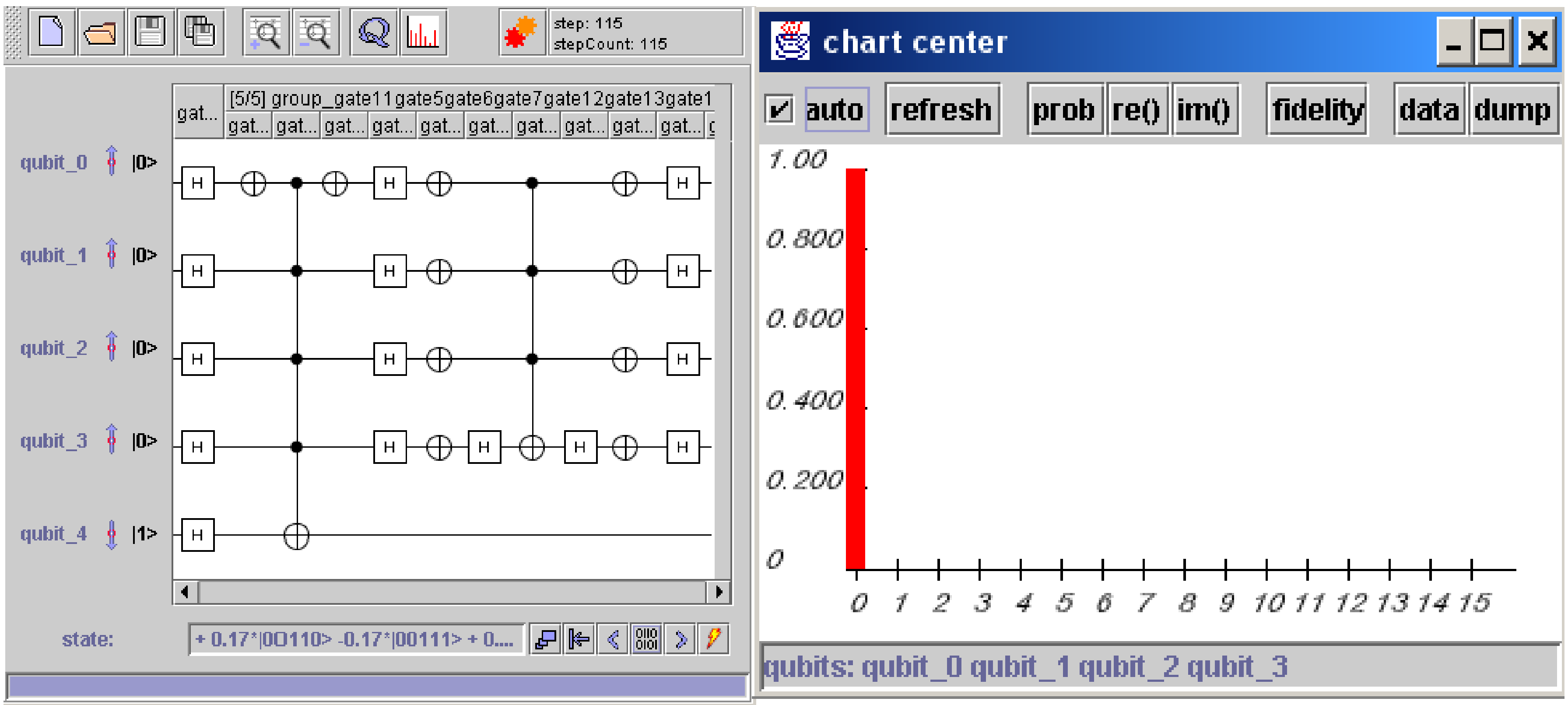
$$= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(-1)^{f(x)}|x\rangle$$

Grover's Quantum Search Algorithm

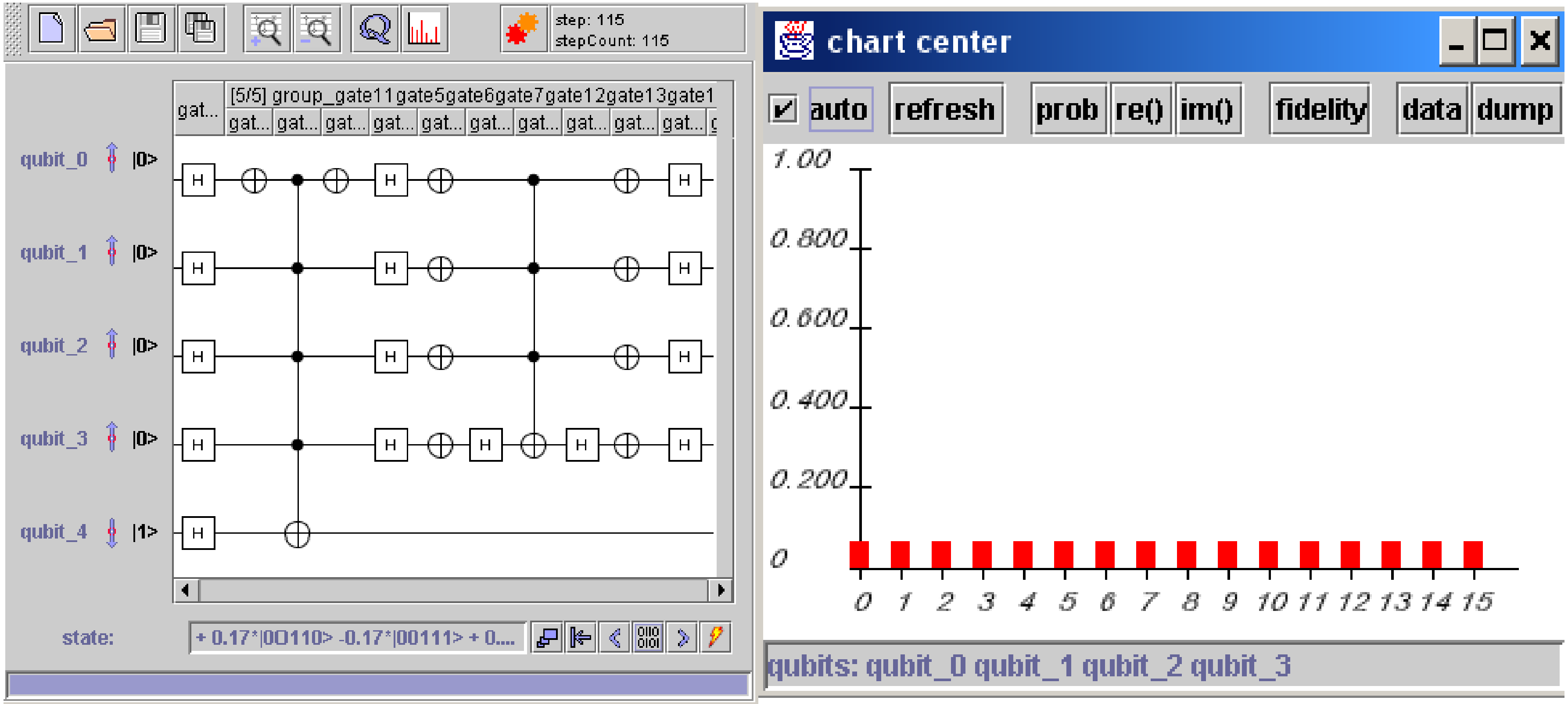
- Given a List L of N=2^n items
  - Step 1 – Prepare a superposition on N items on O(log N)
  - Step 2 – Iterate the Amplitude Amplification for $O\left(\sqrt{N}\right)$
  - Step 3- Measure the quantum register
- Classical Computers require O(N) iteration.



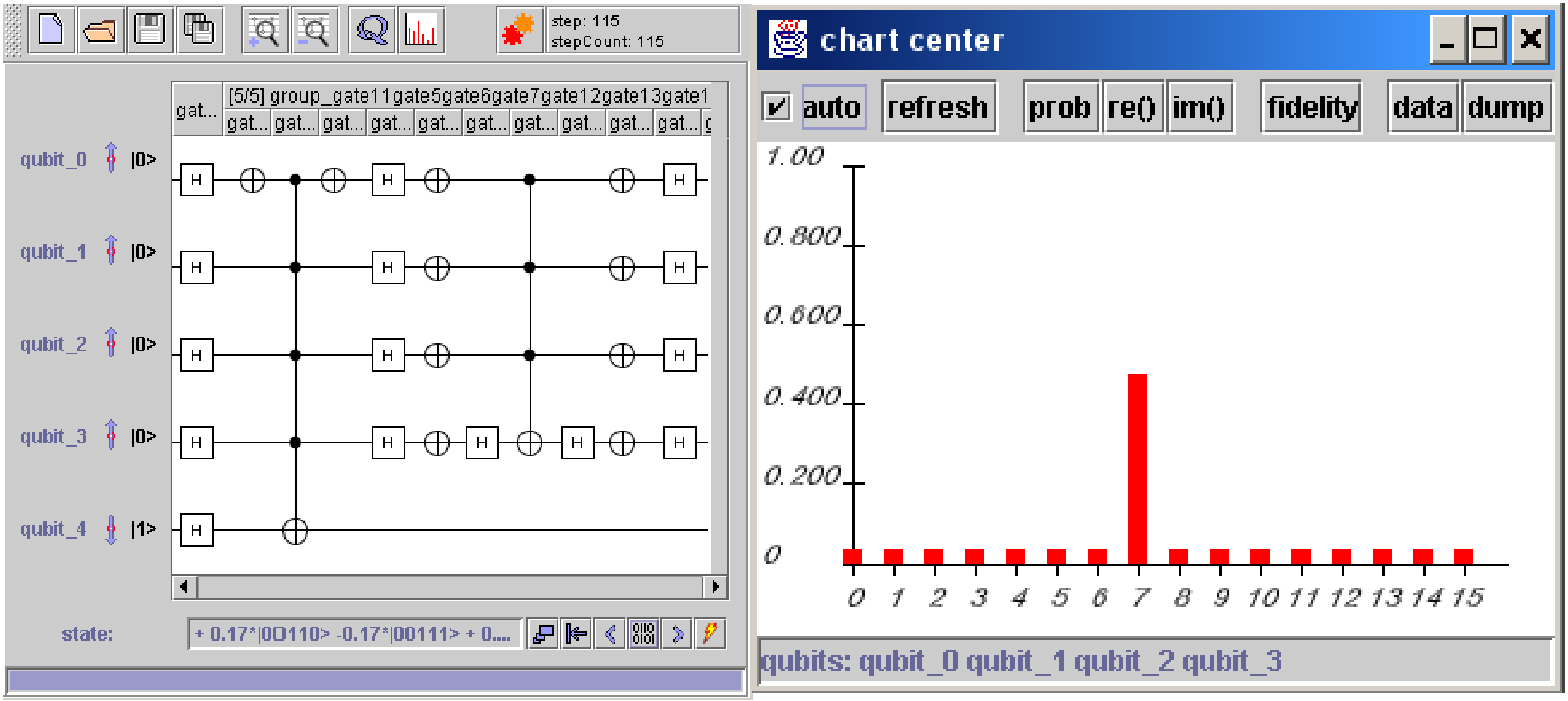
Example: Search for ?.



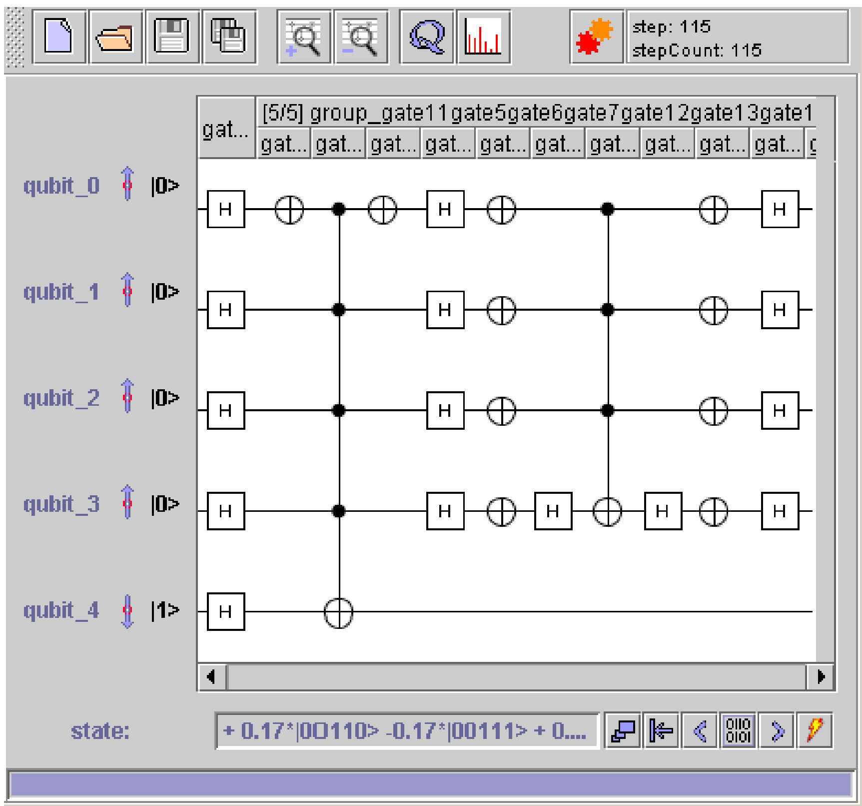
Example: Search for ?.

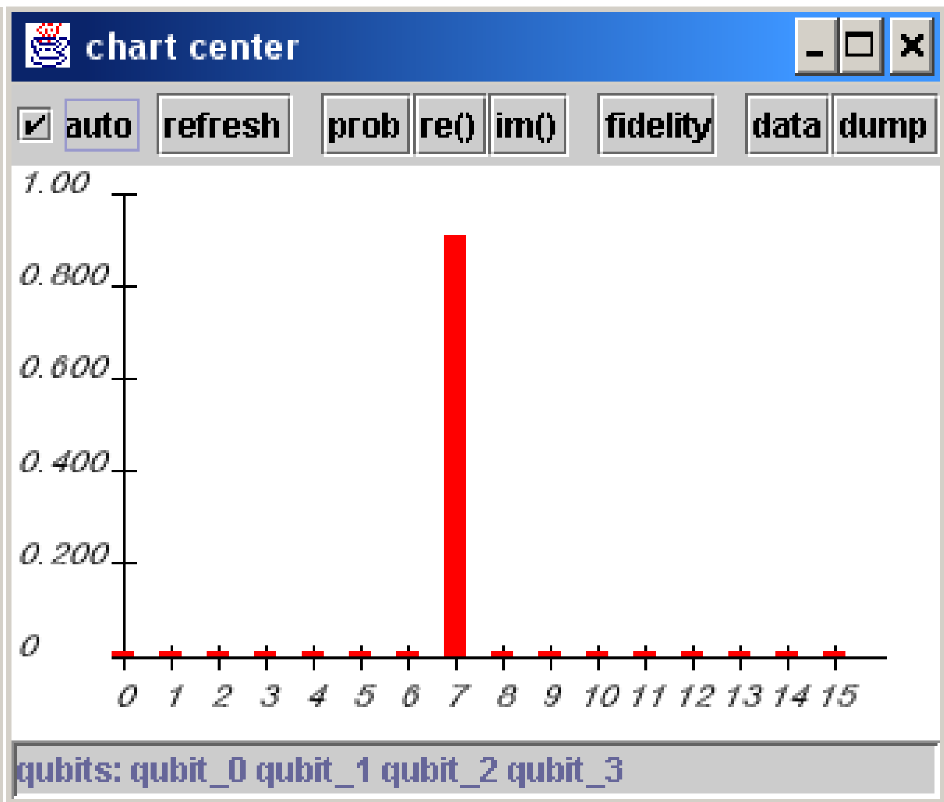


Example: Search for 7.

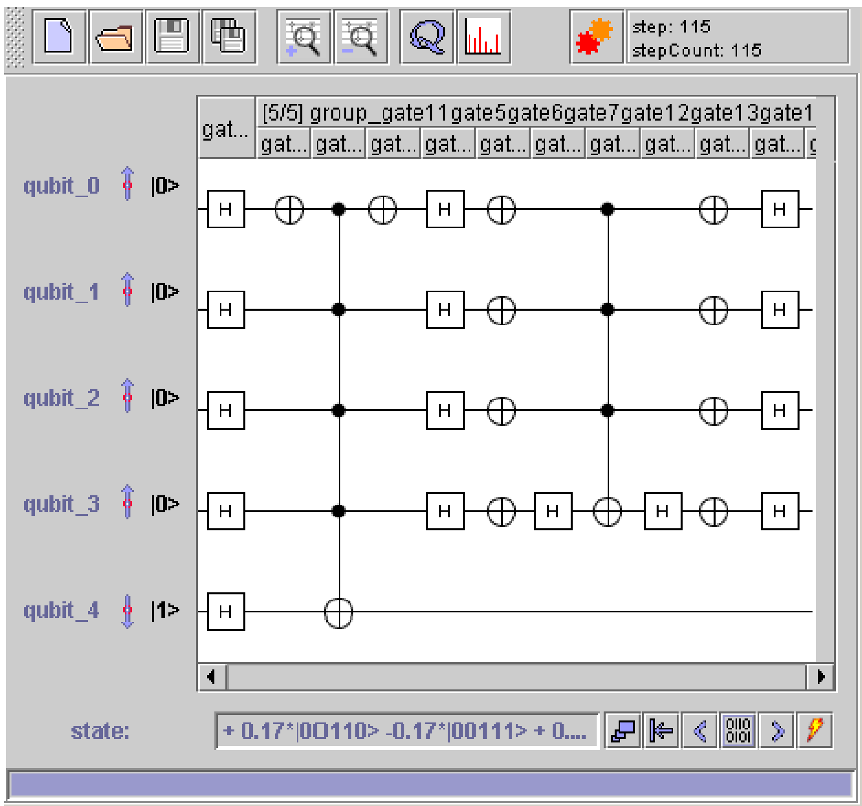


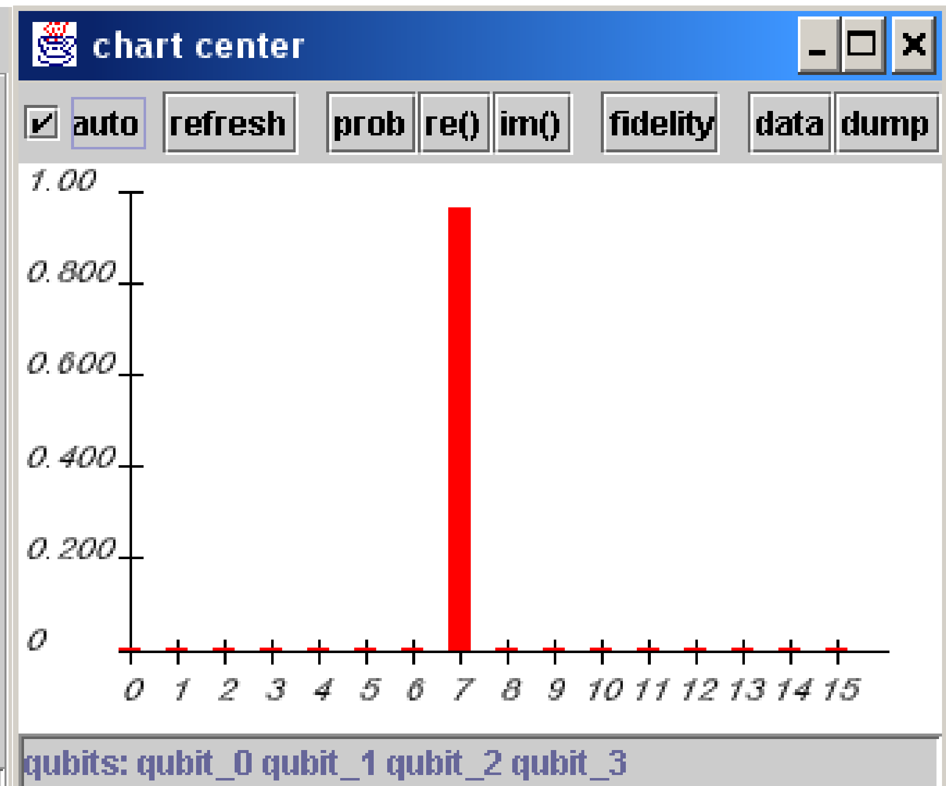
Example: Search for 7.



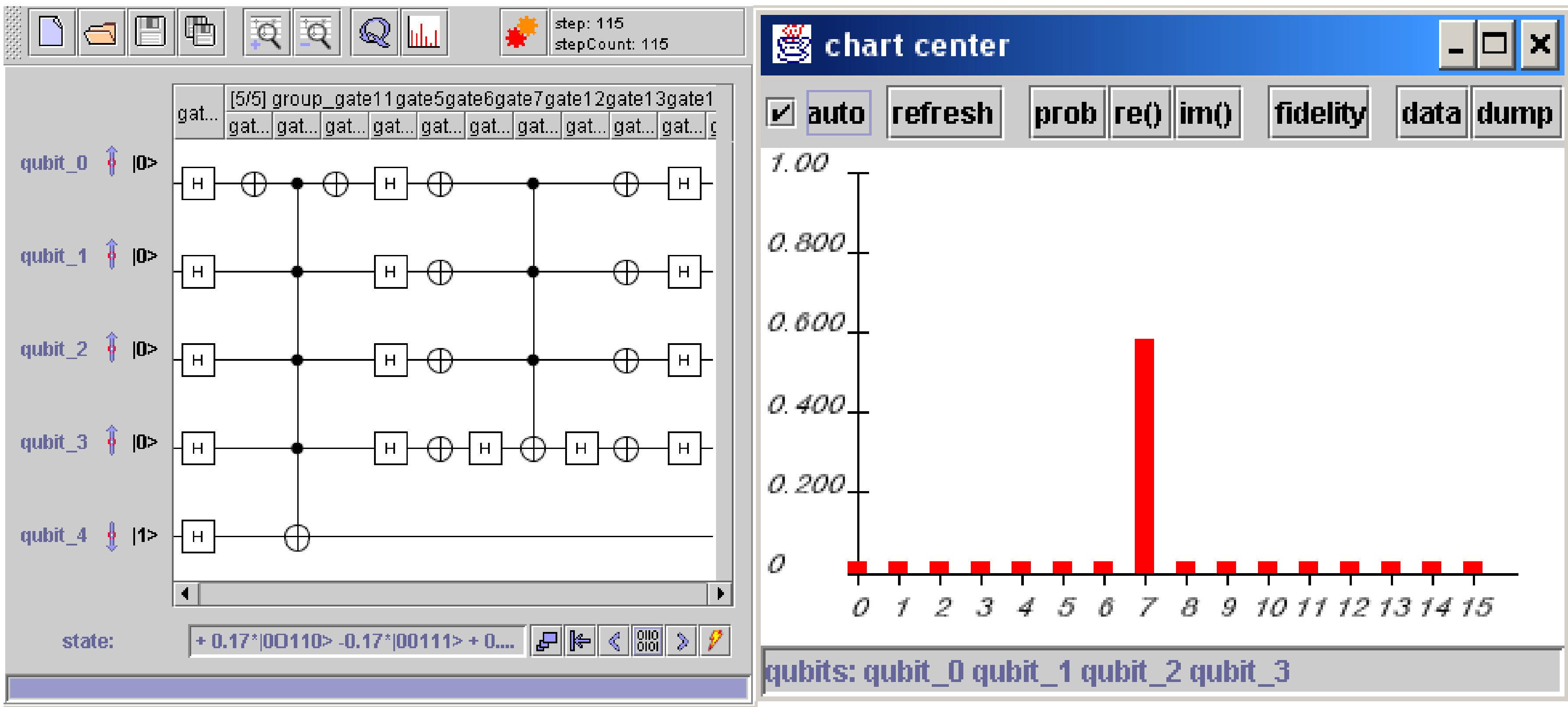


Example: Search for 7.

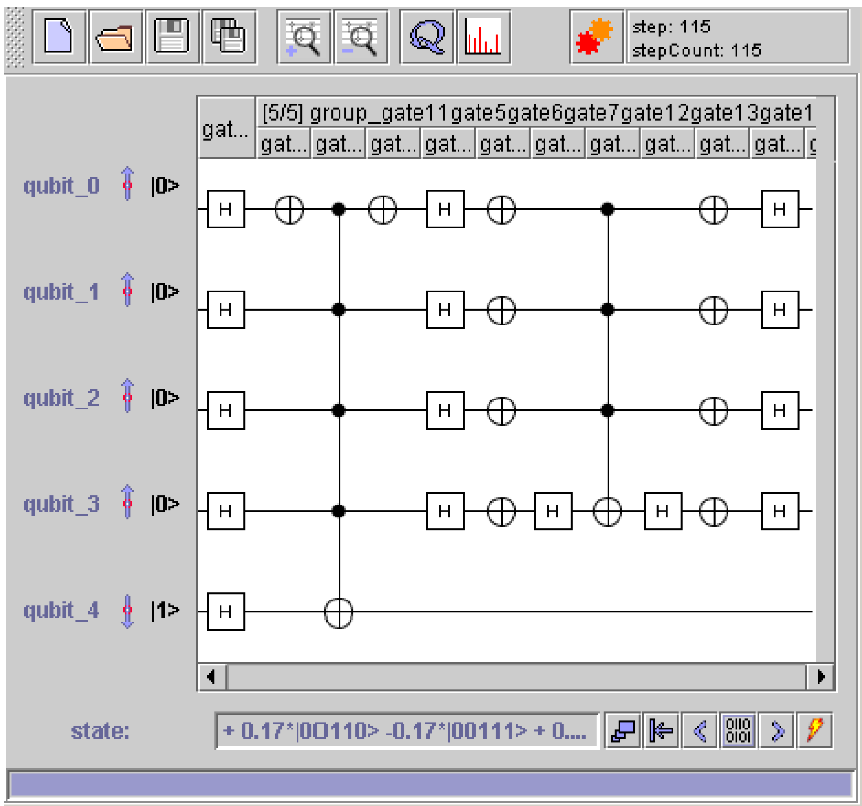


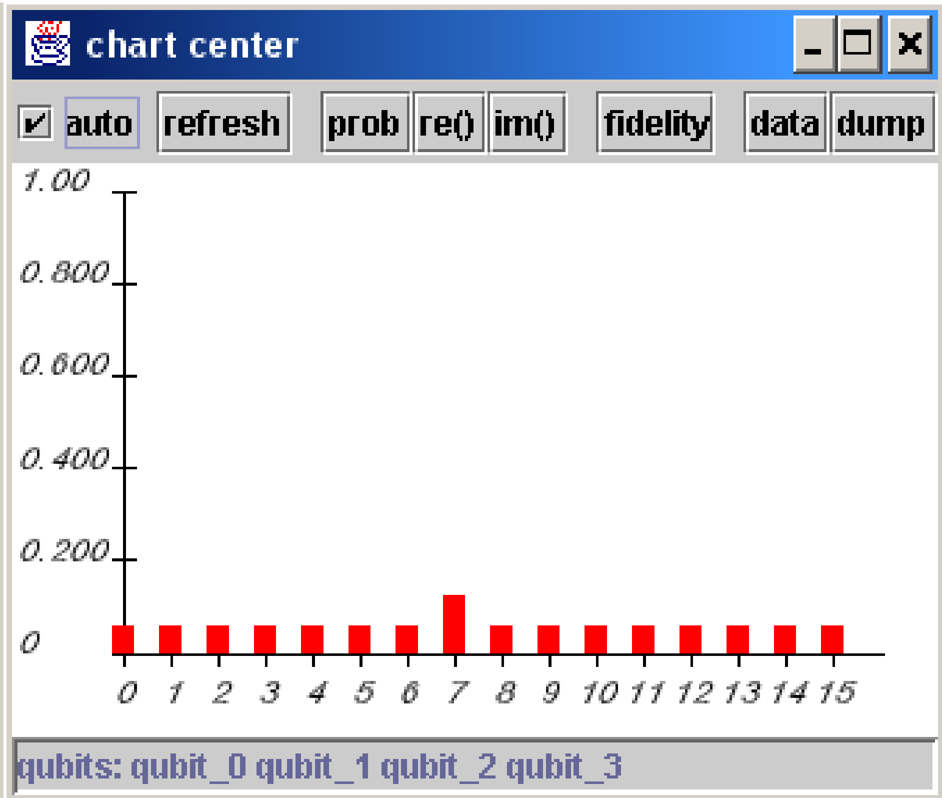


Example: Search for 7.

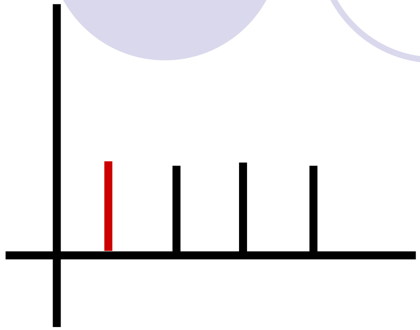


Example: Search for 7.

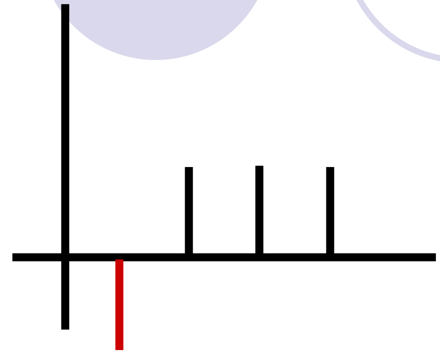




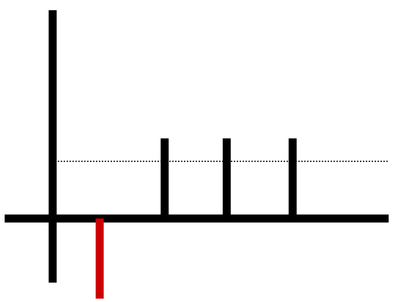
Inversion about the mean



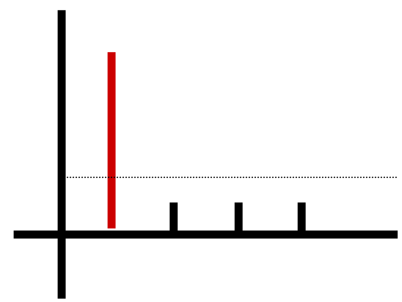
Original Amplitudes



Negate Amplitude



Average of all Amplitudes



Flip all Amplitudes around Avg

Algorithm: Quantum search

1- *Register Preparation.* $\left|W_0^{(G,1)}\right\rangle = |0\rangle^{\otimes n} \otimes |1\rangle$.

2- *Register Initialisation.* $\left|W_1^{(G,1)}\right\rangle = H^{\otimes n+1}\left|W_0^{(G,1)}\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$.

3- *Applying the Oracle and Changing Sign.* $U_f|i\rangle \rightarrow (-1)^{f(i)}|i\rangle$, so that,

$$\left|W_2^{(G,1)}\right\rangle = U_f\left|W_1^{(G,1)}\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle \otimes \left(\frac{|0 \oplus f(i)\rangle - |1 \oplus f(i)\rangle}{\sqrt{2}}\right).$$

$$\left|W_2^{(G,1)}\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}{}''\,|i\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) - \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}{}'\,|i\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right).$$

4- *Inversion about the Mean.* $G = H^{\otimes n}\left(2|0\rangle\langle 0| - I_n\right)H^{\otimes n}$,

$$|\psi\rangle = \sum_{j=0}^{N-1}\alpha_j|j\rangle.$$

$$G|\psi\rangle = \sum_{j=0}^{N-1}\left[-\alpha_j + 2\langle\alpha\rangle\right]|j\rangle,$$

Algorithm: Quantum search

Iterating the Algorithm

$$a_1^G = \frac{N-2M}{N} a_0^G + \frac{2(N-M)}{N} b_0^G, \quad b_1^G = \frac{N-2M}{N} b_0^G - \frac{2M}{N} a_0^G.$$

The system after $q_G \geq 1$ iterations can be written as follows,

$$\left|W^{(G,q_G)}\right\rangle = b_q^G \sum_{i=0}^{N-1}{}'' |i\rangle + a_q^G \sum_{i=0}^{N-1}{}' |i\rangle,$$

such that,

$$M(a_q^G)^2 + (N-M)(b_q^G)^2 = 1,$$

$$a_0^G = b_0^G = \frac{1}{\sqrt{N}},$$
$$a_q^G = \frac{N-2M}{N} a_{q-1}^G + \frac{2(N-M)}{N} b_{q-1}^G, \quad b_q^G = \frac{N-2M}{N} b_{q-1}^G - \frac{2M}{N} a_{q-1}^G.$$

Solving these recurrence relations, the closed forms can be written as follows :

$$a_q^G = \frac{1}{\sqrt{M}} \sin\left(\left(2q_G+1\right)\theta_G\right), \quad b_q^G = \frac{1}{\sqrt{N-M}} \cos\left(\left(2q_G+1\right)\theta_G\right),$$

where $\sin^2(\theta_G) = M/N$ and $0 < \theta_G \leq \pi/2$.

Thank you