# PARTONIC 3D DISTRIBUTIONS: FACTORIZATION, EXTRACTIONS AND MOMENTS

Ignazio Scimemi for CERN 2024, June 24th

*A. Vladimirov, V. Moos, I. S. JHEP 01 (2022) 110, V. Moos, I. S., A. Vladimirov, P. Zurita,JHEP* 05 (2024) 036, *O. del Rio, A. Prokudin, I.S., A. Vladimirov e-Print: 2402.01836, R.F. del Castillo, M. Jaarsma, I. S., W. Waalewijn, JHEP* 02 (2024) 074

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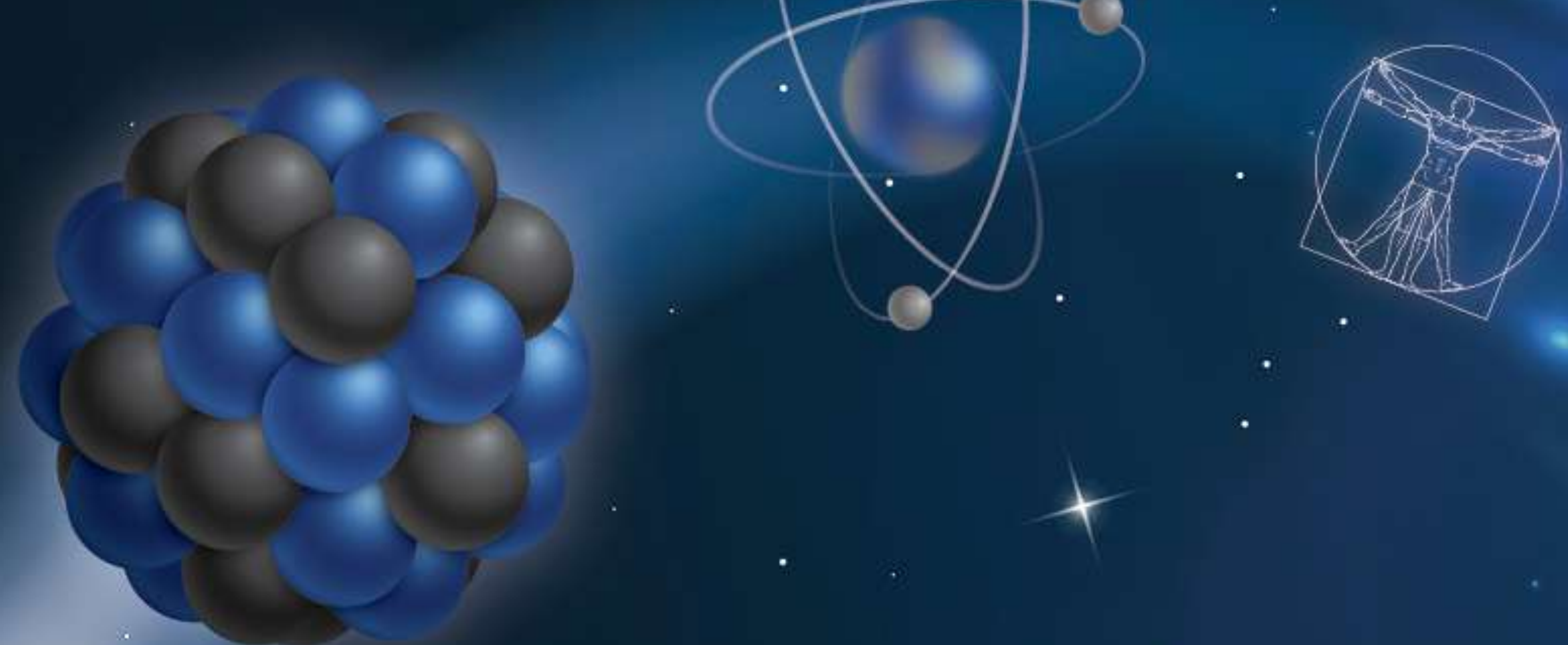
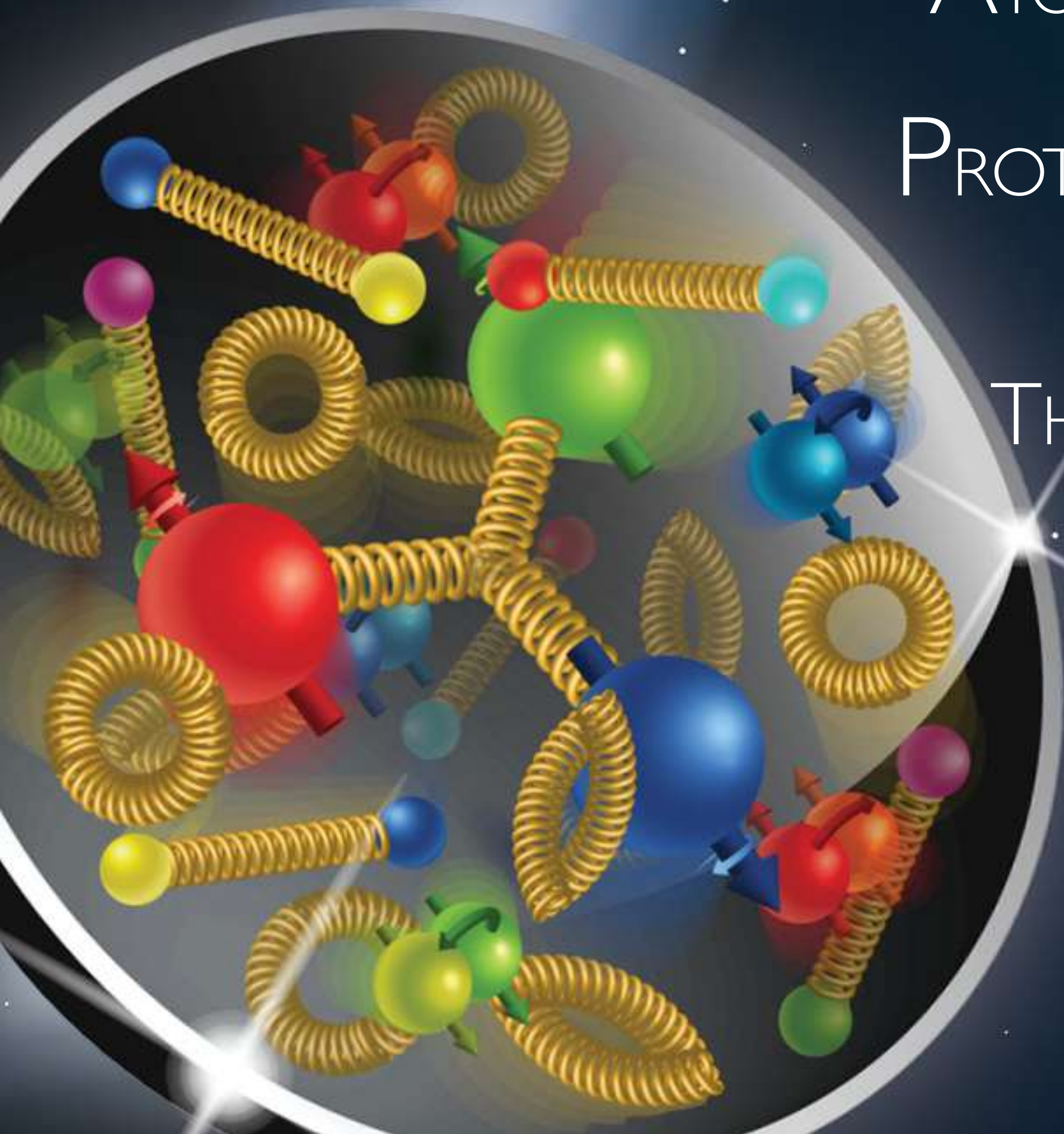




ATOM STRUCTURE IN XXTH CENTURY⇒QM

PROTON STRUCTURE IN XXITH CENTURY⇒

**QCD SOLID STATE**

THE DYNAMICS OF HADRON IS EXPRESSED BY SPECIFIC DISTRIBUTIONS:
PDF, FF, **TMD,** GPD, GTMD,.. WIGNER DISTRIBUTIONS.

**WE HAVE JUST BEGUN**

# *Outline*

- FACTORIZATION FROM BACKGROUND FIELD METHOD
- TMD EXTRACTION FROM DY (ART23)
- TRANSVERSE MOMENTUM MOMENTS

# THE DY CROSS SECTION

The process $$h_1(P_1) + h_2(P_2) \to l(l) + l'(l') + X$$

The cross section $$d\sigma = \frac{2\alpha_{\rm em}^2}{s}\frac{d^3l}{2E}\frac{d^3l'}{2E'}L_{\mu\nu}W^{\mu\nu}\Delta(q)\Delta^*(q)$$

$$\Delta_G(q) = \frac{1}{q^2 + i0}$$

$$J_\mu = e\bar{\psi}\gamma_\mu\psi$$

The tensors $$L_{\mu\nu} = e^{-2}\langle 0 | J_\mu(0) | l, l'\rangle\langle l, l' | J_\nu^\dagger(0) | 0\rangle$$

$$W_{\mu\nu} = e^{-2}\int\frac{d^4x}{(2\pi)^4}e^{-i(x\cdot q)}\sum_X \langle P_1, P_2 | J_\mu^\dagger(x) | X\rangle\langle X | J_\nu(0) | P_1, P_2\rangle$$

# HADRONIC TENSOR EXPANSION FEATURES

$$W^{\mu\nu}_{\rm DY} = \int \frac{d^4y}{(2\pi)^4} e^{-i(yq)} \sum_X \langle p_1, p_2 | J^{\mu\dagger}(y) | X \rangle \langle X | J^{\nu}(0) | p_1, p_2 \rangle$$

**Factorization limits**

$$Q^2 \gg \Lambda^2, \qquad Q^2 \gg \mathbf{q_T}^2 = \text{fixed}$$

$$Q^2 = q^2 = 2q^+q^- - \mathbf{q_T}^2 \quad q_\mu W^{\mu\nu} = 0$$

# INITIAL STEP OF THE EXPANSION IN BACKGROUND FIELD METHOD

For each fermion field we have two copies of QCD fields,
**causal (+, also negative frequency)** and
**anticausal (-, also positive frequency)**

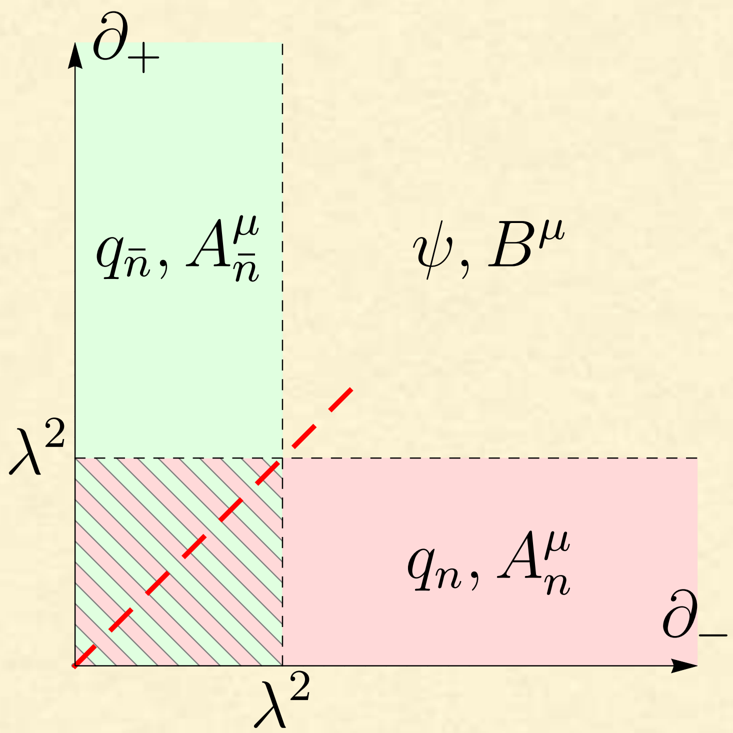
$$W^{\mu\nu}_{\rm DY} = \int \frac{d^4y}{(2\pi)^4} e^{-i(yq)} \int [D\bar{q}^{(+)} Dq^{(+)} DA^{(+)}] \int [D\bar{q}^{(-)} Dq^{(-)} DA^{(-)}]$$
$$\times \Psi^{*(-)}_{p_1} \Psi^{*(-)}_{p_2} \, e^{iS^{(+)}_{\rm QCD} - iS^{(-)}_{\rm QCD}} \, J^{\dagger(-)}_{\mu}(y) J^{(+)}_{\nu}(0) \, \Psi^{(+)}_{p_1} \Psi^{(+)}_{p_2}$$

The hadronic tensors that we consider have two causally-independent sectors which exchange real emissions.

Hadrons are made only
Out of collinear fields

$$\Psi_{p_1} = \Psi_{p_1}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}], \qquad \Psi_{p_2} = \Psi_{p_2}[\bar{q}_n, q_n, A_n]$$

# MODES




Each collinear field has "good" and "bad" components selected using standard (~SCET) projectors

$$q_{\bar{n}}(x) = \xi_{\bar{n}}(x) + \eta_{\bar{n}}(x) \quad \xi_{\bar{n}}(x) = \frac{\bar{n}\!\!\!/ n\!\!\!/}{2} q_{\bar{n}}(x), \qquad \eta_{\bar{n}}(x) = \frac{n\!\!\!/ \bar{n}\!\!\!/}{2} q_{\bar{n}}(x)$$

$$\gamma^+ D_-[A_{\bar{n}}]\xi_{\bar{n}} = -\,D\!\!\!\!/_T[A_{\bar{n}}]\eta_{\bar{n}}$$

$$\gamma^- D_+[A_{\bar{n}}]\eta_{\bar{n}} = -\,D\!\!\!\!/_T[A_{\bar{n}}]\xi_{\bar{n}}$$

$$\Rightarrow \quad \eta_{\bar{n}/n} \sim \lambda\, \xi_{\bar{n}/n}$$

$$A^+_{\bar{n}} \sim 1, \qquad A^{\mu_T}_{\bar{n}} \sim \lambda, \qquad A^-_{\bar{n}} \sim \lambda^2,$$

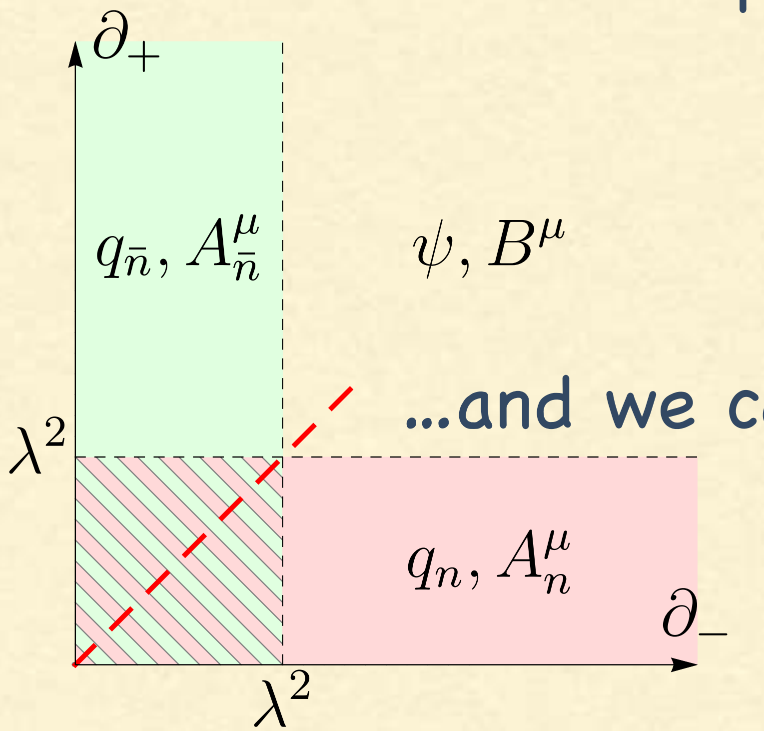
Finally.. $\xi_{\bar{n}/n} \sim \lambda, \qquad \eta_{\bar{n}/n} \sim \lambda^2$

$$A^+_n \sim \lambda^2, \qquad A^{\mu_T}_n \sim \lambda, \qquad A^-_n \sim 1$$

# MODES

There is a momentum scaling of the fields…

$$\{\partial_+, \partial_-, \partial_T\}\, q_{\bar{n}} \lesssim Q\{1, \lambda^2, \lambda\}\, q_{\bar{n}},$$
$$\{\partial_+, \partial_-, \partial_T\}\, A^\mu_{\bar{n}} \lesssim Q\{1, \lambda^2, \lambda\}\, A^\mu_{\bar{n}}$$




…and we can introduce dynamical and background fields…

Dynamical field

$$q^{(\pm)}(x) = \psi^{(\pm)}(x) + q_n^{(\pm)}(x) + q_{\bar{n}}^{(\pm)}(x)$$
$$A_\mu^{(\pm)}(x) = B_\mu^{(\pm)}(x) + A_{n\,\mu}^{(\pm)}(x) + A_{\bar{n}\,\mu}^{(\pm)}(x)$$

Dynamical field

Background Gauge fixation $[\partial_\mu \delta^{AC} + g f^{ABC}(A^{(\pm)B}_{\bar{n}\,\mu} + A^{(\pm)B}_{n\,\mu})]B^{(\pm)\mu\,C} = 0$

# FORMAL RESULT

$$W^{\mu\nu}_{\text{DY(unsub.)}} = \int \frac{d^4y}{(2\pi)^4} e^{-i(yq)}$$

$$\times \int [D\bar{q}^{(+)}_{\bar{n}} Dq^{(+)}_{\bar{n}} DA^{(+)}_{\bar{n}}][D\bar{q}^{(-)}_{\bar{n}} Dq^{(-)}_{\bar{n}} DA^{(-)}_{\bar{n}}] e^{iS^{(+)}_{\text{QCD}}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] - iS^{(-)}_{\text{QCD}}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}]}$$

$$\times \int [D\bar{q}^{(+)}_{n} Dq^{(+)}_{n} DA^{(+)}_{n}][D\bar{q}^{(-)}_{n} Dq^{(-)}_{n} DA^{(-)}_{n}] e^{iS^{(+)}_{\text{QCD}}[\bar{q}_{n}, q_{n}, A_{n}] - iS^{(-)}_{\text{QCD}}[\bar{q}_{n}, q_{n}, A_{n}]}$$

$$\times \Psi^{*(-)}_{p_1}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] \Psi^{*(-)}_{p_2}[\bar{q}_{n}, q_{n}, A_{n}] \, \mathcal{J}^{\mu\nu}_{eff}[\bar{q}_{\bar{n}}, \bar{q}_{n}, \ldots](y) \, \Psi^{(+)}_{p_1}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] \Psi^{(+)}_{p_2}[\bar{q}_{n}, q_{n}, A_{n}]$$

# FORMAL RESULT

The dynamical (hard) degrees of freedom are integrated obtaining

$$\mathcal{J}^{\mu\nu}_{eff}[\bar{q}_{\bar{n}}, \bar{q}_n, \ldots](y) = \int [D\bar{\psi}^{(+)} D\psi^{(+)} DB^{(+)}][D\bar{\psi}^{(-)} D\psi^{(-)} DB^{(-)}] e^{iS^{(+)}_{\rm QCD}[\bar{\psi},\psi,B] - iS^{(-)}_{\rm QCD}[\bar{\psi},\psi,B]} e^{iS^{(+)}_{int} - iS^{(-)}_{int}}$$

$$\times J^{\dagger(-)}_{\mu}[\bar{\psi} + \bar{q}_{\bar{n}} + \bar{q}_n, \ldots](y) J^{(+)}_{\nu}[\bar{\psi} + \bar{q}_{\bar{n}} + \bar{q}_n, \ldots](0)$$

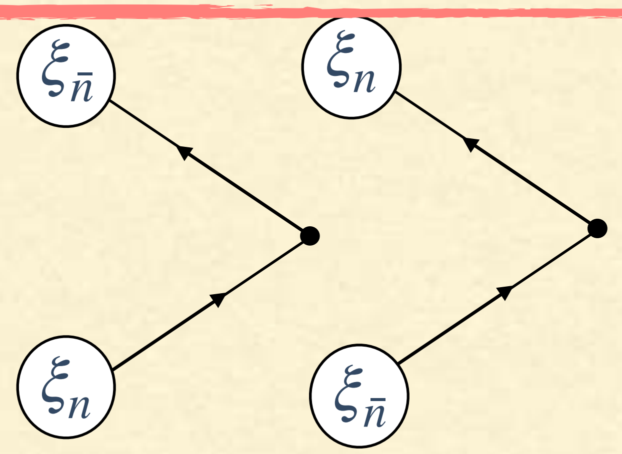
And we expand in a series of operators $\mathcal{J}^{\mu\nu}_{N,k} \sim \lambda^{N+4}$

$$\mathcal{J}^{\mu\nu}_{eff}[\bar{q}_{\bar{n}}, \bar{q}_n, \ldots](y) = \sum_{N=0}^{\infty} \sum_{k} \mathcal{J}^{\mu\nu}_{N,k}[\bar{q}_{\bar{n}}, \bar{q}_n, \ldots](y)$$

Where $N$ *is the power counting* and $k$ *lists the operators*

$$\mathcal{J}^{\mu\nu}_{N,(a,b)}[\bar{q}_{\bar{n}}, \bar{q}_n, \ldots](y) = C^{\mu\nu}_{N,(a+b)}(y) \otimes \mathcal{O}_a[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] \otimes \mathcal{O}_b[\bar{q}_n, q_n, A_n]$$

# LEADING POWER AND LEADING ORDER HADRONIC TENSOR



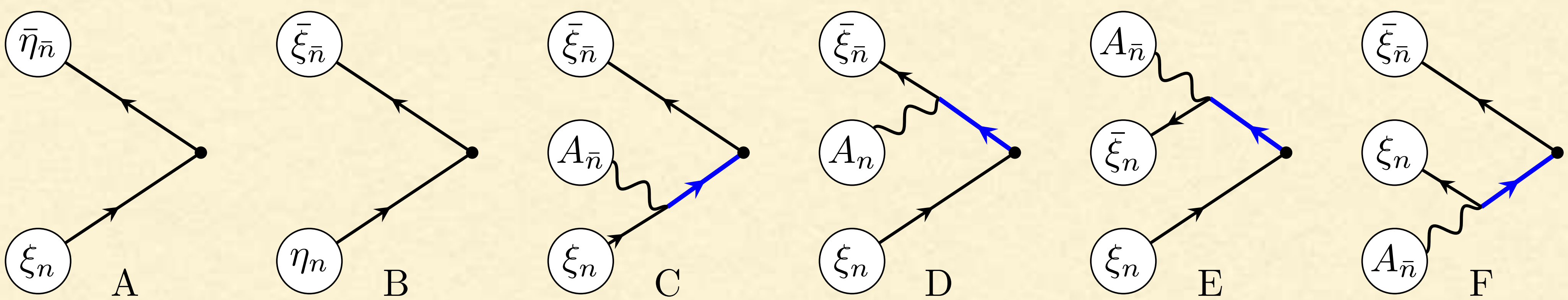

$$\mathcal{J}^{\mu\nu}_{\mathrm{LP}}(y) = \frac{\gamma^{\mu}_{T,ij}\gamma^{\nu}_{T,kl}}{N_c}\left(\mathcal{O}^{li}_{11,\bar{n}}\,\overline{\mathcal{O}}^{jk}_{11,n} + \overline{\mathcal{O}}^{jk}_{11,\bar{n}}\,\mathcal{O}^{li}_{11,n}\right)$$

The $\bar{n}$ and $n$ sector are now separated

$$\mathcal{O}^{ji}_{11,n}(\{y^+,0\},y_T) = \bar{\xi}^{(-)}_{n,i}(y^+\bar{n}+y_T)\,\xi^{(+)}_{n,j}(0), \qquad \mathcal{O}^{ji}_{11,\bar{n}}(\{y^-,0\},y_T) = \bar{\xi}^{(-)}_{\bar{n},i}(y^-n+y_T)\,\xi^{(+)}_{\bar{n},j}(0),$$

$$\overline{\mathcal{O}}^{ji}_{11,n}(\{y^+,0\},y_T) = \xi^{(-)}_{n,j}(y^+\bar{n}+y_T)\,\bar{\xi}^{(+)}_{n,i}(0)\,. \qquad \overline{\mathcal{O}}^{ji}_{11,\bar{n}}(\{y^-,0\},y_T) = \xi^{(-)}_{\bar{n},j}(y^-n+y_T)\,\bar{\xi}^{(+)}_{\bar{n},i}(0)\,.$$

# NEXT-TO-LEADING POWER AT LO



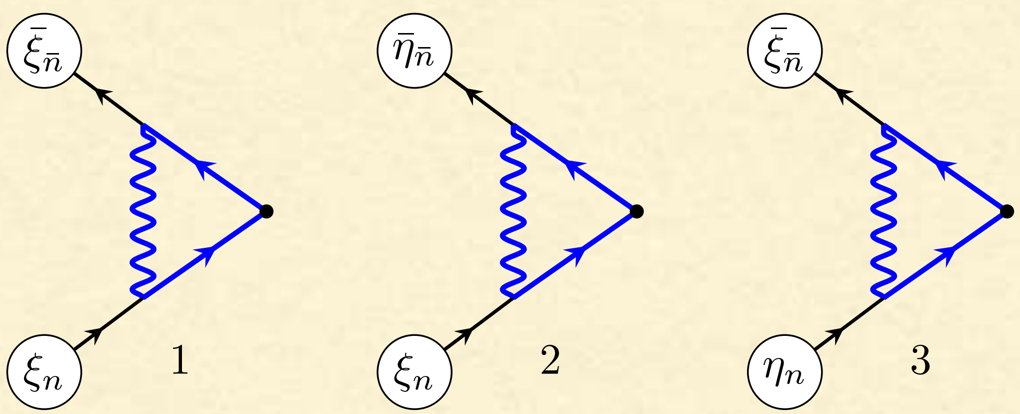

"Bad" component Contributions

(Hard) background contributions

A. Vladimirov, V. Moos, I.S. *JHEP* 01 (2022) 110

# NEXT-TO-LEADING POWER AT NLO

“Bad” component Contributions

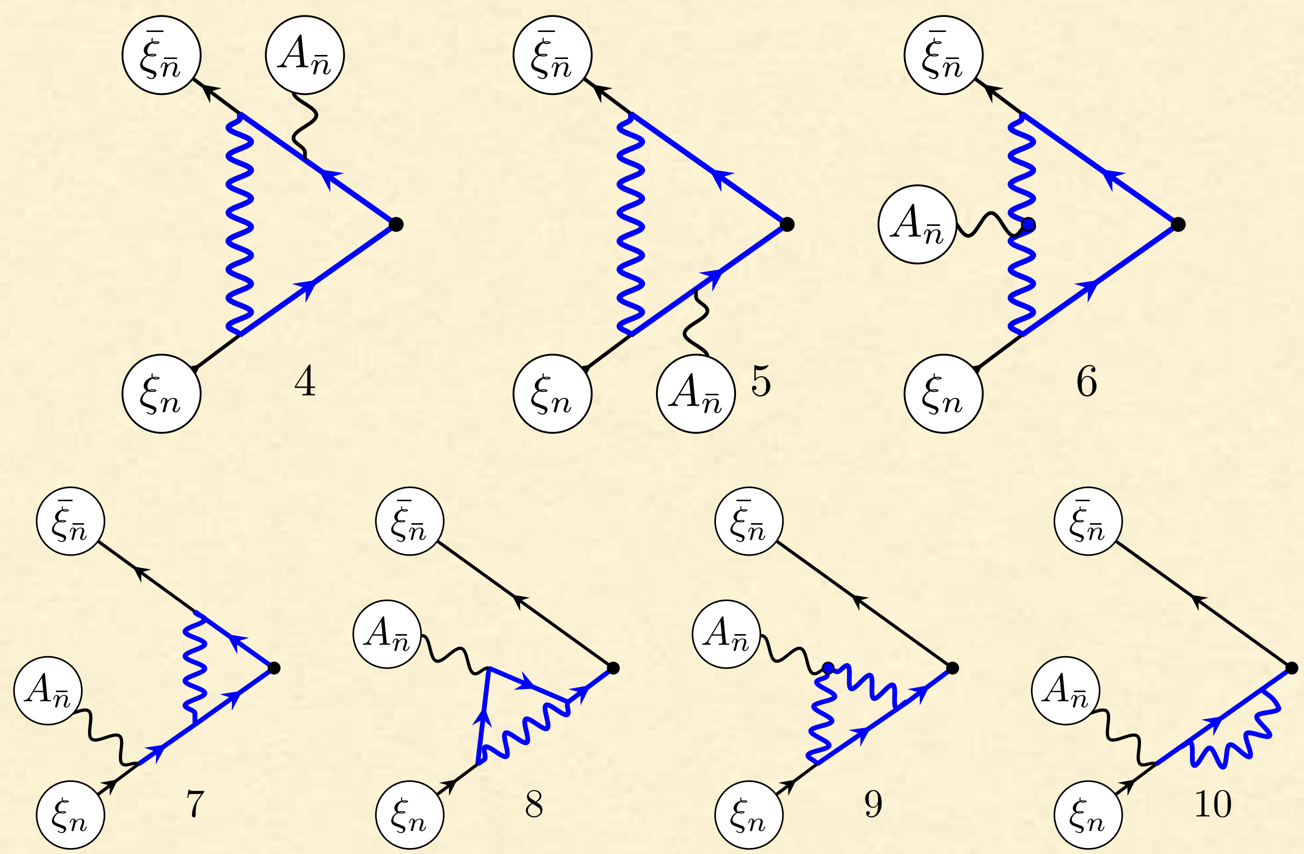



**LP** **NLP**

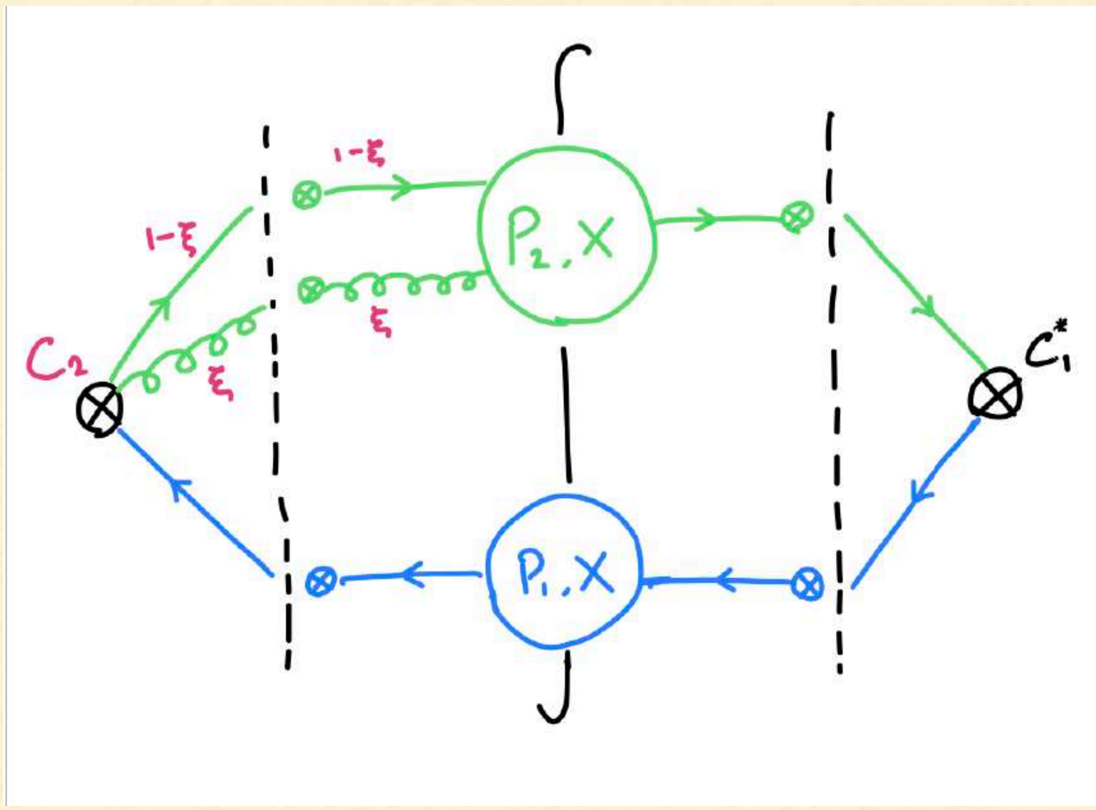
A. Vladimirov, V. Moos, I.S. *JHEP* 01 (2022) 110

Numerical implementation: S. Rodini, L. Rossi, A. Vladimirov E-print: 2405.01162, Honeycomb (C) / Snowflake (FORTRAN)

Hard background contributions



# NEW OBSERVABLES CAN BE STUDIED AT NLP



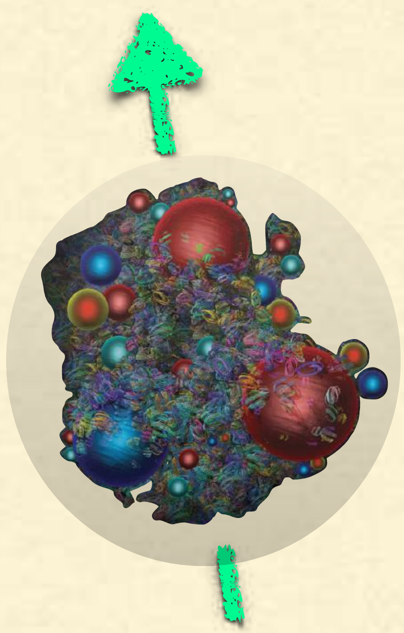

Jets can be produced at higher twist

Di-jet production in $e^+e^-$:
Asymmetries in jet definitions are produced at NLP

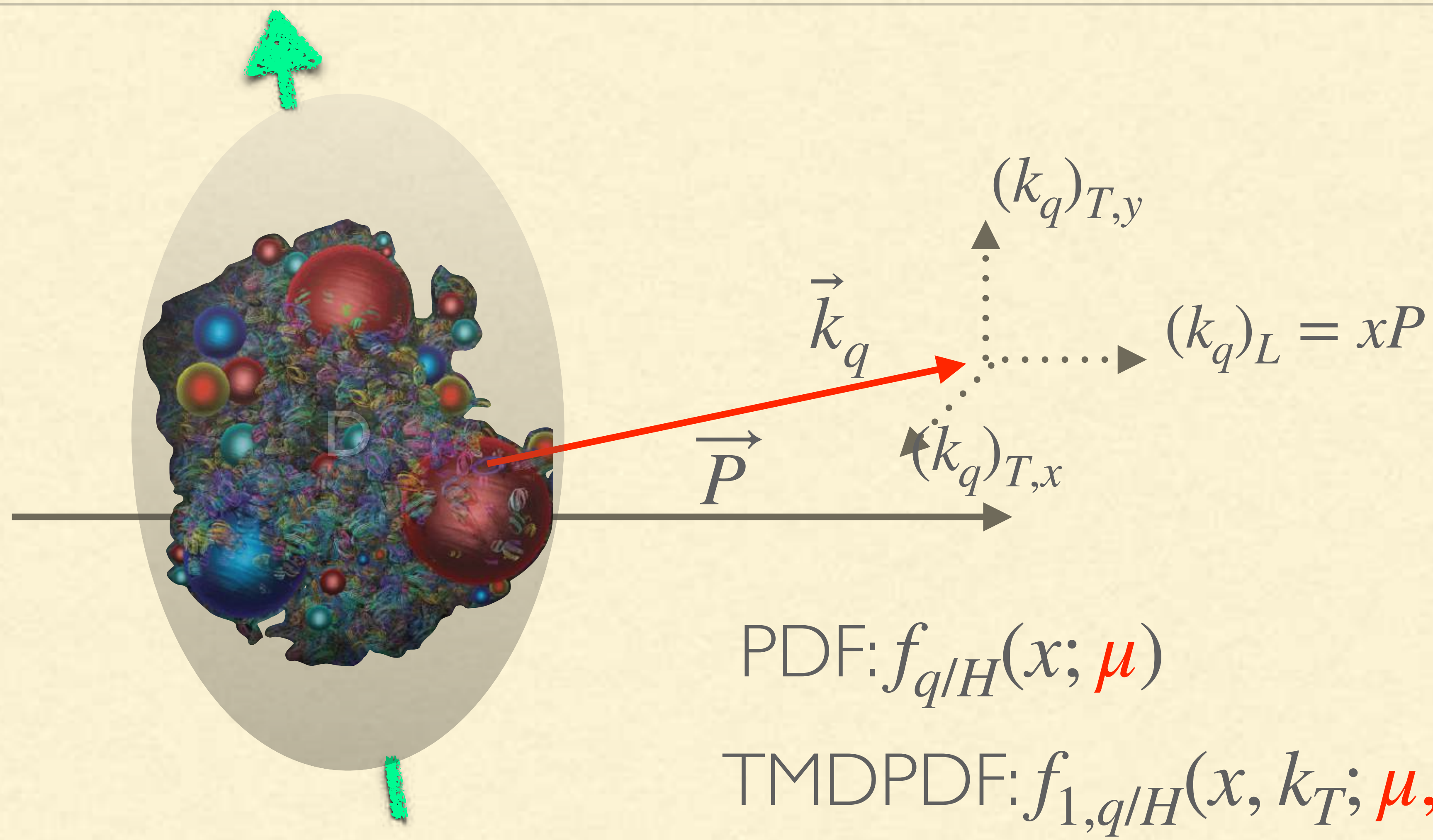
$$
\begin{aligned}
W^{\mu\nu} &= -N_c\, g_T^{\mu\nu}\, H_1(Q^2) \int_0^\infty \mathrm{d}\mathbf{b}^2\, \frac{J_0(|\mathbf{b}||\mathbf{q}|)}{2\pi}\, J_{11}(\mathbf{b}^2)\, J_{11}(\mathbf{b}^2) \\
&\quad + N_c \left[\frac{n^\mu q_T^\nu}{q^+} + \frac{n^\nu q_T^\mu}{q^+}\right] H_1(Q^2) \int_0^\infty \mathrm{d}\mathbf{b}^2\, \frac{J_1(|\mathbf{b}||\mathbf{q}|)}{2\pi|\mathbf{b}||\mathbf{q}|}\, J'_{11}(\mathbf{b}^2)\, J_{11}(\mathbf{b}^2) \\
&\quad + N_c \left[\frac{\bar{n}^\mu q_T^\nu}{q^-} - \frac{n^\mu q_T^\nu}{q^+}\right] \int_0^1 \mathrm{d}\xi\, H_2(\xi, Q^2) \int_0^\infty \mathrm{d}\mathbf{b}^2\, \frac{J_1(|\mathbf{b}||\mathbf{q}|)}{2\pi|\mathbf{b}||\mathbf{q}|} \\
&\qquad \times \left\{ J_{21}(\xi, \mathbf{b}^2)\, J_{11}(\mathbf{b}^2) - J_{11}(\mathbf{b}^2)\, J_{21}(\xi, \mathbf{b}^2) \right\}
\end{aligned}
$$


Pictures from M. Jaarsma talk at SCET2024,
R. F. Del Castillo, *M. Jaarsma*, *I. S.*, *W. Waalewijn*, *JHEP* 02 (2024) 074

# HADRON STRUCTURES



D. Dominguez© CERN




PDF: $f_{q/H}(x; \mu)$

TMDPDF: $f_{1,q/H}(x, k_T; \mu, \zeta)$

Quark Polarization

Time-reversal flip

Nucleon Polarization

| QUARKS | unpolarized | chiral | transverse |
|---|---|---|---|
| U | $f_1$ | | $h_1^\perp$ |
| L | | $g_{1L}$ | $h_{1L}^\perp$ |
| T | $f_{1T}^\perp$ | $g_{1T}$ | $h_{1T}, h_{1T}^\perp$ |

Gluon Polarization

Nucleon Polarization

| GLUONS | unpolarized | circular | linear |
|---|---|---|---|
| U | $f_1^g$ | | $h_1^{\perp g}$ |
| L | | $g_{1L}^g$ | $h_{1L}^{\perp g}$ |
| T | $f_{1T}^{\perp g}$ | $g_{1T}^g$ | $h_{1T}^g, h_{1T}^{\perp g}$ |

# FACTORIZATION FORMULA

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sigma_0 \sum_{f_1,f_2} \int \frac{d^2\boldsymbol{b}}{4\pi} e^{i(\boldsymbol{b}\cdot\boldsymbol{q}_T)} H_{f_1 f_2}(Q,Q)\{R[\boldsymbol{b};(Q,Q^2)]\}^2 F_{f_1\leftarrow h_1}(x_1,\boldsymbol{b}) F_{f_2\leftarrow h_2}(x_2,\boldsymbol{b})$$

- Its range of applicability is provided by $\delta = \frac{q_T}{Q} \ll 1$, fixed-$q_T$, $\delta \sim 0.25$
- We have a non-perturbative evolution kernel, R[], (whose perturbative part is known at N3LO!!). We can work with different schemes (CSS, $\zeta$-prescription).
- We have a re-factorization of TMD at large transverse momentum in Wilson coefficients (now at N3LO!! Gherardo and many others) and PDF (now used at NNLO!!, but N3LO on the way)

$$F_{f\leftarrow h}(x,b) = \sum_{f'} f_{NP}^{f}(x,b) \int_x^1 \frac{dy}{y} C_{f\leftarrow f'}(y, \mathbf{L}_{\mu_{\mathrm{OPE}}}, a_s(\mu_{\mathrm{OPE}}))\, f_{f\leftarrow h}(x/y, \mu_{\mathrm{OPE}})$$

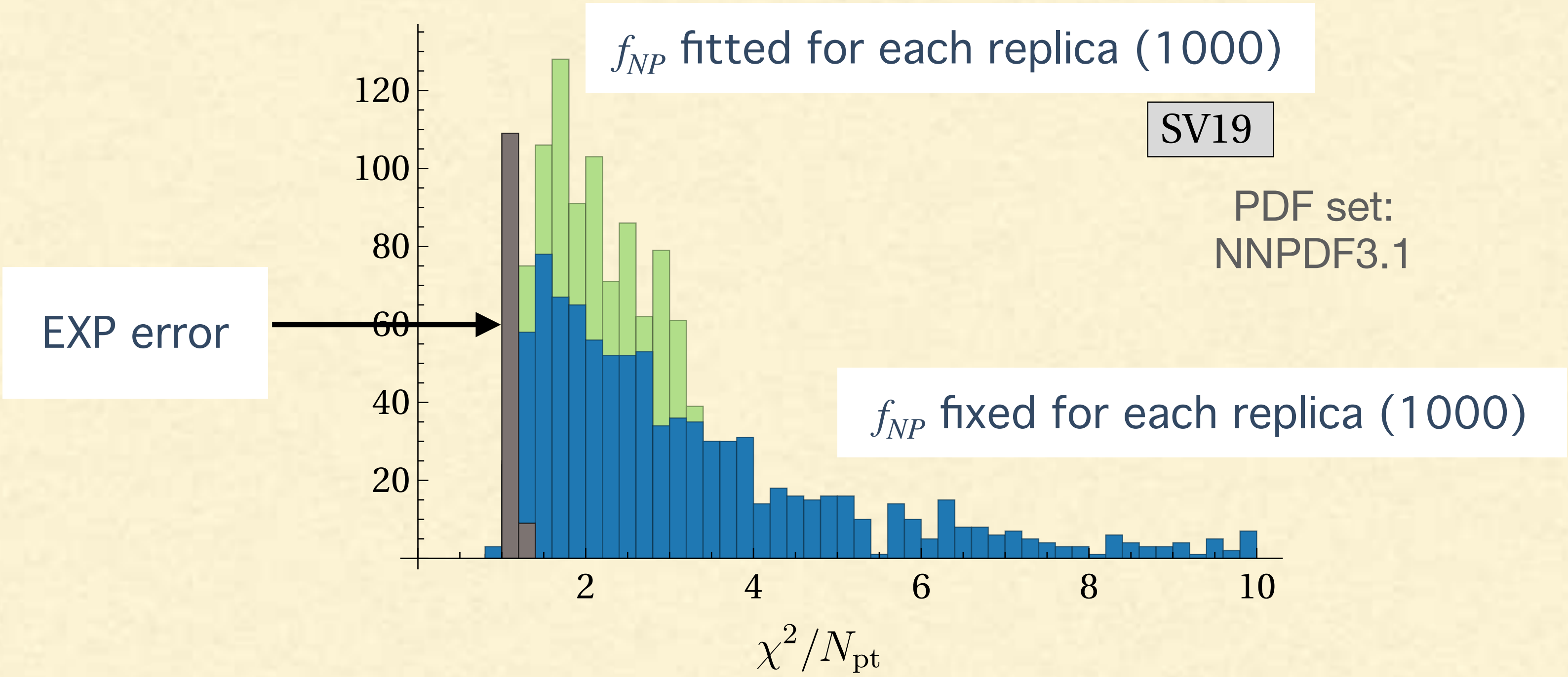
# PDF USAGE IN TMD: SV19 CASE

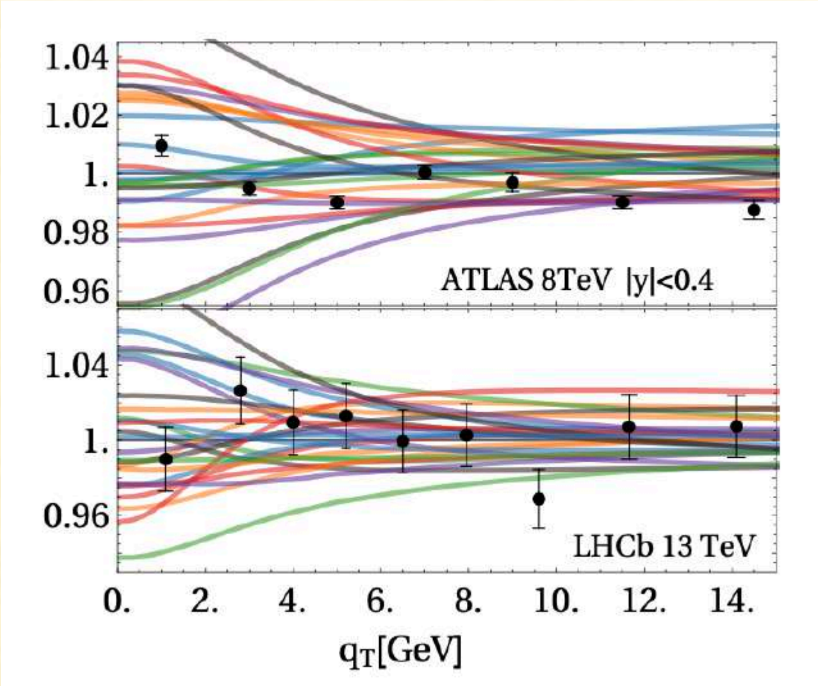
- PDF are just part of a model . Very useful but also problematic: PDF bias

M. Bury, F. Hautmann, S. Leal-Gomez, I. S., A. Vladimirov, PZ, JHEP 10 (2022) 118

| PDF set | $\chi^2_{DY}/N_{pt}$ |
|---|---|
| CT14 | 1,59 |
| HERAPDF2.0 | 0,97 |
| MMHT14 | 1,34 |
| NNPDF3.1 | 1,14 |
| PDF4LHC15 | 1,53 |

- Most of replicas (64%) have $\chi^2$/N>2.
- Each replica has a peculiar shape

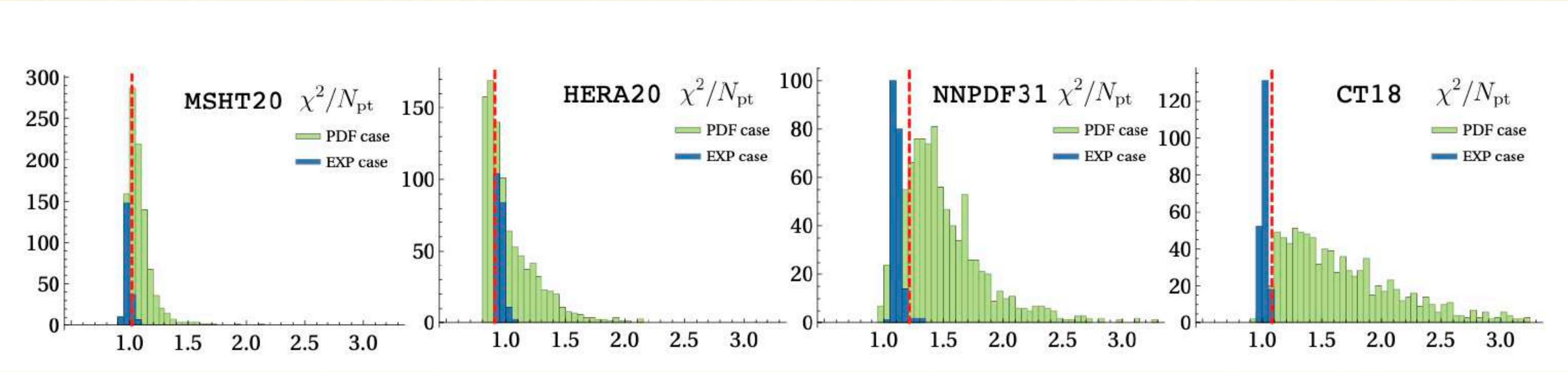


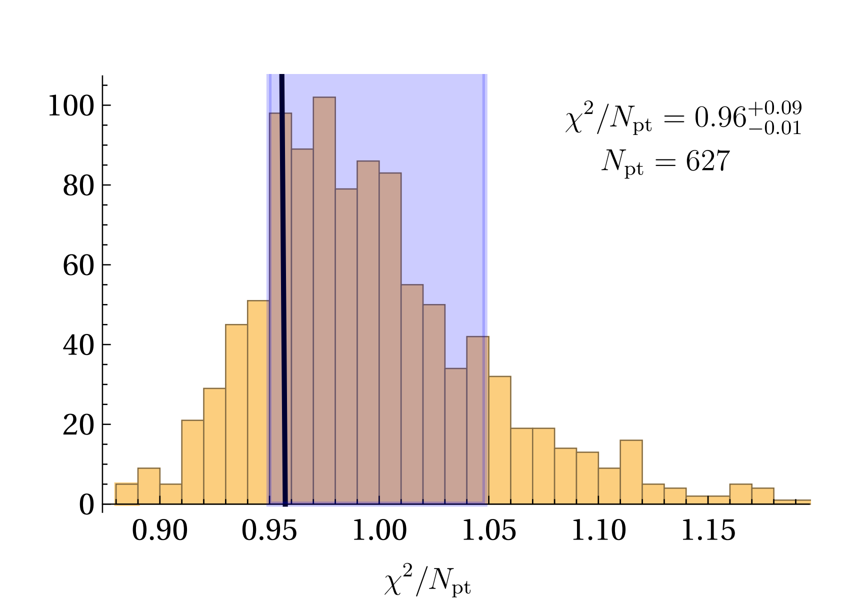



# PDF USAGE IN TMD: SV19 CASE

- PDF bias is strongly reduced using a flavor-dependent $f_{NP}^{f}$

M. Bury, F. Hautmann, S. Leal-Gomez, I. S., A. Vladimirov, PZ, JHEP 10 (2022) 118



# ART23

PUBLIC CODE **ARTEMIDE,**
HTTPS://GITHUB.COM/VLADIMIROVALEXEY/ARTEMIDE-PUBLIC

V. Moos, I. S., A. Vladimirov, P. Zurita, arXiv:2305.07473 [hep-ph]

- All the latest LHC datasets!
- W-boson production! (only Tevatron, $m_T > 50$ GeV)
- Increased perturbative accuracy! ($N^4LL$: highest QCD perturbative precision in a non-perturbative extraction)
- Includes collinear PDF uncertainties!
- TMD flavor dependence included
- A full new fit to Drell-Yan data (627 points)

# RESUMMATION AND $\zeta$-PRESCRIPTION

TMD evolution

$\zeta$-prescription

$$\mu^2 \frac{d}{d\mu^2} F(x,b;\mu,\zeta) = \frac{\gamma_F(\mu,\zeta)}{2} F(x,b;\mu,\zeta)$$

$$\zeta \frac{d}{d\zeta} F(x,b;\mu,\zeta) = -\mathcal{D}(\mu,b) F(x,b;\mu,\zeta)$$

$$F(x,b;\mu_f,\zeta_f) = \exp\left[\int_P \left(\gamma_F(\mu,\zeta)\frac{d\mu}{\mu} - \mathcal{D}(\mu,b)\frac{d\zeta}{\zeta}\right)\right] F(x,b;\mu_i,\zeta_i)$$

The optimal initial condition is identified when $\mathcal{D}(\mu_{saddle}, b) = 0$

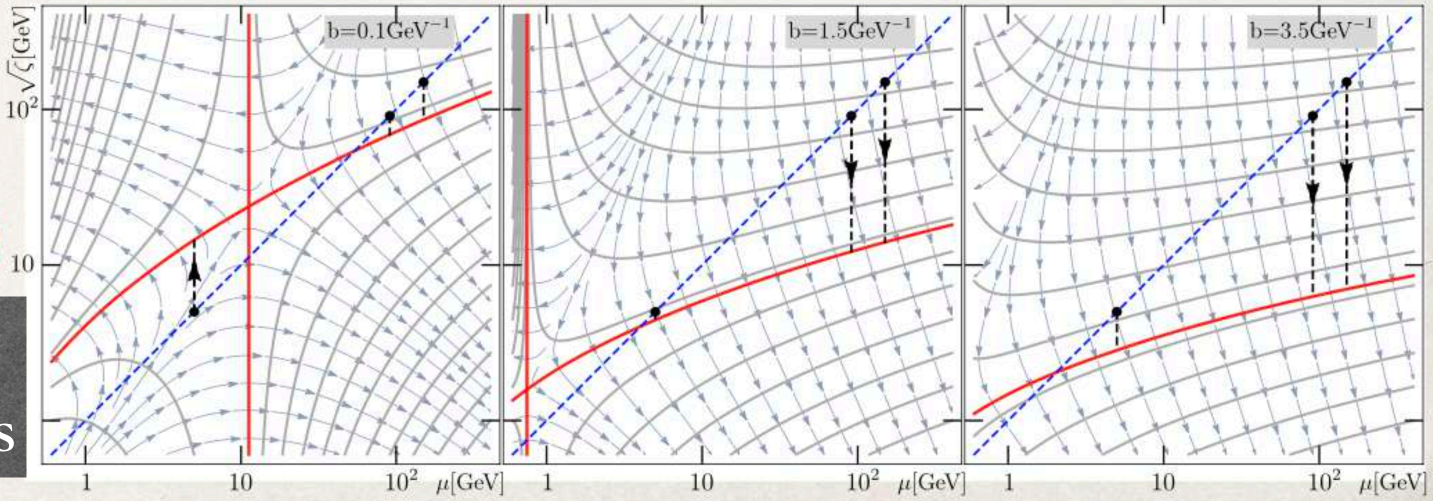
$\gamma_F(\mu_{saddle}, \zeta_{saddle}) = 0$

Path in-dependence is restored changing higher orders in $\gamma_F$

Or defining non-perturbatevely the Null-evolution line

$$R(\mu_f,\zeta_f;\mu_{saddle},\zeta_{saddle}) \equiv R(\mu_f,\zeta_f) = \left(\frac{\zeta_f}{\zeta_{\mu_f}}\right)^{-\mathcal{D}}$$

$$F(x,b,\mu_f,\zeta_f) = R(\mu_f,\zeta_f) F_{optimal}(x,b)$$




PDF and Collins -Soper kernel scales are independent & The optimal TMD is scaleless

# OPTIMAL TMD

The $\zeta$-prescription

$$\begin{cases} \Gamma_{\text{cusp}}(\mu)\ln\left(\dfrac{\mu^2}{\zeta_\mu(b)}\right) - \gamma_V(\mu) = 2\mathscr{D}(b,\mu)\dfrac{d\ln\zeta_\mu(b)}{d\ln\mu^2}. \\ \mathscr{D}(\mu_0, b) = 0, \qquad \gamma_F(\mu_0,\zeta_0) = 0. \end{cases}$$

$$F_{1,q\leftarrow h}(x,b) \equiv F_{1,q\leftarrow h}(x,b,\mu,\zeta_\mu)$$ Scale independence

$$F(x,b;\mu,Q^2) = \left(\frac{Q^2}{\zeta_\mu(b)}\right)^{-\mathscr{D}(b,\mu)} F(x,b)$$ Evolution decoupling

# EVOLUTION KERNEL

Both perturbative and non-perturbative elements are combined.

$$\mathscr{D}(b,\mu) = \mathscr{D}_{\text{small}-\text{b}}(b^*,\mu^*) + \int_{\mu^*}^{\mu} \frac{d\mu'}{\mu'}\Gamma_{\text{cusp}}(\mu') + \mathscr{D}_{\text{NP}}(b),$$

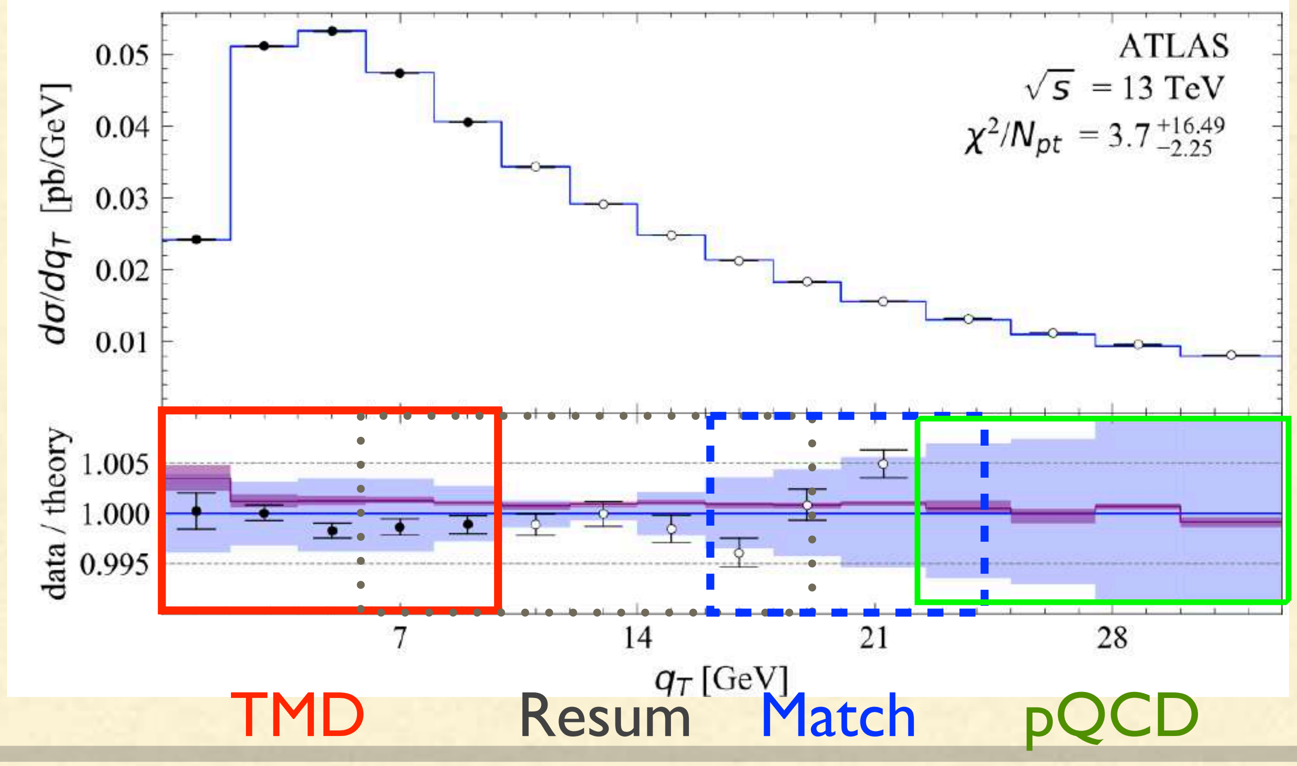
$$\mathscr{D}_{\text{NP}}(b) = bb^* \left[ c_0 + c_1 \ln\left(\frac{b^*}{B_{\text{NP}}}\right) \right]$$

NEW!!

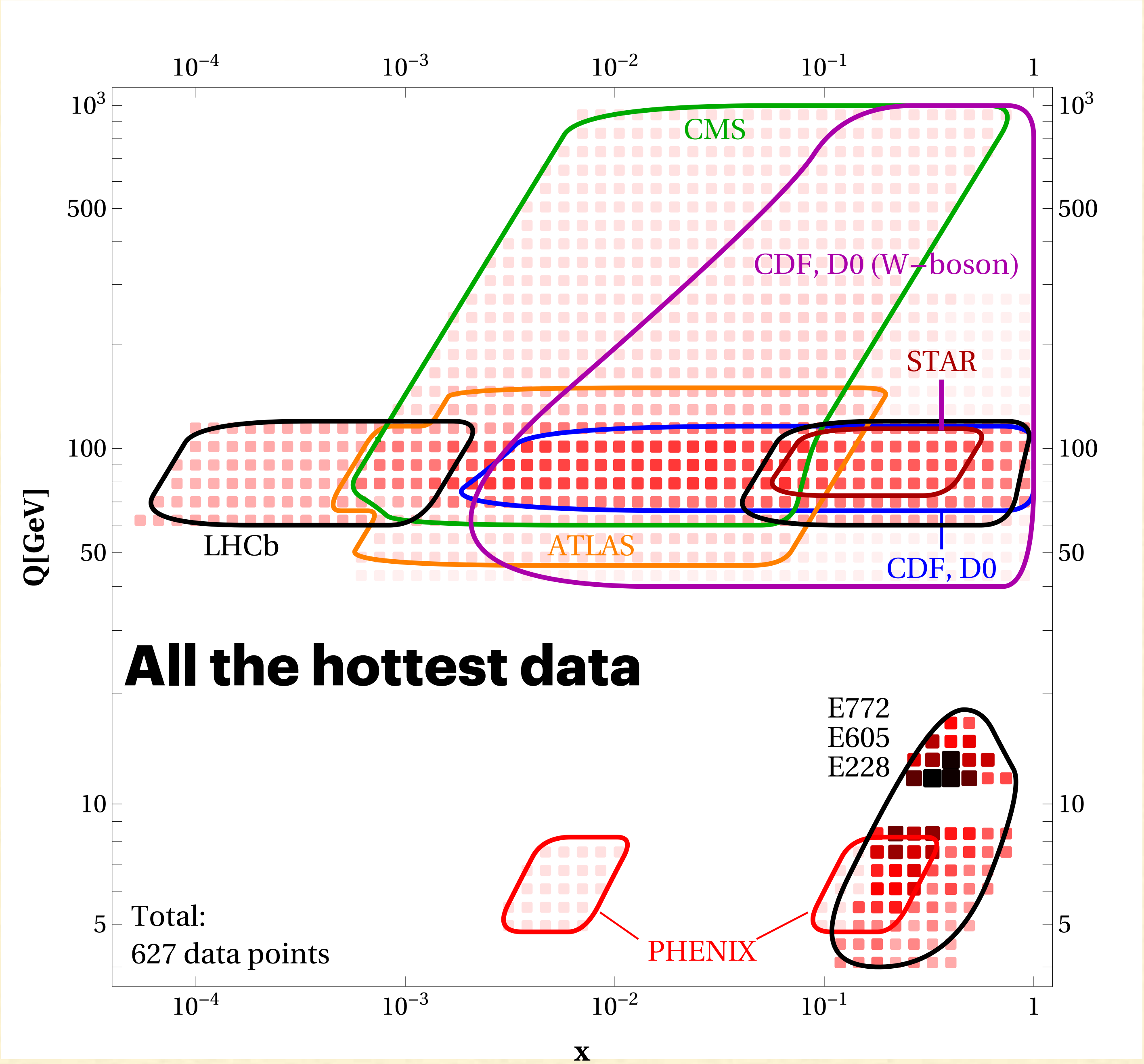
$$b^*(b) = \frac{b}{\sqrt{1+\frac{\vec{b}^2}{B_{\text{NP}}^2}}}, \qquad \mu^*(b) = \frac{2e^{-\gamma_E}}{b^*(b)},$$

# ART23

- Parameterization: $f_{NP}^{f}(x,b) = 1/\cosh[(\lambda_1^f(1-x)+\lambda_2^f x)b]$, $f = u, \bar{u}, d, \bar{d}, sea$
- In total, 13 parameters
- Reference PDFs: MSHT20
- Fitting procedure: construct simultaneous replicas of the **data** **AND** the **PDFs**. Then fit.



# ART23



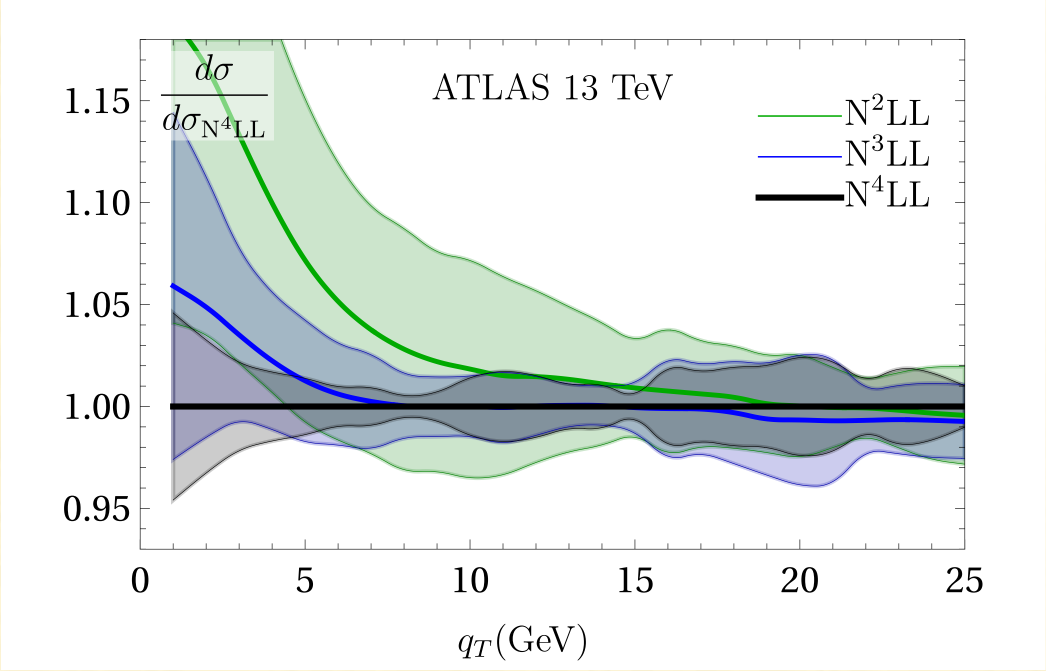

**627 data points**

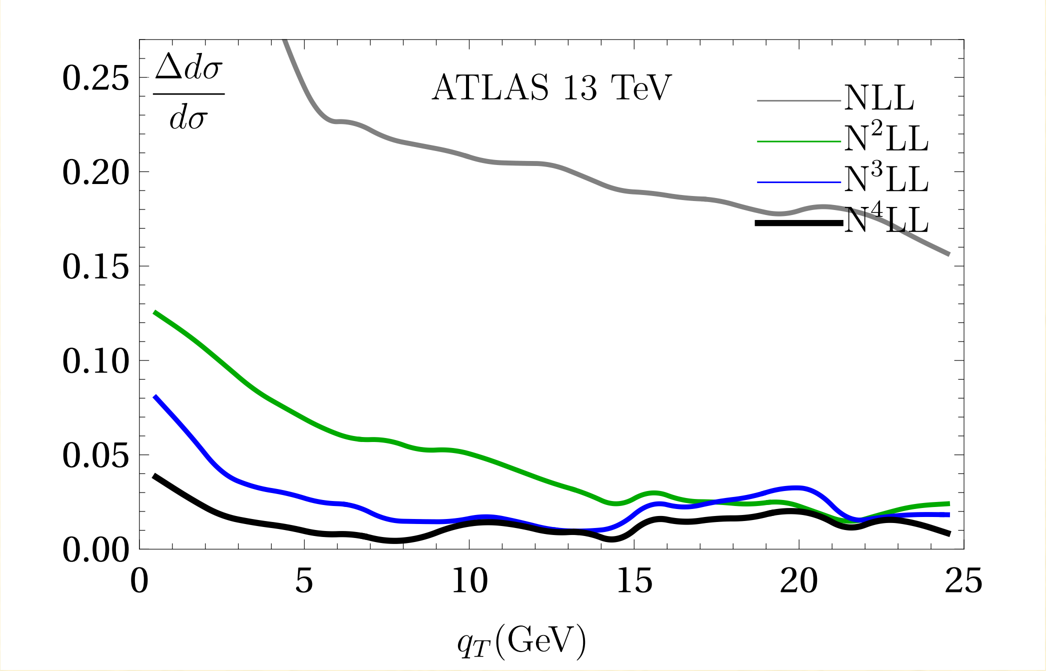
**New in!**

- **PHENIX**: DY data at $\sqrt{s} = 200$ GeV
- **STAR**: Z/γ-boson production at $\sqrt{s} = 510$ GeV (preliminary).
- **CMS** and **LHCb**: y-differential Z-boson production at $\sqrt{s} = 13$ TeV.
- **ATLAS**: high precision differential Z-boson cross-section.
- **CMS**: high-Q neutral-boson production.
- **Tevatron**: W-boson production.

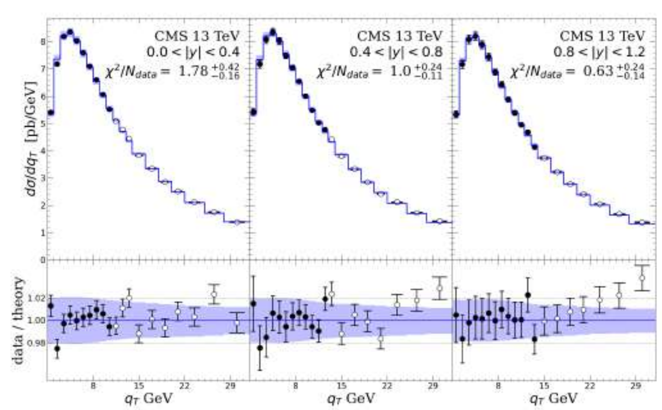
# ART23: RESULTS

- $\chi^2/N_{pt} = 0.93$ (0.957 for the mean prediction), 68%CI (0.950, 1.048)
- Higher data precision plays a key role here.
- Realistic uncertainty bands. Main error from PDF.

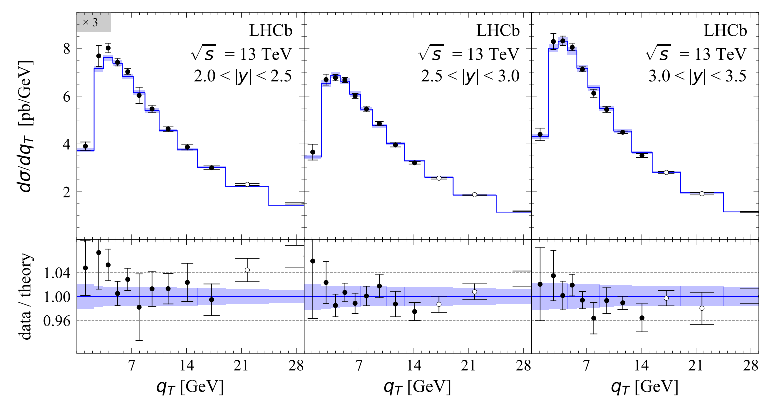


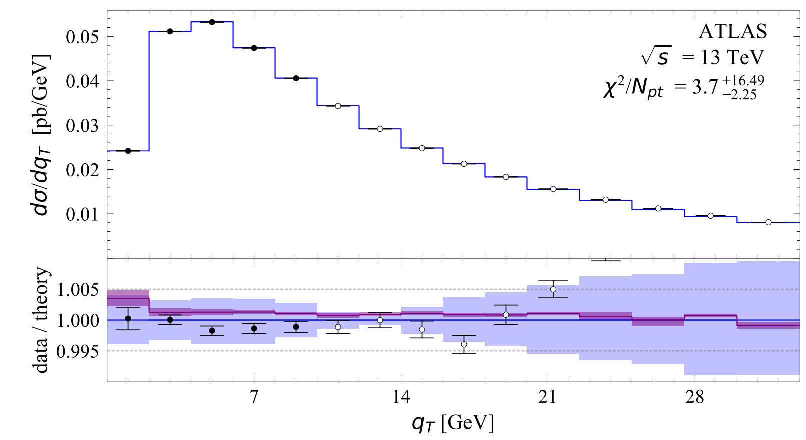
# RESULTS



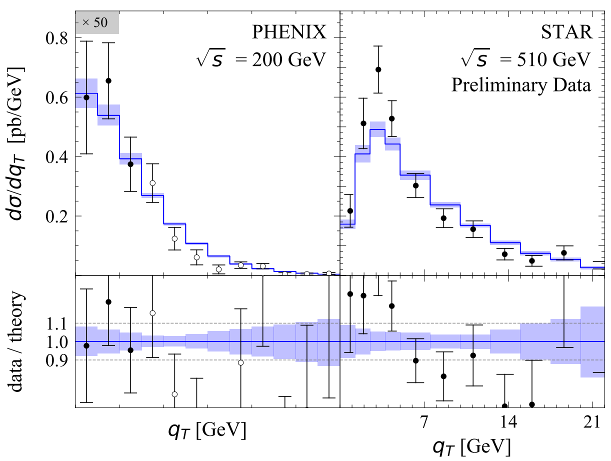
CMS 13 TeV
$0.0 < |y| < 0.4$
$\chi^2/N_{data} = 1.78^{+0.42}_{-0.16}$
CMS 13 TeV
$0.4 < |y| < 0.8$
$\chi^2/N_{data} = 1.0^{+0.24}_{-0.11}$
CMS 13 TeV
$0.8 < |y| < 1.2$
$\chi^2/N_{data} = 0.63^{+0.24}_{-0.14}$
$d\sigma/dq_T$ [pb/GeV]
data / theory
$q_T$ GeV

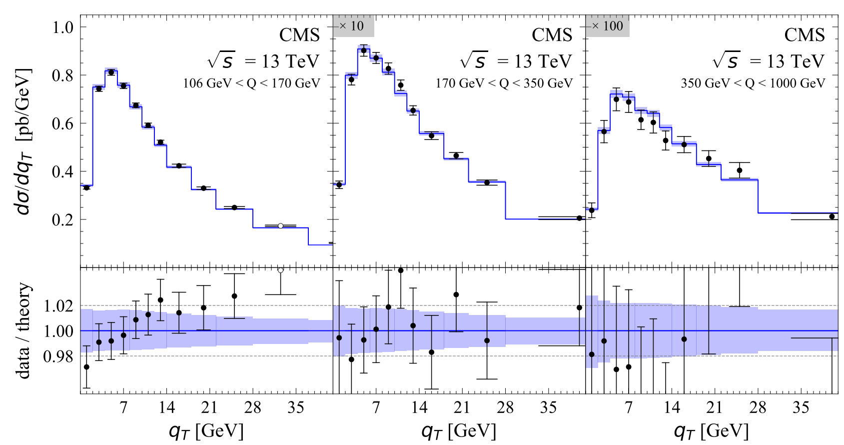
× 3
LHCb
$\sqrt{s}$ = 13 TeV
$2.0 < |y| < 2.5$
LHCb
$\sqrt{s}$ = 13 TeV
$2.5 < |y| < 3.0$
LHCb
$\sqrt{s}$ = 13 TeV
$3.0 < |y| < 3.5$
$d\sigma/dq_T$ [pb/GeV]
data / theory
$q_T$ [GeV]

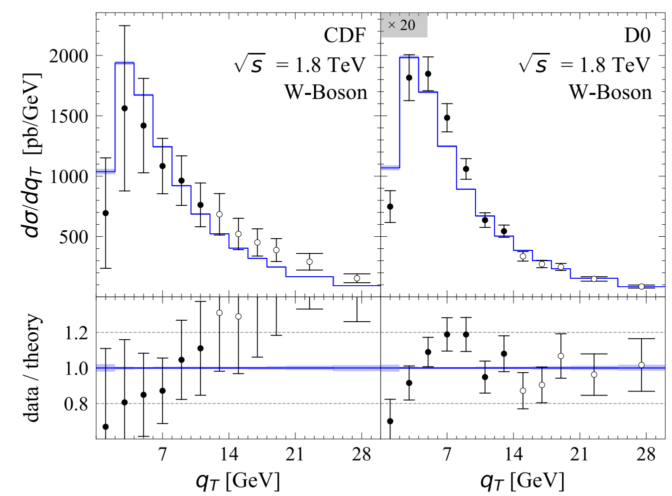
ATLAS
$\sqrt{s}$ = 13 TeV
$\chi^2/N_{pt} = 3.7^{+16.49}_{-2.25}$
$d\sigma/dq_T$ [pb/GeV]
data / theory
$q_T$ [GeV]

× 50
PHENIX
$\sqrt{s}$ = 200 GeV
STAR
$\sqrt{s}$ = 510 GeV
Preliminary Data
$d\sigma/dq_T$ [pb/GeV]
data / theory
$q_T$ [GeV]

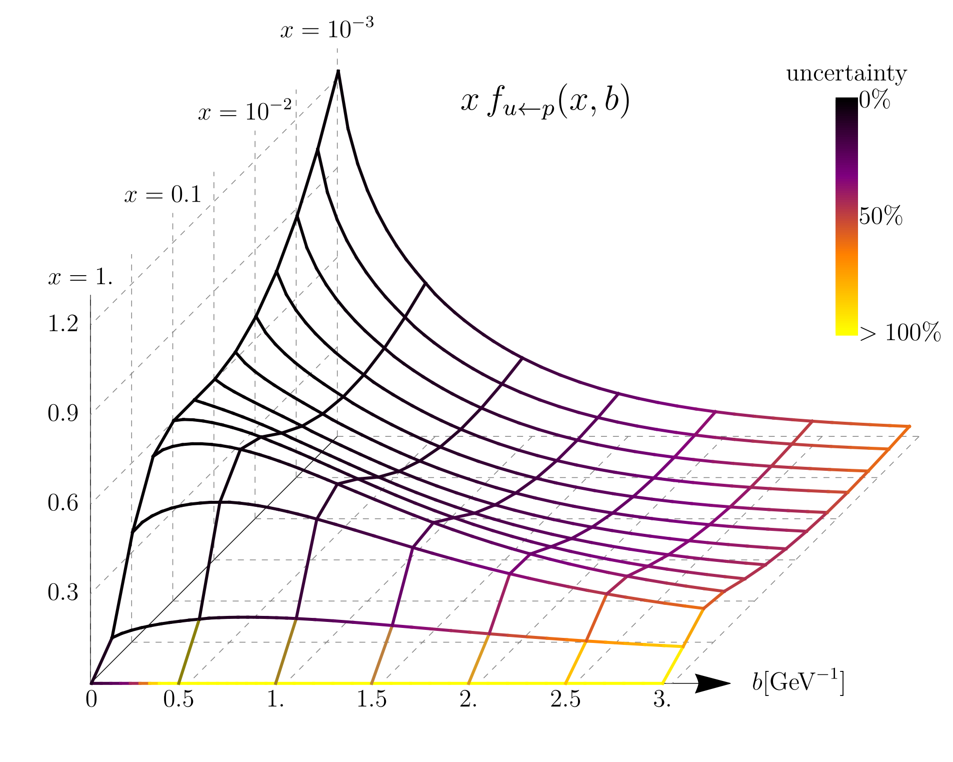



CMS
$\sqrt{s}$ = 13 TeV
106 GeV < Q < 170 GeV
× 10
CMS
$\sqrt{s}$ = 13 TeV
170 GeV < Q < 350 GeV
× 100
CMS
$\sqrt{s}$ = 13 TeV
350 GeV < Q < 1000 GeV
$d\sigma/dq_T$ [pb/GeV]
data / theory
$q_T$ [GeV]

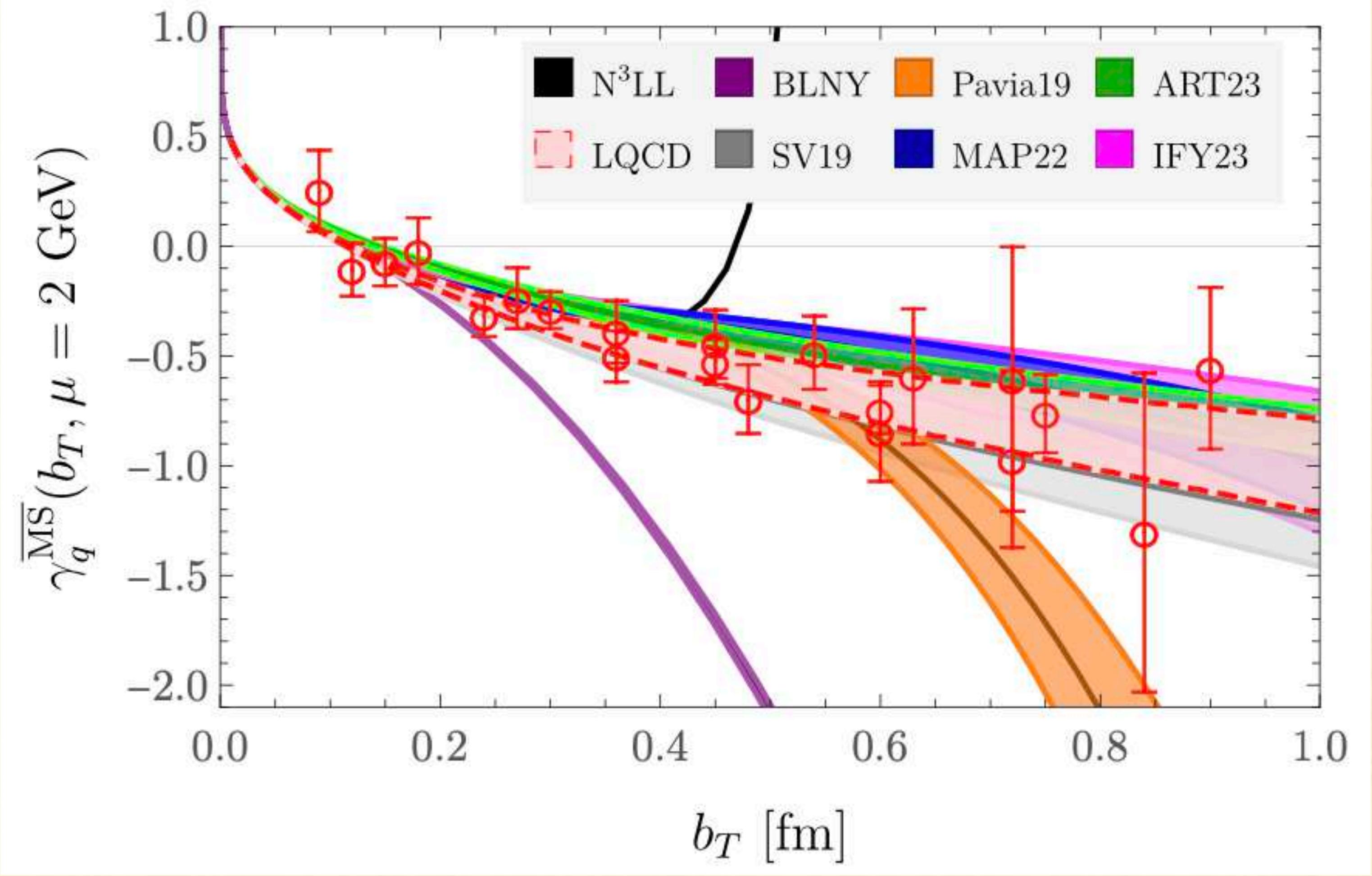
CDF
$\sqrt{s}$ = 1.8 TeV
W-Boson
× 20
D0
$\sqrt{s}$ = 1.8 TeV
W-Boson
$d\sigma/dq_T$ [pb/GeV]
data / theory
$q_T$ [GeV]

# Results in detail



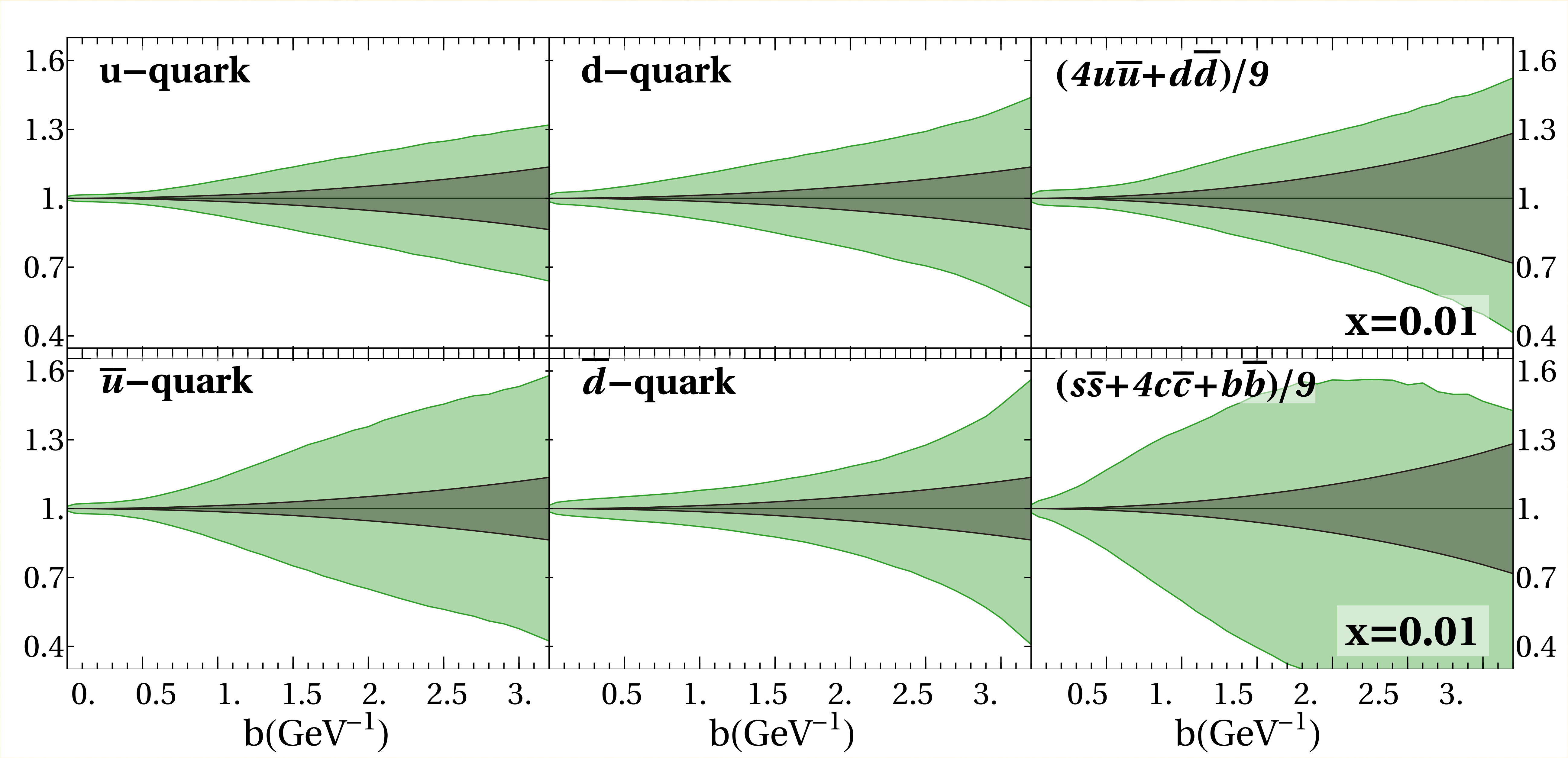

| dataset | $N_{pt}$ | $\chi^2_D/N_{pt}$ | $\chi^2_\lambda/N_{pt}$ | $\chi^2/N_{pt}$ |
|---|---|---|---|---|
| CDF (run1) | 33 | 0.51 | 0.16 | $0.67^{+0.05}_{-0.03}$ |
| CDF (run2) | 45 | 1.58 | 0.11 | $1.59^{+0.26}_{-0.14}$ |
| CDF (W-boson) | 6 | 0.33 | 0.00 | $0.33^{+0.01}_{-0.01}$ |
| D0 (run1) | 16 | 0.69 | 0.00 | $0.69^{+0.08}_{-0.03}$ |
| D0 (run2) | 13 | 2.16 | 0.16 | $2.32^{+0.40}_{-0.32}$ |
| D0 (W-boson) | 7 | 2.39 | 0.00 | $2.39^{+0.20}_{-0.18}$ |
| ATLAS (8TeV, $Q \sim M_Z$) | 30 | 1.60 | 0.49 | $2.09^{+1.09}_{-0.35}$ |
| ATLAS (8TeV) | 14 | 1.11 | 0.11 | $1.22^{+0.47}_{-0.21}$ |
| ATLAS (13 TeV) | 5 | 1.94 | 1.75 | $3.70^{+16.5}_{-2.24}$ |
| CMS (7TeV) | 8 | 1.30 | 0.00 | $1.30^{+0.03}_{-0.01}$ |
| CMS (8TeV) | 8 | 0.79 | 0.00 | $0.78^{+0.02}_{-0.01}$ |
| CMS (13 TeV, $Q \sim M_Z$) | 64 | 0.63 | 0.24 | $0.86^{+0.23}_{-0.11}$ |
| CMS (13 TeV, $Q > M_Z$) | 33 | 0.73 | 0.12 | $0.92^{+0.40}_{-0.15}$ |
| LHCb (7 TeV) | 10 | 1.21 | 0.56 | $1.77^{+0.53}_{-0.31}$ |
| LHCb (8 TeV) | 9 | 0.77 | 0.78 | $1.55^{+0.94}_{-0.50}$ |
| LHCb (13 TeV) | 49 | 1.07 | 0.10 | $1.18^{+0.25}_{-0.01}$ |
| PHENIX | 3 | 0.29 | 0.12 | $0.42^{+0.15}_{-0.10}$ |
| STAR | 11 | 1.91 | 0.28 | $2.19^{+0.51}_{-0.31}$ |
| E288 (200) | 43 | 0.31 | 0.07 | $0.38^{+0.12}_{-0.05}$ |
| E288 (300) | 53 | 0.36 | 0.07 | $0.43^{+0.08}_{-0.04}$ |
| E288 (400) | 79 | 0.37 | 0.05 | $0.48^{+0.11}_{-0.03}$ |
| E772 | 35 | 0.87 | 0.21 | $1.08^{+0.08}_{-0.05}$ |
| E605 | 53 | 0.18 | 0.21 | $0.39^{+0.03}_{-0.00}$ |
| **Total** | 627 | 0.79 | 0.17 | $0.96^{+0.09}_{-0.01}$ |

# ART23: LATTICE COMPARISON




Artur Avkhadiev,1 Phiala E. Shanahan,1 Michael L. Wagman,2 and Yong Zhao: arXive2402.06725

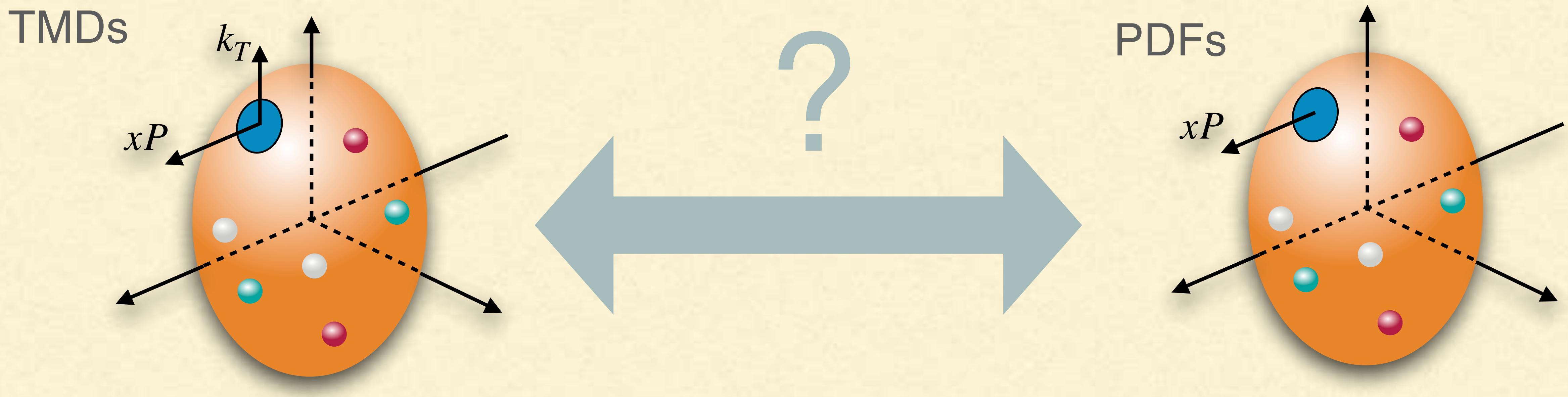
# ART23: RESULTS




Light green: ART23
Dark green: SV19

# WHAT IS THE RELATIONSHIP?

*Oscar del Rio, Alexei Prokudin, I.S., Alexey Vladimirov e-Print: 2402.01836 (2024)*




In principle TMDs are related to PDFs upon integration out the transverse momentum, but what about renormalization scale?

# Evolution

DGLAP EQUATIONS

*Integro-differential equations*

Non diagonal in flavor space

$$\mu^2 \frac{d}{d\mu^2} f_q(x,\mu) = \sum_{f'} \int_x^1 \frac{dy}{y} P_{q\to q'}(y) f_{q'}\left(\frac{x}{y},\mu\right)$$

$\mu$= *UV renormalization scale*

COLLINS-SOPER EQUATIONS

*Double scale differential equations*

Diagonal in flavor space

$$\frac{d \ln \tilde{F}(x, b_T, \mu, \zeta)}{d \ln \mu} = \gamma_F(\mu)$$

$$\frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu)$$

$$\frac{d\tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(\mu)$$

$\zeta$= *Collins-Soper parameter*

**Collins-Soper kernel $\tilde{K}$ is specific for TMDs**

# TRANSVERSE MOMENTUM MOMENTS

O. del Rio, A. Prokudin, I.S., A. Vladimirov e-Print: 2402.01836

- TMMs are weighted integrals with an upper cut-off

$$\mathcal{M}^{[\Gamma]}_{\nu_1\ldots\nu_n}(x,\mu) \equiv \int^{\mu} d^2\vec{k}_T\, \vec{k}_{T\nu_1}...\vec{k}_{T\nu_n} F^{[\Gamma]}(x,k_T)$$

for TMDs in the $\zeta$-prescription which has no scale dependence

$$\mathcal{M}^{*[\Gamma]}_{\nu_1\ldots\nu_n}(x,\mu) \equiv \int^{\mu} d^2\vec{k}_T\, \vec{k}_{T\nu_1}...\vec{k}_{T\nu_n} F^{[\Gamma]}(x,k_T;\mu,\mu^2)$$

for TMDs in the general prescription *For 0-moment: M. Ebert, J. Michel, I. Stewart, Z. Sun, JHEP 07 (2022) 129*

- The upper cut-off becomes the scale at which the collinear functions are evaluated
- TMMs obey DGLAP equations
- We provide a definition for all moments

# TMDS IN $b$-SPACE AND $\mathcal{G}$ OPERATION

TMDs in $b$ space are parametrized as

$$\tilde{F}^{[\gamma^+]}(x,b) = \tilde{f}_1(x,b) + i\epsilon_T^{\mu\nu} b_\mu s_{T\nu} M \tilde{f}_{1T}^{\perp}(x,b),$$

$$\tilde{F}^{[\gamma^+\gamma^5]}(x,b) = \lambda \tilde{g}_1(x,b) + i(b\cdot s_T) M \tilde{g}_{1T}^{\perp}(x,b),$$

$$\tilde{F}^{[i\sigma^{\alpha+}\gamma^5]}(x,b) = s_T^\alpha \tilde{h}_1(x,b) - i\lambda b^\alpha M \tilde{h}_{1L}^{\perp}(x,b)$$
$$+ i\epsilon_T^{\alpha\mu} b_\mu M \tilde{h}_1^{\perp}(x,b) + \frac{M^2}{4}(g_T^{\alpha\mu}\mathbf{b}^2 + 2b^\alpha b^\mu) s_{T\mu} \tilde{h}_{1T}^{\perp}(x,b)$$

# TMDS IN $b$-SPACE AND $\mathcal{G}$ OPERATION

Fourier transformation: angular integrations are trivial

*D. Boer, L. Gamberg, B. Musch, and A. Prokudin, JHEP 10, 021 (2011)*

$$\widetilde{F}^{(n)}(x, b_T; \mu, \zeta) \equiv n! \left(\frac{-1}{M^2 b}\partial_b\right)^n \widetilde{F}(x, b; \mu, \zeta) = \frac{2\pi\, n!}{(bM)^n} \int_0^\infty dk_T\, k_T \left(\frac{k_T}{M}\right)^n J_n(bk_T)\, F(x, k_T; \mu, \zeta)$$

$$\widetilde{f}_1(x, b) \equiv \widetilde{f}_1^{(0)}(x, b), \qquad f_{1T}^{\perp}(x, b) \equiv \widetilde{f}_{1T}^{\perp(1)}(x, b),$$

$$\widetilde{g}_1(x, b) \equiv \widetilde{g}_1^{\perp(0)}(x, b), \qquad \widetilde{g}_{1T}^{\perp}(x, b) \equiv \widetilde{g}_{1T}^{\perp(1)}(x, b),$$

$$\widetilde{h}_1(x, b) \equiv \widetilde{h}_1^{\perp(0)}(x, b), \qquad \widetilde{h}_{1L}^{\perp}(x, b) \equiv \widetilde{h}_{1L}^{\perp(1)}(x, b),$$

$$\widetilde{h}_1^{\perp}(x, b) \equiv \widetilde{h}_1^{\perp(1)}(x, b), \qquad \widetilde{h}_{1T}^{\perp}(x, b) \equiv \widetilde{h}_{1T}^{\perp(2)}(x, b).$$

The superscript $(m)$ determines the large $k_T$ asymptotic $\quad f(x, k_T) \propto \frac{M^{2m}}{(k_T^2)^{m+1}}$

# TMDS IN $b$-SPACE AND $\mathscr{G}$ OPERATION

- Operation $\mathscr{G}$ is defined as

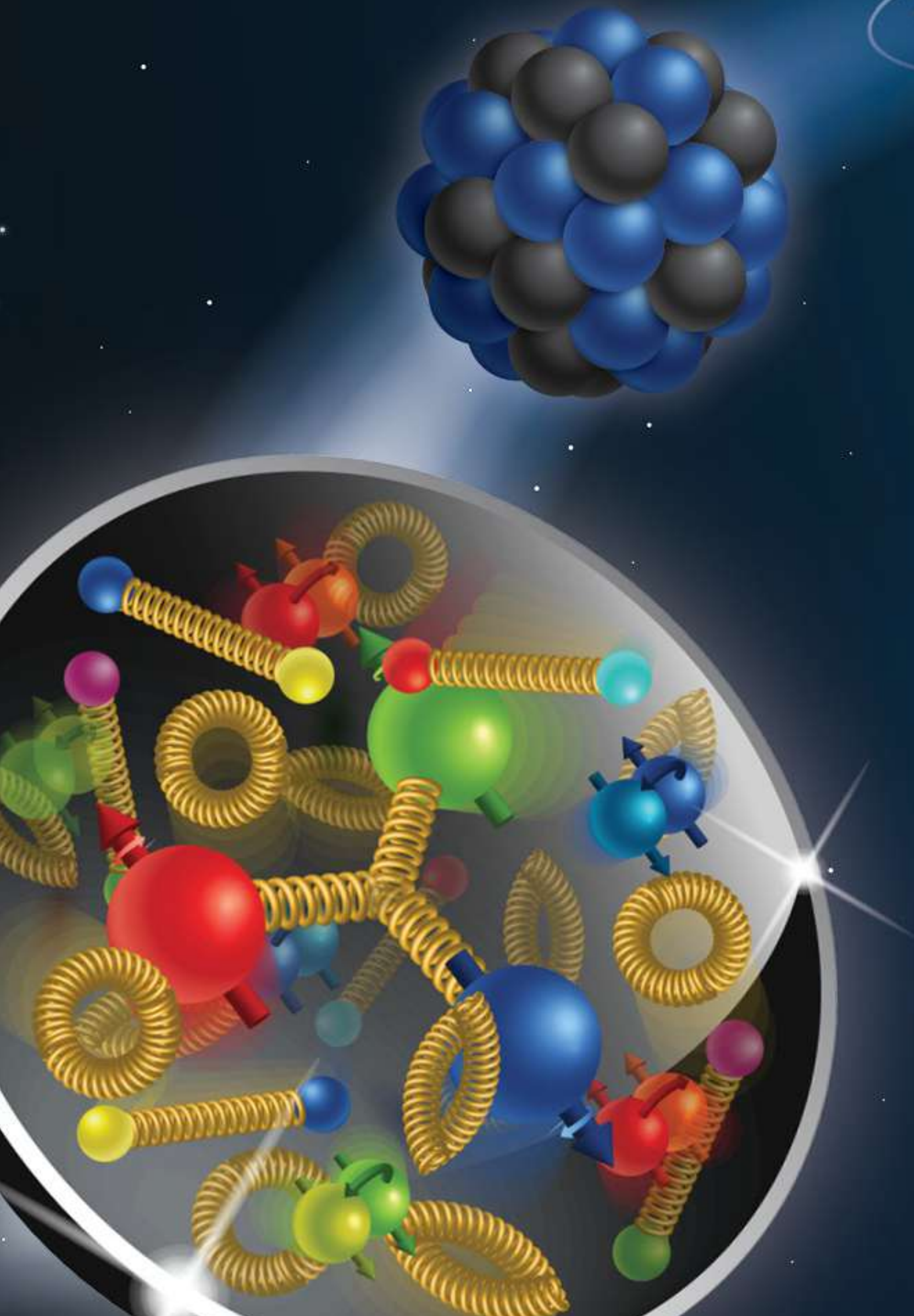
$$\mathcal{G}_{n,m}[f](x,\mu) \;=\; \int^{\mu} d^2\boldsymbol{k}_T \left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^n f(x,k_T)$$

- Without cut-off it corresponds to the conventional $n^{th}$ moment of TMD, $m$ is the corresponding superscript of the TMD $\tilde{f}$

- Its properties: $n = m$ logarithmic divergence, $n = m + l$ power divergence in $\mu$

$$\mathcal{G}_{m,m}[f](x,\mu) \;\propto\; \log(\mu)\;,$$
$$\mathcal{G}_{m+l,m}[f](x,\mu) \;\propto\; \mu^{2l} \text{ for } m+l \geq 0$$

- The logarithmic divergence for $n = m$ is the UV divergence that corresponds to the divergence of the collinear functions



# $0^{th}$ TMM, $1^{st}$ TMM, AND $2^{nd}$ TMM

# ZEROTH TMM

- The $0^{th}$ TMM is

$$
\begin{aligned}
\mathcal{M}^{[\gamma^+]}(x,\mu) = \int^{\mu} d^2\boldsymbol{k}_T F^{[\gamma^+]}(x,k_T) &= \int^{\mu} d^2\boldsymbol{k}_T f_1(x,k_T), \\
\mathcal{M}^{[\gamma^+\gamma_5]}(x,\mu) = \int^{\mu} d^2\boldsymbol{k}_T F^{[\gamma^+\gamma^5]}(x,k_T) &= \lambda \int^{\mu} d^2\boldsymbol{k}_T g_1(x,k_T), \\
\mathcal{M}^{[i\sigma^{\alpha+}\gamma^5]}(x,\mu) = \int^{\mu} d^2\boldsymbol{k}_T F^{[i\sigma^{\alpha+}\gamma^5]}(x,k_T) &= s_T^{\alpha} \int^{\mu} d^2\boldsymbol{k}_T h_1(x,k_T) \\
&\quad - \int^{\mu} d^2\boldsymbol{k}_T \frac{\boldsymbol{k}_T^2}{M^2}\left(\frac{g_T^{\alpha\mu}}{2} + \frac{k_T^{\alpha}k_T^{\mu}}{\boldsymbol{k}_T^2}\right) s_{T\mu} h_{1T}^{\perp}(x,k_T),
\end{aligned}
$$

$\propto \mu^{-2}$ so we drop it

# ZEROTH TMM

- In practice we obtain PDF in a certain (TMD) scheme

$$\begin{aligned}
\mathcal{M}^{[\gamma^+]}(x,\mu) &= \mathcal{G}_0[f_1](x,\mu), \\
\mathcal{M}^{[\gamma^+\gamma^5]}(x,\mu) &= s_L\mathcal{G}_0[g_1](x,\mu), \\
\mathcal{M}^{[i\sigma^{\alpha+}\gamma^5]}(x,\mu) &= s_T^\alpha\mathcal{G}_0[h_1](x,\mu),
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\mathcal{G}_0[f_1](x,\mu) &= q^{(\mathrm{TMD})}(x,\mu)+\mathcal{O}(\mu^{-2}), \\
\mathcal{G}_0[g_1](x,\mu) &= \Delta q^{(\mathrm{TMD})}(x,\mu)+\mathcal{O}(\mu^{-2}), \\
\mathcal{G}_0[h_1](x,\mu) &= \delta q^{(\mathrm{TMD})}(x,\mu)+\mathcal{O}(\mu^{-2}).
\end{aligned}$$

Using Wilson coefficients of small-$b$ and large-$\mu$ asymptotic expansion of Hankel transform one obtains

$$\mathcal{G}_0[F](x,\mu) = \Big\{\mathbf{1}+\alpha_s C_1+\alpha_s^2 C_2$$

$$+\alpha_s^3\Big[\frac{2\zeta_3}{3}\left(P_1\otimes P_1\otimes P_1-3\beta_0 P_1\otimes P_1+2\beta_0^2 P_1\right)+C_3\Big]+\mathcal{O}(\alpha_s^4)\Big\}\otimes f(x,\mu)+\mathcal{O}(\mu^{-2}),$$

*R. Wong, Computers & Mathematics with Applications 3, 271 (1977).*
*R. F. MacKinnon, Mathematics of Computation 26, 515 (1972).*

# ZEROTH TMM

- All scales in the TMD are set to $\mu$ and we have a DGLAP equation

$$\mu^2 \frac{d}{d\mu^2} f^{(\mathrm{TMD})}(x,\mu) = P' \otimes f^{(\mathrm{TMD})}(x,\mu).$$

- Therefore it is the same as PDFs but computed in a different scheme.
- The difference in splitting functions is of order $\alpha_s^2$ and it is calculable

$$P' - P = -\alpha_s^2 \beta_0 C_1 - \alpha_s^3 \left(2\beta_0 C_2 - \beta_0 C_1 \otimes C_1 + \beta_1 C_1\right) + \mathcal{O}(\alpha_s^4).$$

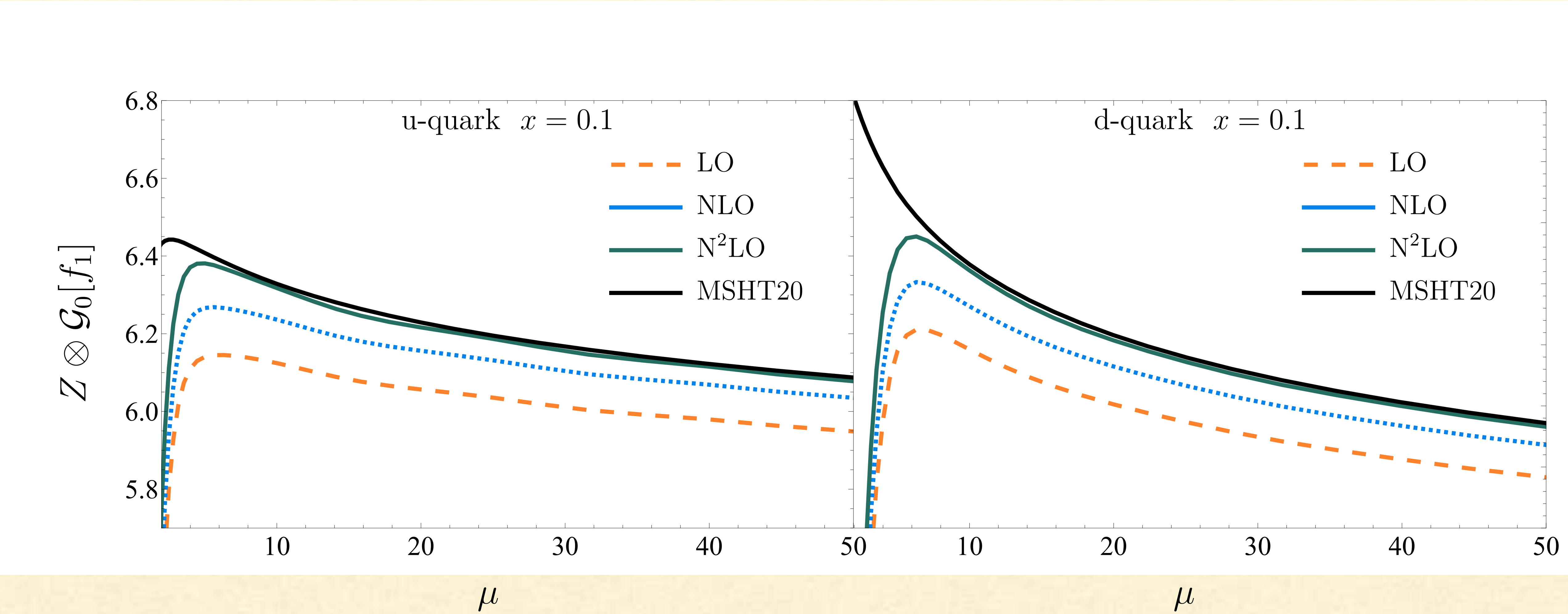
# ZEROTH TMM

- We call this scheme TMD-scheme and the coefficient to transform to $\overline{MS}$ scheme reads

$$f_f^{(\overline{\text{MS}})}(x,\mu) = \sum_{f'} \int_x^1 \frac{dy}{y} Z_{f\leftarrow f'}^{\overline{\text{MS}}/\text{TMD}}(y,\mu) f_{f'}^{(\text{TMD})}\left(\frac{x}{y},\mu\right)$$

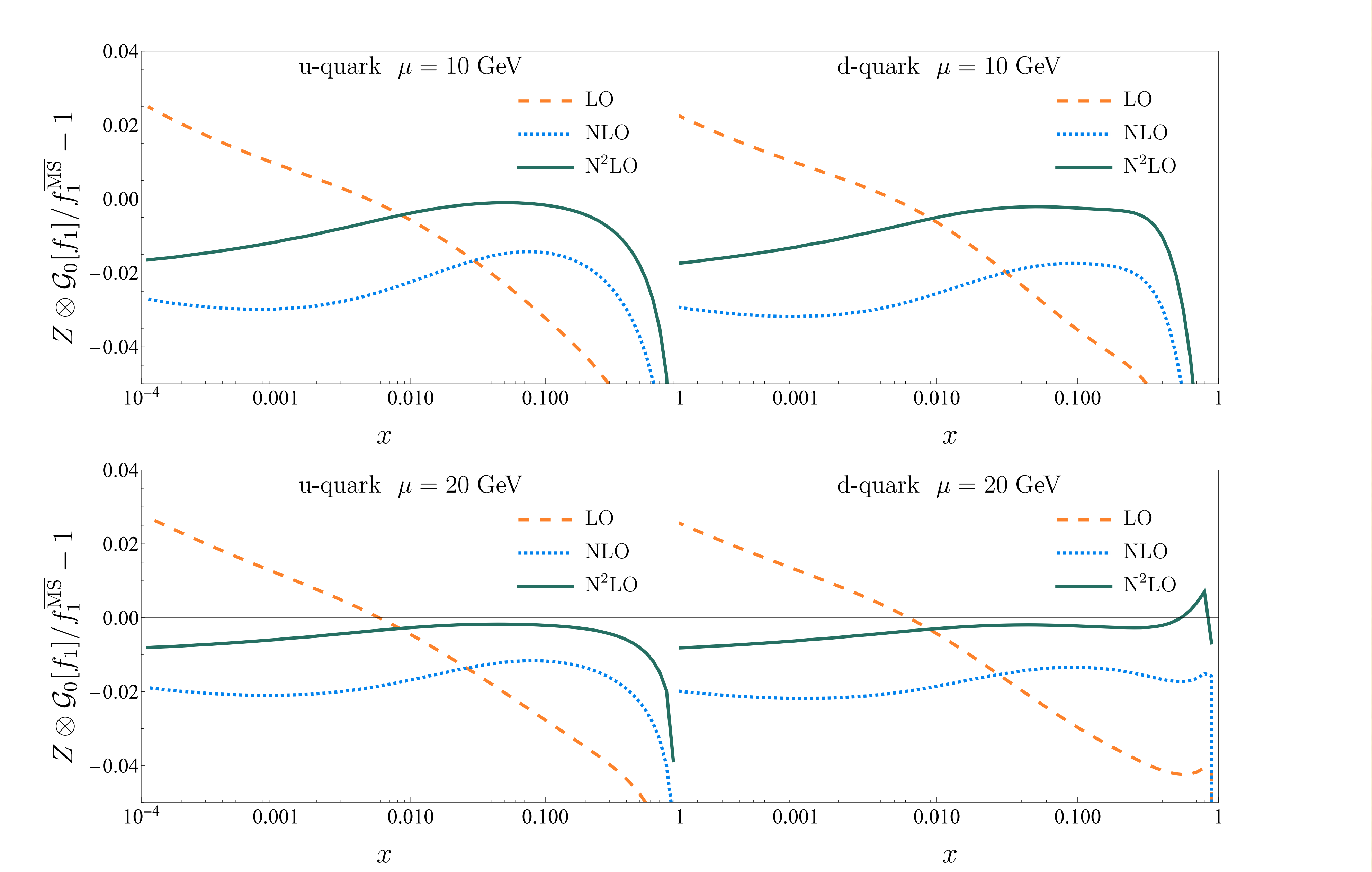
$$Z^{\overline{\text{MS}}/\text{TMD}} = \mathbf{1} - \alpha_s C_1 - \alpha_s^2 (C_2 - C_1 \otimes C_1)$$
$$-\alpha_s^3 \left[ C_3 + C_1 \otimes C_1 \otimes C_1 - C_1 \otimes C_2 - C_2 \otimes C_1 + \frac{2\zeta_3}{3} P_1 \otimes (P_1 - \beta_0 \cdot \mathbf{1}) \otimes (P_1 - 2\beta_0 \cdot \mathbf{1}) \right] + \mathcal{O}(\alpha_s^4)$$

# ZEROTH TMM



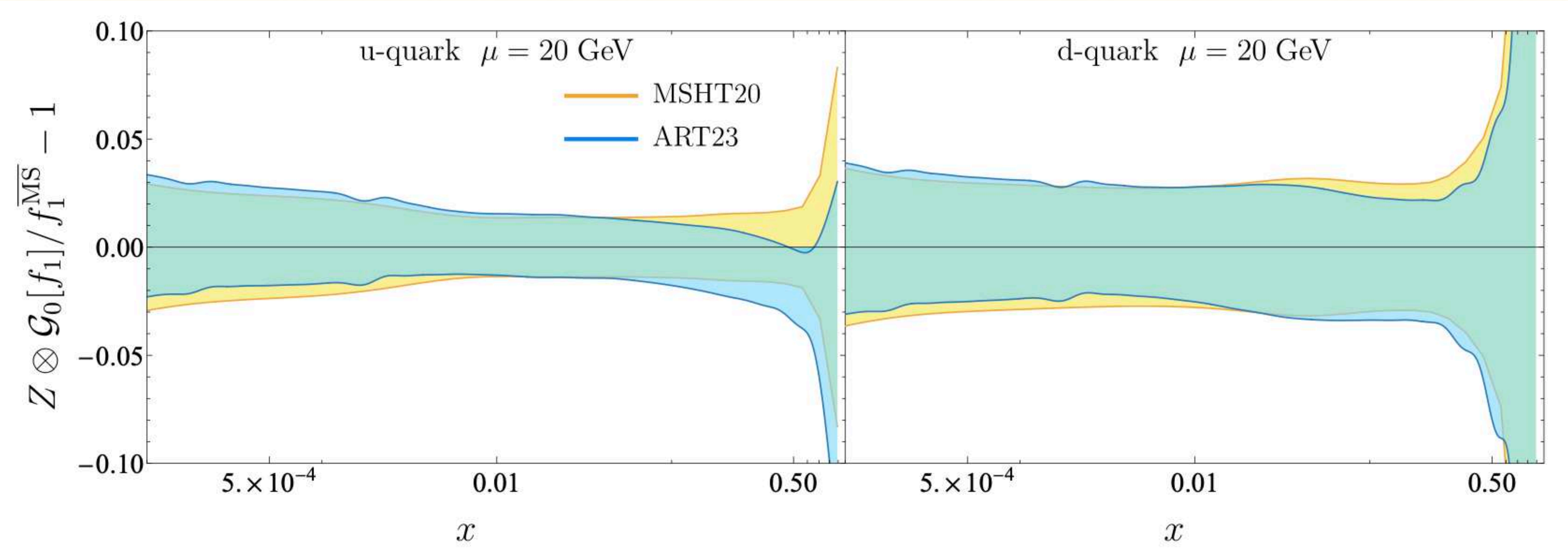

- Above $\mu \geq 5$ GeV the correspondence is quite precise

# ZEROTH TMM




➤ TMDs are from ART 23 extraction *V. Moos, I. Scimemi, A. Vladimirov, and P. Zurita, (2023), arXiv:2305.07473*

# ZEROTH TMM: FROM PDF TO TMD TO PDF




- *We can reproduce the errors: a very nice consistency check.*

# FIRST TMM

- The $1^{st}$ TMM is related to the small-b power expansion of a TMD

$$
\begin{aligned}
\mathcal{G}_1[f_{1T}^{\perp}](x,\mu) &= \pm\frac{\pi}{2}T^{(\text{TMD})}(-x,0,x;\mu)+\mathcal{O}(\mu^{-2}),\\
\mathcal{G}_1[g_{1T}^{\perp}](x,\mu) &= \frac{x}{2}\int_x^1\frac{dy}{y}\Delta q^{(\text{TMD})}(y,\mu)+x\int_{-1}^1 dy_1dy_2dy_3\delta(y_1+y_2+y_3)\int_0^1 d\alpha\delta(x-\alpha y_3)\Big[\\
&\qquad \frac{\Delta T^{(\text{TMD})}(y_{123};\mu)}{y_2^2}+\frac{T^{(\text{TMD})}(y_{123};\mu)-\Delta T^{(\text{TMD})}(y_{123};\mu)}{2y_2y_3}\Big]+\mathcal{O}(\mu^{-2}),\\
\mathcal{G}_1[h_{1L}^{\perp}](x,\mu) &= -\frac{x^2}{2}\int_x^1\frac{dy}{y}\delta q^{(\text{TMD})}(y,\mu)\\
&\quad -x\int_{-1}^1 dy_1dy_2dy_3\delta(y_1+y_2+y_3)\int_0^1 d\alpha\alpha\delta(x-\alpha y_3)H^{(\text{TMD})}(y_{123};\mu)\frac{y_3-y_2}{y_2^2y_3}+\mathcal{O}(\mu^{-2}),\\
\mathcal{G}_1[h_1^{\perp}](x,\mu) &= \mp\frac{\pi}{2}E^{(\text{TMD})}(-x,0,x;\mu)+\mathcal{O}(\mu^{-2}),
\end{aligned}
$$

*I. S., A. Vladimirov, Eur. Phys. J. C 78, 802 (2018), F. Rein, S. Rodini, A. Schäfer, and A. Vladimirov, JHEP 01, 116 (2023)*

- The evolution is of the correct DGLAP-type…

  …With a difference at NLO

$$
\mu^2\frac{d}{d\mu^2}\mathcal{G}_1[F](x,\mu)=R_t\otimes P_t'\otimes t+\mathcal{O}(\alpha_s^2)
$$

$$
P_t'-P_t=\mathcal{O}(\alpha_s^2)
$$

# QIU-STERMAN FUNCTIONS

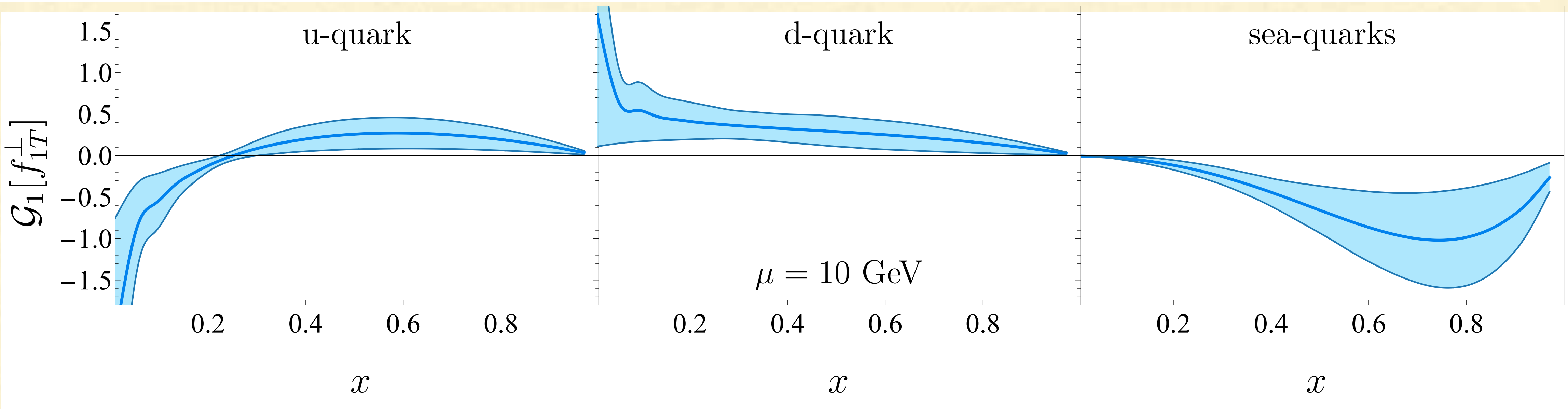
*Oscar del Rio, Alexei Prokudin, Ignazio Scimemi, Alexey Vladimirov e-Print: 2402.01836 (2024)*

- Even though it is not possible to relate the $1^{st}$ TMM of the Sivers functions to the full twist-3 functions with 3 variables $T(x_1, x_2, x_3)$, it is related to Qiu-Sterman functions $T(-x,0,x;\mu)$

$$\langle \boldsymbol{k}_{T,1}^{u} \rangle = -0.011^{+0.011}_{-0.023}\ \text{GeV}, \quad \langle \boldsymbol{k}_{T,1}^{d} \rangle = 0.17^{+0.21}_{-0.17}\ \text{GeV}, \quad \langle \boldsymbol{k}_{T,1}^{sea} \rangle = -0.26^{+0.26}_{-0.32}\ \text{GeV}$$

$$\langle \boldsymbol{k}_{T,1}^{g} \rangle \simeq 0.14^{+0.31}_{-0.14}\ \text{GeV}$$

potentially sizable gluon Sivers function




*Using M. Bury, A. Prokudin, A. Vladimirov, Phys.Rev.Lett. 126 (2021)*

# SECOND TMM

The 2nd moment is power divergent

$$\begin{aligned}
\mathcal{M}^{[\gamma^+]}_{\mu\nu,\text{div}}(x,\mu) &= -g_{T,\mu\nu}M^2\mathcal{G}_{1,0}[f_1], \\
\mathcal{M}^{[\gamma^+\gamma^5]}_{\mu\nu,\text{div}}(x,\mu) &= -\lambda g_{T,\mu\nu}M^2\mathcal{G}_{1,0}[g_1], \\
\mathcal{M}^{[i\sigma^{\alpha+}\gamma^5]}_{\mu\nu,\text{div}}(x,\mu) &= s_{T,\alpha}g_{T,\mu\nu}M^2\mathcal{G}_{1,0}[h_1] + (g_{T,\mu\alpha}s_{T,\nu} + g_{T,\nu\alpha}s_{T,\mu} - g_{T,\mu\nu}s_{T,\alpha})\frac{M^2}{2}\mathcal{G}_2[h_{1T}^{\perp}]
\end{aligned}$$

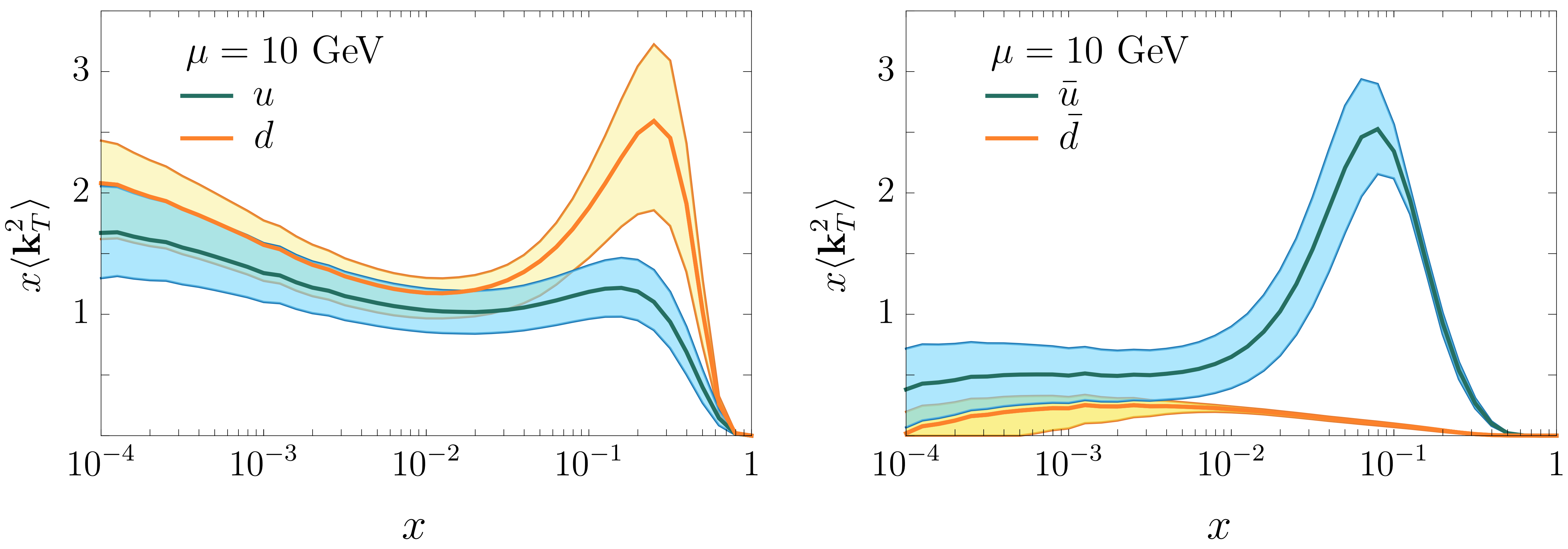
The asymptotic power divergence part is computed analytically ...

$$\mathcal{G}_{n+1,n}[F](x,\mu) = \frac{\mu^2}{2M^2}\text{AS}[\mathcal{G}_{n+1,n}[F]](x,\mu) + \overline{\mathcal{G}}_{n+1,n}[F](x,\mu),$$

... the width of TMDs

$$\langle \boldsymbol{k}_T^2 \rangle = -g_T^{\mu\nu}\mathcal{M}^{[\gamma^+]}_{\mu\nu} = 2M^2\overline{\mathcal{G}}_{1,0}[f_1]$$

# SECOND TMM




$$\langle x\vec{k}_T^2 \rangle_u = 0.52 \pm 0.12 \text{ GeV}^2, \qquad \langle x\vec{k}_T^2 \rangle_d = 1.10 \pm 0.28 \text{ GeV}^2,$$

$$\langle x\vec{k}_T^2 \rangle_{\bar{u}} = 0.42 \pm 0.06 \text{ GeV}^2, \qquad \langle x\vec{k}_T^2 \rangle_{\bar{d}} = 0.024 \pm 0.004 \text{ GeV}^2 .$$

# CONCLUSIONS: SPIN UP!!

- ☑ Factorization is the tool to dig up in proton structure. We can go beyond LP and discover new effects and connect EFT, perturbative QCD, Lattice, Experiments,..
- ☑ ART23 with Artemide reaches N4LL (caveat PDF), flavor dependence of TMD included, latest DY data, complete evaluation of errors (PDF errors!!). SIDIS fit soon.
- ☑ TMM: a robust relations of the 3D and 1D nucleon structures are established, very precious definitions, especially for polarized measurements.

# "THE PROTON IS A CHOCOLATE BOX"

## (INSPIRED BY FORREST GUMP MOVIE)

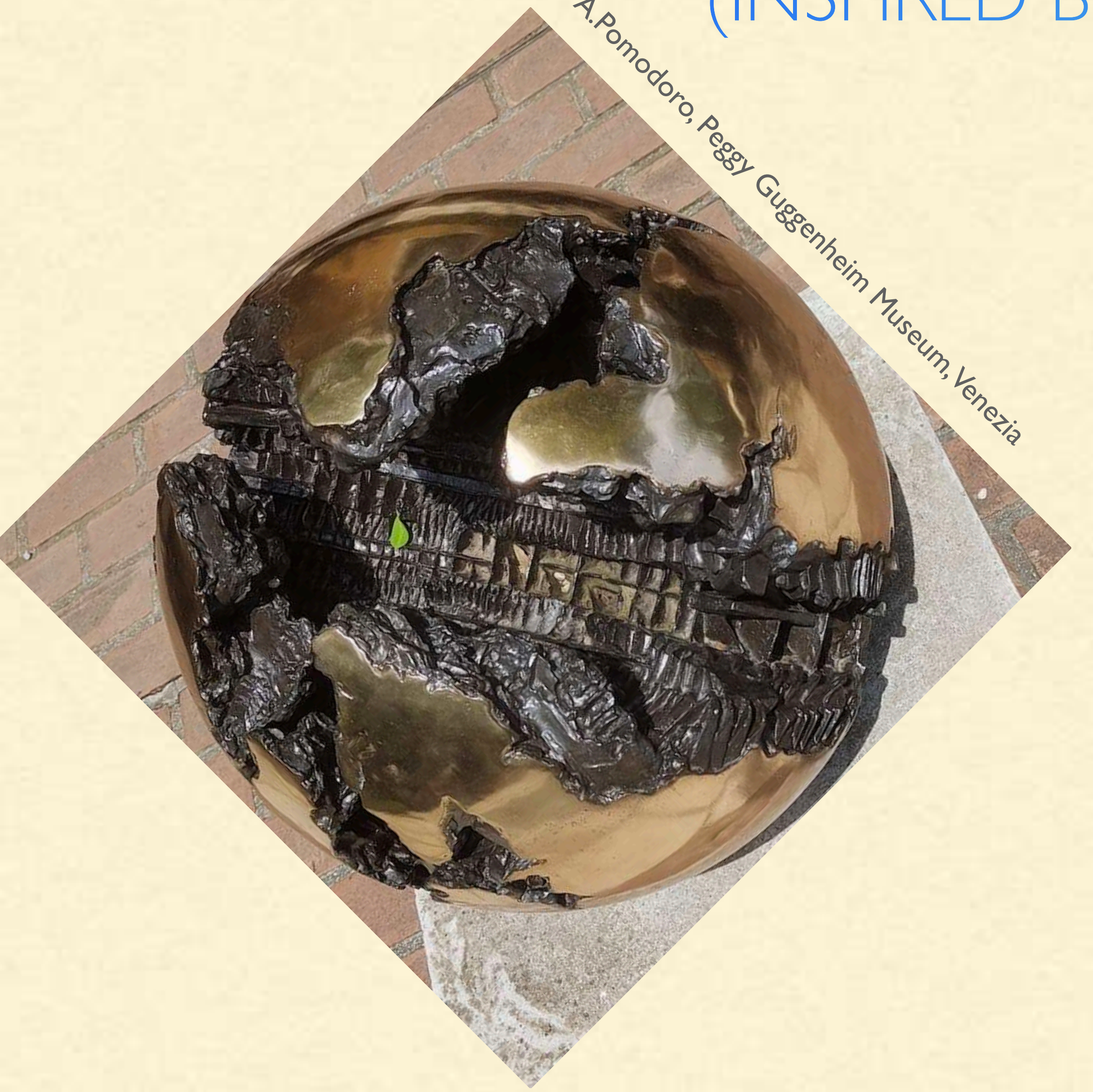

A.Pomodoro, Peggy Guggenheim Museum, Venezia


Foto de Monique Carrati en Unsplash

- EIC@BNL, EICc@HIAF (2030's)?, LHeC? FCCee?
- LHC initiatives (SMOG at LHCb, LHCSpin, etc.)
- Belle and Belle II

# BACK UP SLIDES

# 2-D TMD evolution

## Coupled evolution of TMD ...

TMD (standard) anomalous dimension

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta)$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta)$$

TMD rapidity anomalous dimension

Collinear overlap

$$\zeta \frac{d}{d\zeta} \gamma_F(\mu,\zeta) = -\Gamma(\mu)$$

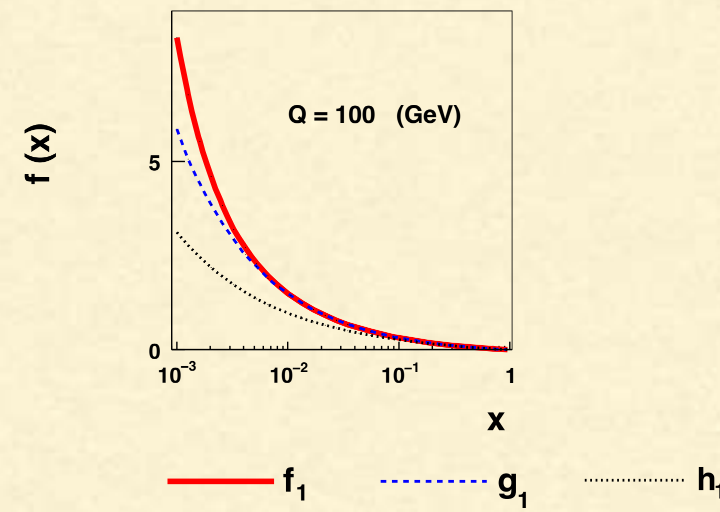
$$\mu \frac{d}{d\mu} \mathcal{D}(\mu,b) = \Gamma(\mu)$$

# PREVIOUS WORKS on moments

- Numerical study

  *A. Bacchetta, A. Prokudin, Nucl.Phys.B 875 (2013) 536-551*

$$f^q(x;\mu,\zeta_F) \equiv 2\pi \int_0^{\mu} k_T dk_T f_{q/P}(x,k_T;\mu,\zeta_F)$$




- Proposed for polarized TMDs

  *L. Gamberg, A. Metz, D. Pitonyak, A. Prokudin Phys.Lett.B 781 (2018) 443-454*

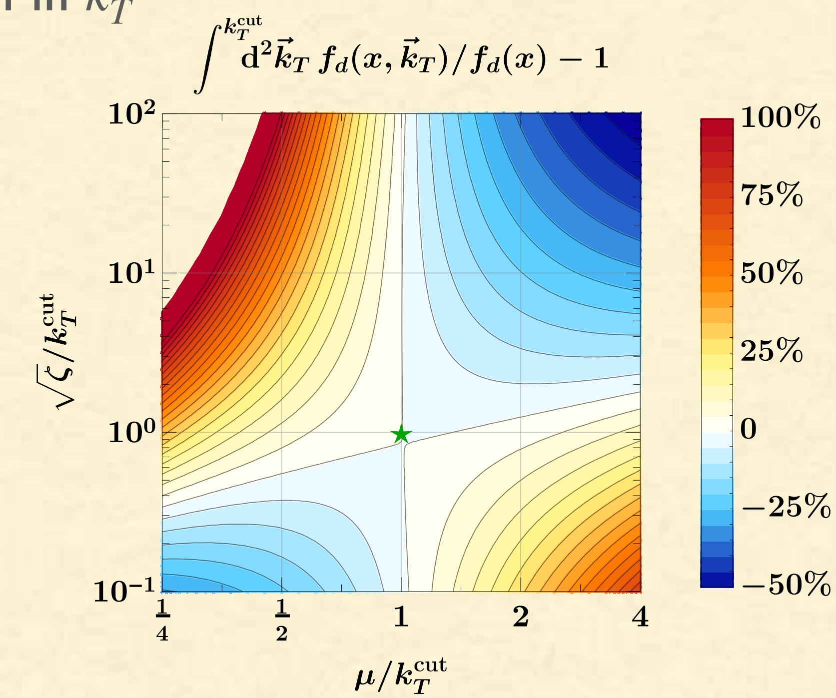
$$\int d^2\mathbf{k}_T \frac{k_T^2}{2M_P^2} f_{1T}^{\perp j}(x,k_T;Q^2,\mu_Q;C_5) \equiv f_{1T}^{\perp(1)\,j}(x;Q^2,\mu_Q;C_5)$$

$b_{min}$ instead of a cut in $k_T$

- Studied in great deal of details in

  *M. A. Ebert, J. K. L. Michel, I. W. Stewart and Z. Sun, JHEP 07 (2022) 129*
  *J. O. Gonzalez-Hernandez, T. Rainaldi, T. C. Rogers Phys.Rev.D 107 (2023) 9, 094029*

$$\int_{k_T \le k_T^{\rm cut}} d^2\mathbf{k}_T\, f_{i/p}\left(x,\mathbf{k}_T,\mu=k_T^{\rm cut},\sqrt{\zeta}=k_T^{\rm cut}\right) \simeq f_i(x,\mu=k_T^{\rm cut})$$



# ZEROTH TMM

- If TMDs are defined in a general scheme (TMD2-scheme), the same conclusions are valid, all scales should be defined by the cut-off

M. A. Ebert, J. K. L. Michel, I. W. Stewart and Z. Sun, JHEP 07 (2022) 129

$$\mu = \mu_{\text{OPE}} = \mu_{\text{TMD}} = \sqrt{\zeta}$$

$$\mu^2 \frac{d}{d\mu^2} f^{(\text{TMD2})}(x,\mu) = \overline{P} \otimes f^{(\text{TMD2})}(x,\mu)$$

$$\overline{P} - P = -\,\alpha_s^2 \beta_0 \overline{C}_1 - \alpha_s^3 \left[ 2\beta_0 \overline{C}_2 - \beta_0 \overline{C}_1 \otimes \overline{C}_1 + \beta_1 \overline{C}_1 - 2\zeta_3 \Gamma_0 \beta_0 \left( P_1 + \left( \frac{\gamma_1}{2} - \frac{2\beta_0}{3} \right) \cdot \mathbf{1} \right) \right] + \mathcal{O}(\alpha_s^4)$$

# FIRST TMM

- The $1^{st}$ TMM is

$$\mathcal{M}_\mu^{[\gamma^+]}(x,\mu) = \int^\mu d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu} F^{[\gamma^+]}(x,k_T) \;=\; -\int^\mu d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu} \epsilon_T^{\rho\nu}\frac{k_{T\rho}s_{T\nu}}{M} f_{1T}^\perp(x,k_T),$$

$$\mathcal{M}_\mu^{[\gamma^+\gamma_5]}(x,\mu) = \int^\mu d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu} F^{[\gamma^+\gamma^5]}(x,k_T) \;=\; -\int^\mu d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu} \frac{(k_T\cdot s_T)}{M} g_{1T}^\perp(x,k_T),$$

$$\mathcal{M}_\mu^{[i\sigma^{\alpha+}\gamma^5]}(x,\mu) = \int^\mu d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu} F^{[i\sigma^{\alpha+}\gamma^5]}(x,k_T) \;=\; \int^\mu d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu} \frac{\lambda k_T^\alpha}{M} h_{1L}^\perp(x,k_T) - \int^\mu d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu} \frac{\epsilon_T^{\alpha\rho}k_{T\rho}}{M} h_1^\perp(x,k_T).$$

- And using $\mathcal{G}$ symbol

$$\mathcal{M}_\mu^{[\gamma^+]}(x,\mu) \;=\; -\epsilon_{T,\mu\nu} s_T^\nu M \mathcal{G}_1[f_{1T}^\perp](x,\mu),$$

$$\mathcal{M}_\mu^{[\gamma^+\gamma^5]}(x,\mu) \;=\; -s_{T\mu} M \mathcal{G}_1[g_{1T}^\perp](x,\mu),$$

$$\mathcal{M}_\mu^{[i\sigma^{\alpha+}\gamma^5]}(x,\mu) \;=\; -\lambda g_{T,\mu\alpha} M \mathcal{G}_1[h_{1L}^\perp](x,\mu) - \epsilon_{T,\mu\alpha} M \mathcal{G}_1[h_1^\perp](x,\mu).$$

- It is related to collinear twist-3 PDFs projected onto Qiu-Sterman type functions $x_1, x_2, x_3 \to x$ with the projection operator $R_t = \pi\delta(x_2)\delta(x_1+x_2+x_3)\delta(x_3-x)$

# SECOND TMM

*Oscar del Rio, Alexei Prokudin, Ignazio Scimemi, Alexey Vladimirov e-Print: 2402.01836 (2024)*

$$\mathcal{M}_{\mu\nu}^{[\gamma^+]}(x,\mu) = \int^{\mu} d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu}\boldsymbol{k}_{T\nu} F^{[\gamma^+]}(x,k_T) \;=\; \int^{\mu} d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu}\boldsymbol{k}_{T\nu} f_1(x,k_T),$$

$$\mathcal{M}_{\mu\nu}^{[\gamma^+\gamma_5]}(x,\mu) = \int^{\mu} d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu}\boldsymbol{k}_{T\nu} F^{[\gamma^+\gamma^5]}(x,k_T) \;=\; \lambda\int^{\mu} d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu}\boldsymbol{k}_{T\nu} g_1(x,k_T),$$

$$\mathcal{M}_{\mu\nu}^{[i\sigma^{\alpha+}\gamma^5]}(x,\mu) = \int^{\mu} d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu}\boldsymbol{k}_{T\nu} F^{[i\sigma^{\alpha+}\gamma^5]}(x,k_T) \;=\; s_T^{\alpha}\int^{\mu} d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu}\boldsymbol{k}_{T\nu} h_1(x,k_T)$$
$$-\int^{\mu} d^2\boldsymbol{k}_T \boldsymbol{k}_{T\mu}\boldsymbol{k}_{T\nu}\frac{\boldsymbol{k}_T^2}{M^2}\left(\frac{g_T^{\alpha\rho}}{2}+\frac{k_T^{\alpha}k_T^{\rho}}{\boldsymbol{k}_T^2}\right) s_{T\rho} h_{1T}^{\perp}(x,k_T)$$

# QIU-STERMAN FUNCTIONS

- Burkardt sum rule: $$\sum_{f=q,\bar{q},g} \int_0^1 dx \mathcal{M}_{\nu,f}^{[\gamma^+]}(x,\mu) = \sum_{f=q,\bar{q},g} \langle \boldsymbol{k}_{T,\nu}^{f} \rangle = 0$$

$\langle \boldsymbol{k}_{T,1}^{u} \rangle = -0.011^{+0.011}_{-0.023}$ GeV, $\langle \boldsymbol{k}_{T,1}^{d} \rangle = 0.17^{+0.21}_{-0.17}$ GeV, $\langle \boldsymbol{k}_{T,1}^{sea} \rangle = -0.26^{+0.26}_{-0.32}$ GeV

$\langle \boldsymbol{k}_{T,1}^{g} \rangle \simeq 0.14^{+0.31}_{-0.14}$ GeV potentially sizable gluon Sivers function