Non-Abelian ferromagnets

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University of Cyprus, 24 May 2024

Based on work with A.P. Polychronakos (City Coll., N.Y.)

- Nucl.Phys. **B994** (2023) 116314, 2305.19345 [hep-th]
- Nucl.Phys. **B996** (2023) 116353, 2306.01051 [hep-th]
- Nucl.Phys. **B999** (2024) 116434, 2310.16887 [hep-th]
- and in progress.

General settings and context – Motivation

Symmetries in physics

Discrete (permutation, lattice) or continuous (rotations, translations, internal)

$SU(N)$ is a continuous symmetry arising in many systems.

- Spin $SU(2)$
- Isospin $SU(2)$
- Flavor $SU(3)$
- Color $SU(3)$
- Grand Unified Theories $SU(5)$,...

Any quantum situation invariant under change among N states.
Defining representation: $N \times N$ unitary matrices.

General motivation I - Some basic math/phys questions

- The total spin of 3 spin-1/2 particles could be either 1/2 or 3/2 with multiplicities 2 and 1, i.e.

$$2 \otimes 2 \otimes 2 = 2 \oplus 2 \oplus 4$$

- Tensor product of n spin-1/2 reps of $SU(2)$: What is the multiplicity $d_{n,j,1/2}$ of the spin-j rep. in the decomposition?

$$\underbrace{2 \otimes 2 \otimes 2 \otimes \cdots \otimes 2}_{n \text{ spin } 1/2} = \sum_j d_{n,j,1/2} \oplus (2j+1) \ .$$

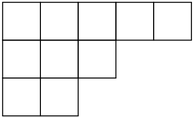
- What about n spin-s reps of $SU(2)$? What is then $d_{n,j,s}$?

$$\underbrace{(2s+1) \otimes (2s+1) \otimes \cdots \otimes (2s+1)}_{n \text{ spin } s} = \sum_j d_{n,j,s} \oplus (2j+1)$$

Relation to random walks [Polychronakos-KS 16]

- Similar questions for $SU(N)$. What is the multiplicity of a general Young Tableau (YT) arising in the decomposition of n fundamentals? Schematically:

$$\underbrace{\square \otimes \square \otimes \cdots \otimes \square}_{n \text{ boxes}} = \sum_{\mathbf{k}} d_{n,\mathbf{k}}$$



- In there anything interesting happening for large n and/or N:
  - If $N = \mathcal{O}(1)$ and $n \gg 1$?
  - If $N, n \gg 1$ with some ratio kept constant?

General motivation II - Physics applications

- $SU(N)$-matrix models:
  - To describe non-perturbative aspects in string theory [Gross-Migdal 90, Douglas-Shenker 90]
  - Aspects of black hole Physics (thermalization, information "paradox" etc) [Kazakov-Kostov-Kutasov 01]
- Large N-expansion of $SU(N)$ gauge theories:
  - Led to a new understanding of the perturbative expansion by reorganizing Feynman diagrams in a topological expansion ['t Hooft 74]
  - Eventually to the AdS/CFT correspondence, a breakthrough in our understanding of QFT and Gravity [Maldacena 97]
- Magnetic systems with $SU(N)$ symmetry in the context of ultracold atoms, spin chains and of interacting atoms on lattice cites and in the presence of magnetic fields.
- Phase transitions for large n and/or N.

Outline

- The $SU(N)$ ferromagnetic model:
  Construction, silent simplifications and essential properties.

- Solution in the thermodynamic limit and finite N.
  Stability and Young tableaux.
  Spontaneous symmetry breaking.

- Phase transitions:
  - $SU(2)$: A single Marie Curie temperature below which spontaneously magnetized occurs; a 2nd order phase transition.
  - $SU(N)$, with $N = 3, 4, \ldots$: More structure and critical temperatures...stable as well unstable phases. Phase transitions are different.

- Turning on magnetic fields.

- Large n, N with N/n^2 fixed. Novel phase structure.

- Concluding remarks.

The $SU(N)$ ferromagnet

Consider n atoms on a lattice with two-body interactions.

- Each atom has N degenerate states $|s\rangle$, $s = 1, 2, \ldots, N$.
- The generic two-body interaction is

$$H_{12} = \sum_{s_1, s_1', s_2, s_2' = 1}^{N} h_{s_1 s_2; s_1' s_2'} \, |s_1\rangle \langle s_1'| \otimes |s_2\rangle \langle s_2'|$$

- Define j_a, $a = 0, 1, ..., N^2 - 1$, the generators of $U(N)$ in the fundamental N-dim rep. (j_0 is the $U(1)$ part). The j_a's form a complete basis for the operators on an N-dim space. Hence,

$$H_{12} = \sum_{a,b=0}^{N^2-1} h_{ab} \, j_{1,a} \, j_{2,b} \ , \quad h_{ab} = h_{ab}^* \ ,$$

where

$$j_{1,a} = j_a \otimes \mathbb{I} \ , \qquad j_{2,a} = \mathbb{I} \otimes j_a \ ,$$

are fundamental $U(N)$ operators on states of atoms 1 and 2.

Assume invariance under change of basis $|s\rangle$:

- Interactions will essentially be the operators exchanging the states of the atoms of the form (up to a constant)

$$H_{12} = c_{12} \sum_{a=1}^{N^2-1} j_{1,a} j_{2,a}$$

- $SU(N)$ emerges from invariance under general changes of basis.
- The full Hamiltonian will be of the form

$$H = \sum_{r,s=1}^{n} c_{\mathrm{r,s}} \sum_{a=1}^{N^2-1} j_{r,a} j_{s,a} \ ,$$

  where $c_{\mathrm{r,s}}$ coupling between atoms r and s.
- Further symmetries and more:
  - Translation invariance: $c_{\mathrm{r},\vec{s}} = c_{\vec{r}-\mathrm{s}}$ and $c_0 = 0$
  - Ferromagnetic: $c_{\mathrm{r}} < 0$

Mean field approximation

- Interactions are assumed reasonably long range.
- Average of neighbors approximated with the full lattice average

$$\sum_{\mathrm{s}} c_{\mathrm{s}}\, j_{\mathrm{r}+\mathrm{s},a} \simeq \Big(\sum_{\mathrm{s}} c_{\mathrm{s}}\Big)\, \frac{1}{n} \sum_{s=1}^{n} j_{s,a} = -\frac{c}{n} J_a \ ,$$

where the total $SU(N)$ generators and average coupling is

$$J_a = \sum_{s=1}^{n} j_{s,a} \ , \quad c = -\sum_{\mathrm{s}} c_{\mathrm{s}} > 0 \ .$$

- Then, the full Hamiltonian becomes [Polychronakos-KS 23]

$$H = -\frac{c}{n} \sum_{a=1}^{N^2-1} \Big(J_a^2 - \sum_{s=1}^{n} j_{s,a}^2 \Big) = -\frac{c}{n} \sum_{a=1}^{N^2-1} J_a^2 + \text{const.}$$
$$= -\frac{c}{n} C_2(J) \ ,$$

where $C_2(J)$ is the quadratic Casimir.

Turning on magnetic fields

We may consider a global external field contributing one-body terms

$$H_B = -\sum_{i=1}^{N-1} B_i H_i$$

where H_i are commuting Cartan generators.

Therefore the total Hamiltonian is [Poly-KS 23]

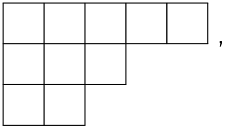
$$\boxed{H = H_I + H_B = -\frac{c}{n} C_2(J) - \sum_{i=1}^{N-1} B_i H_i} \, .$$

Crash course on $SU(N)$ representation theory

- Irreps of $SU(N)$ are labeled by a set of distinct ordered integers $\{k_i\}$

$$k_1 > k_2 > \cdots > k_N \geqslant 0 \ .$$

  The usual Young tableaux (YT)



  is labeled by ℓ_i: the # of boxes in the ith row

$$\ell_i = k_i - k_N + i - N \ , \quad \ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{N-1} \geqslant 0 \ .$$

- The k_i-representation is redundant since we may shift $k_i \to k_i + c$. This is the $U(1)$ charge.
  We fix the redundancy by

$$\sum_{i=1}^{N} k_i = n + \frac{N(N-1)}{2} \ .$$

Basic examples:

- The singlet representation ($n = 0$):

$$\ell_i = 0 \quad \text{or} \quad k_i = N - i \ , \quad \forall\, i = 1, 2, \ldots N \ .$$

- The fundamental representation ($n = 1$):

$$\ell_1 = 1 \quad \text{or} \quad k_1 = N \ , \quad \text{the rest as in singlet}$$

- The symmetric representation ($n = 2$):

$$\ell_1 = 2 \quad \text{or} \quad k_1 = N + 1 \ , \quad \text{the rest as in singlet}$$

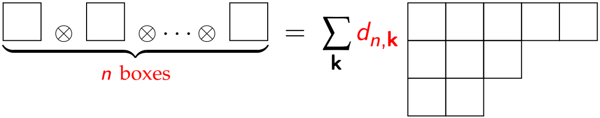
- The antisymmetric representation ($n = 2$):

$$\ell_1 = \ell_2 = 1 \quad \text{or} \quad k_1 = N \, , \ k_2 = N - 1 \ , \quad \text{the rest as in singlet}$$

The multiplicity

What is the multiplicity $d_{n,\mathbf{k}}$ of each irrep $\mathbf{k}$ arising in the decomposition of n fundamentals of $SU(N)$?

Recall that, schematically:



The result is [Poly-KS 23]

$$\boxed{d_{n,\mathbf{k}} = \delta_{k_1+\cdots+k_N,\,n+N(N-1)/2} \prod_{j>i=1}^{N} (S_i - S_j) D_{n,\mathbf{k}}} \; ,$$

where

$$D_{n,\mathbf{k}} = \frac{n!}{\prod_{r=1}^{N} k_r!} \; ,$$

and where S_i acts by replacing k_i by $k_i - 1$.

A closed expression can be also obtained.

$$d_{n;\mathbf{k}} = n!\,\frac{\Delta(\mathbf{k})}{\prod_{i=1}^{N} k_i!}\ , \qquad \sum_{i=1}^{N} k_i = n + \frac{N(N-1)}{2}\ .$$

- where the Vandermonde determinant is

$$\Delta(\mathbf{k}) = \prod_{j>i=1}^{N} (k_i - k_j)\,.$$

- The dimension of the irrep is

$$\mathrm{tr}_{\mathbf{k}}\mathbb{1} = \dim(\mathbf{k}) = \prod_{j>i=1}^{N} \frac{k_i - k_j}{j - i} = \frac{\Delta(\mathbf{k})}{\prod_{s=1}^{N-1} s!}\ ,$$

- The quadratic Casimir is

$$C^{(2)}(\mathbf{k}) = \frac{1}{2}\sum_{i=1}^{N} k_i^2 + \mathrm{const.}\ .$$

For $SU(2)$: We have one-row reps.

$$\ell_1 = k_1 - k_2 - 1 = 2j \ , \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \ .$$

- Then the multiplicity of the j-spin rep arising form the decomposition of n spin-$\frac{1}{2}$ reps is

$$d_{n,j,1/2} = \frac{n!\,(2j+1)}{\left(\frac{n}{2}-j\right)!\,\left(\frac{n}{2}+j+1\right)!} \ ,$$

- The dimension of the irrep is $k_1 - k_2 = 2j + 1$.
- As a check the following identity holds

$$\sum_{j=j_{\mathrm{min}}}^{n/2} (2j+1) d_{n,j} = 2^n \ ,$$

where j_{min} equals 0 or $1/2$ if n is even or odd.

Statistical mechanics of the $SU(N)$ ferromagnet

The partition function

At temperature $T = \beta^{-1}$ this is defined as

$$Z = \sum_{\text{states}} e^{-\beta H} = \sum_{\langle \mathbf{k} \rangle} d_{n;\mathbf{k}}\, e^{\frac{\beta c}{n} C^{(2)}(\mathbf{k})}\, \mathrm{tr}_{\mathbf{k}} e^{\beta \sum_{j=1}^{N} B_j H_j} \;,$$

where $\langle \mathbf{k} \rangle$ denotes ordered integers. Working out the details [Poly-KS]

$$Z = \sum_{\mathbf{k}} \delta_{k_1+\cdots+k_N,n}\, \frac{1}{\Delta(\mathbf{z})} \frac{n!}{\prod_{r=1}^{N} k_r!} \prod_{j>i=1}^{N} \left(S_i^{-1} - S_j^{-1}\right) e^{\frac{\beta c}{2n} \sum_s k_s^2 + \beta B_s k_s} \;.$$

where the Vandermonde determinant

$$\Delta(\mathbf{z}) = \prod_{j>i=1}^{N} (z_j - z_i)\,, \quad z_j = e^{\beta B_j} \;.$$

Thermodynamic limit $n \gg 1$

The rank of the group $N = \mathcal{O}(1)$.

- A typical k_i is of order n, also the exponent in $e^{\frac{\beta c}{2n}\sum_s k_s^2 + \beta B_s k_s}$
- Any prefactor polynomial in n is irrelevant, as is $\Delta(\mathrm{z})$, and $\prod_{j>i}(S_i^{-1} - S_j^{-1})$ which produces subleading factors.
- Apply to $k_r!$ the Stirling approximation.
- In addition,

$$k_i = n x_i \ , \quad c = N T_0 \ ,$$

  introducing intensive quantities x_i and a temperature scale T_0.
- Altogether we obtain

$$Z = \sum_{\mathbf{x}} \delta_{x_1+\cdots+x_N,1}\, e^{-n\beta \mathrm{F}(\mathbf{x}) + \mathcal{O}(n^0)} \ ,$$

  where the free energy of the system is

$$F(\mathbf{x}) = \sum_{i=1}^{N} \left(T x_i \ln x_i - \frac{N T_0}{2} x_i^2 - B_i x_i \right)$$

Equilibrium

Introduce a Lagrange multiplier λ for the condition $\sum_i x_i = 1$ and perform a saddle point analysis:

- The saddle point conditions are

$$\partial_i F_\lambda = T \ln x_i - N T_0 x_i - B_i - \lambda = 0 \ ,$$
$$\sum_i x_i = 1, \quad i = 1, 2, \ldots, N \ .$$

- Eliminating λ

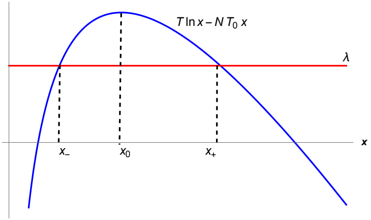
$$T \ln \frac{x_i}{x_N} - N T_0 (x_i - x_N) = (B_i - B_N) \ , \quad i = 1, 2, \ldots, N-1 \ ,$$

- Finding the phases of the system involves:
  - Solving the above conditions
  - Establishing the local and global stability of the solutions
  - Finding phase transition lines between phases (solutions)

Vanishing magnetic fields

The x_i are $N-1$ order parameters; satisfy the common equation

$$T \ln x - N T_0 x = \lambda \ .$$



- $x_i = x_-$ or $x_i = x_+$
- $x_i = 1/N$ (for all i) is always a solution. If stable, paramagnetic phase with unbroken $SU(N)$
- In general M solutions at x_+ and $N-M$ at x_-. If stable, ferromagnetic phase: $SU(N) \to SU(M) \times SU(N-M) \times U(1)$, M-rows YT.

Stability

Stability analysis reveals that the only possible stable states are

- $M = 0$ ($SU(N)$-singlet, paramagnetic)
- $M = 1$ (fully symmetric irrep, ferromagnetic).

Both are captured by the single order parameter x

$$x_1 = \frac{1+x}{N}\,, \quad x_i = \frac{1-x/(N-1)}{N}\ , i = 2, \ldots, N\ ,$$

satisfying

$$(*) \qquad \boxed{T \ln \frac{1+x}{1-x/(N-1)} - T_0 \frac{N}{N-1}\, x = 0}\ .$$

- One-row YT with length $\ell_1 = \frac{x}{N-1} n + \mathcal{O}(1)$.
- Critical temperatures: where stable solutions appear or disappear also satisfy

$$(**) \qquad \boxed{T_c = T_0 (1+x)\big(1 - x/(N-1)\big)}\ .$$

Already two critical temperatures T_0 and $T_c > T_0$.

- For $T > T_c$: only solution is $x = 0$ (stable)
- For $T_0 < T < T_c$: $x = 0$ (stable) and $0 < x_1 < x_c < x_2$ (stable)
- For $T < T_0$: $x = 0$ (unstable) and $x_1 < 0 < x_2$ (stable)

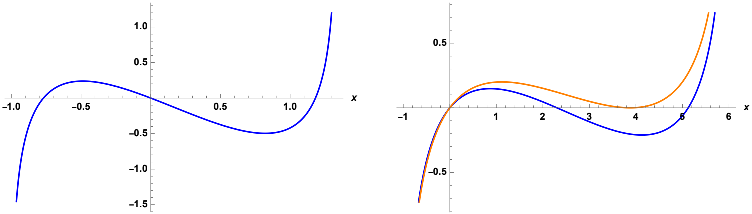


Figure: Plot of the LHS of (*).
L: $T < T_0$. **R:** $T_0 < T < T_c$ (blue) and for $T = T_c$ (yellow).

- Free energy comparison reveals a third critical temperature

$$T_1 = \frac{T_0}{2} \frac{N(N-2)}{(N-1)\ln(N-1)}, \quad T_0 < T_1 < T_c \,.$$

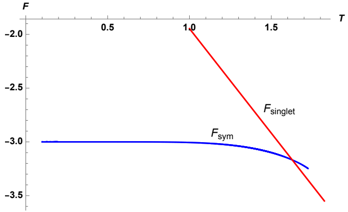
- We get the table for $N \geqslant 3$. Spontaneous magnetization , but not with a single Curie temperature

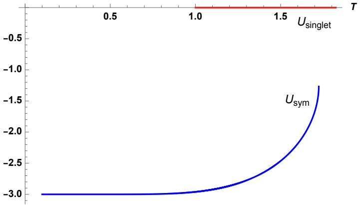
state	$T < T_0$	$T_0 < T < T_1$	$T_1 < T < T_c$	$T_c < T$
singlet	unstable	metastable	stable	stable
1- row	stable	stable	metastable	not a solution

- For $N = 2$, $T_0 = T_1 = T_c \implies$ standard ferromagnetism.

state	$T < T_0$	$T > T_0$
singlet	unstable	stable
1-row	stable	not a solution

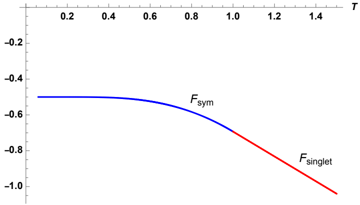
Free energy, internal energy [$SU(N)$ vs $SU(2)$]

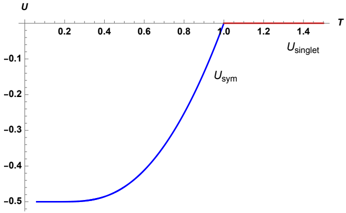




- There is latent heat exchange in the transition between phases
- Hysteresis going up and down in temperature

Compare with ordinary $SU(2)$ ferromagnet: 2nd order phase transition.





Turning on magnetic fields: Linear response (small fields)

Define the magnetizability matrix

$$m_{ij} = \frac{\partial x_i}{\partial B_j} = m_{ji}$$

Then

$$\text{Paramagnetic phase}: \qquad m_{ij} = \frac{1}{N(T - T_0)}\left(\delta_{ij} - \frac{1}{N}\right)$$

Ferromagnetic phase $x_1 \neq 0,\ T \sim T_c$

$$m_{11} \simeq \frac{N-1}{N^2}\,\frac{Q}{\sqrt{T_c - T}} > 0\ , \quad m_{1i} \simeq -\frac{1}{N^2}\,\frac{Q}{\sqrt{T_c - T}} < 0$$

$$m_{ij} \simeq \frac{1}{N^2(N-1)}\,\frac{Q}{\sqrt{T_c - T}} > 0\ , \quad i, j = 2, \ldots, N\ ,$$

where

$$Q = \frac{T_c / T_0}{\sqrt{2x_c(2(N-1)x_c + N - 2)\,T_0}}$$

Turning on magnetic fields: Finite fields

This is the case with the richest phase structure

- Analysis becomes very complicated
- Broken and unbroken phases are hard to quantify
- Full phase diagram is needed to discern critical surfaces

Let's focus on only one component magnetic field B_1.

- Remarkably, if B_1 is large enough then the one-row solution becomes unstable.
- Then, two-row and conjugate one-row states are the stable ones. Hence

$$SU(N) \to SU(N-2) \times U(1) \times U(1) \ .$$

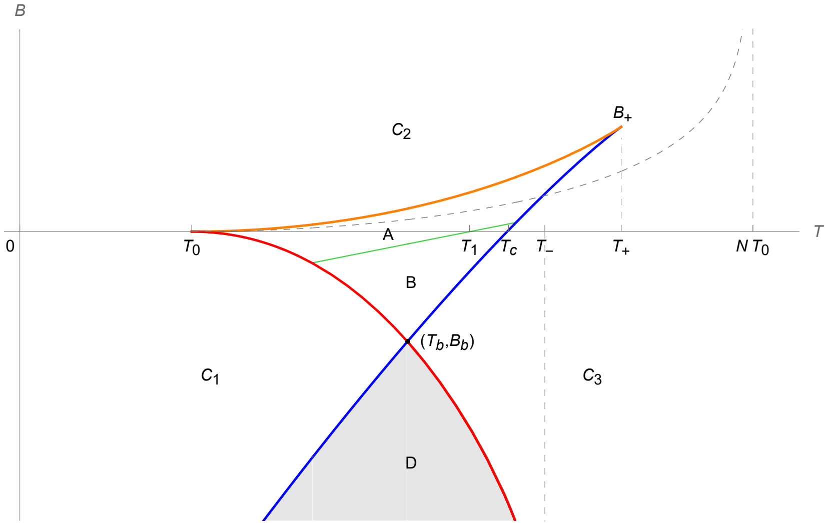


Figure: Thick lines represent phase transitions in the magnetization, the green line is a metastability frontier. Regions A, B are singly marnetized phases, C metastable mixtures of singly and doubly magnetized, and D a doubly magnetized phase. The gray dashed curve represents a crossover.

Double scaling limit

When both $n, N \gg 1$, then the subleading terms we have ignored become important. We think of the k_i as a continuous distribution.

To do that we reformulate the quantities as:

- We define a density
$$\rho_s = \sum_{i=1}^{N} \delta_{s,k_i} \ .$$
- This density ρ_s satisfies the relations
$$\sum_{s=0}^{\infty} \rho_s = N \ , \qquad \sum_{s=0}^{\infty} s\,\rho_s = n + \frac{N(N-1)}{2} \ ,$$
- Then, it can be shown that
$$d_{n,\mathbf{k}} = n! \prod_{t>s=0}^{\infty} (t-s)^{(\rho_s - 1)\rho_t} \ .$$

Analysis for very large temperatures

The Hamiltonian is irrelevant, since $e^{-\beta H} \to 1$.
Consider the entropy-like quantity (its logarithm)

$$m_{w,n;\mathrm{k}} = \left[\,\mathrm{dim}(\mathbf{k})\right]^{w-1} d_{n;\mathbf{k}} = \frac{n!\,\left[\Delta(\mathbf{k})\right]^{w}}{\left(\prod_{s=1}^{N-1} s!\right)^{w-1} \prod_{i=1}^{N} k_i!}\ ,$$

The constant w parametrizes different cases physically and mathematically:

- $w = 1$: # of reps from decomposing n fundamentals.
- $w = 2$: # of states from decomposing n fundamentals.
- $w > 1$: Exotic situations; no clear physical meaning.
- $w < 1$: Unphysical, as entropy decreases with dimensionality.

Calling $\rho(k)$ the continuous version of ρ_s:

- Extremize the functional

$$S_{w,n}[\rho(k)] = \frac{w}{2}\int_0^\infty dk \int_0^\infty dk'\,\rho(k)\rho(k')\ln|k-k'| - \int_0^\infty dk\,\rho(k)\,k(\ln k - 1)$$

- This is subject to the constraints

$$\int_0^\infty dk\,\rho(k) = N\ , \qquad \int_0^\infty dk\,k\,\rho(k) = n + \frac{N^2}{2}\ .$$

- Setting the functional derivative w.r.t. $\rho(k)$ to zero and further differentiating with respect to k we obtain

$$w\int_0^\infty dk'\,\frac{\rho(k')}{k-k'} = \ln k + \lambda\ ,$$

  becomes a standard single-cut Cauchy problem. To solve it we define a resolvent etc... We will skip the details.

Solution of the Cauchy problem

It turns out that the solution has two phases depending on the parameter

$$n_w = \frac{(3w-2)N^2}{4} \ .$$

Then

- Dilute phase $n > n_w$: The density is

  $$\rho(k) = \frac{2}{w\pi} \cos^{-1} \frac{\sqrt{k} + \sqrt{ab/k}}{\sqrt{a} + \sqrt{b}} \ , \qquad a \leqslant k \leqslant b \ ,$$

  where a and b depend on n, N and w.

- Condensed phase $n < n_w$: The density is

  $$\rho(k) = \begin{cases} 1 \ , & 0 < k < a \ , \\ \rho_0(k-a) \ , & a < k < a+b \ , \end{cases}$$

  with

  $$\rho_0(k) = \frac{2}{w\pi} \cos^{-1} \sqrt{\frac{k}{b}} + \frac{2(w-1)}{w\pi} \cos^{-1} \sqrt{\frac{(a+b)k}{(a+k)b}} \ .$$

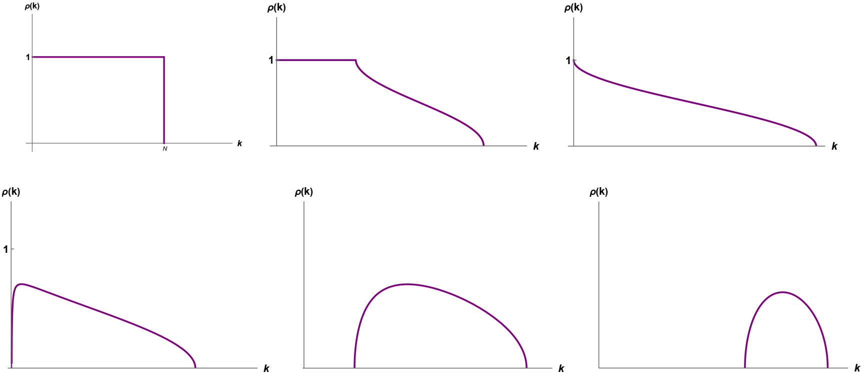


Figure: The distribution $\rho(k)$ for various values of n/N^2. For $n = 0$ (first panel) the distribution is a step function corresponding to the singlet. For $0 < n < N^2/4$ (second panel) the edge of the distribution deforms into an inverse cosine. For $n = N^2/4$ (third panel) the deformation reaches $k = 0$, signaling a phase transition. As soon as n exceeds $N^2/4$ (fourth panel) the left edge of the distribution drops to $\rho(0) = 0$, and as n increases (fifth panel) $\rho(x)$ has support on a positive interval. For $n \gg N^2/4$ (sixth panel) it approaches a Wigner semicircle distribution.

Phase transitions

Consider the entropy functional $S_{w,n}[\rho(k)]$ calculated for the above two solutions.
As a function of n we found that [Poly-KS]:

- Between the two phases it is continuous across $n = n_w$.
- However, higher derivatives w.r.t. n are not, signaling a phase transition.
- Summary of phase transitions for various values of $w \geqslant 1$

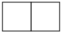
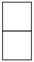
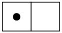
transition	3rd order	4th order	no transition
$w = 1$	✓		
$w > 1\,(w \neq 2)$	(crossover)	✓	
$w = 2$			✓

Concluding remarks

$SU(N)$ ferromagnets display new features:

- Various novel phase transitions
- Metastable phases
- Hysteresis in temperature and magnetic field
- Spontaneous breaking $SU(N) \to SU(N-1) \times U(1)$
- With M magnetic field components, $SU(N) \to SU(N-M-1) \times U(1)^{M+1}$ (generically)
- Admit large-N limit, $N \sim \sqrt{n}$ (shown at very large T)

Future directions:

- Higher representations of $SU(N)$; In particular compose: ,  and 
- Anisotropic couplings h_{ab}, further modified symmetry
- Higher Casimirs, 3-body and higher interactions
- Large-N limit $N \sim \sqrt{n}$, new phases (for finite T)

ΕΥΧΑΡΙΣΤΩ