Hypercontractivity and factorial moment scaling in the symmetry broken phase

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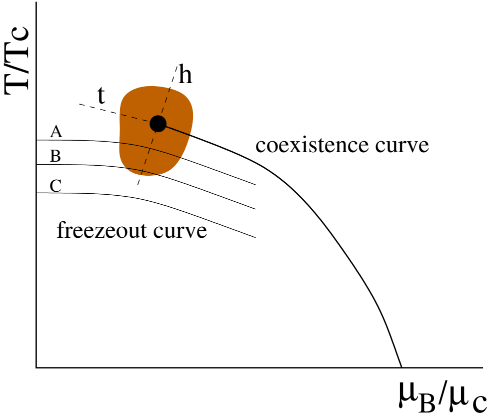
43rd International Symposium on Physics in Collision
Demokritos, Athens, October 22-25

1. Critical region ; scaling properties
2. Measuring fractal dimensions in relativistic ion collisions
3. Intermittency analysis in RHIC
4. Theoretical interpretation of STAR intermittency results
5. Conclusions and outlook

Phase diagram of QCD

A sketch for systems with finite size



Main goal:
Detection/existence of the
QCD Critical Point (CP)

from R. V. Gavai, Contemporary Physics 57, 350 (2016)

Critical region

Scaling laws	$\Leftrightarrow$	Fractal geometry
Order parameter fluctuations (baryon-number density $n_b(\mathbf{r})$)		Random fractal clusters formed by baryon excess
Density-density correlation $C(\mathbf{r},\mathbf{r_0}) = \langle n_b(\mathbf{r}) n_b(\mathbf{r_0}) \rangle$	**local**	Correlation dimension $C(\mathbf{r},\mathbf{r_0}) \propto \vert\mathbf{r}-\mathbf{r_0}\vert^{-(d-d_F)}$
Finite-size scaling (FSS) $\langle N_b \rangle = \langle \int_V n_b(\mathbf{r}) \rangle$	**global**	Mass fractal dimension $\langle N_b \rangle \propto V^{D_F/d}$

critical exponents $\Leftrightarrow$ fractal dimension(s)

(homogeneity: $D_F = d_F$)

Transverse momentum space

In the critical region (large **correlation length**) the **local** scaling:

$$\langle n_b(\mathbf{r}) n_b(\mathbf{r_0}) \rangle \propto |\mathbf{r} - \mathbf{r_0}|^{-(d-d_F)}$$

holds also for large $|\mathbf{r} - \mathbf{r_0}|$

$$\Downarrow$$

Scaling is transferred to **momentum space** for **small momentum differences** (Fourier transform):

$$\lim_{\mathbf{k} \to \mathbf{k}'} \langle n_b(\mathbf{k}) n_b(\mathbf{k}') \rangle \propto |\mathbf{k} - \mathbf{k}'|^{-d_F}$$

A fractal structure in momentum space with $\hat{d}_F = d - d_F$ is **locally** formed!

At **midrapidity** region the momentum space fractal becomes a cartesian product ($d = 3$):

$$\text{Transverse momentum} \otimes \text{Longitudinal momentum}$$

leading to the **transverse momentum** scaling law:

$$\lim_{\mathbf{k}_\perp \to \mathbf{k}'_\perp} \langle n_b(\mathbf{k}_\perp) n_b(\mathbf{k}'_\perp) \rangle \propto |\mathbf{k}_\perp - \mathbf{k}'_\perp|^{-\frac{2d_F}{3}}$$

$$\Downarrow$$

$2d$-fractal in transverse momentum space with $\hat{d}_{F,\perp} = 2 - \frac{2}{3} d_F$

$$\Downarrow$$

Local, power-law distributed, fluctuations in transverse momentum space!

Experimental observation of local, power-law distributed fluctuations

$\Downarrow$

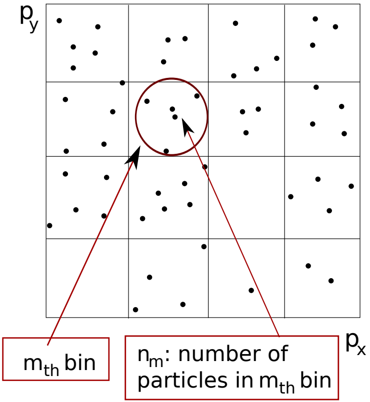
Intermittency in transverse momentum space (net protons at mid-rapidity)

(Critical opalescence in ion collisions)

- Transverse momentum space is partitioned into M^2 cells
- Calculate second factorial moments $F_2(M)$ as a function of cell size $\Leftrightarrow$ number of cells M:

$$F_2(M) \equiv \frac{\sum_m \langle n_m(n_m-1)\rangle}{\sum_m \langle n_m\rangle^2},$$

where $\langle\ldots\rangle$ denotes averaging over events.



For local power-law fluctuations:

$$F_2(M) \propto (M^2)^{\phi_2} \quad \text{for} \quad M^2 \gg 1$$

with $\phi_2 = \frac{1}{2}(2 - \hat{d}_{F,\perp}) \rightarrow$ **Intermittency index**

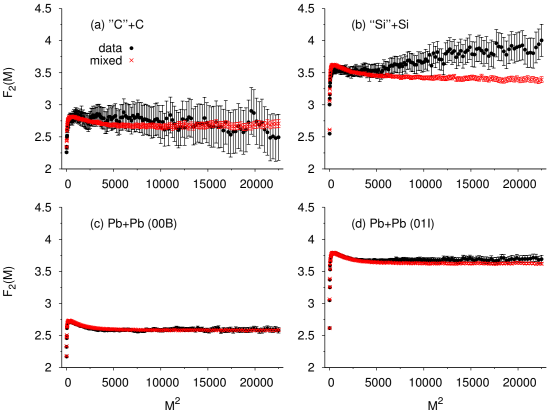
$$\Downarrow$$

Critical fluctuations linked to the **QCD critical point**:

$$\phi_2 = \frac{d_F}{3} = \frac{5}{6} \quad ; \quad \text{with } d_F = \frac{5}{2} \ \text{ for } \ \text{3d Ising}$$

Critical Intermittency $\Rightarrow$ **Measurement of** ϕ_2

Measurement of ϕ_2 in NA49 (SPS, CERN) data



Very small number of critical proton pairs in Si+Si! $\Rightarrow$ need for **very large statistics**

NA49 result:

$$\phi_2^{(Si)} = 0.96^{+0.38}_{-0.25}$$

T. Anticic et al, NA49 Collaboration, Eur. J. Phys. C 75, 587 (2015)

ϕ_2 measurement

$\Downarrow$

advanced techniques

(see N. Davis talk)

https://arxiv.org/abs/2409.14185v1

Factorial moment scaling with order

An **alternative** proposal for critical point search:

- Experimental observables exclusively related to **hadrons** $\Rightarrow$ signal from the phase of broken symmetry

- Study higher order factorial moments in the symmetry broken phase (average number of multiplets of order q per cell and event):

$$F_q(M) \equiv \frac{\sum_m \langle n_m(n_m-1)\dots(n_m-q+1)\rangle}{\sum_m \langle n_m\rangle^q},$$

- **Higher order** factorial moments vs. **second order** one:

$$F_q(M) = (F_2(M))^{\beta_q} \quad ; \quad \beta_q = (q-1)^\nu$$

$$\Downarrow$$

ν is the main observable

Factorial moment scaling with order

Theoretical predictions based on Ginzburg-Landau free energy calculations in the **symmetry broken phase**:

For **second order** transitions $\Rightarrow \nu \approx 1.3$

R.C. Hwa and M.T. Nazirov, Phys. Rev. Lett. 69, 741 (1992)

Generalized to a **universal** exponent $\nu \approx 1.33$
(for first **and** second order transitions)

A.K. Mohanty and S.K. Kataria, Phys. Rev. Lett. 73, 2672 (1994).

Intermittency in RHIC-STAR

Physics Letters B 845 (2023) 138165

Contents lists available at ScienceDirect

Physics Letters B

journal homepage: www.elsevier.com/locate/physletb

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Energy dependence of intermittency for charged hadrons in Au+Au collisions at RHIC

The STAR Collaboration

M.I. Abdulhamid [d], B.E. Aboona [bc], J. Adam [o], L. Adamczyk [b], J.R. Adams [am], I. Aggarwal [ao], M.M. Aggarwal [ao], Z. Ahammed [bj], D.M. Anderson [bc], E.C. Aschenauer [f], S. Aslam [z], J. Atchison [a], V. Bairathi [ba], W. Baker [k], J.G. Ball Cap [v], K. Barish [k], R. Bellwied [v], P. Bhagat [ac], A. Bhasin [ac], S. Bhatta [az], J. Bielcik [o], J. Bielcikova [al], J.D. Brandenburg [am], X.Z. Cai [ax], H. Caines [bm], M. Calderón de la Barca Sánchez [i], D. Cebra [i], J. Ceska [o], I. Chakaberia [af], P. Chaloupka [o], B.K. Chan [j], Z. Chang [aa], A. Chatterjee [q], D. Chen [k], J. Chen [aw], J.H. Chen [t], Z. Chen [aw], J. Cheng [be], Y. Cheng [j], S. Choudhury [t], W. Christie [f], X. Chu [f], H.J. Crawford [h], M. Csanád [r], G. Dale-Gau [m], A. Das [o], M. Daugherity [a], I.M. Deppner [u], A. Dhamija [ao], L. Di Carlo [bl], P. Dixit [x], X. Dong [af], J.L. Drachenberg [a], E. Duckworth [ad], J.C. Dunlop [f], J. Engelage [h], G. Eppley [aq], S. Esumi [bf], O. Evdokimov [m], A. Ewigleben [ag], O. Eyser [f], R. Fatemi [ae], S. Fazio [g], C.J. Feng [ak], Y. Feng [ap], E. Finch [ay], Y. Fisyak [f], F.A. Flor [bm], C. Fu [ab], C.A. Gagliardi [bc], T. Galatyuk [p], T. Gao [aw], F. Geurts [aq], N. Ghimire [bb], A. Gibson [bi], K. Gopal [y], X. Gou [aw], D. Grosnick [bi], A. Gupta [ac], W. Guryn [f], A. Hamed [d], Y. Han [aq], S. Harabasz [p],

Search for the scaling law:

$$F_q(M) \sim \left(F_2(M)\right)^{(q-1)^{1.3(3)}}$$

in **Au+Au collisions** at different energies

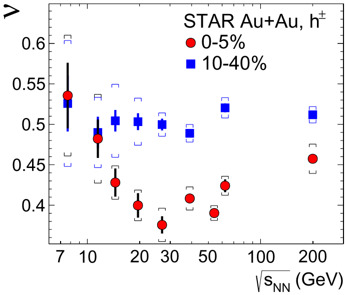
STAR results

- For **all charged particles** it is calculated

$$\Delta F_q(M) = F_q^{(data)}(M) - F_q^{(mixed)}(M)$$

- For $q \leq 6$ a scaling law with $\nu < 1$ for all energies is found:

$$\Delta F_q(M) = (\Delta F_2(M))^{(q-1)^{\nu}}$$



Comments on STAR analysis (procedure)

- The transverse momentum fluctuations analysed in STAR not directly related to order parameter fluctuations:

  All charged particles $\Rightarrow$ dipions ((π^+, π^-)-pairs), protons

- $\Delta F_q(M) = F_q^{(data)}(M) - F_q^{(mixed)}(M)$ **does not remove all** the background even for $q = 2$!

  see T. Anticic et al, NA49 Collaboration, Eur. J. Phys. C 75, 587 (2015)

- Maximum number of cells is limited to $M_{max} = 100 \Rightarrow$ In **critical intermittency** M_{max} is limited only by **experimental resolution**!

Comments on STAR analysis (theory)

In the symmetry broken phase particle distributions have finite moments

$\Downarrow$

A theorem from probability theory:

$F_q(M)$ cannot increase faster than $L^{q\log(q)}$ with increasing q

Based on **hypercontractivity** and **concentration of measure**

M. Vladimirova, S. Girard, H. Nguyen and J. Arbel, Stat. **9**, e318 (2020);

A. Brofas, M. Zampetakis and F.K. Diakonos, https://arxiv.org/abs/2409.19412

$\Downarrow$

The behaviour $F_q \sim (F_2)^{(q-1)^{1.3}}$ can only be **transient** (for $q \leq q_{max}$)!

Comments on STAR analysis (theory)

- In Ginzburg-Landau free energy approach an asymptotic expansion for $q \to \infty$ leads to

$$F_q \sim (F_2)^q \quad ; \quad q \to \infty$$

in accordance with the previous theorem.

- A scaling law of the form:

$$F_q \sim (F_2)^{(q-1)^{1.3}} \quad ; \quad q \leq q_{max}$$

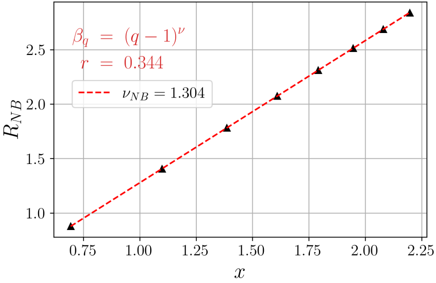
holds for **infinitely many** distributions of **conventional** origin!

$$\Downarrow$$

The scaling law $F_q \sim (F_2)^{(q-1)^{1.3}}$ for $q \leq q_{max}$ **cannot** be used as a **signal** for a critical point!

Comments on STAR analysis (theory)

Example: **Negative Binomial** distribution with real r and **any** p!



$$R_{NB}(q,r) = \ln\left(\frac{\ln(F_{q,NB}(r))}{\ln(F_{2,NB}(r))}\right) \quad ; \quad x = \ln(q-1)$$

Similarly, for mixture of **Poissons**, **discrete Weibull**, etc.

see A. Brofas, M. Zampetakis and F.K. Diakonos, https://arxiv.org/abs/2409.19412

Comments on STAR analysis ($\bar{\nu} < 1$)

STAR: differences of factorial moments!!

$$F_{q,A} = a(q) F_{2,A}^{\beta_q} \quad (A = d,\ m); \quad \Delta F_q = \bar{a}(q) \Delta F_2^{\bar{\beta}_q}$$

with $\beta_q = (q-1)^{\nu}$ and $\bar{\beta}_q = (q-1)^{\bar{\nu}}$

The **amplitude ratio** $\frac{\bar{a}(q)}{a(q)}$ is **important** for ΔF_q **scaling**!

$\Downarrow$

For $F_{2,d} \approx F_{2,m}$, $\Delta F_2 \ll F_{2,A}$ and $\frac{\bar{a}(q)}{a(q)} > \beta_q F_{2,m}^{\beta_q - 1} \Rightarrow \bar{\nu} < 1$

Easily verified through toy-model simulations!

Summary, conclusions and outlook

- The scaling relation relating higher moments with the second moment is **inadequate** for the search of the QCD critical point
- Published STAR intermittency results can be understood through correlations originating from **conventional** distributions.
- Critical Intermittency analysis, based on the **second factorial moment** is the most promising perspective for the detection of the **critical point**:

order parameter fluctuations	**advanced** techniques
	https://arxiv.org/abs/2409.14185v1
$M \gg 1$ (exp. resol.)	**very large** statistics!

Thank you!