BV Pushforward and Applications

Quick recollection

- On finite dimensional superchart, we consider coordinates p_i, q^i

$\mathrm{gh}\, p_i = -\mathrm{gh} q^i - 1$, $\omega = dp_i dq^i$, $\Delta = \frac{\partial}{\partial p_i} \frac{\partial}{\partial q^i}$

- BV integration: ψ odd function of the q's "gauge-fixing fermion"

$$\int_{\mathcal{L}_\psi} f := \int f\Big|_{p_i = (-1)^{\mathrm{gh} p_i} \frac{\partial \psi}{\partial q^i}} dq$$

"$\mathcal{L}_\psi = \{ p_i = (-1)^{\mathrm{gh} p_i} \}$" Lagrangian graph

<u>BV Lemma</u> If $f = \Delta g$, g integrable on $\mathcal{L}_\Psi$, then

$$\int_{\mathcal{L}_\Psi} f = 0$$

<u>BV Theorem</u>

- Ψ_t family of <u>odd</u> functions of the q's parametrized by $t \in I \subset \mathbb{R}$
- f – integrable on every $\mathcal{L}_{\Psi_t}$
  – $\Delta f = 0$

Then

$$\boxed{\frac{d}{dt} \int_{\mathcal{L}_{\Psi_t}} f = 0}$$

"gauge-fixing independence"

Generalization The BV pushforward

Split the coordinates (p,q) into

$$(p_i, q^i)_{i=1,\dots,k} \qquad (p_i, q^i)_{i=k+1,\dots,n}$$

$$\omega = \sum_{i=1}^{k} dp_i\, dq^i + \sum_{i=k}^{n} dp_i\, dq^i$$

$$\omega_1 + \omega_2$$

$$\Delta = \sum_{i=1}^{k} \partial_i \partial^i + \sum_{i=k}^{n} \partial_i \partial^i$$

$$\Delta_1 + \Delta_2$$

Let f be a function of all (p,q)'s

ψ_t be a family of <u>odd</u> functions of the q^i's, $i = k+1, \dots, n$

Assume β integrable on each $\mathcal{L}_{\psi_t}$: $\int_{\mathcal{L}_{\psi_t}} \beta|_{\substack{p_i = (-1)^{q^{i}p_i} \partial_i \psi \\ i > k}}$ is a function of $(p_i, q^i)_{i \leq k}$

Then

1) $$\Delta_1 \int_{\mathcal{L}_{\psi_t}} f = \int_{\mathcal{L}_{\psi_t}} \Delta f$$

2) $$\frac{d}{dt} \int_{\mathcal{L}_{\psi_t}} f = \Delta_1 (\cdots) \quad \text{if } \Delta f = 0$$

The proof follows the same lines as the previous ones

The global version

Data: (M, ω) odd symplectic manifold

i.e.: M (graded) supermanifold, $\omega \in \Omega^2_d(M)$ s.t. $\omega : TM \xrightarrow{\sim} T^*M$

How do we define Δ?

Recall: locally $\Delta f = \partial_i \partial^i f = \frac{1}{2}\left(\frac{\partial}{\partial q^i}\frac{\partial f}{\partial p_i} + \frac{\partial}{\partial p_i}\frac{\partial f}{\partial q^i}\right)$

$= \frac{1}{2} \operatorname{div} X_f$

[Khudaverdian]: $\Delta_\mu f := \frac{1}{2} \operatorname{div}_\mu X_f$

- $\iota_{X_f} \omega = \pm df$

section of the Berezinian line bundle Ber

- μ a $\underbrace{\text{nowhere vanishing Berezinian density}}$

<u>Lemma</u> $\Delta_\mu(fg) = \Delta_\mu f\, g \pm f\, \Delta_\mu g \pm (f,g)$

However, in general $\Delta_\mu^2 \neq 0$!

<u>Def</u> μ is compatible if $\Delta_\mu^2 = 0$

<u>Thm</u> μ compatible structures exist

Rem In the codim case of field theory μ does not exist (tantamount to producing a functional measure)

The definition of Δ_μ goes in parallel with regularization and renormalization.

- $\frac{1}{2}$densities = sections of $\mathrm{Ber}^{\frac{1}{2}}$

Properties

1) $\left.\begin{array}{l}\sigma \ \frac{1}{2}\text{density} \\ \mathcal{L} \subset \mathcal{M} \text{ Lagrangian}\end{array}\right) \Rightarrow \sigma|_{\mathcal{L}}$ density $\rightsquigarrow \int_{\mathcal{L}} \sigma$

2) $\exists$ canonical operator Δ on $\frac{1}{2}$-densities that in local Darboux coords is $\partial_i \partial^i$

[Khudaverdian]

In particular, $\Delta^2 = 0$

BV then now reads 1) $\int_{\mathcal{L}} \Delta\sigma = 0$

2) $\Delta\sigma = 0 \Rightarrow \frac{d}{dt}\int_{\mathcal{L}_t} \sigma = 0$

BV pushforward $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ product of odd sympl. mflds

or more generally $\begin{array}{c}\mathcal{M}\\ \downarrow\\ \mathcal{M}_1\end{array}$ fibration with $\mathcal{M}$, $\mathcal{M}_1$, and typical fiber odd symplectic

+

some conditions on the transition functions ("hedgehog") [C-Mnëv-Reshetikhin 2015]

Back to functions

If σ is a nowhere vanishing $\frac{1}{2}$-density, we get an operator $\Delta^{(\sigma)}$ on functions via

$$\Delta^{(\sigma)} f = \Delta(f\sigma) / \sigma$$

Moreover, $\Delta^{(\sigma)} = \Delta_{\sigma^2}$

Digression: Interpretation via differential forms [Witten]

• [Schwartz] $(\mathcal{M}, \omega) \simeq (\Pi T^*\mathcal{N}, \omega_{can})$

noncanonical symplectomorphism

odd cotangent bundle

"$\sum d f_i \, d q^i$"

i.e., $C^\infty(\mathcal{M}) \simeq \mathcal{V}(\mathcal{N})$ multivector fields

It also follows that $\Gamma(\mathrm{Ber}_{\mathcal{M}}^{\frac{1}{2}}) \simeq \mathcal{V}(\mathcal{N}) \otimes \Omega^{top}(\mathcal{N})$

Now $\mathcal{V}(\mathcal{N}) \otimes \Omega^{top}(\mathcal{N}) \xrightarrow{\sim} \Omega(\mathcal{N})$

$(X \quad , \quad v) \longmapsto \iota_X v$

$\Delta \longleftrightarrow d$!

- Stokes theorem is now part of BV

$C \subseteq N$ submanifold $\rightsquigarrow$ $\Pi N^*C \subset \Pi T^*N$ Lagrangian

- v volume form on N $\rightsquigarrow$ nowhere vanishing $\frac{1}{2}$-density on ΠT^*N

End of the digression

Applications of the BV pushforward

1) Construction of observables

[C-Rossi, 2002] [Mnëv, 2015] [Moshayedi, 2020] [C, 2024]

2) Renormalization à la Wilson [Losev 2006, Mnëv 2006, Costello 2007]

3) Zero modes [C-Felder 2008, C-Mnëv 2008; Gui-Li-Xu, Gui-Li 2021]

Only trees: Merkulov 98, Kontsevich-Soibelman 2000

4) Quasi-isomorphism of L_∞-structures [Mnëv 2006, C-Mnëv 2008, C-Mnëv-Reshetikhin 2015, Jurčo-Raspollini-Sämann-Wolf 2020]

[1) becomes: construction of L_∞-reps]

5) Equivalence of field theories

- Contractible pairs and elimination of auxiliary fields [Henneaux, 1990]
- Parent theories [Barnich-Grigoriev-Semikhatov-Tipunin, 2004]
- Different formulations of gravity [w/ Schiavina, Canepa, Simão 2021]
- Dressing in YM theory [Anselmi, 2022]
- YM theory from a topological field theory [Bonechi-C-Zabzine, 2022]

I will focus on three examples

- Surface observables and 2-knot invariants
- Yang-Mills theory from a topological field theory
- AKSZ formulation of 4D Palatini-Cartan gravity

Surface observables and 2-knot invariants

[$\lambda = 0$: C-Rossi, 2002
$\lambda \neq 0$: C, 2024
(and zero modes)]

4D BF theory with cosmological term:

$$S_{cl}^{BF} = \int_M BF_A - \frac{\lambda}{2} BB$$

M closed, oriented 4-manifold
λ a real parameter

Fields: A · connection 1-form
B · ad-valued 2-form

(an invariant pairing is understood)

- EL equations: $F_A = \lambda B$
- Symmetries:
$$\delta A = -d_A \gamma - \lambda \theta$$
$$\delta B = -d_A \theta + [\gamma, B]$$

The surface observable

$\Sigma \hookrightarrow M$, $\dim \Sigma = 2$

$$\mathcal{O}_{\Sigma,\alpha} = \int D\alpha D\beta \, e^{\frac{i}{\hbar} J_{cl}}$$

$$J_{cl}^{\Sigma} = \int_{\Sigma} \beta d_A \alpha + \frac{\lambda}{2} \beta [\alpha, \alpha] + \beta B$$

Rem 1 ∂J_{cl} can be compensated by the infinitesimal change of variables

$$\alpha \rightsquigarrow \alpha - \theta - [\partial, \alpha], \quad \beta \rightsquigarrow \beta - [\gamma, \beta]$$

Rem 2 "symmetries" $\hat{\partial} \alpha = -d_A p - \lambda [\alpha, p]$, $\hat{\partial} \beta = \lambda [p, \beta]$

$$\hat{\partial} J = \int [\beta, p] (F_A - \lambda B) = 0 \text{ using EL equations}$$

Using the BV p.f. one can fix this

$$S_d^{BF} \rightsquigarrow S_{BV}^{BF}, \qquad J_d^{\Sigma} \rightsquigarrow J_{BV}^{\Sigma}$$

- $$\mathcal{O}_\Sigma(A,B) = \int_{\mathcal{L}_1} e^{\frac{i}{\hbar} J_{BV}^{\Sigma}} \quad \text{is a BV observable}$$

  $\mathcal{L}_1 \subset$ BV space for α, β

- $$\langle \mathcal{O}_\Sigma(A,B) \rangle = \int_{\mathcal{L}} e^{\frac{i}{\hbar} S_{BV}^{BF}} \mathcal{O}_\Sigma \Big/ \int_{\mathcal{L}} e^{\frac{i}{\hbar} S_{BV}^{BF}}$$

is an invariant of the 2-knot Σ

For $\lambda = 0$, these invariants have been studied by Watanabe 2006, 2007 & Leturcq 2020

Details $\quad S_{BV} = \int_M \mathcal{B}F_{\mathcal{A}} - \frac{\lambda}{2}\mathcal{B}\mathcal{B}$

superfields: $\quad \mathcal{A} = c + A + B^+ + \gamma^+ + \phi^+$

$\mathcal{B} = \phi + \gamma + B + A^+ + c^+$

$\rightsquigarrow \quad Q\mathcal{A} = F_{\mathcal{A}} - \lambda\mathcal{B}, \quad Q\mathcal{B} = d_{\mathcal{A}}\mathcal{B}$

$$J_{BV}^{\Sigma} = \int_{\Sigma} b\, d_{\mathcal{A}}\, a + \frac{\lambda}{2} b[a,a] + b\mathcal{B}$$

$a = \psi + \alpha + \beta^+$

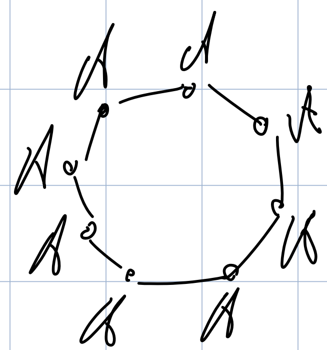
$b = \beta + \alpha^+ + \psi^+$

Lemma $\quad S_{BV} + J_{BV}^{\Sigma}$ satisfies the CME

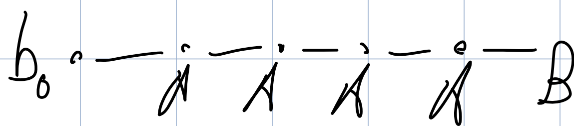
$$\mathcal{O}_\Sigma(A,B) = \int_{\mathcal{L}} e^{\frac{i}{\hbar} S^{\Sigma}_{BV}}$$

may be expanded in Feynman diagrams

For example $(\lambda = 0)$



— = propagator for a, b



b_0 zero mode for b

Yang-Mills theory from a topological field theory [Bonechi-C-Zabzine, 2022]

Again

$$S_{cl} = \int_M BF_A - \frac{\lambda}{2} BB$$

Pick a metric and write $\quad B = B_+ + B_-$ (self/antiselfdual)

Set $B_- = 0 \rightsquigarrow S_{YM} = \int_M B_+ F_A - \frac{\lambda}{2} B_+ B_+$

$B_- = 0$ can be viewed as part of a gauge fixing for a BV pushforward

Result:

$$e^{\frac{i}{\hbar}\left(S_{YM} + O(\hbar)\right)} = \int_{\mathcal{L}_{UV}} e^{\frac{i}{\hbar} S^{BF}_{BV}}$$

In particular, $\mathcal{O}_\Sigma$ induces a surface observable for YM theory [C, 2024]

$$\mathcal{O}^{YM}_\Sigma = \int_{\mathcal{L}_{uv}} e^{\frac{i}{\hbar} S^{BF}_{BV}} \mathcal{O}_\Sigma \Big/ \int_{\mathcal{L}_{uv}} e^{\frac{i}{\hbar} S^{BF}_{BV}} = e^{\frac{i}{\hbar}\int_\Sigma B + \cdots}$$

AKSZ formulation of 4D Palatini–Cartan gravity [Canepa–C, 2024]

$$S_{PC} = \int_M \frac{1}{2} \tilde{e}\tilde{e} F_{\tilde{\omega}}$$

M	4-manifold
Fields: $\tilde{e}$	- coframe
$\tilde{\omega}$	- orthogonal connection

Assume $M = \underset{\text{"space"}}{\Sigma} \times \underset{t}{\underset{\cup}{I}}$. Fix ε_n section of the "fake" tangent bundle

$(\varepsilon_n, \varepsilon_n) = -1$ time-like

Write
$$\tilde{e} = e_n dt + e, \qquad e_n = \mu \varepsilon_n + z_2 e$$
$$\tilde{\omega} = \omega_n dt + \hat{\omega}$$

$$\rightsquigarrow \quad S_{PC} = \int_M \left[e_n e F_{\hat{\omega}} + \frac{1}{2} e e \partial_n \hat{\omega} + \omega_n e d_{\hat{\omega}} e \right] dt$$

Now split $\hat{\omega} = \omega + v$, with $\begin{cases} \varepsilon_n d_\omega e = e\sigma & \text{(arbitrary } \sigma\text{)} \\ e v = 0 \end{cases}$

$$S_{PC} = \int_M \Big[e_n e F_\omega + \frac{1}{2} e e \partial_n \omega dt + \omega_n e d_\omega e$$

$$+ \ L_\xi v \ e d_\omega e + \frac{1}{2} e_n e [v, v] \Big] dt$$

- $\int_M \frac{1}{2} e e \partial_n \omega dt$ analogue to $\int p \dot{q} dt$

- e_n, ω_n: Lagrange multipliers yielding the constraints $\begin{cases} e F_\omega = 0 \\ e d_\omega e = 0 \end{cases}$

Rem) If Σ also has a boundary, there are additional boundary terms (mass, angular momentum...)

Set $\mathfrak{w} = \omega_n + \iota_z v,$ z vector field

(I-dependent, in Σ-directions)

Then $$\boxed{S_{PC} = S_{PCAKSZ} + S_{aux}}$$

- $S_{PCAKSZ}(e, \omega, \mu, z, \mathfrak{w}) =$

$$= \int_M \left[\frac{1}{2} e e \partial_n \omega + (\mu \varepsilon_n + \iota_z e) e F_\omega + \mathfrak{w} e d_\omega e\right] dt$$

- $S_{aux} = \int_M \frac{1}{2} e_n e [v, v] dt$

Idea: integrate out $[\tilde{v}, \tilde{v}]$

Thm Assume $g = \frac{1}{2}(e,e)$ is a nondegenerate metric on each time slice

Then $S_{aux} = \int_M \frac{1}{2} e_n e\,[v, v]\,dt$

is a nondegenerate quadratic form in v

This is a corollary of a theorem in [C - Schiavina, 17]

$$\rightsquigarrow) \quad e^{\frac{i}{\hbar} S_{PCAKPZ}} = \int \mathcal{D}v \; e^{\frac{i}{\hbar} S_{PC}}$$

To deal with symmetries, this has to be extended to a BV pushforward

<u>Result</u>

$$e^{\frac{i}{\hbar} S^{BV}_{PCAKSZ}} = \int_{\mathcal{L}} e^{\frac{i}{\hbar} S^{BV}_{PC}}$$

with S^{BV}_{PCAKSZ} the BV action in [Canepa-C-Schiavina, 20] obtained by the AKSZ method from the BFV action describing the boundary constraints

Thanks