BV quantization in topological/holomorphic QFT

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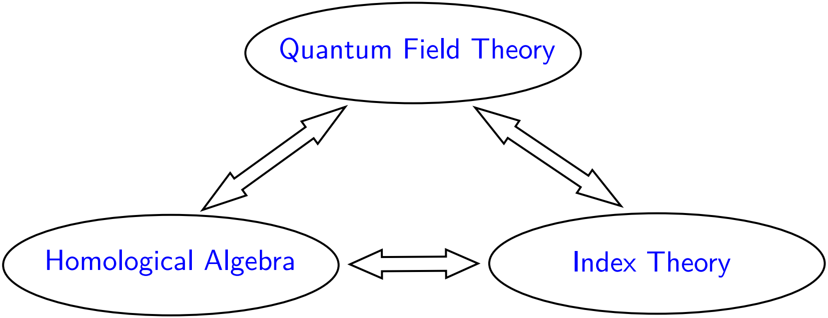
"Gauge Invariance: Quantization and Geometry"
Workshop in memory of Igor Batalin

BV Quantization and Index Theory



Let us first explain how some notions of homological algebra in noncommutative geometry arise naturally in quantum field theory.

Hochschild-Kostant-Rosenberg $\sim$ Renormalization Group Flow

Warm-up Example

Let $A = \mathbb{R}[q^i, p_i]$ be polynomials on the phase space $\mathbb{R}^{2n}$. We have

$$A \otimes A \otimes \cdots \otimes A \to A$$

$$f_0 \otimes f_1 \otimes \cdots \otimes f_m \to f_0 f_1 \cdots f_m$$

Let me first explain how to promote this to a S^1-quantum product

$$f_0 \otimes \cdots \otimes f_m \to \langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{S^1}$$

via quantum mechanics on the phase space

$$S^1 \to \mathbb{R}^{2n}$$

Let us denote the fields

$$(\mathbb{Q}^i(t), \mathbb{P}_i(t)) : S^1 \to \mathbb{R}^{2n}$$

The action is the standard

$$S[\mathbb{Q}, \mathbb{P}] = \int_{S^1} \mathbb{P}_i(t)\dot{\mathbb{Q}}^i(t)dt.$$

The equation of motion

$$\frac{d}{dt}\mathbb{P}_i(t) = 0 \qquad \frac{d}{dt}\mathbb{Q}^i(t) = 0$$

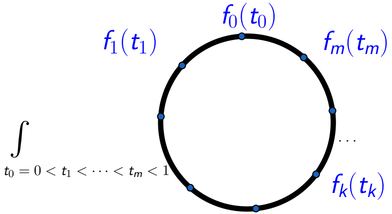
describes constant maps (zero modes)

$$\mathbb{Q}^i(t) = q^i, \quad \mathbb{P}_i(t) = p_i$$

which can be viewed as the classical phase coordinates.

We define the S^1-quantum product by

$$\langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{S^1}(q,p) := \int_{t_0=0<t_1<\cdots<t_m<1} dt_1 \cdots dt_m$$
$$\int [D\mathbb{X} D\mathbb{P}] e^{\frac{i}{\hbar} S[\mathbb{X},\mathbb{P}]} f_0(q+\mathbb{Q}(t_0), p+\mathbb{P}(t_0)) \cdots f_m(q+\mathbb{Q}(t_m), p+\mathbb{P}(t_m))$$



Here the path integral is over the non-zero modes. What kind of structure does this S^1-quantum product have?

A: associative algebra. Hochschild chain complex

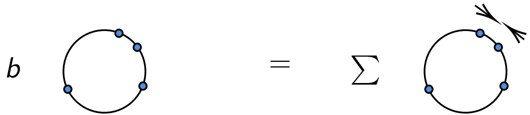
$$(C_\bullet(A), b) = \cdots C_p(A) \xrightarrow{b} C_{p-1}(A) \to \cdots \to C_1(A) \xrightarrow{b} C_0(A)$$

where

$$C_p(A) := A^{\otimes p+1}$$

The Hochschild differential b is

$$b(a_0 \otimes \cdots \otimes a_p) = a_0 a_1 \otimes \cdots \otimes a_p - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_p +$$
$$\cdots + (-1)^{p-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} a_p + (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1}.$$



Hochschild Homology $HH_\bullet(A) = H_\bullet(C_\bullet(A), b)$

Let $A = \mathbb{R}[x^1, \cdots, x^{2n}]$ on the phase space $\mathbb{R}^{2n}$. Here

$$q^1 = x^1, \cdots, q^n = x^n, \quad p_1 = x^{n+1}, \cdots, p_n = x^{2n}$$

Consider

$$\begin{aligned}
\sigma : (C_\bullet(A), b) &\to (\Omega^\bullet_{2n}, 0) \\
f_0 \otimes f_1 \otimes \cdots \otimes f_m &\to \sum_{i_\bullet} \frac{1}{m!} f_0 \partial_{i_1} f_1 \cdots \partial_{i_m} f_m \; dx^{i_1} \cdots dx^{i_m} \\
&= \frac{1}{m!} f_0 df_1 \wedge \cdots \wedge df_m
\end{aligned}$$

Hochschild-Kostant-Rosenberg: σ is a quasi-isomorphism

$$HH_\bullet(A) = \Omega^\bullet_{2n}$$

In general, Hochschild Homology = noncommutative diff forms.

It is now natural to quantize σ to a map $\sigma^\hbar$ by

$$\sigma^\hbar(f_0 \otimes f_1 \otimes \cdots \otimes f_m) := \sum_{i_\bullet} \langle f_0 \otimes \partial_{i_1} f_1 \otimes \cdots \otimes \partial_{i_m} f_p \rangle_{S^1} dx^{i_1} \cdots dx^{i_m}$$

On the other hand, we have the canonical quantization

$$A \quad \overset{\text{quant.}}{\Longrightarrow} \quad W_{2n}^\hbar := \left(\mathbb{R}[x^i][\hbar], \star\right)$$

Here $\star$ = Moyal product.

These two quantizations are related by the following

Theorem (**Gui-L-Xu**, 2021)

The quantized map $\sigma^\hbar$ leads to chain homotopies

$$\begin{array}{rcl} \sigma^\hbar : C_\bullet(W_{2n}^\hbar) & \to & \Omega_{2n}^\bullet((\hbar)) \\ b & \to & \hbar\Delta = \hbar\mathcal{L}_{\omega^{-1}} \\ B & \to & d_{2n} \end{array}$$

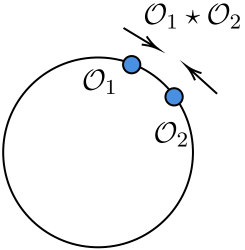
This is BV quantum master equation: $(b + \hbar\Delta)\langle - \rangle_{S^1} = 0$.

- Passing to cohomology $\Longrightarrow$ **Feigin-Felder-Shoikhet** trace
- $\langle - \rangle_{S^1}$ can be glued on a symplectic manifold. Coupling with **Gauss-Manin-Getzler** connection leads to the algebraic index theorem [**Fedosov**,**Nest-Tsygan**]
- $\langle - \rangle_{S^1}$ on symplectic orbifolds (**L-Yang** 2024)
- Stochastic approach ([**L-Wang-Yang]** in preparation)

2d Chiral CFT and elliptic chiral index

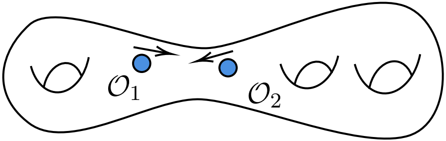
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



Example: $\beta\gamma - bc$ system

$$S[\beta,\gamma] = \int \beta\bar{\partial}\gamma \qquad S[b,c] = \int b\bar{\partial}c$$

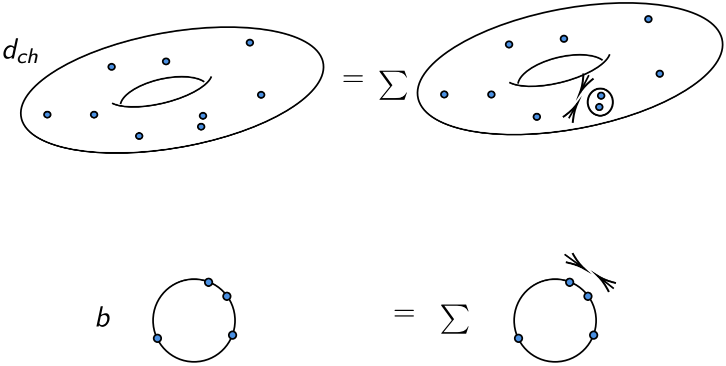
The VOA $\mathcal{V}^{\beta\gamma-bc}$ of $\beta\gamma - bc$ system is the chiral analogue of Weyl/Clifford algebra.

$$\beta(z)\gamma(w) \sim \frac{1}{z-w} + \text{reg.} \qquad b(z)c(w) \sim \frac{1}{z-w} + \text{reg.}$$

It gives rise to a chiral algebra (in the sense of Beilinson and Drinfeld) $\mathcal{A}^{\beta\gamma-bc} = \mathcal{V}^{\beta\gamma-bc} \otimes_{\mathcal{O}_x} \omega_X$ on a Riemann surface $X = \Sigma$.

Elliptic chiral complex

Beilinson and **Drinfeld** defined the chiral homology for general algebraic curves using the Chevalley-Cousin complex. Intuitively



Theorem (Gui-L, 2021)

Let $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. We can construct for $\beta\gamma - bc$ system

$$\langle - \rangle_E : C^{\mathrm{ch}}(E_\tau, \mathcal{A}^{\beta\gamma - bc}) \to \mathcal{A}_E(\!(\hbar)\!)$$

which intertwines the chiral differential d_{ch} with $\hbar\Delta$. Precisely

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_E := \fint_{E_\tau^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$$

- $\mathcal{A}_E$ are functions on zero modes (=copies of $H^\bullet(E_\tau, \mathcal{O}_{E_\tau})$).
- $\langle - \rangle_E$ is a quasi-isomorphism. Chiral analogue of HKR.
- $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is local correlation (via Feynman rules).
- $\fint$ is the regularized integral [**L-Zhou**, CMP 2021].
- The BV trace leads to Witten genus: elliptic chiral index.

Elliptic chiral index (after Douglas-Dijkgraaf)

The partition function of a chiral deformation of conformal field theory by a chiral lagrangian $\mathcal{L}$ is given by

$$\left\langle e^{\frac{1}{\hbar}\int_\Sigma \mathcal{L}} \right\rangle_{2d}$$

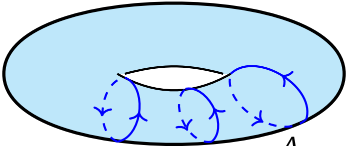
If we quantize the theory on the elliptic curve E_τ,

$$\boxed{\lim_{\bar\tau\to\infty} \left\langle e^{\frac{1}{\hbar}\int_{E_\tau} \mathcal{L}} \right\rangle_E = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - \frac{c}{24}} e^{\frac{1}{\hbar}\oint dz\mathcal{L}}, \quad q = e^{2\pi i\tau}}$$

where the operation $\lim_{\bar\tau\to\infty}$ sends

almost holomorphic modular forms $\implies$ quasi-modular forms.

This can be viewed as a chiral algebraic index.



Theorem (L-Zhou, CMP 2021)

$$\fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right) \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E \quad \textit{lies in} \quad \mathcal{O}_{\mathbf{H}}[\frac{1}{\operatorname{im}\tau}]$$

Let $A_1, \cdots, A_n$ be n disjoint A-cycles on E_τ. Then

$$\lim_{\bar\tau\to\infty} \fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right) \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E$$
$$= \frac{1}{n!} \sum_{\sigma\in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E$$

In particular, $\fint_{E_\tau^n}$ gives a geometric modular completion for quasi-modular forms arising from A-cycle integrals.

Holomorphic Anomaly Equation

Theorem (L-Zhou, CMP 2023)

$$\partial_{\mathbb{Y}} \fint_{E^n} \Psi = \fint_{E^n} \partial_{\mathbb{Y}} \Psi - \sum_{a,b:\, a<b} \fint_{E^{n-\{a\}}} \mathrm{Res}_{z_a=z_b}((z_a - z_b)\Psi)\,.$$

Here Ψ *is an almost-elliptic function, and* $\mathbb{Y} = -\frac{\pi}{\mathrm{im}\,\tau}$*.*

This HAE answers a problem by Dijkgraaf. It also imply the **Yamaguchi-Yau** type HAE for A-cycle integrals as described by **Oberdieck-Pixton**.

Algebraic Index vs Elliptic Chiral Index

1d TQM	2d Chiral CFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
BV QME: $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	BV QME: $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} =$integrals on the compactified configuration spaces of S^1	$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} =$regularized integrals of singular forms on Σ^n
Algebraic Index	Elliptic Chiral Algebraic Index

Joint work with **Zhengping Gui**. arXiv:2112.14572 [math.QA]

Application: Mirror symmetry

Mirror symmetry is about a duality between

symplectic geometry (A-model) $\Longleftrightarrow$ complex geometry (B-model)

$\int_{\mathrm{Map}(\Sigma_g, X)}$ (A-model) $\overset{\text{Fourier transform}}{=\!=\!=}$ $\int_{\mathrm{Map}(\Sigma_g, X')}$ (B-model)

$\downarrow$ localize $\qquad\qquad$ localize $\downarrow$

$\int_{\text{Holomorphic maps}(\Sigma_g, X)}$ $\dashleftarrow\!\dashrightarrow$ $\int_{\text{Constant maps}(\Sigma_g, X')}$???

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

Gromov-Witten Theory $\qquad\qquad$ Hodge theory

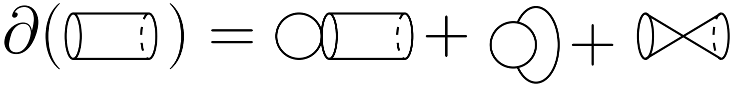
- [**Bershadsky-Cecotti-Ooguri-Vafa**, 1994]: B-twisted topological closed string field theory on Calabi-Yau 3-fold

$$\Longrightarrow \text{Kodaira-Spencer gravity}.$$

- [**Costello-L**, 2012,2015,2016] B-twisted topological closed string-field theory on general Calabi-Yau. Coupling with Witten's holomorphic Chern-Simons in the large N limit

$$\Longrightarrow \text{open-closed BCOV}$$

in the sense of Zwiebach. BV quantum master equation gives Anomaly cancellation mechanism for B-twisted top strings.



Example: Elliptic Curves

Quantum BCOV theory on elliptic curves is completely solved (**L**, JDG 2023) by the chiral deformation of free chiral boson

$$S = \int \partial\phi \wedge \bar{\partial}\phi + \sum_{k\geq 0} \int \eta_k \frac{W^{(k+2)}(\partial_z\phi)}{k+2}$$

where

$$W^{(k)}(\partial_z\phi) = (\partial_z\phi)^k + O(\hbar)$$

are the bosonic realization of the $W_{1+\infty}$-algebra.

Example: Higher genus mirror symmetry on elliptic curves

- Quantum BCOV invariants on elliptic curves via chiral index

$$\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \mathrm{Tr}\, q^{L_0 - \frac{1}{24}} e^{\frac{1}{\hbar}\sum_{k\geq 0} \oint \eta_k \frac{W^{(k+2)}}{k+2}}$$

  Here $W^{(\bullet)}$ are bosonic generators of the $W_{1+\infty}$-algebra.

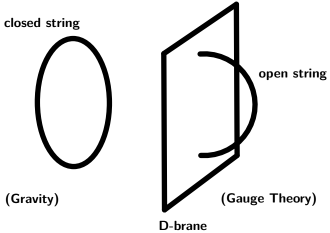
- Coincides with the stationary GW invariants on the mirror computed by **Dijkgraaf** and **Okounkov-Pandharipande**

$$\text{Theorem}: \quad \boxed{\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \langle \mathrm{Stationary} \rangle_E^{\mathrm{GW}}}$$

In this case, we find [**L**, JDG 2023]

Quantum Mirror Symmetry=Boson-Fermion Correspondence

Gauge theory at large N $\Longrightarrow$ Dynamics of Gravity



[**Costello-L**]: Open-closed BCOV leads to the conjectured relations

- BCOV = Twisted Type IIB supergravity
- Open and closed strings are coupled via fashion of Koszul dual

$$QME \Longrightarrow \mathcal{MC}(\text{open} \otimes \text{closed})$$

[**Gui-L-Zeng**, 2022]: a Koszul duality for quadratic chiral algebras

Thank you!