# BOUNDARY CONDITIONS

# OF CHERN - SIMONS

# AND AKSZ

based on joint work(s) with Ján Pulmann, Fridrich Valach and Donald Youmans

# Chern - Simons, as done by Witten

Chern-Simons TQFT on 3-manifolds $\longleftrightarrow$ chiral WZW boundary CFT on $\Sigma = \partial M$

CS: $G$ - cpt Lie group, $\langle , \rangle$ on $\mathfrak{g}$ (level)

$$S(A) = \int_M \frac{1}{2} \langle A, dA \rangle + \frac{1}{6} \langle A, [A, A] \rangle$$

$e^{iS(A)}$ gauge - inv.

chiral WZW:

chiral b.c. : $A|_{\Sigma = \partial M} \in \Omega^{1,0}(\Sigma, \mathfrak{g})$

What if $\langle , \rangle$ is indefinite ?

problem with the chiral b.c. : Hamiltonian not bounded from below

Alternative : $V_+$ - b.c.

"generalized metric" - decomposition $\quad \mathfrak{g} = V_+ \oplus V_-$

$$V_- = V_+^{\perp}, \quad \langle , \rangle \begin{array}{l} \text{pos.} \\ \text{neg.} \end{array} \text{definite on} \begin{array}{l} V_+ \\ V_- \end{array}$$

the $V_+$ - b.c. : $\quad A|_{\Sigma = \partial M} \in \Omega^{0,1}(\Sigma, V_+) \oplus \Omega^{1,0}(\Sigma, V_-)$

classically a conformal b.c., we shall compute:

- conformal anomaly
- RG - flow of $V_+ \subset \mathfrak{g}$
- same for Courant $\sigma$ - model ((generalized) Ricci flow)

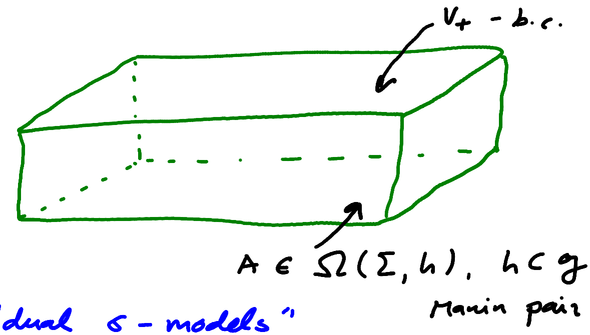
# Why? (i.e. what is my secret agenda)

- Poisson–Lie T-duality

$M = \Sigma \times I$

equiv. to

$\Sigma \to G/H$




different $H$'s – "dual $\sigma$-models"

- everything* should be formulated as boundary conditions of AKSZ models (e.g. Hamiltonian systems on Poisson manifolds = b.c.'s of Poisson $\sigma$-models)

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* no fine print

# BV description of perturbative Chern-Simons

(oh no, not again)

BV space of fields : $\Omega(M, \mathfrak{g})$

BV action $\quad S(A) = \int_M \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A,A]\rangle$

we shall ignore this

propagator = homotopy operator $\quad h : \Omega(M,\mathfrak{g}) \circlearrowleft, \quad dh + hd = id - proj.$

to cohomology

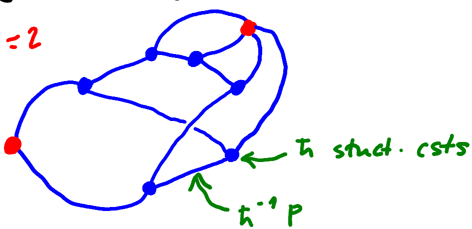
nice : given by a closed 2-form

$P \in \Omega^2(\overline{C}_2(M), \mathfrak{g}\otimes\mathfrak{g}) \qquad \sim$ area form on $S^2$

e.g. $\quad M = \mathbb{R}^3, \quad P = r^* \frac{\text{area form}}{4\pi} \otimes \langle\,,\rangle^{-1}, \quad r(x,y) = \frac{x-y}{\|x-y\|}$

Feynman diagrams $\rightarrow$ correlation forms $G_k \in \Omega(\overline{C}_k(M), (\Lambda\mathfrak{g})^{\otimes k})[[\hbar]]$

$k=2$




$(d + \delta_{CE})\, G_k = 0 \quad$ (+ operadic behavior)

= "pushforward of the measure on $Maps(T[1]M, \mathfrak{g}[1])$ to $(\mathfrak{g}[1])^k$"

# BV for CS with the $V_+$ - b.c.

the $V_+$ - b.c. in the BV space:

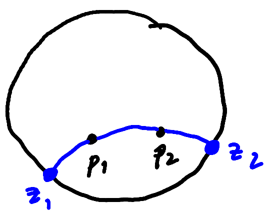
$$\begin{array}{ccccc} 0 & \to & \Omega^{0,1}(\Sigma,V_+)\oplus\Omega^{1,0}(\Sigma,V_-) & \to & \Omega^2(\Sigma,\mathfrak{g})\\ \cap & & \cap & & \cap \\ \Omega^0(\Sigma,\mathfrak{g}) & \to & \Omega^1(\Sigma,\mathfrak{g}) & \to & \Omega^2(\Sigma,\mathfrak{g})\end{array}$$

$\rightsquigarrow$ if $M$ is a handlebody, $\Omega(M,\mathfrak{g})_{V_+\text{-b.c.}}$ is acyclic

explicit propagator for $M = B^3$:

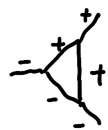
$$\omega_\infty := \frac{1}{2\pi i}\, r^* \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2}$$

$r : \overline{C}_2(B^3) \to C_2(\mathbb{CP}^1)$




$r : (p_1, p_2) \mapsto (z_1, z_2)$

$$P = \underbrace{\omega_\infty \otimes \langle\, ,\rangle^{-1}_{V_-}}_{P_-} + \underbrace{\overline{\omega}_\infty \otimes \langle\, ,\rangle^{-1}_{V_+}}_{P_+}$$

Feynman diags: $D =$  $\mapsto I(D) \otimes L(D)$

form

Lie factor

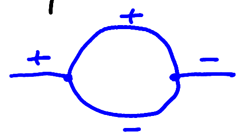
# Divergences, regularization, etc

potential divergence of the integrals:

points collide near the boundary

$$\frac{dz_1\wedge dz_2}{(z_1-z_2)^2}$$

the only divergent 1-loop 1PI diagram:





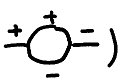
regularization (of the propagator):

$$\omega_g = \frac{1}{2\pi i}\, r^*\left(\frac{dz_1\wedge dz_2}{(z_1-z_2)^2}\, H\big(\mathrm{dist}_g(z_1,z_2)-1\big)\right)$$

$g$ metric in the conformal class

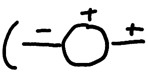
conformal anomaly: $\lim_{\varepsilon\to 0}\; S^{g_1/\varepsilon^2}_{1PI}(A) - S^{g_2/\varepsilon^2}_{1PI}(A) =$

$$= \frac{\hbar}{\pi}\int_{S^2}\log\frac{g_1}{g_2}\,\langle A, L(\ \ )A\rangle + O(\hbar^2)$$



RG-flow

$$\Rightarrow\quad \varepsilon\frac{d}{d\varepsilon}V_+ = \frac{\hbar}{2\pi}L(\ \ ) : V_+\to V_-$$



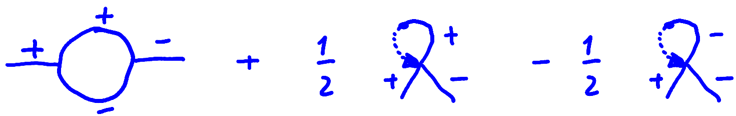
the "generalized Ricci tensor"

## What next

- $V_+$ – b.c. in a Courant $\sigma$-model (again 3d AKSZ model)

  $E \to M$ Courant algebroid, $V_+ \subset E$ generalized metric

  conf. anomaly / RG-flow given by



= the (generalized) Ricci tensor

- the RG-flow is a gradient flow, given by the "generalized string effective action"
  - can we get this functional?
- we ignored tadpoles here – what to do properly?

# What next

- higher-dim AKSZ models

typical b.c.'s : $0, \dots, 0, \Omega^{k}(\partial M), \dots, \Omega^{n}(\partial M)$

$0, \dots, 0, \Omega_{+}^{n/2}(\partial M), \Omega^{n/2+1}(\partial M), \dots \Omega^{n}(\partial M)$

- can we play the same tricks?

what are the "Ricci tensors"?

- "step away from calculations":

$\partial M$ AKSZ target

$LGr(\Sigma, X) = \{$ exact lagr. submanifolds in $Maps(T[1]\Sigma, X)\}$

dg manifold of AKSZ boundary conditions

$L \in LGr(\Sigma, X) \Rightarrow T_L LGr(\Sigma, X) = C^\infty(L)$

RG-flow = vect. field on LGr = a function on each L

= the 1PI action divergence — what is this about?

THANKS!