Correlation Functions and the Homological Perturbation Lemma

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Gauge Invariance: Quantization and Geometry
A workshop in memory of Igor Batalin

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Homotopy Algebras and Quantum Field Theory

- Homotopical methods based on L_∞-algebras and A_∞-algebras have sharpened our understanding of algebraic and kinematic structures of scattering amplitudes and dynamical processes of quantum field theory

- Expresses Feynman diagram techniques in a manner that is mathematically precise and conceptually clear, in contrast to standard textbook approaches (e.g. canonical quantization or path integrals)

- Data of any classical perturbative field theory are completely encoded in a corresponding L_∞-algebra (Hohm & Zwiebach '17; Jurčo, Raspollini, Sämann & Wolf '18; Costello & Gwilliam '18; ...)

- L_∞-algebras are the natural algebraic structure underlying the Batalin-Vilkovisky (BV) formalism, so perturbative QFT is captured by quantum BV theory through the homological perturbation lemma

- 'Integrating out' degrees of freedom in path integral understood as homotopy transfer (Doubek, Jurčo & Pulmann '17; Arvanitakis, Hohm, Hull & Lekeu '20; Elliot & Gwilliam '20; ...)

Homotopy Algebras and Quantum Field Theory

- Scattering amplitudes computed by pulling back cyclic L_∞-structure to its minimal model by quasi-isomorphisms (Nützi & Reiterer '18; Arvanitakis '19; Jurčo, Macrelli, Sämann & Wolf '19; Bonezzi, Chiaffrino, Diaz-Jaramillo & Hohm '23; ...)

- Homological constructions also allow for extensions of standard QFT: Quantum BV formalism makes sense in any closed symmetric monoidal category, e.g. representation category of a triangular Hopf algebra defines braided quantum field theory (Nguyen, Schenkel & Sz '21)

- **In this talk:** Explain formulation and computation of vacuum correlation functions in purely algebraic setting of quantum BV formalism, analogous to setup based on quantum A_∞-algebras inspired by techniques from string field theory (Masuda & Matsunaga '20; Okawa '22; Konosu & Okawa '23; Konosu & Totsuka-Yoshinaka '24; ...)

- Disclaimer: I will not address how to deal with analytic complications of infinite-dimensional vector spaces of fields, distributional nature of correlation functions, or standard loop divergences of quantum field theory

Outline

- L_∞-Algebras
- Quantum BV Formalism
- Homological Perturbation Theory
- Scalar Field Theories
- Algebraic Schwinger-Dyson Equations

Based on [arXiv: 2107.02532, 2302.10713, 2406.02372, 2408.14583]

with D. Bogdanović, M. Dimitrijević Ćirić, N. Konjik, H. Nguyen, B. Nikolić, V. Radovanović, A. Schenkel, G. Trojani

Flat and Curved L_∞-Algebras

- L_∞-algebras organise gauge symmetries and dynamics of classical perturbative field theories through their Maurer-Cartan theory

- Graded antisymm multilinear maps $\ell_n : \wedge^n V \longrightarrow V[2-n]$ of degree $2-n$ for $n \geqslant 0$ on a graded $\mathbb{R}$-vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$:

$$\ell_n(\dots, v, v', \dots) = -(-1)^{|v|\,|v'|}\, \ell_n(\dots, v', v, \dots)$$

- Homotopy Jacobi identites $\mathcal{J}_n(v_1, \dots, v_n) = 0$, $n \geqslant 0$, $v_i \in V$:

$$\mathcal{J}_n = \sum_{i=0}^{n} (-1)^{i\,(n-i)}\, \ell_{n+1-i} \circ \big(\ell_i \otimes \mathbb{1}^{\otimes\, n-i}\big) \circ \sum_{\sigma\in\mathrm{Sh}(i;n)} \mathrm{sgn}(\sigma)\, \sigma$$

- Cyclic: graded inner product $\langle -, - \rangle : V \otimes V \longrightarrow \mathbb{R}[-3]$ satisfying

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n)\rangle = \pm\, \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1})\rangle$$

Flat and Curved L_∞-Algebras

- Flat if $\ell_0 = 0$:

$$\ell_1(\ell_1(v)) = 0 \qquad (V, \ell_1) \text{ is a cochain complex}$$

$$\ell_1(\ell_2(v,w)) = \ell_2(\ell_1(v),w) \pm \ell_2(v,\ell_1(w)) \qquad \ell_2 \text{ is a cochain map}$$

$$\ell_2(v,\ell_2(w,u)) + \text{cyclic} = (\ell_1 \circ \ell_3 \pm \ell_3 \circ \ell_1)(v,w,u) \qquad \text{Jacobi up to coboundary}$$

- Curved if $\ell_0 : \mathbb{R} \longrightarrow V[2]$ is non-zero; $\ell_0(1)$ is the curvature:

$$\ell_1\big(\ell_0(v)\big) = 0 \quad , \quad \ell_1\big(\ell_1(v)\big) = -\ell_2\big(\ell_0(1), v\big) \quad , \quad \dots$$

- Curvature can be used to incorporate source terms or external fields when $\ell_0(1)$ is central:

$$\ell_{n+1}\big(\ell_0(1), v_1, \dots, v_n\big) = 0$$

This ensures ℓ_1 is a differential: $(\ell_1)^2 = 0$

BV Formalism

- Build *derived* space of classical observables of a Lagrangian field theory starting from its cyclic L_∞-algebra $(V, \{\ell_n\}, \langle -,- \rangle)$

- Graded commutative algebra $\mathrm{Sym}\, V[2]$:

$$\varphi\,\psi \;=\; (-1)^{|\varphi|\,|\psi|}\,\psi\,\varphi$$

- Extend cyclic L_∞-structure $(\{\ell_n^{\rm ext}\}, \langle -,- \rangle^{\rm ext})$ to $(\mathrm{Sym}\, V[2]) \otimes V$:

$$\ell_0^{\rm ext}(1) \;=\; 1 \otimes \ell_0(1)$$

$$\ell_n^{\rm ext}(a_1 \otimes v_1, \dots, a_n \otimes v_n) \;=\; \pm\, a_1 \cdots a_n \otimes \ell_n(v_1, \dots, v_n)$$

$$\langle a_1 \otimes v_2, a_2 \otimes v_2 \rangle^{\rm ext} \;=\; \pm\, a_1\, a_2\, \langle v_1, v_2 \rangle$$

- Choose dual bases $e_k \in V$, $e^k \in V^* \simeq V[3]$ and contracted coordinate functions $\xi \;=\; e^k \otimes e_k \;\in\; (\mathrm{Sym}\, V[2]) \otimes V$

BV Formalism

- BV Action $S_{\mathrm{BV}} \in \mathrm{Sym}\, V[2]$ is analogue of curved Maurer-Cartan action:

$$S_{\mathrm{BV}} = \sum_{n\geqslant 0} \frac{(-1)^{\binom{n}{2}}}{(n+1)!} \langle \xi, \ell_n^{\mathrm{ext}}(\xi^{\otimes n})\rangle^{\mathrm{ext}}$$

- (Classical) Master Equation: $\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$,
with the BV antibracket $\{\varphi, \psi\} = \langle \varphi, \psi \rangle$ for $\varphi, \psi \in V[2]$

- $(Q_{\mathrm{BV}})^2 = 0$ where $Q_{\mathrm{BV}} = \{S_{\mathrm{BV}}, -\}$

- Classical observables $\big(\mathrm{Sym}\, V[1]^* \simeq \mathrm{Sym}\, V[2], Q_{\mathrm{BV}}, \{-,-\}_\star\big)$
form a P_0-algebra:

$$-Q_{\mathrm{BV}}\{\varphi, \psi\} = \{Q_{\mathrm{BV}}\varphi, \psi\} + (-1)^{|\varphi|}\{\varphi, Q_{\mathrm{BV}}\psi\} \quad \text{compatibility}$$

$$\{\varphi, \psi\} = (-1)^{|\varphi|\,|\psi|}\{\psi, \varphi\} \quad \text{symmetric}$$

$$\{\varphi, \{\psi, \chi\}\} = \pm\{\psi, \{\chi, \varphi\}\} \pm \{\chi, \{\varphi, \psi\}\} \quad \text{Jacobi identity}$$

$$\{\varphi, \psi\,\chi\} = \{\varphi, \psi\}\,\chi \pm \psi\,\{\varphi, \chi\} \quad \text{Leibniz rule}$$

BV Quantization

- BV Laplacian $\Delta_{\rm BV} : \mathrm{Sym}\, V[2] \longrightarrow (\mathrm{Sym}\, V[2])[1]$:

$$\Delta_{\rm BV}(1) \ = \ 0 \ = \ \Delta_{\rm BV}(\varphi) \quad , \quad \Delta_{\rm BV}(\varphi\,\psi) \ = \ \{\varphi,\psi\}$$

$$\Delta_{\rm BV}(a\,b) \ = \ \Delta_{\rm BV}(a)\,b + (-1)^{|a|}\,a\,\Delta_{\rm BV}(b) + \{a,b\}$$

$$\Delta_{\rm BV}\big(\varphi_1\cdots\varphi_n\big) \ = \ \sum_{i<j} \pm\{\varphi_i,\varphi_j\}\ \varphi_1\cdots\widehat{\varphi}_i\cdots\widehat{\varphi}_j\cdots\varphi_n$$

  Implements Gaussian integration/Wick's Theorem

- Satisfies $(\Delta_{\rm BV})^2 \ = \ 0 \quad , \quad \Delta_{\rm BV}(S_{\rm BV}) \ = \ 0$

- $(Q^\hbar_{\rm BV})^2 \ = \ 0$ where $Q^\hbar_{\rm BV} \ = \ \{S_{\rm BV},-\} + \hbar\,\Delta_{\rm BV}$

- Quantum observables $\big(\mathrm{Sym}\, V[2], Q^\hbar_{\rm BV}\big)$ form an E_0-algebra

Homological Perturbation Theory

- Propagators determine strong deformation retracts of $V[1]^* \simeq V[2]$:

$$\big(H^\bullet(V[2]),0\big) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \big(V[2],\ell_1\big) \circlearrowleft \gamma \qquad \begin{matrix} \pi\,\iota \ = \ \mathbb{1} \ , \ \iota\,\pi - \mathbb{1} \ = \ \ell_1\,\gamma + \gamma\,\ell_1 \\ \gamma^2 \ = \ 0 \ , \ \gamma\,\iota \ = \ 0 \ , \ \pi\,\gamma \ = \ 0 \end{matrix}$$

- Observables: $\big(\mathrm{Sym}\, H^\bullet(V[2]),0\big) \underset{\Pi}{\overset{\mathcal{I}}{\rightleftarrows}} \big(\mathrm{Sym}\, V[2],\ell_1\big) \circlearrowleft \Gamma$

- Maps $\mathcal{I}$ and Π extend ι and π as commutative dg-algebra morphisms:

$$\mathcal{I}([\psi_1]\cdots[\psi_n]) \ = \ \iota[\psi_1]\cdots\iota[\psi_n] \quad , \quad \Pi(\varphi_1\cdots\varphi_n) \ = \ \pi(\varphi_1)\cdots\pi(\varphi_n)$$

- $(\iota\,\pi)^2 \ = \ \iota\,\pi : V[2] \longrightarrow H^\bullet(V[2])$ splits $V[2] \ = \ V[2]^\perp \oplus H^\bullet(V[2])$:

$$\mathrm{Sym}\, V[2] \ = \ \mathrm{Sym}\, V[2]^\perp \otimes \mathrm{Sym}\, H^\bullet(V[2])$$

- Put $\Gamma(\varphi_1^\perp\cdots\varphi_n^\perp\otimes[\psi]) \ = \ \dfrac{1}{n}\displaystyle\sum_{i=1}^n \pm\,\varphi_1^\perp\cdots\varphi_{i-1}^\perp\,\gamma(\varphi_i^\perp)\,\varphi_{i+1}^\perp\cdots\varphi_n^\perp\otimes[\psi]$

Homological Perturbation Theory

- Let $S_{\rm int} \ = \ \langle \xi, \ell_0^{\rm ext}(1)\rangle^{\rm ext} + \sum_{n\geqslant 2} \frac{(-1)^{\binom{n}{2}}}{(n+1)!} \, \langle \xi, \ell_n^{\rm ext}(\xi^{\otimes n})\rangle^{\rm ext}$

- Homological Perturbation Lemma: With $\delta \ = \ \{S_{\rm int}, -\} + \hbar \, \Delta_{\rm BV}$, there is a strong deformation retract

$$\big(\mathrm{Sym}\, H^\bullet(V[2]), \widetilde{\delta}\,\big) \underset{\widetilde{\Pi}}{\overset{\widetilde{\mathcal{I}}}{\rightleftarrows}} \big(\mathrm{Sym}\, V[2], Q_{\rm BV}^\hbar\big) \circlearrowleft \widetilde{\Gamma}$$

where $\widetilde{\Pi} \ = \ \Pi + \Pi\,(\mathbb{1} - \delta\,\Gamma)^{-1}\,\delta\,\Gamma \ = \ \Pi \circ \sum_{k=0}^\infty (\delta\,\Gamma)^k$

- $\langle \varphi_1 \cdots \varphi_n\rangle \ := \ \widetilde{\Pi}\,(\varphi_1 \cdots \varphi_n) \ \in \ \mathrm{Sym}\, H^\bullet(V[2])$ are (smeared) *n*-point correlation functions on space of vacua $H^\bullet(V[2])$ of the field theory

- Evaluated on a particular vacuum this gives the usual numerical correlators of perturbative quantum field theory around this vacuum

Scalar Field Theory

- $V = V_1 \oplus V_2 \;,\; V_1 = V_2 = C^\infty(\mathbb{R}^d) \;,\; \phi \in V_1 \;,\; \phi^+ \in V_2$:

$$\ell_1 = \Box + m^2 \quad , \quad \ell_n(\phi_1, \dots, \phi_n) = (-1)^{\binom{n}{2}} \lambda_n \, \phi_1 \cdots \phi_n$$

- Maurer-Cartan equation:

$$F_\phi = \ell_1(\phi) + \sum_{n \geqslant 2} \frac{(-1)^{\binom{n}{2}}}{n!} \ell_n(\phi^{\otimes n}) = (\Box + m^2)\,\phi + \sum_{n \geqslant 2} \frac{\lambda_n}{n!} \phi^n = 0$$

- With the cyclic inner product $\langle \phi, \phi^+ \rangle = \int \mathrm{d}^d x \; \phi \cdot \phi^+$, the Maurer-Cartan action is:

$$S = \frac{1}{2!} \langle \phi, \ell_1(\phi) \rangle + \sum_{n \geqslant 2} \frac{(-1)^{\binom{n}{2}}}{(n+1)!} \langle \phi, \ell_n(\phi^{\otimes n}) \rangle$$

$$= \int \mathrm{d}^d x \; \frac{1}{2} \phi \, (\Box + m^2)\, \phi + \sum_{n \geqslant 3} \frac{\lambda_{n-1}}{n!} \phi^n$$

Scalar Field Theory

- Plane waves $e_k(x) \;=\; \mathrm{e}^{-\,\mathrm{i}\,k\cdot x} \;=\; \big(e^k(x)\big)^* \quad , \quad \langle e^k, e_p\rangle \;=\; (2\pi)^d\,\delta(k-p)$

- Interactions:

$$S_{\rm int} \;=\; \sum_{n\geqslant 3}\int_{k_1,\dots,k_n}\frac{\lambda_{n-1}}{n!}\,(2\pi)^d\,\delta(k_1+\cdots+k_n)\;e^{k_1}\cdots e^{k_n} \;\in\; \mathrm{Sym}\,V[2]$$

- Deformation retract: $H^\bullet(V[2]) \;=\; 0$ for $m^2>0$:

$$(0,0)\;\underset{0}{\overset{0}{\rightleftarrows}}\;\big(V[2],\ell_1\big)\circlearrowleft_{-G} \qquad G \;=\; (\ell_1)^{-1} \;=\; (\Box+m^2)^{-1}$$

- Correlation functions: $(\mathbb{C},0)\;\underset{\widetilde{\Pi}}{\overset{\widetilde{\mathcal{I}}}{\rightleftarrows}}\;\big(\mathrm{Sym}\,V[2],Q^\hbar_{\rm BV}\big)\circlearrowleft_{\widetilde{\Gamma}}$

$$\tilde G_n(p_1,\dots,p_n) \;:=\; \sum_{k=1}^\infty \Pi\,\big(\hbar\,\Delta_{\rm BV}\,\Gamma+\{S_{\rm int},-\}\,\Gamma\big)^k\,(e^{p_1}\cdots e^{p_n})$$

Only $\Pi(1) \;=\; 1$ is non-zero (as $\pi=0$) — this is a general proposal!

Scalar Field Theory with External Sources

- Couple to external fields $J \in C^\infty(\mathbb{R}^d)$ by adding $\ell_0 : \mathbb{R} \longrightarrow V_2$ with central curvature $\ell_0(1) = J$

- Curved Maurer-Cartan equation: $F_\phi = -\ell_0(1) = -J$

- Curved Maurer-Cartan action:

$$S_J = S + \langle \phi, \ell_0(1) \rangle = S + \int \mathrm{d}^d x \ J \cdot \phi$$

- Curved homological perturbation theory:

$$S_{\rm int}^J = S_{\rm int} + \langle \xi, \ell_0^{\rm ext}(1) \rangle_\star = S_{\rm int} + \int_{k_0} \tilde{J}(k_0) \ e^{k_0}$$

where $\tilde{J}(k) = \int \mathrm{d}^d x \ \mathrm{e}^{-\,\mathrm{i}\, k \cdot x} \, J(x)$

Scalar Field Theory: Examples

1. 4-point function of free scalar field $(\lambda_n = 0)$:

$$\tilde{G}_4^0(p_1, \dots, p_4) = (\hbar\, \Delta_{\rm BV}\, \Gamma)^2 (e^{p_1} \cdots e^{p_4})$$

$$= \tilde{G}_2^0(p_1, p_2)\, \tilde{G}_2^0(p_3, p_4) + \tilde{G}_2^0(p_1, p_2)\, \tilde{G}_2^0(p_2, p_4) + \tilde{G}_2^0(p_1, p_4)\, \tilde{G}_2^0(p_2, p_3)$$

where $\tilde{G}_2^0(p_1, p_2) = -\hbar\, \frac{(2\pi)^d\, \delta(p_1 + p_2)}{p_1^2 + m^2}$; Wick's Theorem

2. 2-point function at 1-loop in $\lambda\,\phi^4$-theory $(\lambda_3 = \lambda)$:

$$\tilde{G}_2(p_1, p_2) = (\hbar\, \Delta_{\rm BV}\, \Gamma)^2 \left\{ S_{\rm int}, \Gamma\, (e^{p_1}\, e^{p_2}) \right\}$$

$$= -\frac{\hbar^2\, \lambda}{2}\, \frac{(2\pi)^d\, \delta(p_1 + p_2)}{(p_1^2 + m^2)\, (p_2^2 + m^2)} \int_k \frac{1}{k^2 + m^2} = -\hbar\, \frac{(2\pi)^d\, \delta(p_1 + p_2)}{p_1^2 + m^2 + \Pi(p_1^2)}$$

This identifies the self-energy

$$\frac{1}{\hbar}\, \Pi = \frac{\lambda}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \frac{1}{p^2 + m^2}$$

Scalar Field Theory: Examples

3. 1-point function at 1-loop in $\lambda\,\phi^3$-theory ($\lambda_2 = \lambda$):

$$\tilde{G}_1(p) \;=\; \hbar\,\Delta_{\rm BV}\,\Gamma\,\{S_{\rm int},\Gamma\,(e^p)\} \;=\; -\hbar\,\frac{\lambda}{2}\,\frac{(2\pi)^d\,\delta(p)}{p^2+m^2}\,\int_k \frac{1}{k^2+m^2}$$

Eliminate tadpoles with curvature: $\ell_0(1) \;=\; Y \in \mathbb{R} \subset V_2$

Adds linear counterterm $Y\,\phi$ in curved Maurer-Cartan action, modifies $S_{\rm int}$ in BV formalism by addition of

$$S_Y \;=\; \langle \xi, \ell_0^{\rm ext}(1)\rangle^{\rm ext} \;=\; Y\,\int_k (2\pi)^d\,\delta(k)\;e^k$$

Full 1-point function at 1-loop:

$$\tilde{G}_1(p) \;=\; -\hbar\,\frac{\lambda}{2}\,\frac{(2\pi)^d\,\delta(p)}{p^2+m^2}\,\int_k \frac{1}{k^2+m^2} + Y\,\frac{(2\pi)^d\,\delta(p)}{p^2+m^2}$$

Cancels all 1-loop tadpole contributions if $Y \;=\; \hbar\,\frac{\lambda}{2}\,\int_k \frac{1}{k^2+m^2}$

Schwinger-Dyson Equations: Textbook Approach

- 'Quantum equations of motion' for correlation functions follow from invariance of path integral measure under infinitesimal variations of fields:

$$0 = \frac{1}{Z} \int \mathscr{D}\phi \, \frac{\delta}{\delta\phi(y)} \left(\phi(x_1) \cdots \phi(x_n) \, \mathrm{e}^{-S/\hbar} \right)$$

$$= \sum_{i=1}^{n} \delta(x_i - y) \, \langle \phi(x_1) \cdots \widehat{\phi(x_i)} \cdots \phi(x_n) \rangle - \frac{1}{\hbar} \left\langle \phi(x_1) \cdots \phi(x_n) \, \frac{\delta}{\delta\phi(y)} S \right\rangle$$

- **Example:** In $\lambda\,\phi^3$-theory after Fourier transformation to momentum space:

$$(p^2 + m^2) \, \langle \tilde{\phi}(p) \, \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_n) \rangle + \frac{\lambda}{2} \int_k \langle \tilde{\phi}(k) \, \tilde{\phi}(p-k) \, \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) \rangle$$

$$= \hbar \sum_{i=1}^{n} (2\pi)^d \, \delta(p + p_i) \, \langle \tilde{\phi}(p_1) \cdots \widehat{\tilde{\phi}(p_i)} \cdots \tilde{\phi}(p_n) \rangle$$

Schwinger-Dyson Equations: Algebraic Approach

Recall: Homological Perturbation Lemma gives a strong deformation retract

$$\big(\mathrm{Sym}\,H^\bullet(V[2]),\widetilde{\delta}\,\big)\ \underset{\widetilde{\Pi}}{\overset{\widetilde{\mathcal{I}}}{\rightleftarrows}}\ \big(\mathrm{Sym}\,V[2],\,Q^{\hbar}_{\rm BV}=\ell_1+\delta\big)\ \circlearrowleft \widetilde{\Gamma}$$

with $\widetilde{\delta} \ = \ \Pi\,(\mathbb{1}-\delta\,\Gamma)^{-1}\,\delta\,\mathcal{I} \quad , \quad \widetilde{\Pi} \ = \ \Pi\,(\mathbb{1}-\delta\,\Gamma)^{-1}$. This implies:

- **Lemma:** $\widetilde{\Pi}\circ Q^{\hbar}_{\rm BV} \ = \ 0$

  Proof: $\widetilde{\Pi}$ is a cochain map: $\widetilde{\Pi}\circ Q^{\hbar}_{\rm BV} \ = \ \widetilde{\delta}\circ\widetilde{\Pi}$. Since only $\Pi(1) \ = \ 1$ is non-zero and $\delta(1) \ = \ \{S_{\rm int},1\}+\hbar\,\Delta_{\rm BV}(1) \ = \ 0$, right-hand side is 0.

- Standard identities obeyed by correlation functions in quantum field theory are corollaries of this (e.g. Ward-Takahashi identities in QED)

- Algebraic Schwinger-Dyson Equations: Precompose with contracting homotopy Γ and use $\ell_1\circ\Gamma+\Gamma\circ\ell_1+\mathbb{1} \ = \ \mathcal{I}\circ\Pi$ to get recursion relation:

$$\widetilde{\Pi} \ = \ \Pi+\widetilde{\Pi}\circ\delta\circ\Gamma$$

Schwinger-Dyson Equations: Wick's Theorem

- In free scalar field theory with $Q_{\rm BV}^{\hbar\,0} = \ell_1 + \hbar\,\Delta_{\rm BV}$ only even-multiplicity correlators are non-zero:

$$\tilde{G}^0_{2n}(p_1,\dots,p_{2n}) \;:=\; \widetilde{\Pi}\,(e^{p_1}\cdots e^{p_{2n}}) \;=\; \widetilde{\Pi}\,(\hbar\,\Delta_{\rm BV}\,\Gamma)\,(e^{p_1}\cdots e^{p_{2n}})$$

- Expanding out right-hand side using definitions gives recursion relations

$$\tilde{G}^0_{2n}(p_1,\dots,p_{2n}) \;=\; \frac{1}{2n}\sum_{i\neq j}\tilde{G}^0_2(p_i,p_j)\;\tilde{G}^0_{2n-2}(p_1,\dots,\widehat{p_i},\dots,\widehat{p_j},\dots,p_{2n})$$

where $\tilde{G}^0_2(p_i,p_j) \;=\; -\hbar\,\dfrac{(2\pi)^d\,\delta(p_i+p_j)}{p_i^2+m^2}$

- Symmetrization of standard equations, due to 'symmetric tensor trick' used in fattening of maps for Homological Perturbation Lemma

- Solution is $\displaystyle\tilde{G}^0_{2n}(p_1,\dots,p_{2n}) \;=\; \frac{1}{n!\,2^n}\sum_{\sigma\in S_{2n}}\prod_{k=1}^{n}\tilde{G}^0_2(p_{\sigma(2k-1)},p_{\sigma(2k)})$

Schwinger-Dyson Equations: Interactions

- With $Q^{\hbar}_{\rm BV} = \ell_1 + \hbar\,\Delta_{\rm BV} + \{S_{\rm int}, -\}$ recursion is

$$\tilde{G}_n(p_1,\dots,p_n) = \widetilde{\Pi}\,(\hbar\,\Delta_{\rm BV}\,\Gamma)\,(e^{p_1}\cdots e^{p_n}) + \widetilde{\Pi}\,\{S_{\rm int}, \Gamma\,(e^{p_1}\cdots e^{p_n})\}$$

- Expanding out right-hand side using definitions gives recursion relations

$$\tilde{G}_n(p_1,\dots,p_n) = \frac{1}{n}\sum_{i\neq j}\tilde{G}^0_2(p_i,p_j)\;\tilde{G}_{n-2}(p_1,\dots,\widehat{p_i},\dots,\widehat{p_j},\dots,p_n)$$

$$-\sum_{r\geqslant 3}\frac{\lambda_{r-1}}{(r-1)!}\,\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i^2+m^2}\int_{k_1,\dots,k_{r-1}}(2\pi)^d\,\delta(k_1+\cdots k_{r-1}-p_i)$$

$$\times\;\tilde{G}_{n+r-2}(p_1,\dots,p_{i-1},k_1,\dots,k_{r-1},p_{i+1},\dots,p_n)$$

- **Example:** $n=2$ in $\lambda\,\phi^3$-theory:

$$\tilde{G}_2(p_1,p_2) = \tilde{G}^0_2(p_1,p_2) + \frac{\lambda}{4}\int_k\frac{\tilde{G}_3(k,p_1-k,p_2)}{p_1^2+m^2} + \frac{\lambda}{4}\int_k\frac{\tilde{G}_3(p_1,k,p_2-k)}{p_2^2+m^2}$$