Gauge invariance: quantization and geometry

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Geometry of Dirac-BFV quantization and quantum cosmology

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Outline

Early attempts: Wheeler-DeWitt equation(s) vs Schroedinger equation, problem of time, etc.

Quantum gravity and Dirac quantization of constrained systems. Dirac quantization is incomplete

Batalin operator quantization within BFV formalism

Dirac quantization as truncation of BFV formalism: projector on the physical states, physical inner product and path integration

Semiclassical approximation: geometry of Dirac-BFV formalism, operator realization of constraints

Lagrangian versus canonical formalisms: one-loop approximation

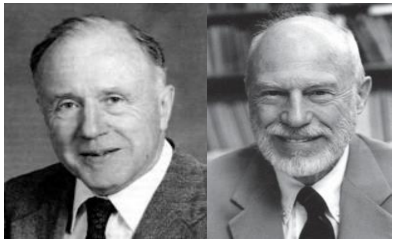
Quantum cosmology: density matrix of the Universe vs no-boundary wavefunction

50-ies and early 60-ies: quantum gravity and Dirac quantization of constrained systems

Hamiltonian and momentum constraints $H_\mu = H_\perp(\mathbf{x}),\ H_i(\mathbf{x})$

$$H_\mu(q,p) = 0 \;\to\; \hat{H}_\mu|\,\Psi\,\rangle = 0.$$

Wheeler-DeWitt equation(s):



Functional coordinate representation of Hamiltonian and momentum constraints

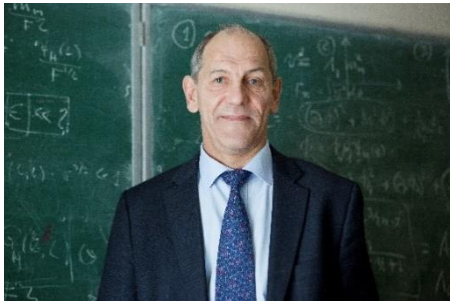
$$q = \gamma_{ab}(\mathbf{x}), \phi(\mathbf{x}), \quad p = \pi^{ab}(\mathbf{x}), p_\phi(\mathbf{x}) \;\to\; \pi^{ab}(\mathbf{x}) = \frac{\hbar}{i}\frac{\delta}{\delta\gamma_{ab}(\mathbf{x})}, \quad p_\phi(\mathbf{x}) = \frac{\hbar}{i}\frac{\delta}{\delta\phi(\mathbf{x})}$$

$$\Big\{-\frac{2\hbar^2}{M_P^2}G_{ab,cd}(\mathbf{x})\frac{\delta^2}{\delta\gamma_{ab}(\mathbf{x})\,\delta\gamma_{cd}(\mathbf{x})} - \frac{M_P^2}{2}\gamma^{1/2}(\mathbf{x})\big({}^3R(\mathbf{x}) - 2\Lambda\big) + H_\perp^{\rm matter}\Big(\phi(\mathbf{x}), \frac{\hbar}{i}\frac{\delta}{\delta\phi(\mathbf{x})}\Big)\Big\}\Psi[\,{}^3\gamma,\phi] = 0,$$

$$\Big\{-2\gamma_{ab}(\mathbf{x})\nabla_c\frac{\delta}{\delta\gamma_{bc}(\mathbf{x})} + H_a^{\rm matter}\Big(\phi(\mathbf{x}), \frac{\hbar}{i}\frac{\delta}{\delta\phi(\mathbf{x})}\Big)\Big\}\Psi[\,{}^3\gamma,\phi] = 0$$

Schroedinger equation from Wheeler-DeWitt equation(s)

$$i\frac{\partial}{\partial t}|\,\Psi(t)\,\rangle = \hat{H}|\,\Psi(t)\,\rangle \quad \Longleftrightarrow \quad \hat{H}_\mu \Psi[\,{}^3\gamma,\phi\,] = 0$$



First attempts to extract time and Schroedinger equation from the Wheeler-DeWitt equations (DeWitt (1967), Lapchinsky-Rubakov (1977))

V.G. Lapchinsky,V.A. Rubakov, Acta Phys. Polon. B10 (1979) 1041

$$\hat{H}_\mu = \hat{H}_\mu^{\text{grav}} + \hat{H}_\mu^{\text{matter}}$$

$$\Psi[\,{}^3\gamma,\phi\,] = \exp\left(\frac{i}{M_P^2}S\left[{}^3\gamma\right]\right)\left|\Psi[\,{}^3\gamma\,]\right\rangle$$

Semiclassical gravity factor

quantum state of matter parametrically depending on 3-metric

$$|\,\Psi(t)\,\rangle = \left|\Psi[\,{}^3\gamma_{ab}(t)\,]\right\rangle$$

Classical solution of vacuum Einstein equations

$$\hat{H}_\mu \Psi[\,{}^3\gamma,\phi\,] = 0 \quad \Rightarrow \quad i\frac{\partial}{\partial t}|\,\Psi(t)\,\rangle = \hat{H}_{\text{matter}}|\,\Psi(t)\,\rangle + O\big(1/M_P^2\big)$$

graviton loops

$$\hat{H}_{\text{matter}} = \int d^3x\,\left(N^\perp \hat{H}_\perp^{\text{matter}} + N^a \hat{H}_a^{\text{matter}}\right)$$

Semiclassical expansion in $1/M_P^2$, no back reaction, tree-level gravitational field as a whole – source of time evolution

Classical theory of gauge constrained systems: canonical formalism

In GR --- Dirac, Arnowitt-Deser-Misner (ADM)

$$S=\int dt\left\{p_i\dot q^i-H_0(q,p)-N^\mu H_\mu(q,p)\right\}\;\rightarrow\;\frac{\delta S}{\delta N^\mu}=-H_\mu(q,p)=0$$

ADM surface integral
(0 *in spatially closed cosmology)*

constraints

DeWitt condensed notations

In GR:

$$\begin{cases} q^i=\gamma_{ab}(\mathbf{x}),\phi(\mathbf{x}), \quad p_i=\pi^{ab}(\mathbf{x}),p_\phi(\mathbf{x}) \\ p_i\dot q^i=\int d^3x\,\big(\pi^{ab}(\mathbf{x})\dot\gamma_{ab}(\mathbf{x})+p_\phi(\mathbf{x})\dot\phi(\mathbf{x})\big) \\ N^\mu H_\mu=\int d^3x\,\big(N^\perp(\mathbf{x})H_\perp(\mathbf{x})+N^a(\mathbf{x})H_a(\mathbf{x})\big) \end{cases}$$

Constraints algebra
(first-class constraints)

$$\{H_\mu,H_\nu\}=U^\lambda_{\mu\nu}H_\lambda,$$
$$\{H_0,H_\nu\}=U^\lambda_{0\,\nu}H_\lambda$$

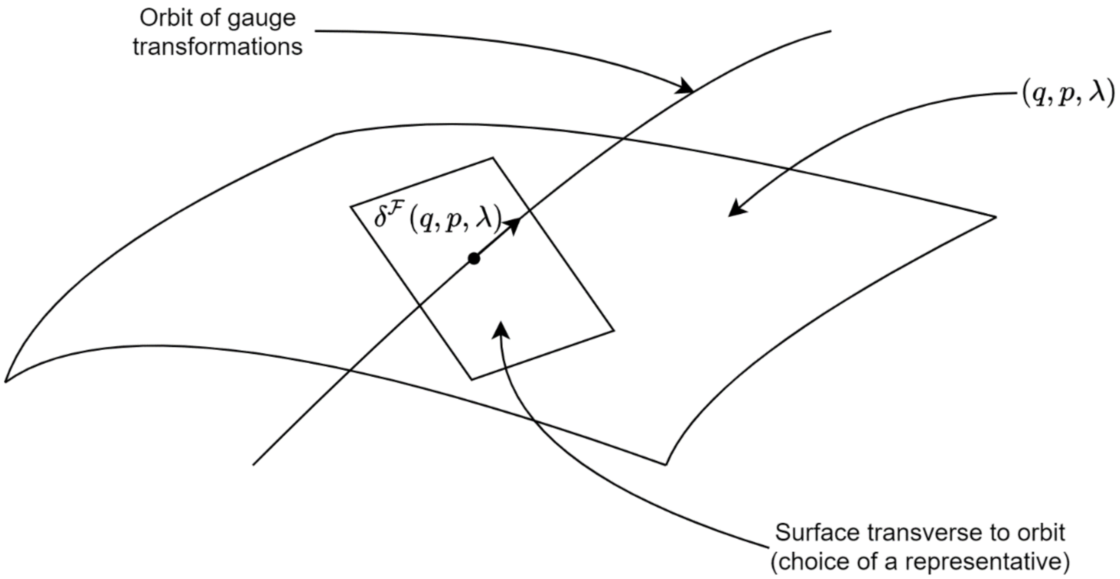
Local gauge invariance of the action:

canonical transformation

non-canonical

$$\left.\begin{array}{l} \delta^{\mathcal{F}}\begin{bmatrix} q \\ p \end{bmatrix} = \left\{\begin{bmatrix} q \\ p \end{bmatrix}, H_\mu\right\}\mathcal{F}^\mu \\ \delta^{\mathcal{F}} N^\mu = \dot{\mathcal{F}}^\mu - U^\mu_{\alpha\beta} N^\alpha \mathcal{F}^\beta - U^\mu_{0\,\nu}\mathcal{F}^\nu \end{array}\right] \qquad \delta^{\mathcal{F}} S = 0$$

local parameter



Gauge fixation – choice of the representative of the equivalence class on the orbit of gauge group by imposing the gauge

$$\chi^\mu(q, p) = 0$$

for uniqueness of the representative should be independent of the Lagrangian multipliers

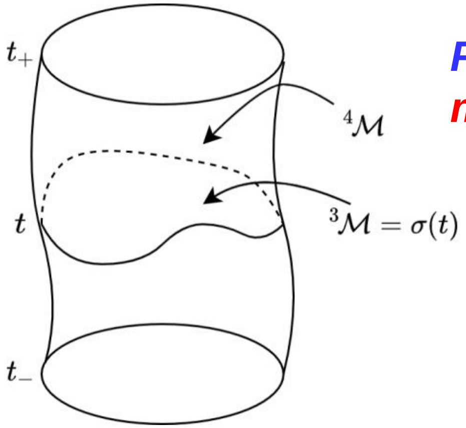
$$\delta^{\mathcal{F}}\chi^{\mu} = \{\chi^{\mu}, H_{\nu}\}\mathcal{F}^{\nu} \neq 0 \;\rightarrow\; \underbrace{J^{\mu}_{\nu} \equiv \det\{\chi^{\mu}, H_{\nu}\}}\neq 0$$

Canonical Faddeev-Popov operator is invertible

The inverse of Faddeev-Popov operator

Recovery of unique Lagrange multipliers

$$\frac{d}{dt}\chi^{\mu}(q,p) = \{\chi^{\mu}, H_0\} + \{\chi^{\mu}, H_{\nu}\}N^{\nu} = 0 \;\;\rightarrow\;\; N^{\mu} = -J^{-1}{}^{\mu}_{\nu}\{\chi^{\mu}, H_0\}$$



For $H_0 = 0$ ***lapse and shift functions should be nonzero --- moving in spacetime spatial section*** $\sigma(t)$

⇓

explicitly time-dependent gauge $\chi^{\mu}(q) = 0 \;\rightarrow\; \chi^{\mu}(q,t) = 0$

$$N^{\mu} = -J^{-1}{}^{\mu}_{\nu}\frac{\partial\chi^{\nu}}{\partial t} \neq 0$$

This is the problem of frozen time formalism and its solution

More general theories --- hierarchy of structure functions

$$H_\mu, U^\alpha_{\mu\nu} \to G = \left\{H_\mu, U^\alpha_{\mu\nu}, U^{\alpha\beta}_{\mu\nu\lambda}, \ldots\right\}$$

$$\{H_\mu, \{H_\sigma, H_\lambda\}\} + \mathrm{cycle}(\mu, \sigma, \lambda) = 0$$

E.S.Fradkin, T.Fradkina
Batalin, Fradkin, Vilkovisky

Mechanism of origin of higher order structure functions

$$\left(\{H_\mu, U^\alpha_{\sigma\lambda}\} + U^\beta_{\sigma\lambda} U^\alpha_{\mu\beta} + \mathrm{cycle}(\mu, \sigma, \lambda)\right) H_\alpha = 0$$

$$\{H_\mu, U^\alpha_{\sigma\lambda}\} + U^\beta_{\sigma\lambda} U^\alpha_{\mu\beta} + \mathrm{cycle}(\mu, \sigma, \lambda) = U^{\alpha\beta}_{\mu\nu\lambda} H_\beta, \qquad U^{\alpha\beta}_{\mu\nu\lambda} = -U^{\beta\alpha}_{\mu\nu\lambda}$$

Dirac quantization scheme

Quantization

$$(q, p, H_\mu, H_0) \to (\hat{q}, \hat{p}, \hat{H}_\mu, \hat{H}_0)$$

Quantum Dirac constraints

$$H_\mu(q,p) = 0 \ \to \ \hat{H}_\mu |\Psi\rangle = 0.$$

Poisson bracket algebra to operator algebra

$$\{H_\mu, H_\nu\} = U^\lambda_{\mu\nu} H_\lambda \to [\hat{H}_\mu, \hat{H}_\nu] = i\hbar\,\hat{U}^\lambda_{\mu\nu}\hat{H}_\lambda,$$

$$\{H_0, H_\nu\} = U^\lambda_{0\,\nu} H_\lambda \to \ ...$$

Consistency conditions (generalization of Poisson bracket algebra)

$$[\hat{H}_\mu, \hat{H}_\nu] = i\hbar\,\hat{U}^\lambda_{\mu\nu}\hat{H}_\lambda$$

Stands to the left of constraint operators

Higher order structure functions

$$U^{\alpha\beta}_{\mu\nu\lambda}, \cdots \ \to \ \hat{U}^{\alpha\beta}_{\mu\nu\lambda}, \cdots$$

$$[\,\hat{H}_\mu, \hat{U}^\alpha_{\sigma\lambda}] + i\hbar\hat{U}^\beta_{\sigma\lambda}\hat{U}^\alpha_{\mu\beta} + \mathrm{cycle}(\mu,\sigma,\lambda) = i\hbar\hat{U}^{\alpha\beta}_{\mu\nu\lambda}\hat{H}_\beta, \qquad \hat{U}^{\alpha\beta}_{\mu\nu\lambda} = -\hat{U}^{\beta\alpha}_{\mu\nu\lambda}$$

Beyond classical theory --- quantization?

No Hamiltonian, no time, no Schroedinger equation, no inner product, operator ordering ???

WDW equation is the ``most useless" equation in theoretical physics?

Dirac quantization scheme is not complete !

$$\hat{H}_\mu | \Psi \rangle = 0 \;\rightarrow\; \Psi(q) \sim \delta(\hat{H}_\mu)\Phi(q)$$

For instance, problem of physical inner product

$$\langle \Psi' \,|\, \Psi \rangle_{\rm phys} \overset{?}{=} \int dq\, \Psi'^*(q)\,\Psi(q) \sim \int dq \left[\delta(\hat{H}_\mu)\right]^2 = \infty$$

Resolution of these difficulties --- BFV

Batalin operator quantization within BFV formalism

Canonical relativistic phase space:

$$q^i, p_i \quad \rightarrow \quad Q^I, P_I = q^i, p_i;\ N^\mu, \pi_\mu;\ \underbrace{C^\mu, \mathcal{P}_\mu;\ \bar{C}_\mu, \bar{\mathcal{P}}^\mu}_{\text{ghosts}},$$

$$[Q^I, P_J] = i\,\delta^I_J, \qquad \hbar = 1$$

$$[A, B] \equiv AB - (-1)^{n(A)n(B)} BA$$

Grassmann parity: $n(q) = n(N) = 0, \quad n(C) = n(\mathcal{P}) = n(\bar{C}) = n(\bar{\mathcal{P}}) = 1$

$$||\,Q\,\rangle\rangle \equiv ||\,q, N, C, \bar{C}\,\rangle\rangle, \quad \hat{Q}^I||\,Q\,\rangle\rangle = Q^I||\,Q\,\rangle\rangle, \quad \boldsymbol{\Psi}(Q) = \langle\langle\,Q\,||\,\boldsymbol{\Psi}\,\rangle\rangle,$$

BFV inner product: $\langle\langle\,\boldsymbol{\Psi}_1\,||\,\boldsymbol{\Psi}_2\rangle\rangle = \int dQ\,\boldsymbol{\Psi}_1{}^*(Q)\,\boldsymbol{\Psi}_2(Q)$

Hermiticity: $C^{\mu\dagger} = C^\mu, \quad \mathcal{P}^\dagger_\mu = -\mathcal{P}_\mu, \quad \bar{C}^\dagger_\mu = -\bar{C}_\mu, \quad \bar{\mathcal{P}}^{\mu\dagger} = \bar{\mathcal{P}}^\mu$

Nilpotent Grassmann BRST operator

$$[\hat{\Omega}, \hat{\Omega}] \equiv 2\hat{\Omega}^2 = 0$$

$$\hat{\Omega} = \pi_\alpha \bar{\mathcal{P}}^\alpha + C^\mu \hat{H}_\mu + \frac{1}{2} C^\nu C^\mu \hat{U}^\lambda_{\mu\nu} \mathcal{P}_\lambda + \ldots, \qquad \hat{\Omega}^\dagger = \hat{\Omega}$$

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$$\hat{H}_\mu - \hat{H}^\dagger_\mu = i\hat{U}^\lambda_{\mu\lambda} + \ldots$$

Unitarizing Hamiltonian

$$\hat{\mathcal{H}}_\Phi = \hat{\mathcal{H}}_0 + \frac{1}{i}[\hat{\Phi}, \hat{\Omega}]$$

BRS extension of $H_0, \qquad [\hat{\mathcal{H}}_0, \hat{\Omega}] = 0$

Gauge fermion

$$\hat{\Phi} = \mathcal{P}_\mu N^\mu + \bar{C}_\mu \chi^\mu(q), \qquad n(\hat{\Phi}) = 1$$

Unitary evolution operator

$$i\hbar\frac{\partial}{\partial t}\hat{\boldsymbol{U}}_{\Phi}(t,t_-) = \hat{\mathcal{H}}_{\Phi}\hat{\boldsymbol{U}}_{\Phi}(t,t_-), \quad \hat{\boldsymbol{U}}_{\Phi}(t_-,t_-) = \mathbb{I}$$

Physical states

$$\hat{\Omega}\,||\,\boldsymbol{\Psi}\rangle\rangle = 0$$

Main property: gauge independence of physical matrix elements

$$[\,\hat{\Omega}, \hat{\boldsymbol{U}}_{\Phi}(t,t_-)\,] = 0$$

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$$\hat{\Omega}\,||\,\boldsymbol{\Psi}_{1,2}\rangle\rangle = 0 \quad \to \quad \delta_{\Phi}\langle\langle\,\boldsymbol{\Psi}_1\,||\,\hat{\boldsymbol{U}}_{\Phi}(t_+,t_-)\,||\,\boldsymbol{\Psi}_2\rangle\rangle = 0$$

Path integral representation

$$\boldsymbol{U}_{\Phi}(\,t_+,Q_+|\,t_-,Q_-) \equiv \langle\langle\,Q_+\,||\,\hat{\boldsymbol{U}}_{\Phi}(t_+,t_-)\,||\,Q_-\,\rangle\rangle$$

$$= \int_{Q(t_\pm)=Q_\pm} D[\,Q,P\,]\,\exp\left\{i\int_{t_-}^{t_+}dt\,\left(P_I\dot{Q}^I - \mathcal{H}_0 - \{\Phi,\Omega\}\right)\right\}$$

BRS extension of H_0

quantum measure

Liouville measure

$$D[\,Q,P\,] = \prod_t dQ(t)\prod_{t^*} dP(t^*)$$

Dirac quantization as truncation of BFV formalism

$$||\boldsymbol{\Psi}\rangle\rangle \to ||\boldsymbol{\Psi}\rangle\rangle' = ||\boldsymbol{\Psi}\rangle\rangle + \hat{\Omega}\,||\boldsymbol{\Phi}\rangle\rangle$$

Batalin-Marnelius
gauge fixing

$$||\boldsymbol{\Psi}\rangle\rangle \to ||\boldsymbol{\Psi}_{BM}\rangle\rangle :$$

$$\hat{\mathcal{P}}_\mu||\boldsymbol{\Psi}_{BM}\rangle\rangle = 0,\ \ \hat{N}^\mu||\boldsymbol{\Psi}_{BM}\rangle\rangle = 0$$

$$\Downarrow$$

Quantum Dirac
constraints

$$0 = \frac{1}{i}\,[\,\hat{\Omega}, \hat{\mathcal{P}}_\mu]\,||\boldsymbol{\Psi}_{BM}\rangle\rangle = (\hat{H}_\mu + C^\nu \hat{U}^\lambda_{\nu\mu}\hat{\mathcal{P}}_\lambda)\,||\boldsymbol{\Psi}_{BM}\rangle\rangle = \hat{H}_\mu\,||\boldsymbol{\Psi}_{BM}\rangle\rangle\rangle$$

Independence of
ghosts

$$[\hat{\Omega}, N^\mu]\,||\boldsymbol{\Psi}_{BM}\rangle\rangle = \frac{\partial}{\partial C^\mu}\boldsymbol{\Psi}_{BM}(Q) = 0, \qquad \frac{\partial}{\partial \bar{C}_\mu}\boldsymbol{\Psi}_{BM}(Q) = 0$$

Truncation:
BFV to Dirac

$$\boldsymbol{\Psi}_{BM}(Q) = \langle\, q\,|\,\Psi\,\rangle\,\delta(N) \equiv \Psi(q)\,\delta(N),$$

$$\hat{\Omega}\boldsymbol{\Psi}_{BM}(Q) = 0 \quad \to \quad \hat{H}_\mu\Psi(q) = 0$$

Dirac
wavefunction

Physical inner product

$$\int dQ \,|\,\boldsymbol{\Psi}_{BM}(Q)\,|^2 \sim \int dN\,[\delta(N)]^2 \int dC\, d\bar{C} = \infty \times 0 \quad \textbf{?}$$

Gauge independent regularization of inner product

$$\langle\langle\,\boldsymbol{\Psi}'\,||\,\boldsymbol{\Psi}\rangle\rangle_{\rm phys} = \langle\langle\,\boldsymbol{\Psi}'_{BM}||\,e^{[\,\hat{\Phi},\hat{\Omega}\,]}||\,\boldsymbol{\Psi}_{BM}\rangle\rangle$$

$$[\hat{\Omega},[\hat{\Phi},\hat{\Omega}]] = 0, \quad \hat{\Omega}\,||\,\boldsymbol{\Psi}_{BM}\rangle\rangle = 0 \quad \Rightarrow \quad \frac{\delta}{\delta\Phi}\langle\langle\,\boldsymbol{\Psi}'\,||\,\boldsymbol{\Psi}\rangle\rangle_{\rm phys} = 0$$

Gauge fermion and Faddeev-Popov operator

$$\hat{\Phi} = \mathcal{P}_\mu N^\mu + \bar{C}_\mu \hat{\chi}^\mu \quad \Rightarrow \quad \hat{J}^\mu_\nu = \frac{1}{i}\,[\,\hat{\chi}^\mu, \hat{H}_\nu]$$

Physical inner product

$$\langle\langle\,\boldsymbol{\Psi}'\,||\,\boldsymbol{\Psi}\rangle\rangle_{\rm phys} = \int dq\,\Psi'^*(q)\,\hat{M}\left(q,\frac{\partial}{i\partial q}\right)\Psi(q) = \langle\,\Psi'\,|\,\hat{M}\,|\,\Psi\,\rangle$$

operator-valued measure

Physical inner product measure in the coordinate gauges (q-gauges)

$$\hat{M} = \int dN\, dC\, d\bar{C}\, \delta(N)\, e^{-i\bar{C}_\mu \hat{J}^\mu_\nu C^\nu + \hat{\chi}^\mu \frac{\partial}{\partial N^\mu}}\, \delta(N)$$

$$= \int d\pi\, dC\, d\bar{C}\, e^{-i\bar{C}_\mu \hat{J}^\mu_\nu C^\nu + i\pi_\mu \hat{\chi}^\mu}$$

$$= \delta(\hat{\chi})\ \mathrm{det}\, \hat{J}^\mu_\nu \Big(1 + O(\,[\,\hat{\chi}, \hat{J}\,]\,)\Big)$$

multi-loop corrections

$$\langle \Psi' \,|\, \Psi \,\rangle_{\mathrm{phys}} = \int dq\, \Psi'^*(q)\, \delta(\chi(q))\ \mathrm{det}\, \hat{J}^\mu_\nu\ \Psi(q) + O(\hbar)$$

Projector on the space of physical states and path integral

$$\Psi(q)=\int dN\,\boldsymbol{\Psi}(q,N,C,\bar{C})\,\Big|_{C=0}$$
$$U(q_+,q_-)=\int dN_+\,dN_-\,\boldsymbol{U}_\Phi(t_+,Q_+|\,t_-,Q_-)\,\Big|_{C_\pm=0}$$

$$U(q_+,q_-)\equiv\langle\langle\boldsymbol{\Psi}_+||\,\hat{\boldsymbol{U}}_\Phi(t_+,t_-)\,||\boldsymbol{\Psi}_-\rangle\rangle,$$
$$\boldsymbol{\Psi}_\pm(q,N,C,\bar{C})=\delta(q-q_\pm)\delta(C)\delta(\bar{C}),\qquad \hat{\pi}_\alpha||\boldsymbol{\Psi}_\pm\rangle\rangle=0,\qquad \hat{\Omega}\,||\boldsymbol{\Psi}_\pm\rangle\rangle=0$$

Gauge and time independence

$$\delta_\Phi U(q_+,q_-)=0$$
$$i\frac{\partial}{\partial t_+}U(q_+,q_-)=\langle\langle\boldsymbol{\Psi}_+||\,\frac{1}{i}\,[\hat{\Phi},\hat{\Omega}\,]\,\hat{\boldsymbol{U}}_\Phi(t_+,t_-)\,||\boldsymbol{\Psi}_-\rangle\rangle=0$$

Quantum Dirac constraints

$$\hat{H}_\mu\hat{U}=0,\qquad \hat{U}\,\overleftarrow{\hat{H}}{}^{\dagger}_{\mu}=0$$

$$\Downarrow$$

Projector on the space of physical states

$$U(q,q')=\langle\,q\,|\,\hat{U}\,|\,q'\,\rangle,\qquad \hat{U}=\text{const}\times\prod_\mu\delta(\hat{H}_\mu)$$

Choice of gauge fermion for relativistic gauge:

$$\hat{\Phi} = \mathcal{P}_\mu N^\mu + \bar{C}_\mu \widehat{\chi}^\mu$$

$$\Downarrow$$

$$U(q_+,q_-) = \int D[Q,P]\,\mathrm{exp}\left[i \int\limits_{t_-}^{t_+} dt \left(p_i \dot{q}^i - N^\mu H_\mu + \mathcal{P}_\mu \dot{C}^\mu + \bar{\mathcal{P}}_\mu \dot{\bar{C}}^\mu \right.\right.$$

$$\left.\left. + \pi_\mu \underbrace{(\dot{N}^\mu - \chi^\mu)}_{\text{relativistic gauge conditions}} - \bar{C}_\mu J^\mu_\nu C^\nu - \mathcal{P}_\alpha(\bar{\mathcal{P}}^\alpha + U^\alpha_{\mu\nu} N^\mu C^\nu) \right) \right] \Bigg|_{q(t_\pm)=q_\pm,\ \ C_\pm = 0}$$

$N_\pm$ integrated over

Transition to unitary gauge: equivalence to quantization in a physical sector

$$\chi^\mu \to \frac{\chi^\mu}{\varepsilon},\quad \pi_\mu \to \varepsilon \pi_\mu,\quad \bar{C}_\mu \to \varepsilon \bar{C}_\mu;\quad \varepsilon \to 0$$

$$U(q_+,q_-) = \int\limits_{q(t_\pm)=q_\pm} D[q,p]\,DN \underbrace{\left(\prod_{t_+>t>t_-} \delta(\chi)\ \mathrm{det}\, J^\mu_\nu \right)}_{\text{Faddeev-Popov gauge-fixing factor (missing at } t_\pm\text{)}} \mathrm{exp}\left[i \int_{t_-}^{t_+} dt \left(p_i \dot{q}^i - N^\mu H_\mu \right) \right]$$

Semiclassical approximation

Operator realization in coordinate representation

$$H_\mu \overset{?}{\to} \hat{H}_\mu : \quad [\hat{H}_\mu, \hat{H}_\nu] = i\hbar\,\hat{U}^\lambda_{\mu\nu}\hat{H}_\lambda, \quad \hat{H}_\mu - \hat{H}^\dagger_\mu = i\hbar\,\hat{U}^\lambda_{\mu\lambda} + \ldots$$

$$\hat{H}_\mu = \mathcal{N}_W \left\{ H_\mu + \frac{i\hbar}{2} U^\nu_{\mu\nu} + O(\hbar^2) \right\},$$

Weyl ordering

semiclassical Weyl symbols

$$\hat{U}^\lambda_{\mu\nu} = \mathcal{N}_W \left\{ U^\lambda_{\mu\nu} - \frac{i\hbar}{2} U^{\lambda\sigma}_{\mu\nu\sigma} + O(\hbar^2) \right\}$$

V.Krykhtin & A.B., Class. Quantum Grav. 10 (1993) 1957

A.B. Class. Quantum Grav. 10 (1993) 1985; gr-qc/9612003

Geometrical propertries of this realization ?

Covariance under diffeomorphisms of q-space

Contact canonical transformations
in phase space

$$q^i = q^i(q'), \;\; p_i = p_{k'} \frac{\partial q^{k'}}{\partial q^i}, \;\; G(q,p) = G'(q',p')$$

$$\hat{G} = \{\hat{H}_\mu, \hat{U}^\alpha_{\mu\nu}, \hat{U}^{\alpha\beta}_{\mu\nu\lambda}, \ldots\} \qquad\qquad \hat{G} = \mathcal{N}_W \tilde{G}(q,p)$$

Weyl symbol of $\hat{G}$

$$q \to q', \quad \hat{G} \to \hat{G}',$$

A.B., gr-qc/9612003

$$\hat{G} \to \hat{G}' = \left|\frac{\partial q'}{\partial q}\right|^{-1/2} \hat{G} \left|\frac{\partial q'}{\partial q}\right|^{1/2}$$

$$\Psi'(q') = \left|\frac{\partial q}{\partial q'}\right|^{1/2} \Psi(q) \;\Leftrightarrow\; \big\langle \Psi_1 | \Psi_2 \big\rangle = \int dq\, \Psi_1^*(q) \Psi_2(q) = \big\langle \Psi_1' | \Psi_2' \big\rangle,$$

½-weight density

Transformation of the constraint basis

$$H'_\mu = \Omega^{\,\nu}_{\mu} H_\nu, \;\; \Omega^\nu_\mu = \Omega^\nu_\mu(q,p), \;\; \det\Omega^\nu_\mu \neq 0,$$

$$\hat{H}'_\mu = \hat{\Omega}^{-1/2}\hat{\Omega}^\nu_\mu \hat{H}_\nu \hat{\Omega}^{1/2}, \quad |\Psi'\rangle = \hat{\Omega}^{-1/2}|\Psi\rangle,$$

$$\hat{\Omega} = \det\hat{\Omega}^\mu_\nu, \quad \hat{\Omega}^\nu_\mu = \mathcal{N}_W\left\{\Omega^\nu_\mu + O(\hbar^2)\right\}$$

A.B., gr-qc/9612003

Invariant physical observables

$$\{\mathcal{O}_I, H_\mu\} = U^{\lambda}_{I\,\mu} H_\lambda,$$

$$\{\mathcal{O}_I, \mathcal{O}_J\} = U^L_{IJ}\mathcal{O}_L + U^\lambda_{IJ} H_\lambda, \;\; U^L_{IJ} = \text{const}$$

$$\mathcal{O} \to \hat{\mathcal{O}}, \quad [\hat{\mathcal{O}}_I, \hat{H}_\mu] = i\hbar\hat{U}^{\lambda}_{I\,\mu}\hat{H}_\lambda,$$

$$\hat{\mathcal{O}}_I = \mathcal{N}_W\left\{\mathcal{O}_I + \frac{i\hbar}{2}U^{\lambda}_{I\,\lambda} + \frac{i\hbar}{2}U^{J}_{IJ} + O(\hbar^2)\right\},$$

$$\hat{U}^{\lambda}_{I\,\mu} = \mathcal{N}_W\left\{U^{\lambda}_{I\,\mu} - \frac{i\hbar}{2}U^{\lambda\sigma}_{I\,\mu\sigma} + O(\hbar^2)\right\}$$

Semiclassical solution of quantum Dirac constraints

one-loop prefactor

$$\boldsymbol{U}(q,q') = \Big[\,\boldsymbol{P}(q,q') + O(\hbar)\,\Big]\, \mathsf{exp}\left[\frac{i}{\hbar}\boldsymbol{S}(q,q')\right]$$

$$\widehat{H}_\mu \boldsymbol{U}(q,q') = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} H_\mu\Big(q, \partial\boldsymbol{S}/\partial q\Big) = 0, \\ \\ \underbrace{\frac{\partial}{\partial q^i}(\nabla^i_\mu \boldsymbol{P}^2) = U^\lambda_{\mu\lambda}\boldsymbol{P}^2}_{\text{“continuity” equations}}, \quad \nabla^i_\mu \equiv \left.\frac{\partial H_\mu}{\partial p_i}\right|_{p=\partial S/\partial q} \end{array} \right.$$

Hamilton-Jacobi equation

“continuity” equations

Degenerate matrix: $S_{ik'} = \dfrac{\partial^2 S(q,q')}{\partial q^i \, \partial q^{k'}}$

$$\boxed{\frac{\partial}{\partial q^{k'}} H_\mu\big(q, \partial S/\partial q\big) = 0}$$

$$\nabla^i_\mu S_{ik'} = 0, \quad S_{ik'} \nabla^{k'}_\nu = 0, \quad \nabla^{k'}_\nu \equiv \frac{\partial H_\nu(q',p')}{\partial p'_k}\bigg|_{p'=-\partial S/\partial q''}$$

$$S_{ik'} \;\to\; F_{ik'} = S_{ik'} + \underbrace{\chi^\mu_i c_{\mu\nu} \chi^\nu_{k'}}_{\textit{gauge-breaking term}}, \qquad \underbrace{\chi^\mu_i \equiv \frac{\partial \chi^\mu(q)}{\partial q^i}, \qquad \chi^\nu_{k'} \equiv \frac{\partial \chi^\nu(q')}{\partial q^{k'}}, \qquad c_{\mu\nu}}_{\textit{gauge matrices}}$$

Solution of continuity equation $\quad \boldsymbol{P}(q,q') = \left[\dfrac{\det F_{ik'}}{J\, J' \det c_{\mu\nu}}\right]^{1/2}$

A.B. Phys.Lett.
B 241 (1990) 201

$$\left.\begin{array}{l} J \equiv \det J^\mu_\nu(q) \neq 0, \quad J^\mu_\nu(q) = \chi^\mu_i \nabla^i_\nu, \\ J' \equiv \det J^\mu_\nu(q') \neq 0, \quad J^\mu_\nu(q') = \chi^\mu_{i'} \nabla^{i'}_\nu \end{array}\right\}$$

Faddeev-Popov determinants for gauges $\chi^\mu(q), \chi^\nu(q')$

Recognize in this expression the analogue of the one-loop effective action with the contribution of the Faddeev-Popov determinant.

Properties:

1) gauge independence of the prefactor

$$\delta_\chi \boldsymbol{P}(q,q')=0, \quad \delta_c \boldsymbol{P}(q,q')=0$$

2) closure of the Lie bracket algebra

$$\nabla^i_\mu \frac{\partial \nabla^k_\nu}{\partial q^i} - \nabla^i_\nu \frac{\partial \nabla^k_\mu}{\partial q^i} = U^\lambda_{\mu\nu} \nabla^k_\lambda$$

3) the ‘’continuity” equation for the prefactor

$$\frac{\partial}{\partial q^i}(\nabla^i_\mu \boldsymbol{P}^2) = U^\lambda_{\mu\lambda} \boldsymbol{P}^2$$

Hamiltonian reduction to the physical sector

$$S=\int dt\,\Big\{p_i\dot{q}^i-H_0(q,p)-N^\mu H_\mu(q,p)\Big\},\qquad \underbrace{i=1,\dots n,\qquad \mu=1,\dots m}$$

formal counting (in field theory always both are infinite)

Solution of first-class constraints and gauges
(ultralocal in time, but nonlocal in space)

$$\left.\begin{array}{l} H_\mu(q,p)=0 \\[2ex] \chi^\mu(q,p)=0 \end{array}\right] \Rightarrow \quad \begin{array}{l} q^i=e^i(\xi^A,\pi_A) \\ p_i=P_i(\xi^A,\pi_A),\qquad A=1,\dots n-m \end{array}$$

$n-m=\#$ *physical degrees of freedom*

$$S_{\rm phys}[\,\xi,\pi\,]=\int dt\,\Big[\,\pi_A\dot{\xi}^A-H_{\rm phys}(\xi,\pi)\,\Big]$$

$$H_{\rm phys}(\xi,\pi)=H_0\Big(e(\xi,\pi),P(\xi,\pi)\Big)-P_i(\xi,\pi)\frac{\partial e^i(\xi,\pi,t)}{\partial t}$$

if $\chi^\mu=\chi^\mu(q,p,t)$
$\to e^i=e^i(\xi,\pi,t)$

Hamiltonian reduction and semiclassical inner product

“Moving” physical subspace

$$\Sigma(t)\,: q^i = e^i(\xi,t),\ \ \chi^\mu(e^i(\xi,t),t) \equiv 0$$

$$d^{n-m}\xi = d^n q\, \delta\Big((\chi(q,t)\Big)\, \mathcal{M}$$

measure factor

Physical Hamilton-Jacobi function

$$S(t,\xi|t',\xi') = S(q,q')\Big|_{q=e(\xi,t),\ q'=e(\xi',t')},$$

$$\frac{\partial^2 S(t,\xi|t',\xi')}{\partial\xi^A\,\partial\xi^{B'}} = S_{ik'}\,\frac{\partial e^i}{\partial\xi^A}\,\frac{\partial e^{k'}}{\partial\xi^{B'}}\Big|_{q=e(\xi,t),\ q'=e(\xi',t')}$$

Semiclassical unitary evolution operator in the physical sector (Pauli-Van Vleck)

$$U_{\mathrm{phys}}(t,\xi|t',\xi') \equiv \left[\det\frac{i}{2\pi\hbar}\,\frac{\partial^2 S(t,\xi|t',\xi')}{\partial\xi^A\,\partial\xi^{B'}}\right]^{1/2} e^{\frac{i}{\hbar}S(t,\xi|t',\xi')} + O(\hbar)$$

Unitary map between physical and Dirac-Wheeler-DeWitt semiclassical evolution operators

$$U_{\mathsf{phys}}(t,\xi|t',\xi') = \mathsf{const}\,\left(\frac{J}{\mathcal{M}}\right)^{1/2} \boldsymbol{P}(q,q')\left(\frac{J'}{\mathcal{M}'}\right)^{1/2}\Bigg|_{q=e(\xi,t),\,q'=e(\xi',t')}$$

Unitary map between wavefunctions

A.B. Class. Quantum Grav. 10 (1993) 1985; gr-qc/9612003

$$\Psi_{\mathsf{phys}}(t,\xi) = \mathsf{const}\,\left(\frac{J}{\mathcal{M}}\right)^{1/2} \Psi(q)\Bigg|_{q=e(\xi,t)}$$

Unitarity --- conservation of physical inner product

$$\Big(\Psi_1^{\mathsf{phys}}(t)\,|\,\Psi_2^{\mathsf{phys}}(t)\Big)_{\mathsf{phys}} \equiv \int d\xi\,\Psi_1^{\mathsf{phys}\,*}(\xi,t)\,\Psi_2^{\mathsf{phys}}(\xi,t)$$

$$= \int dq\,\Psi_1^*(q)\,\underbrace{J_\chi\,\delta\Big(\chi(q,t)\Big)}_{\textit{measure}}\,\Psi_2(q) \equiv \langle\,\Psi_1\,|\,\Psi_2\,\rangle_{\mathsf{phys}}$$

Gauge independence of physical inner product

$$\frac{\delta}{\delta\chi}\int dq\,\Psi_1^*(q)\,J_\chi\,\delta(\chi)\,\Psi_2(q) = 0 \;\; ?$$

Exercise on Stokes theorem

$$\int dq\,\Psi_1^*(q)\,J_\chi\,\delta(\chi)\,\Psi_2(q) = \int_\Sigma \omega^{(n-m)},$$

A.B., Class. Quant.
Grav. 10 (1993) 1985

$$\omega^{(n-m)} = \frac{dq^{i_1}\wedge\ldots\wedge dq^{i_{n-m}}}{(n-m)!}\epsilon_{i_1\ldots i_n}\Psi_2^*\,\nabla_1^{i_{n-m+1}}\ldots\nabla_m^{i_n}\,\Psi_1$$

$m = \text{codim}\,\Sigma$ **-- *# of continuity type equations:***

closed form

$$\left.\begin{array}{l} \dfrac{\partial}{\partial q^i}\left(\nabla_\mu^i\,\Psi_2^*\,\Psi_1\right) = U_{\mu\lambda}^\lambda\Psi_2^*\,\Psi_1 \\[2ex] \nabla_\mu^i\dfrac{\partial\nabla_\nu^k}{\partial q^i} - \nabla_\nu^i\dfrac{\partial\nabla_\mu^k}{\partial q^i} = U_{\mu\nu}^\lambda\nabla_\lambda^k \end{array}\right\} \to \quad d\omega^{(n-m)} = 0$$

Lie bracket algebra of vector fields

$\mathcal{M}_{12}$ ***forms a cobordism of*** Σ_1 ***and*** Σ_2

$$\partial\mathcal{M}_{12} = \Sigma_1 \bigcup \Sigma_2, \qquad \dim\mathcal{M}_{12} = \dim\Sigma_{1,2} + 1$$

$$\Downarrow$$

$$\int_{\Sigma_2}\omega^{(n-m)} - \int_{\Sigma_1}\omega^{(n-m)} = \int_{\mathcal{M}_{12}} d\omega^{(n-m)} = 0$$

Dirac constraints operators are Hermitian with respect to physical inner product

$$\langle\, \hat{H}_\mu \Psi_1 |\, \Psi_2 \,\rangle_{\rm phys} - \langle\, \Psi_1 |\, \hat{H}_\mu \Psi_2 \,\rangle_{\rm phys}$$

multi-loop orders

$$= \langle\, \Psi_1 |\, \hat{H}^\dagger_\mu J\, \delta(\chi) |\, \Psi_2 \,\rangle - \langle\, \Psi_1 |\, J\, \delta(\chi) \hat{H}_\mu |\, \Psi_2 \,\rangle = 0(\hbar)$$

Conjugation with respect to auxiliary BFV inner product $\hat{H}^\dagger_\mu = \hat{H}_\mu - i\hbar U^\lambda_{\mu\lambda}$

A.B., unpublished e-print gr-qc/9612003

The same is true for invariant physical observables subject to classical algebra:

$$\langle\, \hat{\mathcal{O}}^\dagger_I \Psi_1 |\, \Psi_2 \,\rangle_{\rm phys} - \langle\, \Psi_1 |\, \hat{\mathcal{O}}_I \Psi_2 \,\rangle_{\rm phys} = O(\hbar)$$

Lagrangian versus canonical formalisms

Lagrangian variables

$$g^a = (q^i(t), N^\mu(t)), \qquad S\,[\,g\,] = \int_{t_-}^{t_+} dt\, L(q, \dot{q}, N)$$

Lagrangian gauge generators

$$R^a_\mu \frac{\delta S\,[\,g\,]}{\delta g^a} = R_\mu{}^a\Big(\frac{d}{dt}\Big) \frac{\delta S\,[\,g\,]}{\delta g^a(t)}, \quad \mu \to (\mu, t),$$

$$R^i_\mu = \delta(t - t')\, \frac{\partial T_\mu}{\partial p_i}\bigg|_{p = p^0(q, \dot{q}, N)}, \qquad R^\alpha_\mu = \Big(\delta^\alpha_\mu \frac{d}{dt} - U^\alpha_{\lambda\mu} N^\lambda\Big)\, \delta(t - t').$$

Lagrangian path integral

$$\boldsymbol{U}(q_+, q_-) = \int Dg\, DC\, D\bar{C}\, \mathrm{exp}\, \frac{i}{\hbar} \Big\{ \Big(S\,[\,g\,] - \frac{1}{2} \chi^\mu c_{\mu\nu} \chi^\nu \Big) + \bar{C}_\mu Q^\mu_\nu C^\nu \Big\}.$$

Relativistic gauge fixing:

$$\frac{1}{2} \chi^\mu c_{\mu\nu} \chi^\nu = \frac{1}{2} \int_{t_-}^{t_+} dt\, \chi^\mu(g, \dot{g})\, c_{\mu\nu} \chi^\nu(g, \dot{g})$$

gauge-breaking term

$$\chi^\mu = \chi^\mu(g, \dot{g}), \qquad a^\mu_\nu = -\frac{\partial \chi^\mu}{\partial \dot{N}^\nu}, \quad \mathrm{det}\, a^\mu_\nu \neq 0$$

$$Q^\mu_\nu = \frac{\delta \chi^\mu}{\delta g^a} R^a_\nu, \qquad Q^\mu{}_\nu\Big(\frac{d}{dt}\Big)\delta(t - t') = \Big(- a^\mu_\nu \frac{d^2}{dt^2} + \ldots \Big)\delta(t - t')$$

dynamical ghost operator

Boundary conditions on integration variables

$$q^i(t_\pm) = q^i_\pm, \;\; C^\mu(t_\pm) = 0, \;\; \bar{C}_\nu(t_\pm) = 0,$$

$$-\infty < N^\mu(t_\pm) < +\infty$$

Lagrange multipliers at the boundaries are integrated over

Quantum Dirac constraints for the path integral

$$\int Dg\, DC\, D\bar{C}\, \frac{\delta}{\delta N^\mu(t_+)} \Big(\textbf{\textit{full path integral integrand}} \Big) = 0$$

$$\Downarrow$$

$$\hat{H}_\mu\Big(q, \frac{\hbar}{i}\frac{\partial}{\partial q}\Big)\, \boldsymbol{U}(q, q') = 0$$

H.Leutwyler (1964)
A.B. (1980)
Hartle-Hawking (1983)

Boundary value problem for the saddle point configuration:

$$\frac{\delta S\,[\,g\,]}{\delta g^a(t)} = 0, \qquad\qquad\qquad\qquad g = g(\,t\,|\,q_+, q_-)$$
$$\chi^\mu(g,\dot g) = 0,\;\; t_- \le t \le t_+, \quad \Longrightarrow$$
$$q(t_\pm) = q_\pm$$

Hamilton-Jacobi function --- on-shell action $\boldsymbol{S}(q,q') = S\,[\,g(\,t\,|\,q_+, q_-)\,]$

One-loop prefactor $\boldsymbol{P}(q_+, q_-) = \left.\dfrac{\mathrm{Det}\,Q^\mu_\nu}{(\,\mathrm{Det}\,F_{ab}\,)^{1/2}}\right|_{g=g(\,t\,|\,q_+,q_-)}$

Hessian of the action

$$F_{ab} = S_{ab} - \chi^\mu_a\, c_{\mu\nu}\chi^\nu_b, \qquad S_{ab} \equiv \frac{\delta^2 S\,[\,g\,]}{\delta g^a\,\delta g^b},$$
$$\chi^\mu_a \equiv \frac{\delta\chi^\mu}{\delta g^a} = \chi^\mu{}_a\Big(\frac{d}{dt}\Big)$$

Specification of functional determinants:

$$\delta \ln \mathrm{Det}\, F_{ab} = G^{ba}\delta F_{ab},$$

$$\delta \ln \mathrm{Det}\, Q^{\mu}_{\nu} = Q^{-1\,\nu}_{\mu}\,\delta Q^{\mu}_{\nu}$$

Mixed Dirichlet-Neumann problem

$$F_{ca}(d/dt)G^{ab}(t,t') = \delta^{b}_{c}\delta(t-t'),$$
$$G^{ib}(t_{\pm},t') = 0,$$
$$\chi^{\mu}{}_{a}\Big(\frac{d}{dt_{\pm}}\Big)G^{ab}(t_{\pm},t') = 0$$

Dirichlet problem

$$Q_{\mu}{}^{\alpha}(d/dt)\,Q^{-1\,\beta}_{\alpha}(t,t') = \delta^{\beta}_{\mu}\delta(t-t'),$$
$$Q^{-1\,\beta}_{\alpha}(t_{\pm},t') = 0$$

A.B.,Nucl. Phys. B 520 (1998) 533

$$(\,\mathrm{Det}\, F_{ab}\,)^{-1/2} = \mathrm{Const}\left[\frac{\det\left(\boldsymbol{S}_{ik'} + \chi^{\mu}_{i}c_{\mu\nu}\chi^{\nu}_{k'}\right)}{\det c_{\mu\nu}}\right]^{1/2}$$

$$\mathrm{Det}\, Q^{\mu}_{\nu} = \mathrm{const}\left(\det J^{\mu}_{\nu}\,\det J'^{\mu}_{\nu}\right)^{-1/2}, \quad J^{\mu}_{\nu} = \chi^{\mu}_{i}\,\nabla^{i}_{\nu}, \quad J'^{\mu}_{\nu} = \chi^{\mu}_{i'}\nabla^{i'}_{\nu}$$

Gauge conditions matrices are built of basis functions of gauge and ghost operators

$$\left.\frac{\mathrm{Det}\, Q^{\mu}_{\nu}}{(\,\mathrm{Det}\, F_{ab}\,)^{1/2}}\right|_{g=g(\,t\,|\,q_{+},q_{-})} = \left[\frac{\det F_{ik'}}{J\,J'\det c_{\mu\nu}}\right]^{1/2}$$

Analogue of Pauli-Van Vleck relation for gauge theories

All this is really not working!

$$U^{\lambda}_{\mu\lambda} = ?$$

$$U^{c\mathrm{x}''}_{a\mathrm{x}\; b\mathrm{x}'} = \delta^c_b \partial_a \delta(\mathrm{x}, \mathrm{x}') \delta(\mathrm{x}', \mathrm{x}'') - (a\mathrm{x} \leftrightarrow b\mathrm{x}')$$

$$\int d^3x' \, U^{b\mathrm{x}''}_{a\mathrm{x}\; b\mathrm{x}'} \Big|_{\mathrm{x}'=\mathrm{x}''} \sim \delta(0) = \infty$$

+ *other UV divergences*

We need other 4D covariant formalism to be able to regularize and renormalize the theory --- path integral method

Let us begin with minisuperspace applications in which this problem does not at all arise

Quantum cosmology: initial conditions for cosmological inflation

Spatially homogeneous (minisuperspace) metric

$$ds^2 = -N^2 dt^2 + a^2(t)\sigma_{ab} dx^a\, dx^b$$

$$q^i,\, p_i,\, N^\mu\, H_\mu \mapsto\; a(t),\, p(t),\, N(t),\, H_\perp(a,p)$$

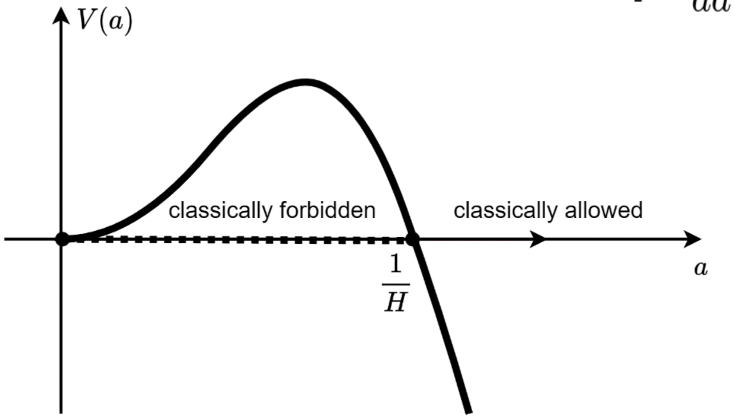
$$H_\perp(a,p) = \frac{1}{24\pi^2 M_P^2}\left[-\frac{p^2}{a} + (12\pi^2 M_P^2)^2 a(H^2 a^2 - 1)\right]$$

$$H^2 = \frac{\Lambda}{3}$$

Hubble constant in terms of the cosmological constant

Minisuperspace Wheeler-DeWitt equation

$$\left[-\frac{d^2}{da^2} - (12\pi^2 M_P^2)^2\, a^2(H^2 a^2 - 1)\right]\psi(a) = 0$$



Stationary Schroedinger equation in the potential V(a) at the energy level E=0

Underbarrier tunneling via a bounce solution

Action of underbarrier bounce --- euclideanized gravitational action

$$I[g^E_{\mu\nu}] = -\frac{M_P^2}{2}\int d^4x\, g^{1/2}(R-2\Lambda)\Big|_{\text{Euclidean metric}} = -iS[g_{\mu\nu}]\Big|_{N\to iN^E}$$

in ADM parametrization

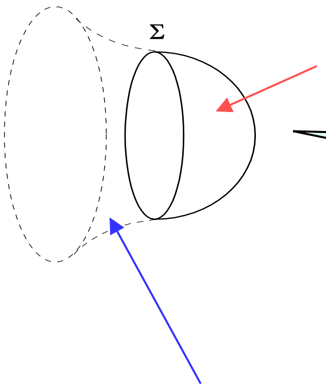
$$g_{\mu\nu} \to g^E_{\mu\nu}:\ N \to iN^E$$

$$I = -\frac{4\pi^2 M_P^2}{H^2} < 0$$

Negative Euclidean action

4D Euclidean hemisphere

$$ds^2 = d\tau^2 + a_E^2(\tau)\, d\Omega^2_{(3)}$$
$$a_E(\tau) = \frac{1}{H}\sin(H\tau),$$



Euclidean spacetime

no tunneling, really: "birth from nothing"

Analytic continuation to Lorentzian signature de Sitter geometry:

$$\tau = \pi/2H + it$$
$$a_L(t) = \frac{1}{H}\cosh(Ht)$$

Hartle-Hawking no-boundary wavefunction

$$\Psi \sim e^{-I} = \exp\left[+\frac{4\pi^2 M_P^2}{H^2}\right]$$

Chaotic inflation --- effective cosmological and Hubble constants are generated by the inflaton scalar field potential

$$\Lambda = 3H^2 \quad \to \quad \frac{V(\phi)}{M_P^2} = 3H^2(\phi)$$

Probability of inflation:

$$P(\phi) \simeq |\Psi(\phi)|^2\Big|_{a=1/H(\phi)} \simeq \left(\exp\left[\frac{4\pi^2 M_P^2}{H^2(\phi)}\right]\right)^2 = \exp\left[\frac{24\pi^2 M_P^4}{V(\phi)}\right] \to \infty, \quad V(\phi) \to 0$$

Point of nucleation of the Lorentzian spacetime from the Euclidean one

Most probable at the minimum
of inflaton potential $H_{eff} \to 0$
-- insufficient amount of inflation

Why the Hartle-Hawking wavefunction is called no-boundary ?

Euclidean quantum gravity path integral for no-boundary wavefunction (Hartle-Hawking)

$$\Psi_{no-boundary}[{}^3\gamma,\varphi] = \int D[{}^4g,\phi]\, e^{-I_{\mathcal{M}}[{}^4g,\phi]}\Big|_{{}^3g(\partial\mathcal{M})={}^3\gamma,\,\phi(\partial\mathcal{M})=\varphi}$$

Euclidean metric and matter regular on $\mathcal{M}$, includes all gauge fixing details

Euclidean action

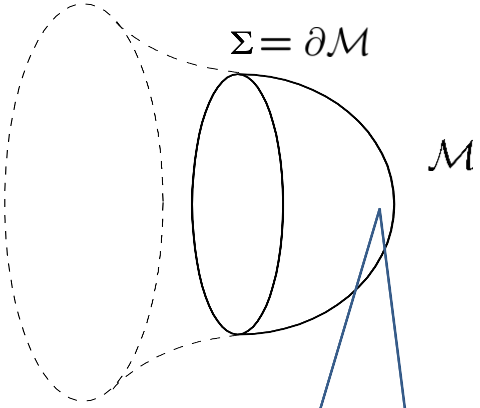
$$I_{\mathcal{M}}[g_{\mu\nu},\phi] = -\frac{M_P^2}{2}\int_{\mathcal{M}} d^4x\, g^{1/2}(R-2\Lambda) + S_{\text{matter}}[g_{\mu\nu},\phi]$$

Semiclassically:

$$\Psi_{\text{no}-\text{boundary}}[{}^3\gamma,\varphi] = e^{-I_{\mathcal{M}}[{}^4g_0,\phi_0]}$$

$${}^4g_0\big|_{\partial M} = {}^3\gamma,\quad \phi_0\big|_{\partial M} = \varphi$$

Saddle point –- solution subject to boundary data



no initial Cauchy boundary, just regularity

The alternative --- microcanonical density matrix of the Universe

$$\hat{\rho} = \frac{1}{Z} \sum_{\text{all } |\boldsymbol{\Psi}\rangle} w_{\Psi} | \boldsymbol{\Psi} \rangle \langle \boldsymbol{\Psi} |, \qquad w_{\Psi} = 1$$

A.B., Phys. Rev. Lett. 99, 071301 (2007)

sum over "everything" that satisfies the Wheeler-DeWitt equation $\hat{H}_{\mu} |\boldsymbol{\Psi}\rangle = 0$

Motivation:

A simplest analogy in unconstrained system with a conserved Hamiltonian $\hat{H}$ ***Is the microcanonical density matrix with a fixed energy E***

$$\hat{\rho} \sim \delta(\hat{H} - E)$$

Spatially closed cosmology does not have freely specifiable constants of motion. The only conserved quantities are the Hamiltonian and momentum constraints H_{μ}***, all having a particular value --- zero.***

An ultimate equipartition in the full set of states of the theory --- "Sum over Everything". Creation of the Universe from Everything is conceptually more appealing than creation from Nothing, because the democracy of the microcanonical equipartition better fits the principle of Occam razor, preferring to drop redundant assumptions, than the selection of a concrete state.

This is the projector onto the subspace of quantum gravitational constraints

Partition function

$$\hat{\rho} = \frac{1}{Z}\prod_{\mu}\delta(\hat{H}_\mu), \qquad Z = \mathrm{Tr}_{\mathrm{phys}}\prod_{\mu}\delta(\hat{H}_\mu)$$

$$\langle\, q\,|\prod_{\mu}\delta(\hat{H}_\mu)\,|\,q'\,\rangle = U(q,q'), \qquad q = (\gamma_{ab},\phi)$$

$$\mathrm{Tr}_{\mathrm{phys}}\hat{U} = \int dq\,\hat{M}_+\,U(q_+,q_-)\,\Big|_{q_\pm=q}$$

$$Z = \int dq\,\hat{M}_{t_+}\int\limits_{q(t_\pm)=q} D[\,q,p\,]\,DN\,\Big(\overbrace{\prod_{t_+>t>t_-} M_t}^{\text{Faddeev-Popov gauge-fixing factors (missing at } t_\pm)}\Big)\,\mathrm{exp}\Big[\,iS[\,q,p,N\,]\Big]$$

$$= \int\limits_{\mathrm{periodic}} D[\,q,p\,]\,DN\,\Big(\prod_{\mathrm{all}\ t} M_t\Big)\,\mathrm{exp}\Big[\,iS[\,q,p,N\,]\Big]$$

$$= \int\limits_{\mathrm{periodic}} D[\,g_{\mu\nu},\phi\,]\;e^{iS[\,g_{\mu\nu},\phi\,]}$$

inner product measure is absorbed into path imtegral measure

↑

Lorentzian and covariant in relativistic gauge

No need for inner product and its measure – path integral takes care of it automatically!

Absence of periodic Lorentzian histories

rotation of integration contours over fields (or time)

$$Z = \int_{\text{periodic}} D[\,g_{\mu\nu},\Phi\,]\, e^{-S[\,g_{\mu\nu},\Phi\,]}$$

Euclideanization of the theory:

Euclideanization of the metric and its gravitational action can be attained by the transition

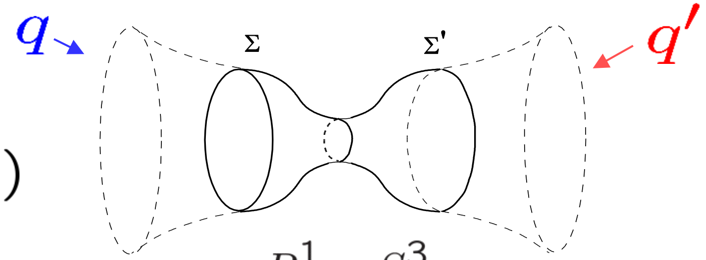
$$N \to -iN$$

$$iS \to -S_{\text{Euclidean}} \equiv -I$$

Euclidean path integral and its saddle points

Hartle-Hawking state as a vacuum member of the microcanonical ensemble:

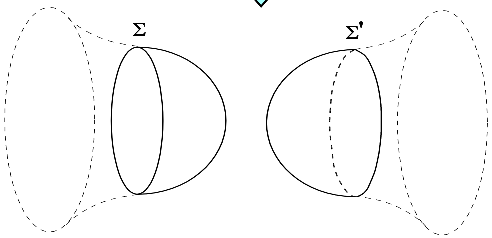
$$\hat{\rho}_{\text{mixed}} = \rho_{\text{mixed}}(q, q')$$



$R^1 \times S^3$

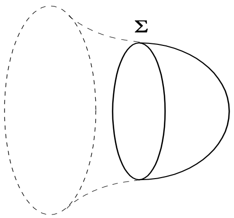
pinching a tubular spacetime

$$|\boldsymbol{\Psi}_{HH}\rangle\langle\boldsymbol{\Psi}_{HH}| = \boldsymbol{\rho}_{HH}(q, q')$$



$D^4 \cup D'^4$

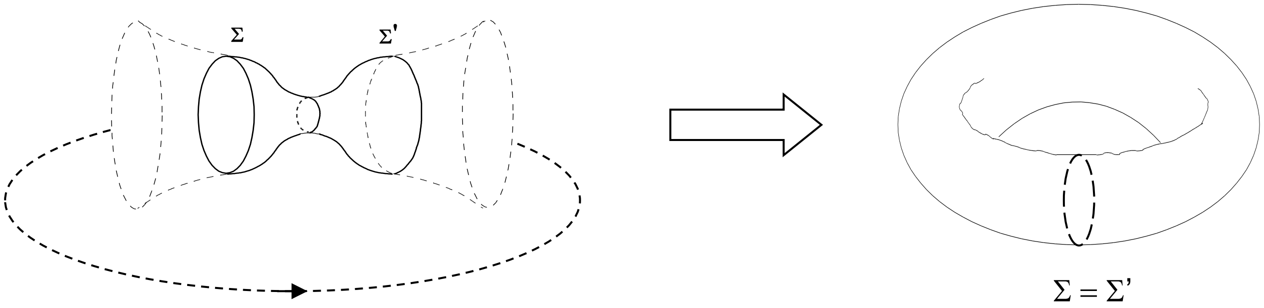
$$\Longleftrightarrow \quad |\boldsymbol{\Psi}_{HH}\rangle = \boldsymbol{\Psi}_{HH}(q)$$



D^4

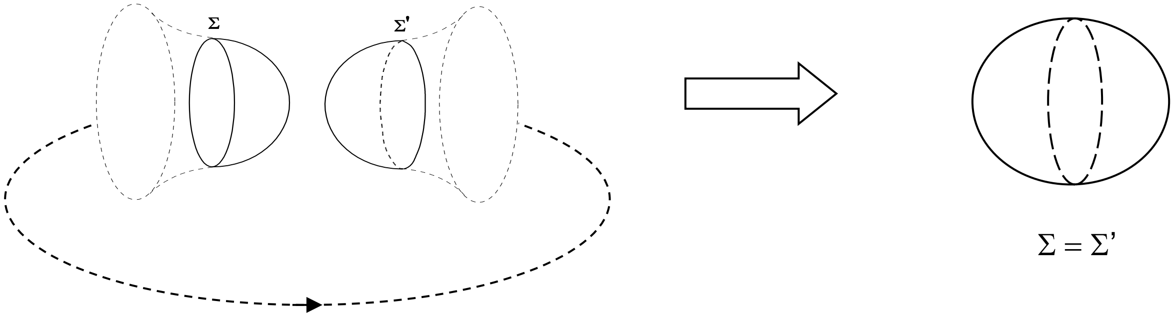
density matrix representation of a pure Hartle-Hawking state

Transition to statistical sums



$R^1 \times S^3$ $S^1 \times S^3$

thermal instantons



$D^4 \cup D'^4$ S^4

Hartle-Hawking (vacuum) instanton

Inflationary model driven by the trace anomaly of Weyl invariant fields --- CFT driven cosmology

$$S[\,g_{\mu\nu},\Phi\,] = -\frac{M_P^2}{2}\int d^4x\,g^{1/2}\,(R-2\Lambda) + S_{CFT}[\,g_{\mu\nu},\Phi\,]$$

Λ -- primordial cosmological constant

⇩

Omission of graviton loops

$$S_{\rm eff}[g_{\mu\nu}] = -\frac{M_P^2}{2}\int d^4x\,g^{1/2}(R-2\Lambda) + \varGamma_{CFT}[g_{\mu\nu}],$$

$$e^{-\varGamma_{CFT}[\,g_{\mu\nu}]} = \int D\Phi\,e^{-S_{CFT}[\,g_{\mu\nu},\Phi\,]}$$

Recovery of $\varGamma_{CFT}$ from the conformal anomaly on a static Einstein Universe (anomaly, Casimir energy and free energy contributions)

$$g_{\mu\nu}\frac{\delta\varGamma_{CFT}}{\delta g_{\mu\nu}} = \frac{1}{64\pi^2}g^{1/2}\left(\beta E + \alpha\Box R + \gamma C^2_{\mu\nu\alpha\beta}\right)$$

Gauss –Bonnet term

Weyl

$\beta = \sum_s \beta_s\,\mathbb{N}_s,\quad \mathbb{N}_s$ − # of spin s fields, β_s − spin-dependent coefficients

β -- *critically important parameter (overall coefficient of Gauss-Bonnet term in conformal anomaly)*

Effective Friedmann equation for saddle points of the path integral:

$$\frac{\delta S_{\rm eff}[\,a,N\,]}{\delta N(\tau)} = 0$$

$$\frac{1}{a^2} - \frac{\dot{a}^2}{a^2} = \frac{\varepsilon}{3M_\pm^2(\varepsilon)},$$

Effective Friedmann equation

$$M_\pm^2(\varepsilon) = \frac{M_P^2}{2}\Big(1 \pm \sqrt{1 - \frac{\beta}{6\pi^2 M_P^4}\,\varepsilon}\,\Big),$$

Effective Planck mass

$$\varepsilon = M_P^2 \Lambda + \frac{1}{2\pi^2 a^4}\sum_\omega \frac{\omega}{e^{\eta\,\omega} - 1},$$

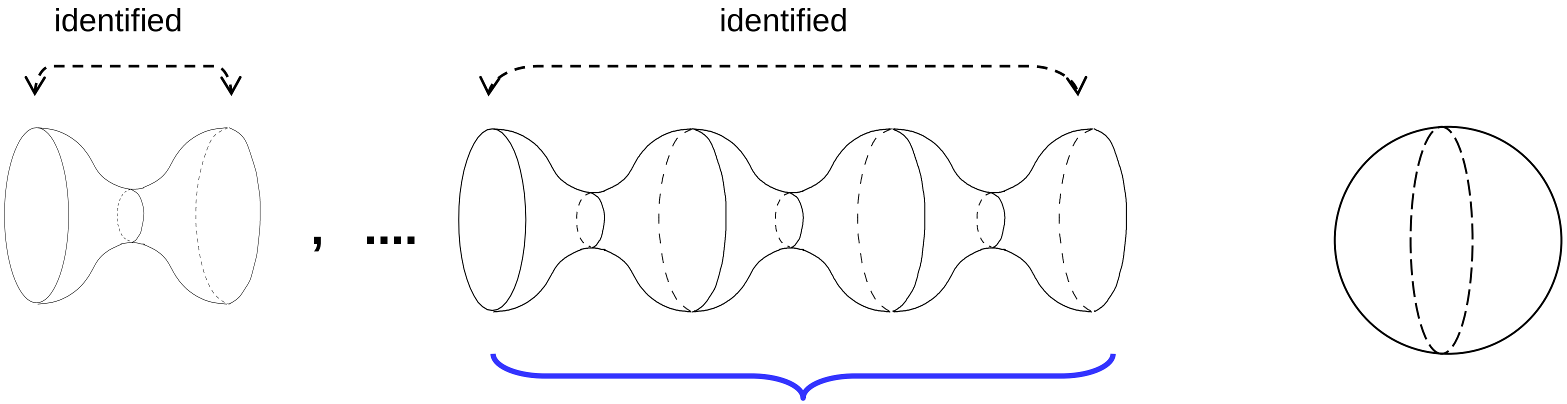
Energy density = Λ + radiation of CFT particles -- sum over field oscillators with frequencies (eigenvalues of Laplacian on S^3)

$$\eta = \int_{S^1} \frac{d\tau\, N}{a}$$

Inverse temperature in units of conformal time period on S^1

Existence of the quasi-thermal stage preceding the inflation

Saddle point solutions --- set of periodic (thermal) garland-type instantons with oscillating scale factor ($S^1 \times S^3$) and the vacuum Hartle-Hawking instantons (S^4)



1- fold, k=1

k- folded garland, k=1,2,3,…

S^4

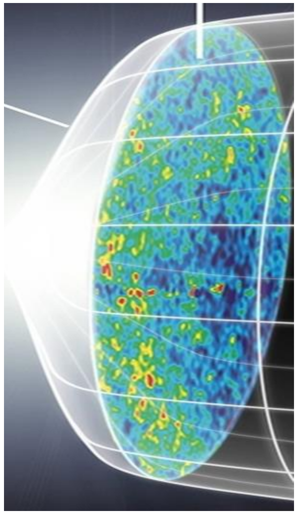
does not contribute: ruled out by infinite positive Euclidean action (effect of conformal anomaly)

UV bounded cosmological constant range:

$$\Lambda_{min} < \Lambda < \Lambda_{max} = \frac{12\pi^2}{\beta} M_P^2$$

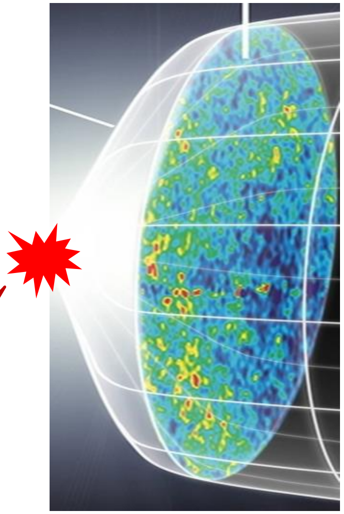
Initial thermal state with the primordial temperature T_{prim} of matter

Standard inflation scenario versus Density matrix scenario



Vacuum,
absolute zero
temperature

Inflation, hot
big-bang
→ relic radiation



Thermal state,
primordial
temperature T_{prim}

Inflation, $T_{prim} \to 0$,
hot big-bang
→ relic radiation

“SOME LIKE IT HOT” (SLIH) scenario

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Cosmological landscape from nothing: some like it hot

A O Barvinsky[1] and A Yu Kamenshchik[2,3]

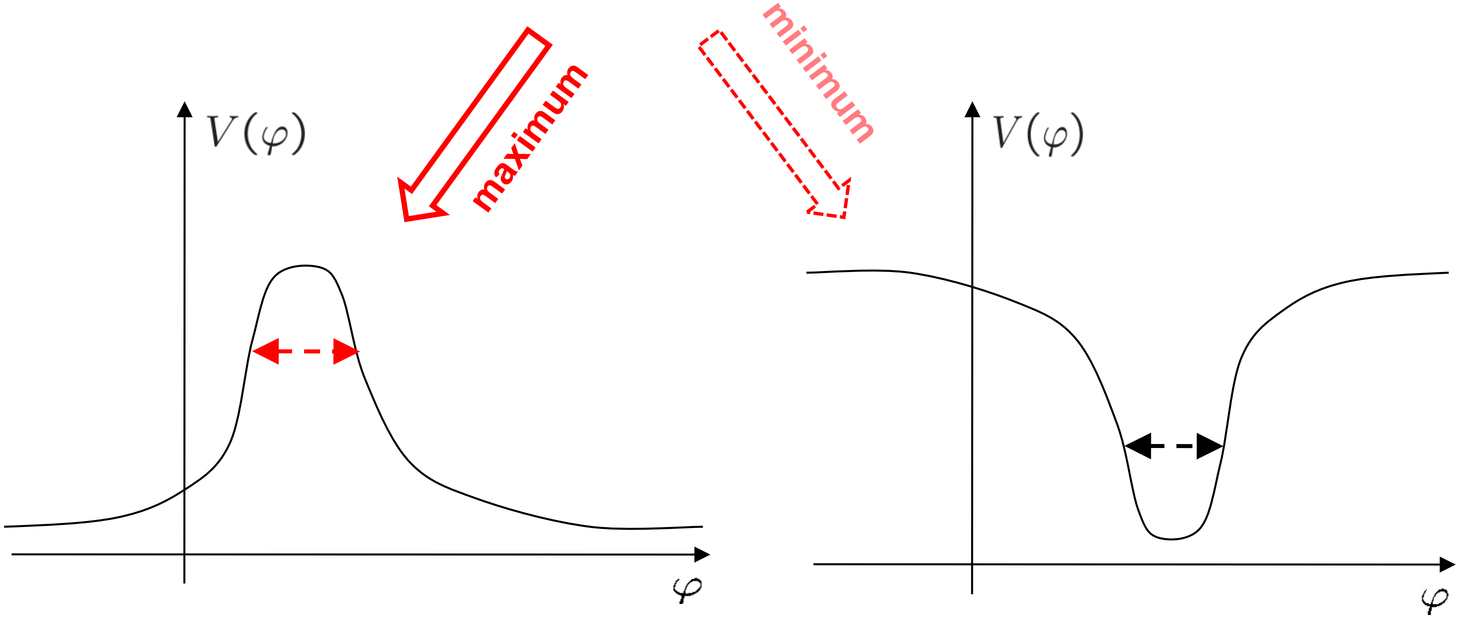
Known inflation paradigm retracted the BB concept by replacing it with the initial vacuum state.

“SOME LIKE IT HOT” (SLIH) scenario recovers a new incarnation of Hot Big Bang -- it incorporates effectively thermal state at the onset of the cosmological evolution.

So how does SLIH scenario matches with inflation?

Selection of inflaton potential maxima as initial conditions for inflation

Critical feature: $$\frac{d}{d\tau} a^3 \dot{\phi} = a^3 \frac{\partial V}{\partial \phi} \Rightarrow \oint d\tau \, a^3 \frac{\partial V}{\partial \phi} = 0 \Rightarrow \frac{\partial V}{\partial \phi} \gtrless 0$$ Potential extremum ``inside" instanton



classically forbidden (underbarrier) oscillation

classically allowed (overbarrier) oscillation --- ruled out because of necessity of underbarrier oscillations

Predictions of microcanonical cosmological initial conditions:

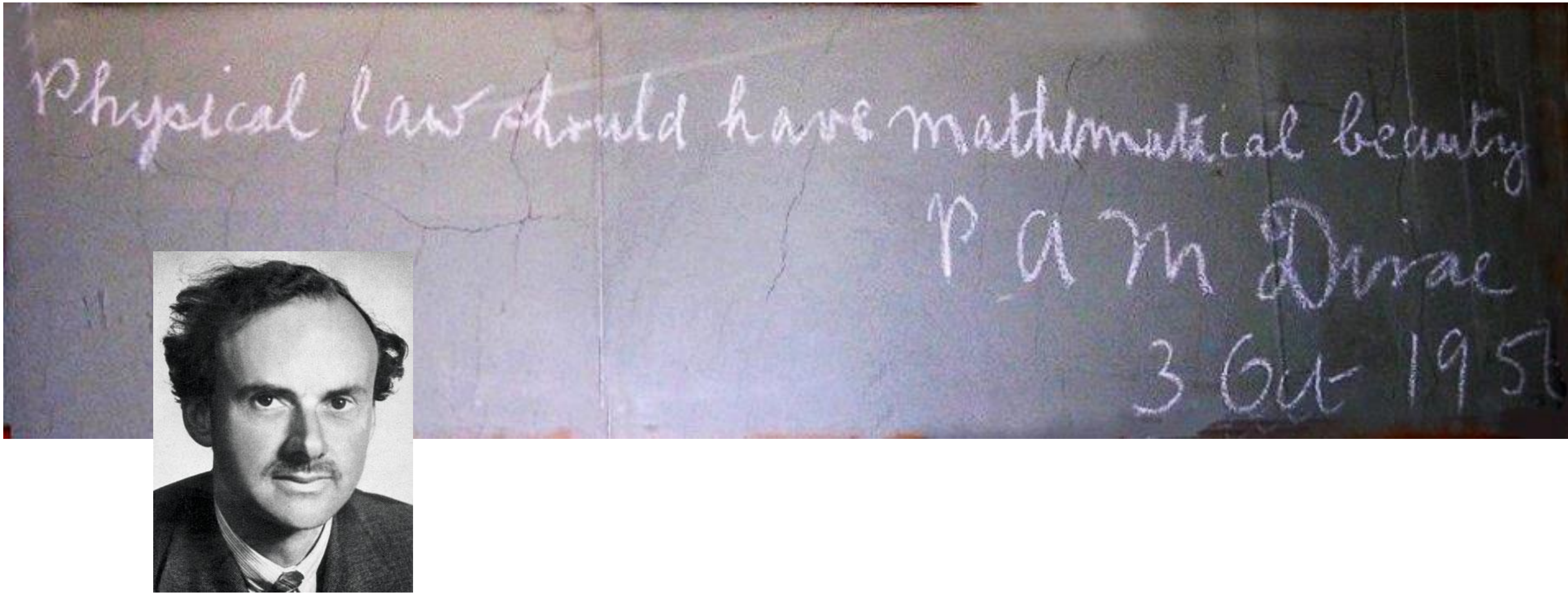
CFT driven cosmology: *suppression of no-boundary instantons; quasi-thermal stage preceding inflation and UV bounded range of its energy scale*

New type of hill-top inflation, $\Lambda \to V(\phi)$ *– selection of inflaton potential* $V(\phi)$ *maxima*

Mechanism of hill-top potential: *origin of non-minimal Higgs inflation and* R^2 *gravity*

Conformal higher spin fields (CHS): *solution of hierarchy problem -- origin of the Universe is the subplanckian phenomenon; justification of semiclassical expansion and 1/N-expansion*

Thermally corrected CMB spectrum: *observable signature of the primordial thermal epoch*



Physical law should have mathematical beauty

P A M Dirac

3 Oct 195

THANK YOU!