Homotopy Algebra Perspective on Quantum Field Theory

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based on work done in collaboration with

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and on the work of many others

Outline

- From Lie Algebras to L_∞-Algebras
- Homotopy Maurer–Cartan Theory
- Applications in Quantum Field Theory
- Conclusions

From Lie Algebras to L_∞-Algebras

Lie Algebras

Lie algebras (bracket picture):

- Vector space $\mathfrak{g}$
- Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$
- Basis e_a defines the structure constants $f_{ab}{}^c$ via $[\mathsf{e}_a, \mathsf{e}_b] = f_{ab}{}^c \mathsf{e}_c$

Lie algebras (Chavelley–Eilenberg picture):

- Dual vector space $(\mathfrak{g}[1])^*$ (all elements have degree 1)
- Basis ξ^a (of degree 1) are coordinate functions on $\mathfrak{g}[1]$
- Vector field $Q := -\frac{1}{2} f_{ab}{}^c \xi^a \xi^b \frac{\partial}{\partial \xi^c}$ of degree 1 on $\mathfrak{g}[1]$, and $Q^2 = 0$ $\Leftrightarrow$ Jacobi identity

L_∞-Algebras

L_∞-algebras (Chavelley–Eilenberg picture):

- Graded vector space $\mathfrak{L} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{L}_i$ with basis e_a and the dual $(\mathfrak{L}[1])^*$ with basis ξ^a
- Vector field $Q := \sum_i \pm\frac{1}{i!} f_{a_1\cdots a_i}{}^b \xi^{a_1}\cdots\xi^{a_i}\frac{\partial}{\partial\xi^b}$ of degree 1 on $\mathfrak{L}[1]$, and $Q^2 = 0 \Leftrightarrow$ homotopy Jacobi identity
- The constants $f_{a_1\cdots a_i}{}^b$ define brackets $\mu_i(\mathsf{e}_{a_1},\ldots,\mathsf{e}_{a_i}) =: f_{a_1\cdots a_i}{}^b\mathsf{e}_b$

L_∞-algebras (bracket picture):

- Graded vector space $\mathfrak{L} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{L}_i$
- Degree $2-i$ graded antisymmetric multilinear brackets $\mu_i : \mathfrak{L}\times\cdots\times\mathfrak{L}\to\mathfrak{L}$ subject to the homotopy Jacobi identity

$$\sum_{j+k=i}\sum_{\sigma(j;i)} \pm\mu_{k+1}(\mu_j(X_{\sigma(1)},\ldots,X_{\sigma(j)}),X_{\sigma(j+1)},\ldots,X_{\sigma(i)}) = 0$$

with $\sigma(j;i)$ the $(j,i-j)$-unshuffles i.e. $\sigma\in\mathsf{S}_i$ with $\sigma(1)<\cdots<\sigma(j)$ and $\sigma(j+1)<\cdots<\sigma(i)$

L_∞-Algebras

L_∞-algebras (bracket picture):

- $\mu_1^2 = 0$ making $(\mathfrak{L}, \mu_1)$ into a complex

$$\cdots \xrightarrow{\mu_1} \mathfrak{L}_{-1} \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \cdots$$

- μ_1 is a derivation for the bracket μ_2
- $\mu_2(\mu_2(X,Y),Z) \pm \mu_3(\mu_1(X),Y,X) + \text{cyclic} = \pm\mu_1(\mu_3(X,Y,Z))$
  i.e. the Jacobi identity is violated in a controlled way

Special cases:

- Lie algebras: $\mathfrak{L} = \mathfrak{L}_0$ and $\mu_i = 0$ for $i \neq 2$
- graded Lie algebras: $\mu_i = 0$ for $i \neq 2$
- differential graded Lie algebras: $\mu_i = 0$ for $i > 2$

L_∞-algebras are generalisations of differential graded Lie algebras

Cyclic L_∞-Algebras

Lie algebras:

- An inner product is a map $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that is non-degenerate, symmetric, bilinear, and cyclic $\langle X, [Y, Z] \rangle = \langle Z, [X, Y] \rangle$
- Dually, it is given by a symplectic form ω of degree 2 on $\mathfrak{g}[1]$ such that $\mathcal{L}_Q \omega = 0$

L_∞-algebras:

- An inner product or cyclic structure is a map $\langle -, - \rangle : \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$ of degree -3 that is non-degenerate, graded symmetric, bilinear, and cyclic $\langle X_1, \mu_i(X_2, \ldots, X_{i+1} \rangle = \pm \langle X_{i+1}, \mu_i(X_1, \ldots, X_i) \rangle$
- Dually, it is given by a symplectic form ω of degree -1 on $\mathfrak{L}[1]$ such that $\mathcal{L}_Q \omega = 0$

Morphisms of L_∞-Algebras

Lie algebras:

- Given two Lie algebras $(\mathfrak{g}, [-,-])$ and $(\mathfrak{g}', [-,-]')$, a morphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ satisfies $\phi([X,Y]) = [\phi(X), \phi(Y)]'$
- Dually, we simply have $\phi \circ Q = Q' \circ \phi$

L_∞-algebras:

- Dually, we again have $\phi \circ Q = Q' \circ \phi$
- In the bracket picture, for two L_∞-algebras $(\mathfrak{L}, \mu_i)$ and $(\mathfrak{L}', \mu_i')$, a morphism $\phi : \mathfrak{L} \to \mathfrak{L}'$ is collection of graded antisymmetric multilinear maps $\phi_i : \mathfrak{L} \times \cdots \times \mathfrak{L} \to \mathfrak{L}'$ of degree $1-i$ subject to

$$\sum_{j+k=i} \sum_{\sigma(j;i)} \pm \phi_{k+1}(\mu_j(X_{\sigma(1)}, \ldots, X_{\sigma(j)}), X_{\sigma(j+1)}, \ldots, X_{\sigma(i)})$$

$$= \sum_{j=1}^{i} \tfrac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma(k_1,\ldots,k_{j-1};i)}$$

$$\pm \mu_j'\Big(\phi_{k_1}\big(X_{\sigma(1)}, \ldots, X_{\sigma(k_1)}\big), \ldots, \phi_{k_j}\big(X_{\sigma(k_1+\cdots+k_{j-1}+1)}, \ldots, X_{\sigma(i)}\big)\Big)$$

- A morphism is called a quasi-isomorphism provided ϕ_1 induces an isomorphism $H^\bullet_{\mu_1}(\mathfrak{L}) \cong H^\bullet_{\mu_1'}(\mathfrak{L}')$

Homotopy Maurer–Cartan Theory

Homotopy Maurer–Cartan Theory

- For $(\mathfrak{L}, \mu_i)$ an L_∞-algebra, we call $a \in \mathfrak{L}_1$ a gauge potential and define its curvature as

$$f := \mu_1(a) + \tfrac{1}{2}\mu_2(a,a) + \cdots = \sum_{i \geqslant 1} \tfrac{1}{i!}\mu_i(a, \ldots, a)$$

- Due to the homotopy Jacobi identity, f satisfies the Bianchi identity

$$\mu_1(f) + \mu_2(a,f) + \cdots = \sum_{i \geqslant 0} \tfrac{1}{i!}\mu_{i+1}(a, \ldots, a, f) = 0$$

- For $c_0 \in \mathfrak{L}_0$, gauge transformations act as

$$\delta_{c_0} a := \mu_1(a) + \mu_2(a, c_0) + \cdots = \sum_{i \geqslant 0} \tfrac{1}{i!}\mu_{i+1}(a, \ldots, a, c_0),$$

$$\delta_{c_0} f = \mu_2(f, c_0) + \cdots = \sum_{i \geqslant 0} \tfrac{1}{i!}\mu_{i+2}(a, \ldots, a, f, c_0),$$

and there are higher gauge transformations with $c_{-k} \in \mathfrak{L}_{-k}$ and

$$\delta_{c_{-k-1}} c_{-k} := \sum_{i \geqslant 0} \tfrac{1}{i!}\mu_{i+1}(a, \ldots, a, c_{-k-1})$$

Homotopy Maurer–Cartan Theory

- The equation $f = 0$ is called the Maurer–Cartan equation and solutions to this equation are called Maurer–Cartan elements

- For $(\mathfrak{L}, \mu_i, \langle -, - \rangle)$ a cyclic L_∞-algebra, the Maurer–Cartan equation follows from the gauge-invariant action functional

$$S := \tfrac{1}{2}\langle a, \mu_1(a)\rangle + \tfrac{1}{3!}\langle a, \mu_2(a,a)\rangle + \cdots = \sum_{i\geqslant 0} \tfrac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle$$

- A morphism $\phi : (\mathfrak{L}, \mu_i) \to (\mathfrak{L}', \mu_i')$ acts as on a gauge potential and its curvature as

$$a \mapsto a' := \sum_{i\geqslant 1} \tfrac{1}{i!}\phi_i(a,\ldots,a) \quad \Rightarrow \quad f \mapsto f' = \sum_{i\geqslant 0} \tfrac{1}{i!}\phi_{i+1}(a,\ldots,a,f)$$

- Provided a is a Maurer–Cartan element, gauge equivalence classes $[a]$ are mapped to gauge equivalence classes $[a']$ and so, for quasi-isomorphisms, the corresponding moduli spaces are isomorphic

- A morphism is called cyclic provided $\langle X, Y\rangle = \langle \phi_1(X), \phi_1(Y)\rangle'$ and $\sum_{j+k=i}\langle \phi_j(X_1,\ldots,X_i), \phi_k(X_{j+1},\ldots,X_i)\rangle' = 0$ and so, $S[a] = S'[a']$

Example: Yang–Mills Theory

- Let M be a 4-dimensional compact oriented Riemannian manifold without boundary and let $\mathfrak{g}$ be a simple Lie algebra with inner product $\langle -, - \rangle_{\mathfrak{g}}$. The following data constitutes a cyclic L_∞-structure:

$$\underbrace{\Omega^1(M,\mathfrak{g})}_{=:\mathfrak{L}_1} \xrightarrow{\mu_1 := \mathrm{d}_M \star \mathrm{d}_M} \underbrace{\Omega^3(M,\mathfrak{g})}_{=:\mathfrak{L}_2}$$

  with

$$\mu_2(A_1, A_2) := \mathrm{d}_M \star [A_1, A_2] + [A_1, \star \mathrm{d}_M A_2] + [A_2, \star \mathrm{d}_M A_1],$$
$$\mu_3(A_1, A_2, A_3) := [A_1, \star [A_2, A_3]] + \text{cyclic}$$

  and

$$\langle \omega_1, \omega_2 \rangle := \int_M \langle \omega_1, \omega_2 \rangle_{\mathfrak{g}}$$

- The Maurer–Cartan action becomes $S = \frac{1}{2} \int_M \langle F, \star F \rangle_{\mathfrak{g}}$

Applications in Quantum Field Theory

Application I: Batalin–Vilkovisky Formalism

Batalin–Vilkovisky Formalism

BV formalism in a nutshell:

- Resolve the quotient space of observables:
  - Introduce ghosts to resolve gauge redundancy ('BRST')
  - Introduce anti-fields to resolve equations of motion
  - Differential Q_{BV} encodes gauge symmetries and equations of motion

$$Q_{\mathsf{BV}}\phi \;=\; Q_{\mathsf{BRST}}\phi + \cdots \quad \text{and} \quad Q_{\mathsf{BV}}\phi^+ \;=\; \pm\frac{\delta S_{\mathsf{BRST}}}{\delta\phi} + \cdots$$

- BV field space $\mathfrak{L}_{\mathsf{BV}}[1] := T^*[-1](\mathfrak{L}_{\mathsf{BRST}}[1])$ is a graded vector space that comes with a natural symplectic form $\omega_{\mathsf{BV}} := \delta\phi^+ \wedge \delta\phi$ of degree -1, and Q_{BV} is Hamiltonian with Hamiltonian S_{BV} and $Q_{\mathsf{BV}}^2 = 0 \Leftrightarrow \{S_{\mathsf{BV}}, S_{\mathsf{BV}}\}_{\mathsf{BV}} = 0$
- Dually, we obtain a cyclic L_∞-algebra $(\mathfrak{L}_{\mathsf{BV}}, \mu_i, \langle -,- \rangle)$
- BV action is a Maurer–Cartan action

BV formalism can be applied to any theory but it is essentially the only way when quantising theories with higher gauge symmetries

Yang–Mills Theory in the Batalin–Vilkovisky Formalism

- Let M be a compact oriented Riemannian manifold without boundary and let $\mathfrak{g}$ be a simple Lie algebra with inner product $\langle -, - \rangle_{\mathfrak{g}}$. Consider

$$\underbrace{\Omega^0(M,\mathfrak{g})}_{=:\mathfrak{L}_0 \ni c} \xrightarrow{\mu_1 := \mathrm{d}_M} \underbrace{\Omega^1(M,\mathfrak{g})}_{=:\mathfrak{L}_1 \ni A} \xrightarrow{\mu_1 := \mathrm{d}_M \star \mathrm{d}_M} \underbrace{\Omega^3(M,\mathfrak{g})}_{=:\mathfrak{L}_2 \ni A^+} \xrightarrow{\mu_1 := \mathrm{d}_M} \underbrace{\Omega^4(M,\mathfrak{g})}_{=:\mathfrak{L}_3 \ni c^+}$$

  with

$$\mu_2(c_1, c_2) := [c_1, c_2], \quad \mu_2(c, A) := [c, A], \quad \mu_2(c, A^+) := [c, A^+],$$
$$\mu_2(c, c^+) := [c, c^+], \quad \mu_2(A, A^+) := [A, A^+],$$
$$\mu_2(A_1, A_2) := \mathrm{d}_M \star [A_1, A_2] + [A_1, \star \mathrm{d}_M A_2] + [A_2, \star \mathrm{d}_M A_1],$$
$$\mu_3(A_1, A_2, A_3) := [A_1, \star[A_2, A_3]] + \text{cyclic}$$

  and $\langle \omega_1, \omega_2 \rangle := \pm \int_M \langle \omega_1, \omega_2 \rangle$

- Then, with $a = c + A + A^+ + c^+$, the Maurer–Cartan action becomes

$$S = \int_M \left\{ \tfrac{1}{2} \langle F, \star F \rangle_{\mathfrak{g}} - \langle A^+, \nabla c \rangle_{\mathfrak{g}} + \tfrac{1}{2} \langle c^+, [c, c] \rangle \right\}$$

Relative L_∞-Algebras and Homotopy Maurer–Cartan Theory

Cyclic L_∞-algebras are suitable for theories on manifolds without boundary or when considering fields with appropriate fall-off. What about theories where we have boundaries?

- A relative L_∞-algebra is a pair of L_∞-algebras, $(\mathfrak{L}, \mu_i)$ and $(\mathfrak{L}^\partial, \mu_i^\partial)$, and a morphism $\phi : (\mathfrak{L}, \mu_i) \to (\mathfrak{L}^\partial, \mu_i^\partial)$ between them
- It is called cyclic provided it comes with a map $\langle -, - \rangle_{\mathfrak{L}} : \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$ of degree -3 that is non-degenerate, graded symmetric, and bilinear as well as a map $\langle -, - \rangle_{\mathfrak{L}^\partial} : \mathfrak{L}^\partial \times \mathfrak{L}^\partial \to \mathbb{R}$ of degree -2 that is bilinear such that $(X_1, \ldots, X_{i+1}) \mapsto [X_1, \ldots, X_{i+1}]_{\mathfrak{L}}$ with

$$[X_1, \ldots, X_{i+1}]_{\mathfrak{L}} := \langle X_1, \mu_i(X_2, \ldots, X_{i+1}) \rangle_{\mathfrak{L}} + \sum_{j+k=i+1} \langle \phi_j(X_1, \ldots, X_i), \phi_k(X_{j+1}, \ldots, X_{i+1}) \rangle_{\mathfrak{L}^\partial}$$

  is non-degenerate and cyclic.
- The Maurer–Cartan action now reads as

$$S := \sum_{i \geqslant 0} \frac{1}{(i+1)!} [a, \ldots, a]_{\mathfrak{L}}$$

Example: Yang–Mills Theory

- Let M be a compact oriented Riemannian manifold with boundary ∂M and let $\mathfrak{g}$ be a simple Lie algebra with inner product $\langle -,-\rangle_{\mathfrak{g}}$. Take $(\mathfrak{L},\mu_i)$ as before but because of ∂M, $\langle -,-\rangle_{\mathfrak{L}}$ is not cyclic
- For $(\mathfrak{L}^\partial,\mu_i^\partial)$ we take

$$\underbrace{\Omega^0(\partial M,\mathfrak{g})}_{=:\mathfrak{L}_0^\partial\ni\gamma}\xrightarrow{\mu_1^\partial}\underbrace{\Omega^1(\partial M,\mathfrak{g})\oplus\Omega^2(\partial M,\mathfrak{g})}_{=:\mathfrak{L}_1^\partial\ni(\alpha,\beta)}\xrightarrow{\mu_1^\partial}\underbrace{\Omega^3(\partial M,\mathfrak{g})}_{=:\mathfrak{L}_2\ni\alpha^+}$$

with

$$\mu_1^\partial(\gamma):=(\mathrm{d}_{\partial M}\gamma,0),\quad \mu_1^\partial(\alpha,\beta):=\mathrm{d}_{\partial M}\beta,$$
$$\mu_2(\gamma_1,\gamma_2):=[\gamma_1,\gamma_2],\quad \mu_2(\gamma,(\alpha,\beta)):=([\gamma,\alpha],[\gamma,\beta]),$$
$$\mu_2(\gamma,\alpha^+):=[\gamma,\alpha^+],$$
$$\mu_2((\alpha_1,\beta_1),(\alpha_2,\beta_2)):=[\alpha_1,\beta_2]+[\alpha_2,\beta_1]$$

and

$$\langle\gamma,\alpha^+\rangle_{\mathfrak{L}^\partial}:=\int_{\partial M}\langle\gamma,\alpha^+\rangle_{\mathfrak{g}},\quad \langle(\alpha_1,\beta_1),(\alpha_2,\beta_2)\rangle_{\mathfrak{L}^\partial}:=\int_{\partial M}\langle\alpha_1,\beta_2\rangle_{\mathfrak{g}}$$

Example: Yang–Mills Theory

- The morphism $\phi : (\mathfrak{L}, \mu_i) \to (\mathfrak{L}^\partial, \mu_i^\partial)$ is now

$$\phi_1(c) := c|_{\partial M}\,, \quad \phi_1(A) := (A, \star \mathrm{d}_M A)|_{\partial M}\,, \quad \phi_1(A^+) := A^+|_{\partial M}\,,$$
$$\phi_2(A_1, A_2) := \star[A_1, A_2]|_{\partial M}$$

- Then, with $a = c + A + A^+ + c^+$, the Maurer–Cartan action becomes

$$\begin{aligned} S &= \sum_{i \geqslant 0} \frac{1}{(i+1)!}[a, \ldots, a]_{\mathfrak{L}} \\ &= \int_M \left\{ \tfrac{1}{2}\langle F, \star F \rangle_{\mathfrak{g}} - \langle A^+, \nabla c \rangle_{\mathfrak{g}} + \tfrac{1}{2}\langle c^+, [c, c] \rangle \right\} \end{aligned}$$

Applications in Quantum Field Theory

Application II: Perturbation Theory and Scattering Amplitudes

Homological Perturbation Theory

Homotopy Transfer:

- Start from a deformation retract, that is, two quasi-isomorphic complexes $(\mathfrak{L}, \mu_1)$ and $(\mathfrak{L}', \mu_1')$ with

$$\mathsf{h} \circlearrowright (\mathfrak{L}, \mu_1) \underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}} (\mathfrak{L}', \mu_1'),$$

$$1 = \mathsf{e} \circ \mathsf{p} + \mathsf{h} \circ \mu_1 + \mu_1 \circ \mathsf{h}, \quad \mathsf{p} \circ \mathsf{e} = 1$$

  where h is of degree -1 and called a contracting homotopy
- Consider higher products $\mu_{i>1}$ on $\mathfrak{L}$ as perturbation
- Recursive prescription as how this generates higher products $\mu_{i'>1}$ on $\mathfrak{L}'$ so that $(\mathfrak{L}, \mu_i)$ and $(\mathfrak{L}', \mu_i')$ are quasi-isomorphic

Applications:

- For $\mathfrak{L}' := H^\bullet_{\mu_1}(\mathfrak{L})$: recover minimal model and tree-level Feynman diagram expansion
- Introducing another perturbation $\mathrm{i}\hbar\Delta_{\mathsf{BV}}$ yields loop-level Feynman diagram expansion
- Recursive character underlies Berends–Giele-type recursion relations which exist for all field theories

Colour-Stripping as Factorisation

C_∞-algebra $\otimes$ L_∞-algebra $=$ L_∞-algebra

Explicit formulas:

$$\hat{\mathfrak{L}} := \mathfrak{C} \otimes \mathfrak{L} = \bigoplus_{k\in\mathbb{Z}} \hat{\mathfrak{L}}_k, \quad \hat{\mathfrak{L}}_k := \bigoplus_{i+j=k} \mathfrak{C}_i \otimes \mathfrak{L}_j,$$

$$\hat{\mu}_1(c_1 \otimes \ell_1) := \mathrm{d}c_1 \otimes \ell_1 \pm c_1 \otimes \mu_1(\ell_1)$$

$$\vdots$$

Examples:

- For $\mathfrak{C} = \Omega^\bullet(M^3)$, $\mathfrak{L} = \mathfrak{g}$ Lie algebra
  $\rightarrow S$ for $\hat{\mathfrak{L}}$ is the action for Chern–Simons theory
- For $\mathfrak{C} = \Omega^\bullet(M^d)$, $\mathfrak{L} = \mathfrak{L}_{-d+3} \oplus \cdots \oplus \mathfrak{L}_0$
  $\rightarrow S$ for $\hat{\mathfrak{L}}$ is d-dimensional higher Chern–Simons theory

Colour-stripping in scattering amplitudes for a general gauge theory:
$\mathfrak{L} = \mathfrak{C} \otimes \mathfrak{g}$ with kinematic C_∞-algebra $\mathfrak{C}$ and colour Lie algebra $\mathfrak{g}$

Strictification

Rendering a field theory cubic:

- Simpler to analyse field theories with only cubic vertices
- Any L_∞-algebra is quasi-isomorphic to a strict L_∞-algebra, that is, a differential graded Lie algebra
- This is called strictification

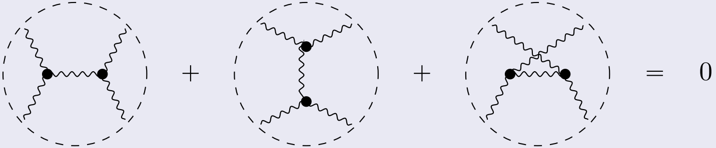
Examples:

- The 2nd-order formulation of Yang–Mills theory $S_{\mathrm{YM}_2} = \frac{1}{2}\int_M \langle F, \star F\rangle_{\mathfrak{g}}$ is quasi-isomorphic to the 1st-order formulation $S_{\mathrm{YM}_1} = \int_M \langle B, \star\big(F - \frac{1}{2}B\big)\rangle_{\mathfrak{g}}$ for $B \in \Omega^2(M, \mathfrak{g})$
- More later on ...

Strictification is used in the context of colour–kinematics duality

Colour–Kinematics Duality

Colour–kinematics duality of scattering amplitudes states that one can arrange them such that the colour-stripped vertex is Lie-like, e.g. Jacobi:



Thus, vertices (i.e. cubic terms in action) should ideally look like

$$g_{ad} f^{d}_{bc}\; g_{il} k^{l}_{jk}\; \Phi^{ai} \Phi^{bj} \Phi^{ck}$$

with

- g_{ad} and f^{d}_{bc} metric and structure constants of gauge Lie algebra
- g_{il} and k^{l}_{jk} metric and structure constants of kinematic Lie algebra

What is the kinematic Lie algebra homotopy algebraically?

Kinematic Lie algebra

- Factorise, i.e. colour-strip, the differential graded Lie algebra as $\mathfrak{L} = \mathfrak{C} \otimes \mathfrak{g}$ with $(\mathfrak{C}, \mathsf{d}, \mathsf{m}_2)$ a differential graded commutative algebra, d the kinematic operator, and m_2 the interactions
- Deformation retract

$$\mathsf{h} \circlearrowright (\mathfrak{C}, \mathsf{d}) \underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}} (H^\bullet_{\mathsf{d}}(\mathfrak{C}), 0)$$

$$1 = \mathsf{e} \circ \mathsf{p} + \mathsf{d} \circ \mathsf{h} + \mathsf{h} \circ \mathsf{d}, \quad \mathsf{p} \circ \mathsf{e} = 1$$

  with h the propagator
- Write h as $\mathsf{h} =: \frac{\mathsf{b}}{\blacksquare}$ so that $\blacksquare = \mathsf{b} \circ \mathsf{d} + \mathsf{d} \circ \mathsf{b}$
- If b is a second-order differential operator, the derived bracket

$$\{X, Y\} := \mathsf{b}(\mathsf{m}_2(X, Y)) + m_2(\mathsf{b}(X), Y) \pm m_2(X, \mathsf{b}(Y))$$

  is a (shifted) Lie bracket
- The derived bracket maps fields to fields: kinematic Lie bracket

Colour–Kinematics Duality from BV$^{\blacksquare}$-Algebras

Algebraic structures:

- $(\mathfrak{C}, \{-,-\})$: Gerstenhaber algebra
- $(\mathfrak{C}, \mathsf{d}, \mathsf{b}, \mathsf{m}_2)$ with $\mathsf{d} \circ \mathsf{b} + \mathsf{b} \circ \mathsf{d} = 0$ is a differential graded BV algebra

A BV$^{\blacksquare}$-algebra is a differential graded commutative algebra $\mathfrak{C}$ with a differential b of degree -1 that is a second-order differential operator with $\mathsf{d} \circ \mathsf{b} + \mathsf{b} \circ \mathsf{d} = \blacksquare$

A theory exhibits colour–kinematics duality, if its L_∞-algebra is quasi-isomorphic to a differential graded Lie algebra $\mathfrak{L} = \mathfrak{C} \otimes \mathfrak{g}$ with $\mathfrak{C}$ a differential graded commutative algebra such that $\mathfrak{C}$ admits a BV$^{\square}$-algebra structure with $\blacksquare = \square$

Examples

Biadjoint scalar field theory, $\mathsf{b} = [1]$

$$S := \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \varphi_{a\bar{a}} \Box \varphi^{a\bar{a}} - \tfrac{\lambda}{3!} f_{abc} f_{\bar{a}\bar{b}\bar{c}} \varphi^{a\bar{a}} \varphi^{b\bar{b}} \varphi^{c\bar{c}} \right\}$$

Self-dual Yang–Mills theory in light-cone gauge, $\mathsf{b} = [1]$

$$S := \int \mathrm{d}^4 x \left\{ \tfrac{1}{2} \langle \phi, \Box \phi \rangle_{\mathfrak{g}} + \tfrac{1}{3!} \varepsilon^{\alpha\beta} \langle \phi, [\partial_{\alpha\dot{2}} \phi, \partial_{\beta\dot{2}} \phi] \rangle_{\mathfrak{g}} \right\}$$

Chern–Simons theory, for harmonic forms, $\mathsf{b} = \pm \star \mathrm{d} \star$

$$S := \int \left\{ \tfrac{1}{2} \langle A, \mathrm{d}A \rangle_{\mathfrak{g}} + \tfrac{1}{3!} \langle A, [A, A] \rangle_{\mathfrak{g}} - \langle A^+, \nabla c \rangle_{\mathfrak{g}} + \tfrac{1}{2} \langle c^+, [c, c] \rangle_{\mathfrak{g}} \right\}$$

Applications

Idea: Look for Chern–Simons-type formulations of field theories

For Yang–Mills theory:

- Holomorphic Chern–Simons theory on twistor space (self-dual sector)
- Q-Chern–Simons theory on pure spinor space
- Chern–Simons-like formulation on harmonic superspace

The first two: organising principles for colour–kinematics duality

(just as superspaces for supersymmetry)

Applications in Quantum Field Theory

Application III: Yang–Mills Theory via Twistors

Yang–Mills Constraint System

- Consider Euclidean $\mathcal{N} = 3$ superspace $\mathbb{R}^{4|12}_{\rm cpl} := \mathbb{R}^{4|0} \times \mathbb{C}^{0|12}$ with coordinates $(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}, \theta^{i\alpha})$ and set

$$D^i_{\dot\alpha} := \partial^i_{\dot\alpha} + \theta^{i\alpha}\partial_{\alpha\dot\alpha}, \quad D_{i\alpha} := \partial_{i\alpha} + \eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}$$

  and so

$$[D_{i\alpha}, D^j_{\dot\alpha}] = 2\delta_i{}^j\partial_{\alpha\dot\alpha}$$

- For $\mathfrak{g}$ a Lie algebra, the covariantisation

$$[\nabla^i_{(\dot\alpha}, \nabla^j_{\dot\beta)}] = 0, \quad [\nabla_{i(\alpha}, \nabla_{j\beta)}] = 0, \quad [\nabla_{i\alpha}, \nabla^j_{\dot\alpha}] = 2\delta_i{}^j\nabla_{\alpha\dot\alpha}$$

  is the constraint system of $\mathcal{N} = 3$ SYM theory; it is equivalent to the equations of motion of $\mathcal{N} = 3$ SYM theory on $\mathbb{R}^4$

Cauchy–Riemann Ambitwistors

- Consider $F := \mathbb{R}^{4|12}_{\rm cpl} \times \mathbb{C}P^1 \times \mathbb{C}P^1$ with $\lambda_{\dot\alpha}$ and μ_α as coordinates on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and which comes with a quaternionic structure $(\lambda_{\dot\alpha}, \mu_\alpha) \mapsto (\hat\lambda_{\dot\alpha}, \hat\mu_\alpha)$
- Define

$$
\begin{gathered}
T^{0,1}_{\rm CR} F := \operatorname{span}\{\hat E_{\rm F}, \hat E_{\rm L}, \hat E_{\rm R}, \hat E^i, \hat E_i\},\\
\hat E_{\rm F} := \mu^\alpha \lambda^{\dot\alpha} \partial_{\alpha\dot\alpha}, \quad \hat E_{\rm L} := |\lambda|^2 \lambda_{\dot\alpha} \frac{\partial}{\partial \hat\lambda_{\dot\alpha}}, \quad \hat E_{\rm R} := |\mu|^2 \mu_\alpha \frac{\partial}{\partial \hat\mu_\alpha},\\
\hat E^i := \lambda^{\dot\alpha} D^i_{\dot\alpha}, \quad \hat E_i := \mu^\alpha D_{i\alpha}
\end{gathered}
$$

  which is an integrable CR structure with $[\hat E_i, \hat E^j] = 2\delta_i{}^j \hat E_{\rm F}$
- Let $A \in \Omega^{0,1}_{\rm CR} \otimes \mathfrak{g}$. Under the assumption that there is a gauge in which $\hat E_{\rm L} \lrcorner A = 0 = \hat E_{\rm R} \lrcorner A$, the CR holomorphic Chern–Simons equation

$$
\bar\partial_{\rm CR} A + \tfrac{1}{2}[A, A] = 0
$$

  on F is equivalent to the $\mathcal{N} = 3$ SYM constraint system on $\mathbb{R}^{4|12}_{\rm cpl}$

Twisted CR Structure

- Consider the CR holomorphic and antiholomorphic coordinates $\eta_i := \eta_i^{\dot\alpha}\lambda_{\dot\alpha}$, $\theta^i := \theta^{i\alpha}\mu_\alpha$, $\bar\eta_i := -\frac{\eta_i^{\dot\alpha}\hat\lambda_{\dot\alpha}}{|\lambda|^2}$, and $\bar\theta^i := -\frac{\theta^{i\alpha}\hat\mu_\alpha}{|\mu|^2}$ and the new basis

$$T^{0,1}_{\rm CR}F = {\rm span}\{\hat E'_{\rm F}, \hat E'_{\rm L}, \hat E'_{\rm R}, \hat E'^i, \hat E'_i\},$$
$$\hat E'_{\rm F} := \hat E_{\rm F},\quad \hat E'_{\rm L} := \hat E_{\rm L} + \bar\theta^i\eta_i\hat E_{\rm F},\quad \hat E'_{\rm R} := \hat E_{\rm R} - \theta^i\bar\eta_i\hat E_{\rm F},$$
$$\hat E'^i := \hat E^i - \bar\theta^i\hat E_{\rm F},\quad \hat E'_i := \hat E_i - \bar\eta_i\hat E_{\rm F}$$

  with $[\hat E'_{\rm L}, \hat E'_{\rm R}] = 2\theta^i\eta_i\hat E'_{\rm F}$

- Set $g := {\rm e}^{\bar\theta^i\eta_i E_{\rm W} + \theta^i\bar\eta_i E_{\hat{\rm W}}}$ with $E_{\rm W} := \frac{\mu^\alpha\hat\lambda^{\dot\alpha}}{|\lambda|^2}\partial_{\alpha\dot\alpha}$ and $E_{\hat{\rm W}} := -\frac{\hat\mu^\alpha\lambda^{\dot\alpha}}{|\mu|^2}\partial_{\alpha\dot\alpha}$ and define the twisted CR structure

$$T^{0,1}_{\rm CR,\,tw}F := {\rm span}\{\hat V_{\rm F}, \hat V_{\rm L}, \hat V_{\rm R}, \hat V^i, \hat V_i\},$$
$$\hat V_{\rm F} := g\hat E'_{\rm F}g^{-1} = \hat E_{\rm F},$$
$$\hat V_{\rm L} := g\hat E'_{\rm L}g^{-1} = \hat E_{\rm L} + \theta^i\eta_i E_{\hat{\rm W}},\quad \hat V_{\rm R} := g\hat E'_{\rm R}g^{-1} = \hat E_{\rm R} + \theta^i\eta_i E_{\rm W},$$
$$\hat V^i := g\hat E'^i g^{-1} = \bar\partial^i,\quad \hat V_i := g\hat E'_i g^{-1} = \bar\partial_i$$

  with $[\hat V_{\rm L}, \hat V_{\rm R}] = 2\theta^i\eta_i\hat V_{\rm F}$

Quasi-Isomorphy

- Let $\Omega^{0,\bullet}_{\rm CR,\,tw,\,red}$ be those elements of $\Omega^{0,\bullet}_{\rm CR,\,tw}$ that do not have CR antiholomorphic fermionic directions and that depend CR holomorphically on the fermionic coordinates
- The differential graded Lie algebras $(\Omega^{0,\bullet}_{\rm CR}\otimes\mathfrak{g},\bar\partial_{\rm CR},[-,-])$, $(\Omega^{0,\bullet}_{\rm CR,\,tw}\otimes\mathfrak{g},\bar\partial_{\rm CR,\,tw},[-,-])$, and $(\Omega^{0,\bullet}_{\rm CR,\,tw,\,red}\otimes\mathfrak{g},\bar\partial_{\rm CR,\,tw,\,red},[-,-])$ are all quasi-isomorphic
- Hence,

$$\bar\partial_{\rm CR,\,tw}A+\tfrac12[A,A]=0\quad\text{with}\quad \hat V^i\lrcorner A=0=\hat V_i\lrcorner A$$

is equivalent to the $\mathcal{N}=3$ SYM constraint system on $\mathbb{R}^{4|12}_{\rm cpl}$

- Define the twisted CR holomorphic volume form

$$\Omega_{\rm CR,\,tw}:=v^{\rm F}\wedge v^{\rm W}\wedge v^{\hat{\rm W}}\wedge v^{\rm L}\wedge v^{\rm R}\otimes v_1v_2v_3v^1v^2v^3$$

and so

$$S:=\int\Omega_{\rm CR,\,tw}\wedge\left\{\tfrac12\langle A,\bar\partial_{\rm CR,\,tw}A\rangle+\tfrac1{3!}\langle A,[A,A]\rangle\right\}$$

Semi-Classical Equivalence

BV action for twisted CR holomorphic Chern–Simons theory:

$$S_{\mathrm{CRCS}} := \int \Omega_{\mathrm{CR,\,tw}} \wedge \Big\{ \tfrac{1}{2}\langle A, \bar\partial_{\mathrm{CR,\,tw}} A\rangle + \tfrac{1}{3!}\langle A, [A,A]\rangle - \langle A^+, \bar\nabla_{\mathrm{CR,\,tw}} c\rangle + \tfrac{1}{2}\langle C^+, [C,C]\rangle \Big\}$$

BV action for first-order $\mathcal{N}=3$ supersymmetric Yang–Mills theory:

$$S_{\mathrm{YM}_1} := \int \Big\{ \langle B, \star F\rangle - \tfrac{1}{2}\langle B, \star B\rangle - \langle A^+, \nabla c\rangle - \langle B^+, [B,c]\rangle + \tfrac{1}{2}\langle c^+, [c,c]\rangle \Big\} + \text{‘}\mathcal{N}=3 \text{ completion’}$$

The theories described by S_{CRCS} and S_{YM_1} are quasi-isomorphic via homotopy transfer, that is, S_{YM_1} is obtained from S_{CRCS} by integrating out infinitely many auxiliary fields

Conclusions

A Dictionary

The Homotopy Algebraic Perspective on perturbative QFT:

Perturbative QFT	Homotopy Algebra
fields of ghost number n	elements of degree $1-n$ in an L_∞-algebra
action principle	cyclic L_∞-algebra
free part of the action	differential μ_1
interaction parts	higher products $\mu_{i>1}$
semi-classical equivalence	L_∞-quasi-isomorphism
Feynman diagram expansion	homological perturbation theory $(\mathsf{h}, \mathsf{p}, \mathsf{e})$
propagator	contracting homotopy h
gauge fixing	embedding $\mathsf{e} + \ldots$
scattering amplitudes	Maurer–Cartan action for minimal model
Berends–Giele recursions	L_∞-quasi-morphism to minimal model
colour-stripping	factorising L_∞-algebra
$\vdots$	$\vdots$

Action and scattering amplitudes on equal footing: L_∞-algebras

Further Applications

- The double copy i.e. gauge theory $\otimes$ gauge theory $=$ gravity can be understood via homotopy algebras in terms tensor products of BV$^{■}$-algebras
- Quasi-isomorphisms are not necessarily obtained by homotopy transfer, however, one can always construct a span of L_∞-algebras $\mathfrak{L}_1 \leftarrow \mathfrak{L} \rightarrow \mathfrak{L}_2$ such that the arrows are homotopy transfers; for instance, T-duality can be understood this way
- L_∞-algebras are the gauge algebras of higher gauge theory and the infinitesimal versions of higher groups $\rightarrow$ higher differential geometry
- Higher structures appear also in other contexts such as fluid dynamics where incompressible fluid flows in $d \geqslant 3$ dimensions can be understood via higher symplectic geometry
- ...

Thank You!