# Silicon Instrumentation Summer School - Intro to Analog Electronics Notes

by Dan Weatherill

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## 1 Kirchoff's Current & Voltage laws

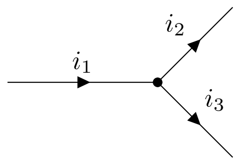



**Figure 1.** Kirchoff's Current law

**Kirchoff's Current Law (KCL)** - currents entering and exiting a **node** sum to 0:

$$i_1 - i_2 - i_3 \;=\; 0$$

this is **always true** (it's a statement of conservation of charge) - see Figure 1.

**Kirchoff's Voltage Law (KVL)** - voltages round a closed loop sum to zero, or **equivalently** any two points connected only by a "wire" are at the same voltage. Only true if there is no external magnetic flux coupling to the circuit!!

$$V_1 - V_2 - V_3 \;=\; 0$$

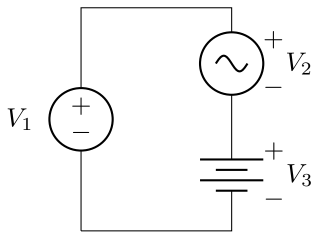



**Figure 2.** Kirchoff's Voltage law

Tend to use KCL in "by hand" analysis, but either is fine. When using KCL to convert a circuit to a matrix (e.g. for SPICE) it's called "node analysis". When using KVL to convert a circuit to a matrix, it's called "mesh analysis".

**It doesn't matter which way round you define currents and voltages provided your entire circuit is consistent** - current flows from positive to negative voltage (see Figure 3).

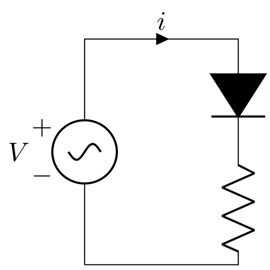



**Figure 3.** consistent arrangement of voltage $V$ and current $i$ in a circuit diagram

# 2 Complex Impedance and Passive Circuits

Sinusoidal voltages (or currents) can be written as complex numbers (without loss of generality due to Fourier synthesis):

$$\tilde{V}(t) = V_0 \exp(j\omega t)$$

(with $j^2 = -1$, angular frequency $\omega = 2\pi f$, amplitude $V_0$). The measured voltage on an instrument is then given by $\Re\{\tilde{V}\}$ (the real part). Complex impedance $\tilde{Z}$ is defined (analogous to Ohm's law) as:

$$\begin{aligned} \tilde{V} &= \tilde{I}\tilde{Z} \\ \underbrace{\tilde{Z}}_{\text{impedance}} &= \underbrace{R}_{\text{resistance}} + j \underbrace{X}_{\text{reactance}} \end{aligned}$$

Each passive component then has its own complex impedance. A resistor is purely resistive, capacitors and inductors are purely reactive. Complex impedances can be easily derived from the component's constitutive equation. For the capacitor:

$$\begin{aligned} \underbrace{Q}_{\text{charge}} &= \underbrace{C}_{\text{capacitance}} \times \underbrace{V}_{\text{voltage}} \\ \Rightarrow \int I dt &= C \times V \\ \Rightarrow I &= C\frac{dV}{dt} \\ \Rightarrow \tilde{I} &= Cjw\tilde{V} \\ \therefore \tilde{Z}_C &= \frac{1}{j\omega C} \end{aligned}$$

and for the inductor, similarly:

$$\begin{aligned} \underbrace{\Phi}_{\text{flux linkage}} &= \underbrace{L}_{\text{inductance}} \times \underbrace{I}_{\text{current}} \\ \int V dt &= L \times I \\ \Rightarrow \tilde{V} &= j\omega L \times \tilde{I} \\ \therefore Z_L &= j\omega L \end{aligned}$$

complex impedances combine like resistances, in series:

$$\tilde{Z}_{\text{series}} = \tilde{Z}_1 + \tilde{Z}_2$$

and in parallel:

$$\frac{1}{\tilde{Z}_{\|}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2}$$

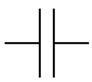
| Component | Symbol | Constitutive Equation | Complex Impedance |
|---|---|---|---|
| Resistor |  | $V = I \times R$ | $Z_R = R$ |
| Capacitor |  | $Q = C \times V$ | $Z_C = \frac{1}{j\omega C}$ |
| Inductor |  | $\Phi = L \times I$ | $Z_L = j\omega L$ |

**Table 1.** Summary of complex impedances of components

The “bode plot” is a plot of the magnitude and phase of the complex impedance of a circuit. It gives the information about how signals of different frequencies will be filtered by the circuit.

Complex impedance can become difficult when we start talking about **power**. Because there are now several possible measurements, summarised in table 2. The instantaneous power flowing through a circuit is as per a DC circuit the real parts of the voltage multiplied by real part of current. This is called **active power**. The **complex power** is given by $\tilde{V}\tilde{I}^*$ (where * represents the complex conjugate), and contains information both about active power and **reactive power** (the imaginary part). Reactive power transfers no net energy to the load but represents cyclical transfer of energy. The **apparent power** is the magnitude of the complex power. It is measured in Volt-Amperes, and it is important because it is the amount of power that your circuit must be able to handle! The **power factor** is the ratio of active power to apparent power (i.e. it is the cosine of the phase angle of the complex power). When designing power supplies or considering power loads required, the power factor must be taken into account - a low power factor circuit will draw more current, for the same amount of active / “useful” power transfer. Improper power factor correction is the cause of many problems in power engineering, up to and including many grid-scale blackouts.

| **Quantity** | **Name** | **Units** |
|---|---|---|
| $\Re\{\tilde{V}\} \times \Re\{\tilde{I}\}$ | active power | Watt (W) |
| $\tilde{V} \times \tilde{I}^*$ | Complex Power | Volt-Ampere (VA) |
| $\Im\{\tilde{V} \times \tilde{I}^*\}$ | Reactive power | Volt-Ampere reactive (var) |
| $\lvert\tilde{V} \times \tilde{I}^*\rvert$ | Apparent power | Volt-Ampere (VA) |
| $\frac{\Re\{\tilde{V}\} \times \Re\{\tilde{I}\}}{\lvert\tilde{V} \times \tilde{I}^*\rvert}$ | power factor | dimensionless |

**Table 2.** Power with complex signals

## 2.1 examples

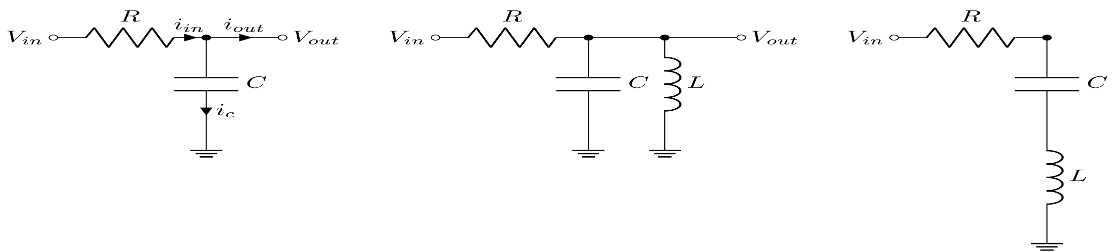



**Figure 4.** passive filter examples. Left - RC low pass filter. Middle - Parallel resonant LC filter. Right - series resonant LC filter

Looking at the examples in Figure 4. On the left we have the humble RC filter. We can work out the response using complex impedances using Kirchoff’s current law. For simplicity we assume the output is high impedance (i.e. $i_{\text{out}} \approx 0$) for now:

$$\begin{aligned}
V_{\text{in}} - i_{\text{in}} R &= V_{\text{out}} \\
V_{\text{out}} - \frac{i_c}{j\omega C} &= 0 \\
i_{\text{in}} &= i_c \quad (i_{\text{out}} \approx 0) \\
\Rightarrow i_{\text{in}} &= V_{\text{out}} j\omega C \\
\Rightarrow V_{\text{in}} &= V_{\text{out}}(1 + j\omega \text{RC}) \\
\Rightarrow \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{1}{1 + j\omega RC}
\end{aligned}$$

note the form of that expression, it will become **very** familiar. For the middle circuit, we can combine the $C$ and $L$ in parallel:

$$\frac{1}{Z_{\mathrm{LC}}} = \frac{1}{j\omega L} + j\omega C$$

and the analysis proceeds as before. Try these for yourself. You will find that both the other circuits are **resonant** (i.e. they have strongly peaked impedances). These can be used as selective filters, or as e.g. radio tuning circuits with some slight extra work.

# 3 Linear Systems Theory (Review)

The Fourier transform of a quantity and its inverse are given by:

$$\begin{aligned}\hat{F}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \\ f(t) &= \int_{-\infty}^{\infty} \hat{F}(\omega)e^{j\omega t}d\omega\end{aligned}$$

Some results on Fourier Transforms are so useful we need them very often:

| **Operation** | **time domain** | **frequency domain** |
|---|---|---|
| integration | $\int f(t)dt$ | $\frac{1}{j\omega}\hat{F}(\omega)$ |
| differentiation | $\frac{df}{dt}$ | $j\omega\hat{F}(\omega)$ |
| time shift | $f(t+t_0)$ | $e^{-jt_0\omega}\hat{F}(\omega)$ |
| convolution | $f*g \equiv \int_{-\infty}^{\infty} f(t)g(t-\tau)d\tau$ | $\hat{F}(\omega)\hat{G}(\omega)$ |

**Table 3.** Fourier Transform ingredients

the last line (convolution) is important because it gives a link between filtering circuits in frequency domain and impulse response in time domain. For example, if we have an output signal $\tilde{v}_{\mathrm{out}}(t)$, input signal $\tilde{v}_{\mathrm{in}}(t)$, then some circuit with **impulse response** $h$ gives:

$$\begin{aligned}\tilde{v}_{\mathrm{out}}(t) &= (h*\tilde{v}_{\mathrm{in}})(t) \\ \Rightarrow \hat{V}_{\mathrm{out}}(\omega) &= \hat{H}(\omega)\hat{V}_{\mathrm{in}}(\omega)\end{aligned}$$

the quantity $\hat{H}(\omega)$ is known as the **transfer function** of the circuit, and (at least for passive networks and linear active ones) can be always written as a **rational** function, like:

$$\hat{H}(\omega) = \frac{n_0 + n_1 s + n_2 s^2 + \cdots}{d_0 + d_1 s + d_2 s^2 + \cdots} \text{where } s = j\omega \tag{1}$$

note we have used the Laplace transform notation $s$ there even though we're using Fourier Transforms. This is justified in electronics because actual signals are real valued, therefore Fourier Transforms always hermitian, but it can need care when dealing with complex impedances and making sure you are in the right domain.

Anyway, the factorization of the numerator of (1) gives the **zeros** of the function (the values of $s$ where the imaginary part of transfer function goes to zero!). And the factorization of the denominator gives the so-called **poles** of the function (where it may blast off to infinity, or else have a instantaneous phase shift). Because it's just the imaginary part, it is not necessary that this causes the magnitude of the output to head to infinity (though that can happen if it's an even power of $s$ of course!). Knowing the poles and zeros of a passive filter is equivalent to knowing **everything** about it. Each component added with a different value will add either a pole or a zero to the transfer function.

# 4 Linear Amplifiers (op amps)

The most useful and widely used type of linear amplifier is the "operational amplifier" or op-amp. They are internally complicated objects (see e.g. the half century old design of the OP741 in Figure 5), but the principles apply to other types of linear or differential amplifiers. Note we will not touch any non-inverting configurations in the course but they are equally as important as inverting configurations, there are a couple of problems given at the end on them.

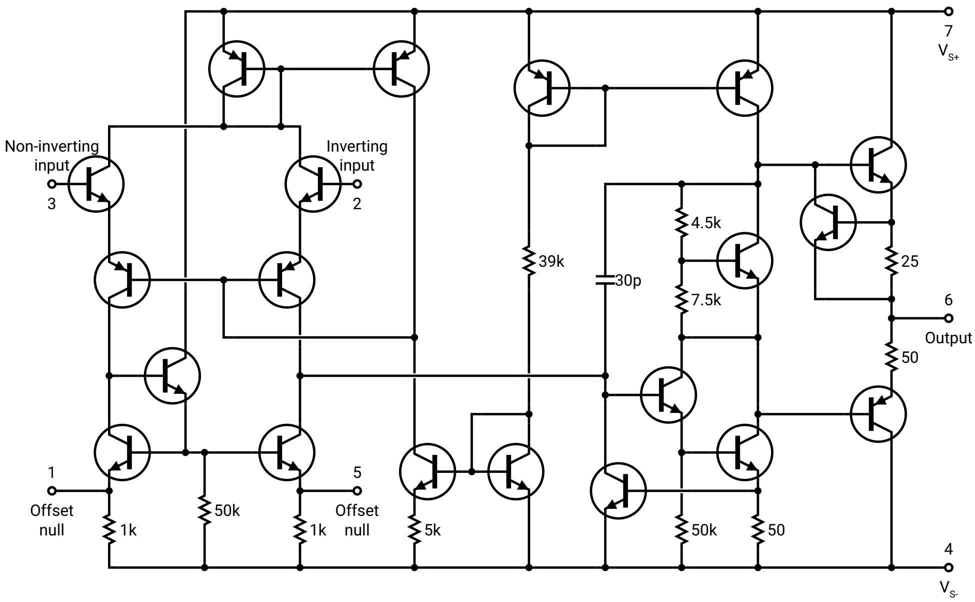



**Figure 5.** Internal schematic of the "classic" OP741 op-amp.

## 4.1 Basics

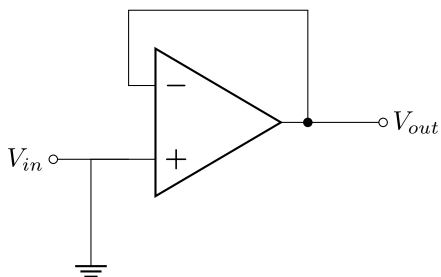



**Figure 6.** Op-amp in source follower configuration

The transfer function of an ideal differential amplifier ("op amp") is given by:

$$V_{\text{out}} = A(\omega)(V_+ - V_-)$$

where $V_{\text{out}}$ is output voltage, $V_+$ and $V_-$ are non-inverting and inverting input voltages respectively, and $A(\omega)$ is the **open loop gain**. Note this on its own violates conservation of energy, but that's fine because the op-amp has its own separate power supply. Some op-amps are designed to be so-called "**fully compensated"** and if that is the case, then $A(\omega)$ has the form:

$$A(\omega) = \frac{A_0}{1 + j\frac{\omega}{\omega_c}}$$

$A_0$ and $\omega_c$ are rarely specified in the datasheet, but often their product $A_0\omega_c$ is. This quantity is called the **Gain-Bandwidth Product** (see section 4.7). When op-amps are used as filters or amplifiers, the circuit topology will be designed with **negative feedback** of some sort. Consider the simplest possible arrangement in Figure 6, the "source follower". With the op-amp transfer function, we get:

$$\begin{aligned}
V_{\text{out}} &= A(V_+ - V_-)\\
\Rightarrow V_{\text{out}} &= A(V_{\text{in}} - V_{\text{out}})\\
\Rightarrow V_{\text{out}}(1 + A) &= AV_{\text{in}}\\
\Rightarrow \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{A}{1 + A} \approx 1 \text{ (for } A \gg 1)
\end{aligned}$$

If $A \gg 1$, then we see that the voltage gain approaches unity. In general, for **any** arrangement with negative feedback, Kirchoff's current law + the op-amp transfer function will give arrangements like $\frac{A}{1+A}$ in the final gain equation, and which for an ideal op-amp with very high open loop gain ($A \gg 1$), will all cancel out. This is where the **Golden Rule #2** comes from: "assume $V_+$ is at the same voltage as $V_-$". This is equivalent to saying that $A \to \infty$ as it turns out. **Golden Rule #1** is "assume the inputs draw no current", which means they are very high impedance. The principle of **negative feedback** is used in many types of amplifiers, not just with op-amps.

## 4.2 Feedback - Inverting Op Amp

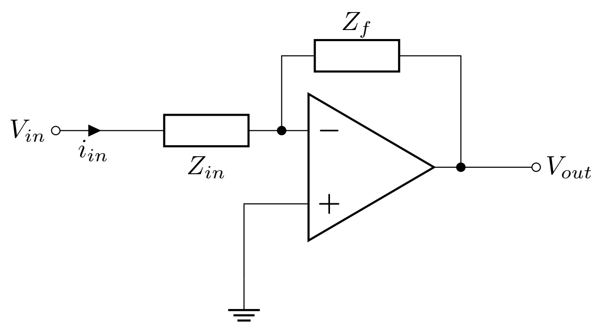



**Figure 7.** Op amp in inverting configuration

Now consider the example in Figure 7. The impedances round the feedback $Z_F$ and at the input $Z_i$ are general. To work out the gain, use Kirchoff's current law:

$$\begin{aligned}
V_{\text{out}} - i_f Z_f &= V_-\\
V_{\text{out}} &= A(V_+ - V_-) = -\text{AV}_-
\end{aligned}$$

$$\begin{aligned} V_{\text{out}} - i_f Z_f &= -\frac{V_{\text{out}}}{A} \\ V_{\text{out}}\left(1+\frac{1}{A}\right) &= i_f Z_f \end{aligned}$$

we now invoke "golden rule #1" and say that the inputs draw no current (i.e. $i_+ = i_- = 0$). Most op-amps are built to make this a good assumption (though be careful and read the data sheet!). Therefore:

$$\begin{aligned} V_{\text{in}} - i_{\text{in}} Z_{\text{in}} &= V_- \\ i_{\text{in}} &= -i_f \quad (\text{inputs draw no current}) \\ V_{\text{in}} + i_f Z_{\text{in}} &= V_- \\ V_{\text{in}} + i_f Z_{\text{in}} &= V_{\text{out}} - i_f Z_f \\ \Rightarrow V_{\text{in}} - V_{\text{out}} &= -i_f (Z_{\text{in}} + Z_f) \\ \Rightarrow V_{\text{in}} - V_{\text{out}} &= -V_{\text{out}}\left(1+\frac{1}{A}\right)\frac{Z_{\text{in}}+Z_f}{Z_f} \\ \Rightarrow V_{\text{in}} &= V_{\text{out}}\left(1-\left(1+\frac{1}{A}\right)\frac{Z_{\text{in}}+Z_f}{Z_f}\right) \\ \Rightarrow A_V = \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{1}{\left(1-(1+\frac{1}{A})\frac{Z_{\text{in}}+Z_f}{Z_f}\right)} = \frac{-Z_f}{Z_{\text{in}}+\frac{1}{A}(Z_{\text{in}}+Z_f)} \\ \Rightarrow A_V &\approx -\frac{Z_f}{Z_{\text{in}}} \quad (A \gg 1) \end{aligned}$$

note that if we used both "Golden Rules" we could get to this much faster:

$$\begin{aligned} V_{\text{out}} - i_f Z_f &= V_- = V_+ = 0 \quad (\text{using golden rule 2, assume } V_- = V_+) \\ V_{\text{in}} - i_{\text{in}} Z_{\text{in}} &= V_- = 0 \\ \Rightarrow i_{\text{in}} = -i_f &= \frac{V_{\text{in}}}{Z_{\text{in}}} \,(\text{golden rule 1, inputs draw no current}) \\ \Rightarrow V_{\text{out}} + V_{\text{in}}\frac{Z_f}{Z_{\text{in}}} &= 0 \\ \Rightarrow \frac{V_{\text{out}}}{V_{\text{in}}} &= -\frac{Z_f}{Z_{\text{in}}} \end{aligned}$$

in the case of $Z_f$ and $Z_i$ being resistors, we can see that this amplifier inverts and amplifies (or attenuates) the input voltage. Now consider what happens when $Z_f$ is instead a capacitor $C_f$ (with $Z_{\text{in}}$ a resistor $R_{\text{in}}$):

$$A_V = -\frac{1}{j\omega R_{\text{in}} C_f}$$

the circuit is now acting as a **low pass filter** (at higher $\omega$ the gain drops). Consider using Kirchoff's laws, "golden rules" but **not** complex impedance to analyze this circuit (same derivation as above but using $Q = C \times V$ for capacitor):

$$\begin{aligned} V_{\text{in}} - i_{\text{in}} R_{\text{in}} &= V_- = 0 \\ \Rightarrow V_{\text{in}} + i_f R_{\text{in}} &= 0 \qquad i_f = \frac{-V_{\text{in}}}{R_{\text{in}}} \\ V_{\text{out}} - \underbrace{\frac{1}{C_f}\int i_f dt}_{Q=C\times V} &= 0 \end{aligned}$$

$$
\begin{aligned}
V_{\text{out}} + \frac{1}{C_f R_{\text{in}}} \int V_{\text{in}} dt &= 0 \\
\Rightarrow V_{\text{out}} &= -\frac{1}{C_f R_{\text{in}}} \int V_{\text{in}}\, dt
\end{aligned}
$$

for obvious reasons, then, this circuit is also known as an "integrator". If we put the capacitor at $Z_{\text{in}}$ instead, we'd get a "differentiator". It is possible to also use non-linear cirucit elements (like diodes) to get logarithmic and exponential responses. From the theory on linear systems & Fourier Transforms, we can see that low pass filtering and integration are equivalent (since $\mathcal{F}\{\int f(t)dt\} = \frac{1}{j\omega}\hat{F}(\omega)$.

## 4.3 Charge Sensitive Amplifier

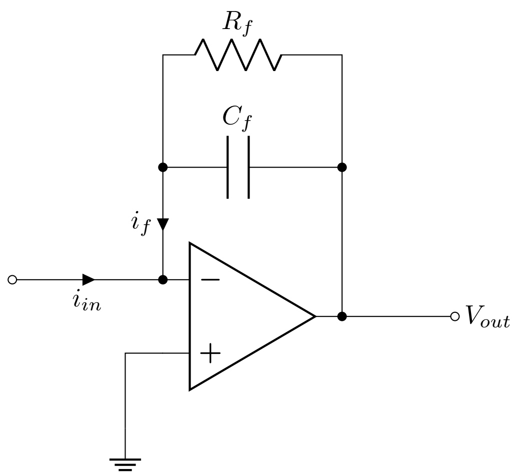



**Figure 8.** inverting charge sensitive amplifier / transconductance amplifier

Now we consider the case with a resistor and capacitor in parallel round the feedback (Figure 8). We also remove the input resistor $R_{\text{in}}$. Note now there cannot be a sensible concept of "voltage gain" (try the maths yourself if you like!). But what we can write down is the ratio between the output voltage and the input current $i_{\text{in}}$. The feedback impedance is now:

$$
\begin{aligned}
\frac{1}{Z_f} &= \frac{1}{R_f} + j\omega C_f \\
\Rightarrow Z_f &= \frac{R_f}{1 + j\omega R_f C_f}
\end{aligned}
$$

and so we get:

$$
\begin{aligned}
V_{\text{out}} - i_f Z_f &= V_- = -\frac{V_{\text{out}}}{A} \\
\Rightarrow V_{\text{out}}\left(1 + \frac{1}{A}\right) &= i_f Z_f \\
\Rightarrow V_{\text{out}}\left(1 + \frac{1}{A}\right) &= -i_{\text{in}} Z_f \\
\Rightarrow \frac{V_{\text{out}}}{i_{\text{in}}} &= -\frac{R_f}{1 + j\omega R_f C_f}\left(\frac{A(\omega)}{1 + A(\omega)}\right)
\end{aligned}
$$

If we just have a resistor round the feedback $R_f$, we call this a "transimpedance" amplifier, since it converts an input current into an output voltage along with some signal gain. If on the other hand we just have the capacitor (i.e. $R_f \gg 1$ like an open circuit), then we end up with

$$\Rightarrow \frac{V_{\text{out}}}{i_{\text{in}}} = -\frac{1}{j\omega C_f}\left(\frac{A(\omega)}{1+A(\omega)}\right)$$

so that we are back to just the integrator. Except now we're integrating the input current, rather than an input voltage. Such an amplifier is called a "transconductance" amplifier, since it gives an output voltage in response to an input **charge** rather than a current. This is the basis of the **charge sensitive amplifier** used in many particle (and astro) detector electronics frontends. It is worth considering the interaction with a semiconductor detector in this configuration. We model the detector as basically being like a pulsed current source with some capacitance. Thus, the two diagrams shown in Figure 9 are equivalent.

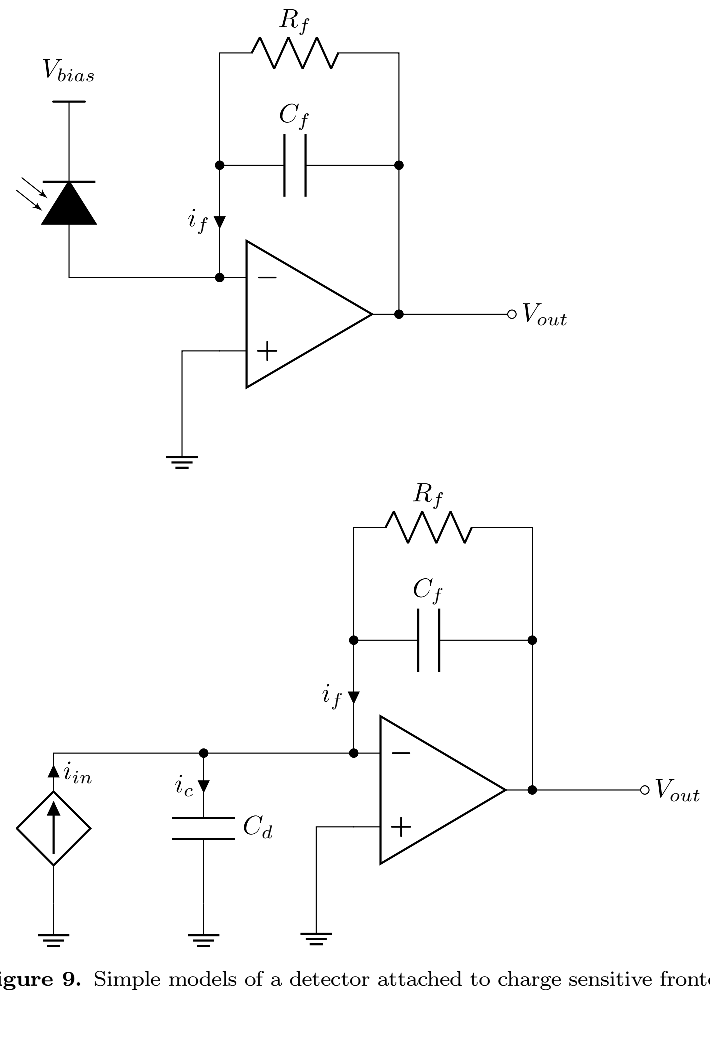



**Figure 9.** Simple models of a detector attached to charge sensitive frontend.

We can work out the changes that happen due to the detector's impedance (here, just capacitance) quite easily. From above we had that:

$$\Rightarrow V_{\text{out}}\left(1+\frac{1}{A}\right) = i_f Z_f$$

but we now have (from KCL) that $i_f + i_{\text{in}} = i_c$. We also know (simple $Q = \text{CV}$ or $V = \text{IZ}$) that $V_- = \frac{i_c}{j\omega C}$. Putting this together with $V_{\text{out}} = -AV_-$ we have:

$$
\begin{aligned}
\frac{i_c}{j\omega C_d} &= -\frac{V_{\text{out}}}{A} \\
i_f + i_{\text{in}} + \frac{V_{\text{out}} j\omega C_d}{A} &= 0 \\
\therefore V_{\text{out}}\left(1 + \frac{1}{A} + \frac{j\omega C_d Z_f}{A}\right) &= -i_{\text{in}} Z_f \\
\Rightarrow \frac{V_{\text{out}}}{i_{\text{in}}} &= -\frac{AZ_f}{1 + A + j\omega C_D Z_f}
\end{aligned}
$$

as can be seen, increasing the detector capacitance reduces the voltage to current gain at high frequencies, and introduces an extra pole into the transfer function. Let's also analyse this via "superposition". In a linear circuit, we may take any voltage (or current) node, and work out the contributions from each active current or voltage source to it in the circuit. We may then add these all together separately and the result is the same as if we analysed the entire circuit. **Of course this doesn't work for non-linear circuits! (which is all of them to some degree)**. Anyway, consider firstly replacing the input current source with a high impedance (so no current flows) - the left panel of Figure 10. We can regard the voltage at the inverting input $v_-$ as the answer to a simple voltage dividier problem:

$$
v_-^{(1)} = V_{\text{out}} \frac{Z_d}{Z_d + Z_f}
$$

(here we have just used $Z_d$ as shorthand for the capacitance, since it could also have other components). Instead, with the current source turned on but the output voltage source of the op amp turned off. Then we have a simple parallel impedance problem:

$$
\begin{aligned}
v_-^{(2)} &= i_{\text{in}} Z \\
\Rightarrow v_-^{(2)} &= i_{\text{in}} \frac{1}{\frac{1}{Z_f} + \frac{1}{Z_d}} \equiv i_{\text{in}} \frac{Z_f Z_d}{Z_f + Z_d}
\end{aligned}
$$

summing the two situations together, we get:

$$
\begin{aligned}
V_- &= \underbrace{V_{\text{out}} \frac{Z_d}{Z_d + Z_f}}_{v_-^{(1)}} + \underbrace{i_{\text{in}} \frac{Z_f Z_d}{Z_f + Z_d}}_{v_-^{(2)}} \\
\Rightarrow -\frac{V_{\text{out}}}{A} &= \underbrace{V_{\text{out}} \frac{Z_d}{Z_d + Z_f}}_{v_-^{(1)}} + \underbrace{i_{\text{in}} \frac{Z_f Z_d}{Z_f + Z_d}}_{v_-^{(2)}} \\
\therefore V_{\text{out}}\left(\frac{1}{A} - \beta\right) &= -i_{\text{in}} \beta Z_f \\
\Rightarrow \frac{V_{\text{out}}}{i_{\text{in}}} &= -Z_f \frac{A\beta}{1 - A\beta}, \text{with } \beta = \frac{Z_d}{Z_f + Z_d}
\end{aligned}
$$

this analysis is presented because it gives the "standard form" for stability analysis (see Section 4.9). $\beta$ is often called the "feedback factor" in amplifier analysis. However, don't mistake it for $\beta$ as used for the current gain of a transistor! Going back to the previous form, the full gain for the

RC feedback loop then is:

$$\frac{V_{\text{out}}}{i_{\text{in}}} = -\frac{AR_f}{(1+j\omega R_f C_f)(1+A)+j\omega C_d R_f}$$

from this you can deduce a few things. If we want the biggest signal possible (we usually do for noise reasons!) then you want the lowest possible $C_D$ (so make sure the detector's fully depleted!), the highest possible $R_f$, and the lowst possible $C_f$. You might ask here why have $R_f$ at all since we don't want transimpedance but charge sensitivity. Fair question, the answer to which is, if we didn't have it, the amplifier would soon reach the voltage rails and never reset! More clever reset implementations (e.g. using a transistor switch) are possible, which improve performance over this version but are more complicated. From the above equations we can also get the input impedance:

$$Z_{\text{in}} = \frac{V_-}{i_{\text{in}}} = \frac{Z_f}{(A+1)+j\omega Z_f C_d}$$

for best performance we want $Z_{\text{in}}$ as low as possible, this is consistent with lowest possible detector capacitance, and also highest possible op amp gain (as expected).

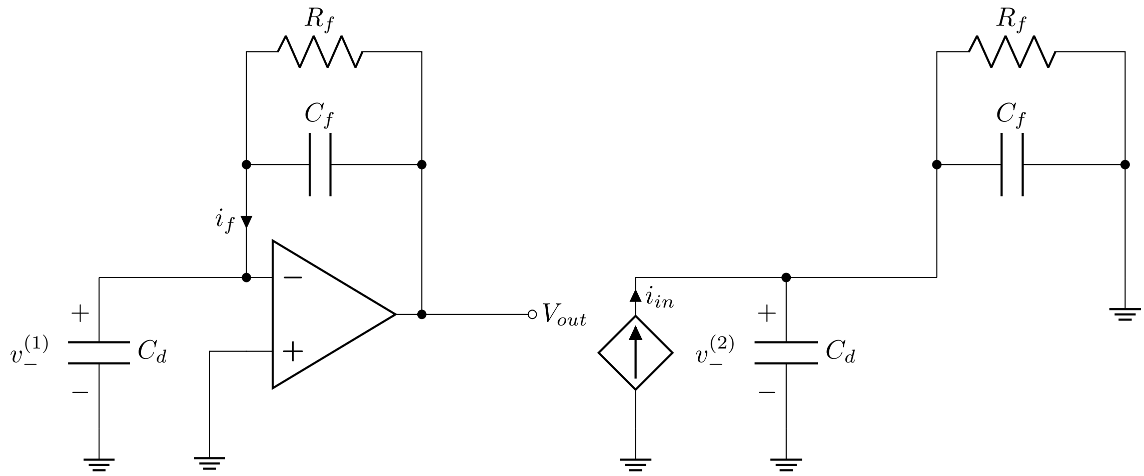



**Figure 10.** superposition analysis of the CSA. Firstly with input current source switched off (open circuit), and secondly with output voltage source switched off (closed circuit).

Before leaving CSA, let's do one more thing, explicitly calculate the charge to voltage gain. Noting that charge is the time integral of current, we can then w.l.o.g in the frequency domain say that $\hat{Q} = \frac{1}{j\omega}\hat{I}$ (remember that low pass filtering and integration are equivalent). So, start from gain equation:

$$\begin{aligned}
\Rightarrow \frac{V_{\text{out}}}{i_{\text{in}}} &= -\frac{AZ_f}{1+A+j\omega C_D Z_f} \\
\Rightarrow \frac{V_{\text{out}}}{Q_{\text{in}}} = \frac{V_{\text{out}} j\omega}{i_{\text{in}}} &= -\frac{j\omega A Z_f}{1+A+j\omega Z_f C_D} \\
\Rightarrow \frac{V_{\text{out}}}{Q_{\text{in}}} &= \frac{AZ_f}{j\omega(1+A)-\omega^2 Z_f C_D}
\end{aligned}$$

again, we can see that the charge gain is affected not only by the detector capacitance and the feedback impedance $Z_f$, but also by the finite open loop gain of the op-amp.

## 4.4 2nd order Bandpass filter ("CRRC shaper")

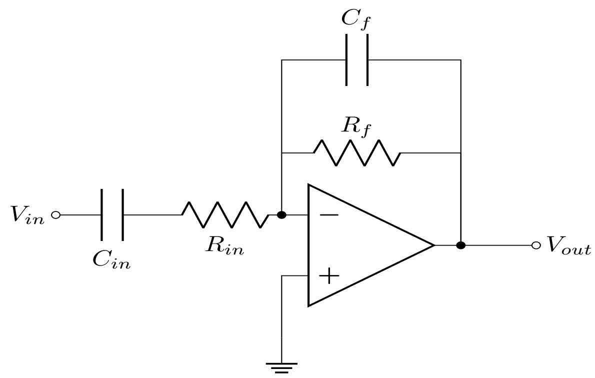



**Figure 11.** 2nd order shaping amplifier "CRRC"

Final examples are more complicated filters. Firstly, sticking with inverting configuration. What happens if we make the arrangement in Figure 11 We now have:

$$\begin{aligned} Z_{\text{in}} &= R_{\text{in}} + \frac{1}{j\omega C_{\text{in}}} \\ Z_f &= \frac{R_f}{1 + j\omega R_f C_f} \end{aligned}$$

and so, the voltage gain will be:

$$\begin{aligned} A_V &= -\frac{Z_f}{Z_{\text{in}}} \\ \Rightarrow A_V &= -\frac{j\omega C_{\text{in}} R_f}{(1 + j\omega C_{\text{in}} R_{\text{in}})(1 + j\omega R_f C_f)} \end{aligned}$$

The resulting transfer function is quite complicated (and 2nd order, it goes up to $\omega^2$ in the denominator). It can be used to make low-pass, high-pass, band-pass or band-cut filters, with steeper slopes than the 1st order integrator/differentiator. It is mentioned here because it is a common fixture in particle and nuclear physics readouts, where we usually call it the "CR-RC" shaper.

## 4.5 Non-Inverting Amplifier Configuration

Just for completeness we mention here the non-inverting standard configuration.

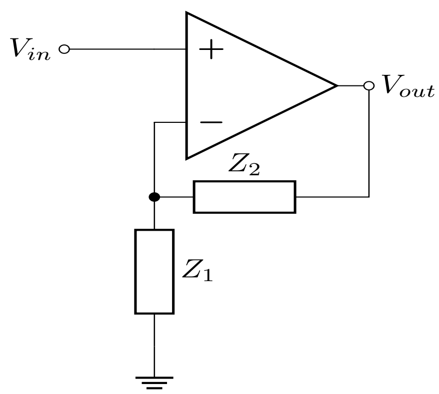



**Figure 12.** non-inverting op amp configuration

we will just do the quickest derivation, using both golden rules and KCL:

$$\begin{aligned}
V_{\text{in}} &= V_+ = V_- \quad (\text{GR2}: V_+ = V_-) \\
V_- &= V_{\text{out}} \frac{Z_1}{Z_1 + Z_2} \quad (\text{voltage divider}) \\
\therefore V_{\text{in}} &= V_{\text{out}} \frac{Z_1}{Z_1 + Z_2} \\
\Rightarrow \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{Z_1 + Z_2}{Z_1} \equiv 1 + \frac{Z_2}{Z_1}
\end{aligned}$$

note that this circuit has a minimum absolute gain of 1 (whereas the inverting configuration can go lower in principle, but careful of stability - see Section 4.9).

## 4.6 Op amp linear current source

A powerful use of the op amp is as a "linearizer" or "control loop" for other circuitry. We are not going into transistor theory today but consider the circuit in Figure 13. Both the diode $D_1$ and the MOSFET $Q_1$ are highly non-linear devices whose voltage and current depend on each other in complicated ways. The only fact that we do need to know here, is that the MOSFET doesn't take any current through its gate contact (the one connected to the op amp output). Therefore, the same current flows through the diode and the MOSFET.

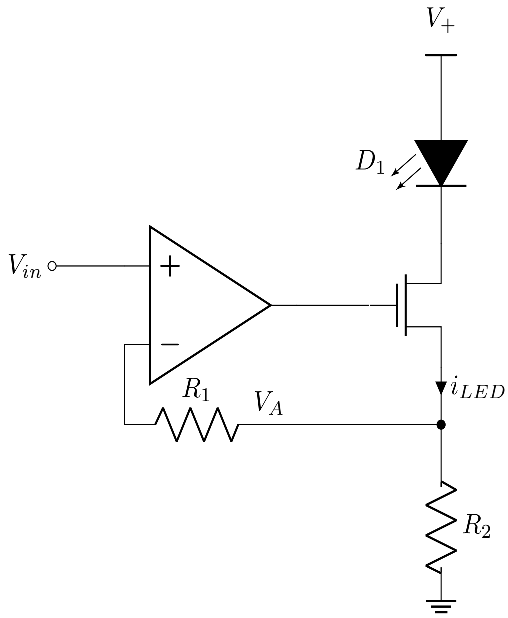



**Figure 13.** linear current source with an op amp

assuming the inputs draw no current (as usual), then we can write using KCL:

$$\begin{aligned}
V_A &= i_{\text{LED}} R_2 \\
V_A &= \frac{R_2}{R_1 + R_2} V_- \\
\therefore V_A &= \frac{R_2}{R_1 + R_2} V_{\text{in}} \quad (V_+ = V_- = V_{\text{in}}, \text{via golden rule}) \\
\Rightarrow i_{\text{LED}} R_2 &= \frac{R_2}{R_1 + R_2} V_{\text{in}} \\
\Rightarrow i_{\text{LED}} &= \frac{V_{\text{in}}}{R_1 + R_2}
\end{aligned}$$

in the full analysis there are some extra details, but see what happened there. Without knowing the transfer functions of the transistor or the LED, we have used an op-amp to linearly put a current through the LED that depends on our input voltage. The op amp will "magically" adjust its output to ensure that this remains true (at least, within constraints of bandwidth, power rails, etc etc).

## 4.7 Gain-Bandwidth Product

Recall that a fully compensated op-amp has an open loop gain of:

$$A(\omega) = \frac{A_0}{1+j\frac{\omega}{\omega_c}}$$

Note from previous section that inverting gain of a purely resistive feedback configuration is:

$$A_V = \frac{-R_f}{R_{\text{in}}+\frac{1}{A}(R_{\text{in}}+R_f)} \approx -\frac{R_f}{R_{\text{in}}} \text{ for } A \gg 1$$

$A_V$ is called "closed loop gain". Bear in mind that open loop gain always falls with frequency. Consider:

$$\begin{aligned}
A_V(\omega) &= -\frac{R_f}{R_{\text{in}}}\frac{1}{1+\frac{1}{A}\frac{R_{\text{in}}+R_f}{R_{\text{in}}}} \\
\Rightarrow A_V(\omega) &= -\frac{R_f}{R_{\text{in}}}\frac{1}{1+\frac{1}{A\beta}} \equiv -\frac{R_f}{R_{\text{in}}}\frac{A\beta}{1+A\beta} \\
\Rightarrow A_V(\omega) &= -\frac{R_f}{R_{\text{in}}}\left(\frac{\frac{A_0}{1+j\frac{\omega}{\omega_c}}\beta}{1+\frac{A_0}{1+j\frac{\omega}{\omega_c}}\beta}\right) \equiv -\frac{R_f}{R_{\text{in}}}\left(\frac{A_0\beta\left(1+A_0\beta+j\frac{\omega}{\omega_c}\right)}{(1+A_0\beta)^2+\left(\frac{\omega}{\omega_c}\right)^2}\right) \\
|A_V(\omega)| &= -\frac{R_f}{R_{\text{in}}}\frac{A_0\beta}{\sqrt{(1+A_0\beta)^2+\left(\frac{\omega}{\omega_c}\right)^2}} \\
|A_V(\omega)| &\approx -\frac{R_f}{R_{\text{in}}}\frac{A_0\omega_c\beta}{\sqrt{(A_0\omega_c\beta)^2+\omega^2}}
\end{aligned}$$

note that approximation is much better since $A_0$ is a constant, and for a good op amp it actually is $\gg 1$ (typically in the millions!). Although in a general configuration $\beta$ can depend on frequency, note that for the resistive combination it is necessarily $|\beta| < 1$, and this is generally true for complex combinations as well except where the denominator of $\beta$ goes to zero and causes the whole expression to blow up. The open loop gain then can be considered the "limiting gain" of an amplifier configuration. **It is a fairly general rule that as we increase bandwidth $\boldsymbol{\omega}$, we decrease gain.** Note further:

$$\begin{aligned}
A(\omega) &= \frac{A_0}{1+j\frac{\omega}{\omega_c}} = \frac{A_0\left(1-j\frac{\omega}{\omega_c}\right)}{1+\left(\frac{\omega}{\omega_c}\right)^2} \\
|A(\omega)|^2 &= \frac{A_0^2}{1+\left(\frac{\omega}{\omega_c}\right)^2} \\
\Rightarrow |A(\omega)|^2\left(1+\left(\frac{\omega}{\omega_c}\right)^2\right) &= A_0^2 \\
\Rightarrow |A(\omega)|^2\omega^2 &\approx A_0^2\omega_c^2 \\
\Rightarrow |A(\omega)|\omega &\approx A_0\omega_c
\end{aligned}$$

$A_0\omega_c$ is (roughly) a **constant** for a particular amplifier design. From the above we can see that, since all configurations are ultimately limited by open loop gain, it is the case that the gain multiplied by bandwidth of an op-amp circuit is approximately constant, and hence the quantity

$A_0\omega_c$ is known as the **gain-bandwidth product (GBP)**. In general, higher bandwidth means less stability (see Section 4.9), and as such we have the somewhat un-intuitive result that **op amps are more stable at higher closed loop gain**. Indeed, there are some (mainly specialist high speed) op-amps that are designed to not be able to operate at low gains. An op amp that can operate at all possible non-inverting gains is advertised as “unity gain stable”.

## 4.8 Comparators (& relaxation oscillator)

All amplifier configurations we looked at have negative feedback, otherwise the high value of $A$ means that the output goes off to infinity. What if we want that to happen? Then we have a **comparator**. An ideal comparator is the identical component to an ideal op-amp. In the real world, we can make various design choices differently when we want the thing to efficiently swing from rail to rail rather than do linear amplification. Many comparators are **open collector output** to facilitate interfacing to whatever output level you want. Don’t worry about that for now.

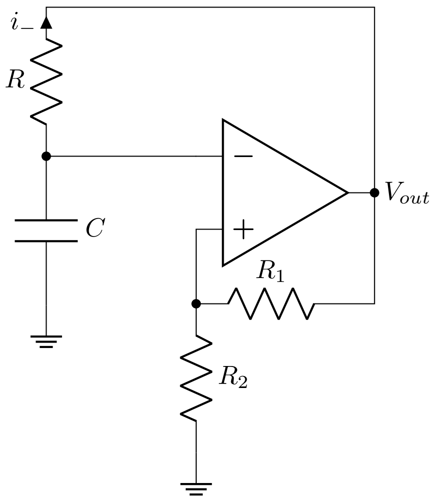



**Figure 14.** Relaxation Oscillator circuit using a comparator

Consider the circuit in Figure 14. You can solve it exactly quite easily (but no golden rule #2, since this is an oscillator!). First work out the voltage at the inverting terminal. Assuming inputs draw no current, the branch round the inverting side is just a potential divider.

$$\begin{aligned}
V_- &= V_{\text{out}}\frac{Z_C}{Z_C+R}\\
V_+ &= V_{\text{out}}\frac{R_2}{R_1+R_2}\\
V_{\text{out}} &= A(V_+ - V_-)\\
\Rightarrow 1 &= A\left(\frac{R_2}{R_1+R_2} - \frac{1}{1+j\omega \text{RC}}\right)
\end{aligned}$$

what does this mean? Note we derived a condition just between frequency and open loop gain here. This result can be interpreted using Barkhausen Stability (see Section 4.9) but it’s probably easier here to derive it without using complex impedances instead:

$$\begin{aligned}
V_- - \frac{1}{C}\int i_- dt &= 0\\
V_- + \frac{1}{RC}\int (V_- - V_{\text{out}})dt &= 0\\
V_{\text{out}} &= A\left(\frac{R_2}{R_1+R_2}V_{\text{out}} - V_-\right)\\
\therefore \frac{1}{RC}\int V_{\text{out}}\left(-\frac{1}{A} + \frac{R_2}{R_1+R_2} - 1\right)dt &= \frac{V_{\text{out}}}{A} - \frac{R_2}{R_1+R_2}V_{\text{out}}
\end{aligned}$$

if you try solving that differential equation you'll get oscillations. A simpler way to see it intuitively is to see that when $V_{\text{out}}$ goes high (when $V_+ > V_-$), then the capacitor $C$ will charge up with time constant $RC$. When the voltage $V_-$ gets high enough ($V_- > V_+$), the output voltage will flip to negative (see op amp transfer function), and then the capacitor starts discharging again. The output therefore basically looks like a square wave. This is the simplest **non-linear oscillator** circuit that you can build just about.

It is often necessary to add **hysteresis** to a comparator circuit, because otherwise it can flip up and down rapidly sensitive to environmental noise.

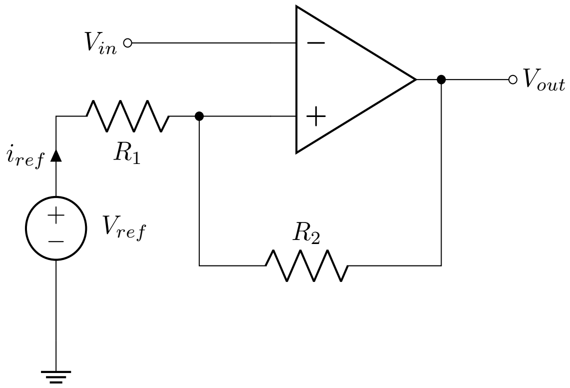



**Figure 15.** comparator with hysteresis

looking at the circuit in Figure 15, applying Kirchoff Current Law we get:

$$
\begin{aligned}
V_{\text{ref}} - i_{\text{ref}} R_1 &= V_+ \\
V_{\text{ref}} - i_{\text{ref}}(R_1 + R_2) &= V_{\text{out}} \\
V_{\text{ref}} + \frac{V_+ - V_{\text{ref}}}{R_1}(R_1 + R_2) &= V_{\text{out}} \\
\Rightarrow V_{\text{out}} + V_{\text{ref}}\left(\frac{R_1 + R_2}{R_1} - 1\right) &= \left(\frac{V_{\text{out}}}{A} + V_{\text{in}}\right)\frac{R_1 + R_2}{R_1}
\end{aligned}
$$

what we are doing here is slightly offsetting the reference voltage into the non-inverting input. It depends exactly on the specific comparator and its input offset voltage how much hysteresis we get, but the function of $R_1$ and $R_2$ here is to allow the "upswing" voltage to be different to the "downswing" voltage. This circuit is extremely common, but clearly a full analysis isn't possible assuming "perfect" op amps and comparators. **Always consult the datasheet!** Using a comparator is the simplest way to convert analog pulses from your amplifiers into **timing pulses**, and we can enter the digital domain. Note this is **not** a good analog to digital converter, since we don't have any vertical resolution at all. Often in particle physics though, we use a circuit like this, and, knowing the pulse shape, we can convert a timing measurement (length of the pulse) into a guess of how high it was. You also need to care about **timewalk** in that case. That is beyond the scope of these notes but ask if interested.

## 4.9 Barkhausen Stability

As mentioned above, in general the higher the bandwidth of the circuit, the lower the gain. But signals of all frequencies get injected into our circuits (e.g. via RF interference or other noise). For **any** negative feedback amplifier, there is a mathematical theorem called the **Barkhausen Stability Criterion** which specifies conditions that are **sufficient** for a circuit to **not oscillate**. However, many people erroneously use it as a condition for a circuit to oscillate, because it often works (Interestingly - there is no known simple criteria that guarantees a circuit **will** oscillate! But trust me op amps mostly will if you don't obey Barkhausen). The technical statement is:

**Theorem 1.** *A circuit may sustain steady state oscillations if a pair of complex poles exist in its transfer function on the imaginary axis.*

that statement is mostly useless for normal people. A better one is probably this, though it's equivalent:

**Theorem 2.** *For an amplifier which has an open loop gain of $A$ and a feedback fraction $\beta$, it is possible to oscillate at some frequency $\omega_{\text{osc}}$ iff at $\omega_{\text{osc}}$ both:*

$$\begin{aligned}|\beta A| &= 1\\ \angle\beta A &= 2\pi n\end{aligned}$$

in practice, it's even simpler: The loop gain goes down as frequency goes up. The phase lag of the output with respect to the input also increases. If there is **any** frequency where the loop gain is unity and the phase lag is more than $\pi$ then we can get oscillations and instability. Why just $\pi$ and not $2\pi$? Well, in the case of the op amp, because it's differential we get a phase lag of $\pi$ "for free", so the so-called **phase margin** we have left is only $\pi$. Consider it this way: the transfer function of an op amp is:

$$\begin{aligned}V_{\text{out}} &= A(\omega)(V_+ - V_-)\\ \Rightarrow V_{\text{out}} &= A(\omega)(V_+ - f(V_{\text{out}}))\end{aligned}$$

(we've connected the inverting terminal by some feedback loop). Note that if the function $f$ introduces a phase lag of $\pi$, it would be equivalent to write:

$$V_{\text{out}} = A(\omega)(V_+ + |f(V_{\text{out}})|)$$

(i.e. we turn negative feedback into **positive feedback**. Hence, the output will go to infinity. The best way to study and check for Barkhausen stability is to look at the **Bode plot**, and make sure you have some phase margin left when the value of $A(\omega)$ drops below unity.

## 4.10 Miller Effect

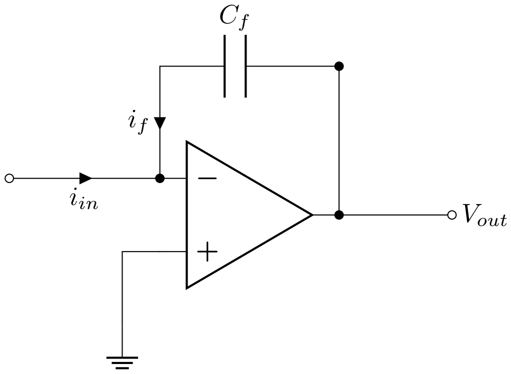



**Figure 16.** Demonstration of the Miller Effect

Consider the arrangement in Figure 16. It is a simple transconductance amplifier like we had above. What is the effective input impedance (i.e, what does the impedance of this circuit "look like" to the input circuitry?) Using KCL (yet again):

$$\begin{aligned}
V_{\text{in}} = V_- &= V_{\text{out}} - \frac{i_f}{j\omega C_f} \\
\Rightarrow V_{\text{in}} &= -V_{\text{in}} A(\omega) - \frac{i_f}{j\omega C_f} \\
\Rightarrow V_{\text{in}} &= \frac{i_{\text{in}}}{j\omega C_f(1 + A(\omega))} \\
\Rightarrow Z_{\text{in}} = \frac{V_{\text{in}}}{i_{\text{in}}} &= \frac{1}{j\omega C_f(1 + A(\omega))}
\end{aligned}$$

not that much of a surprise. But notice: this impedance looks just like a capacitor, but one that has a capacitance of $C_f(1+A)$, rather than just $C_f$. It is a general property of negative feedback systems that capacitances in the feedback loop are made to "look like" **much bigger** capacitances from the input impedance perspective. This is called the **Miller Effect**, and the apparent capacitance $C_f(1+A)$ is called the **Miller Capacitance**. For a charge sensitive amplifier, this is usually fine. But in many cases, consider that you might not want to effectively put a huge capacitor straight on your input! It may well affect the input circuitry and affect its bandwidth. Indeed, in more sophisticated amplifier designs, we use things like the "cascode" arrangement to negate the Miller effect.

## 4.11 Noise Sources and Amplifier Noise Gain

There are several sources of noise in an electronic system, briefly the important ones:

- **Johnson Noise** - thermal noise that occurs in all conductors. It provides an rms noise voltage of $v_{\text{rms}} = \sqrt{4k_BTRB}$ where $R$ is resistance, $T$ is temperature in Kelvin, $k_B$ is Boltzmann's constant, $B$ is bandwidth. The 4 is difficult to derive (but possible, look it up if you want). Related is so-called "kTC" noise, which is Johnson noise across a capacitor, where it produces an rms noise charge of $q_{\text{rms}} = \sqrt{k_BTC}$ on a capacitor $C$. The main cause of noise in many low-noise readout systems (particularly in astro, but also in particle sometimes)

- **shot noise** - white / gaussian noise arising as a result of the quantisation of charge. Characterised by the fact that electron arrival is a Poisson process, and hence its mean is equal to its variance. You need a very low noise system in general to be limited by this, but it is the ultimate noise limit in all detector systems (that don't use some kind of quantum effects to reduce the fluctuations below shot-noise statistics)

- **microphonic noise** - noise arising as a result of mechanical or acoustic vibrations changing circuit parameters, e.g. a capacitor with parallel plates vibrating and that modulates the capacitance. Rare nowadays with SMD components but worth knowing about

- **RTS / flicker/ "1/f" noise** - Noise which has a correlated power spectrum (usually proportional to $\frac{1}{\omega}^{\alpha}$, where $\alpha$ may actually be 1, but usually is >1. Caused by various mechanisms, e.g. channel modulation in transistors. We do not fully understand the origin of this in all cases. Your amplifiers are made of transistors and will have this

- **EMI** and **RFI** - coupling of your circuit to external electromagnetic emissions, which may be of the form of free electric fields (rare in household items, common in particle physics detectors!), free magnetic fields (are there any motors nearby?), or mobile phones (look out for 868MHz, 1.8GHz, 2.4GHz/5GHz for the GSM, 4G/LTE/5G and WiFi/bluetooth bands specifically. Likely you need to get rid of this by filtering, shielding, or clever circuit design.

In general, the noise spectrum of basically any amplifier system will be of the form:

$$P = K\left(1+\left(\frac{1}{\omega}\right)^{\alpha}\right)$$

which has a white noise term (Johnson noise and shot noise) + a 1/f term that dominates at low frequencies. The frequency where the dominant noise changes from 1/f to white noise is generally called the "corner frequency".

In amplifier systems, especially op amps there are a couple of specific considerations. Noise signals are by their nature random in time, and so they do not have well defined phase relationships with other signals. The result is that the signals add up in **power**, not in voltage. In an op-amp, we specify the noise properties as the **input referred noise**. This means, we imagine a fictitious voltage source **directly** at one of the inputs, and work out what the consequences are.

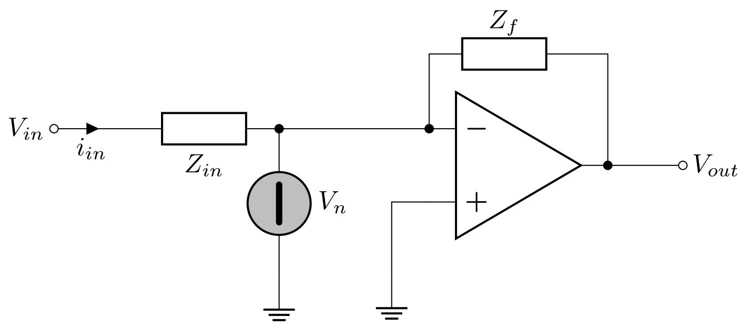



**Figure 17.** amplifier with voltage noise source at the inverting input

Looking at Figure 17, although we worked out the closed loop signal gain above to be:

$$A_V \approx -\frac{Z_f}{Z_{\text{in}}}$$

we can separately define the "noise gain" as the output signal due to just the fictitious noise source at the input. In this process, we regard all other input voltages as set to ground (c.f. the superposition principle above):

$$\begin{aligned}
V_{\text{out}} - i_{\text{in}} Z_f &= V_- = V_n \\
V_n - i_{\text{in}} Z_{\text{in}} &= 0 \quad (V_{\text{in}}\,\text{is ground for this purpose}) \\
\Rightarrow V_{\text{out}} - \frac{V_n}{Z_{\text{in}}} Z_f &= V_n \\
\Rightarrow V_{\text{out}} &= V_n\left(1+\frac{Z_f}{Z_{\text{in}}}\right) \\
\Rightarrow \frac{V_{\text{out}}}{V_n} \equiv A_{\text{noise}} &= 1+\frac{Z_f}{Z_{\text{in}}}
\end{aligned}$$

the keen among you will recognise this as the gain of a non-inverting configuration. We would have got the same result if we placed the noise source at the non-inverting input. The noise gain of the amplifier is always non-inverting, as a consequence of the above fact that noise sources do not have coherent phase relationships. Using the input-referred noise of the op amp specified in the data sheet, you can then use the noise gain equation to work out how much noise you expect at the output.

The **most important thing** to know about noise gain in op amps, is that since it's always positive and thus has a phase lag of 0 by default, it is **usually** the thing which limits the Barkhausen stability, rather than the signal gain. You will note also that by carefully selecting $Z_f$ and $Z_{\text{in}}$ it's possible to manipulate the noise gain **separately** from the signal gain. It's also possible to add extra components that will appear in the noise gain equation but **not** in the signal gain. This is particularly useful to improve the stability of the amplifier (by increasing the noise gain). You will end up buying more phase margin at high frequencies, and hence better stability, but at the expense of higher output noise. This is called by engineers "forcing the noise gain". It is an advanced topic that I mention here only because you will come across it pretty soon in detector frontends.

## 4.12 The Shannon-Nyquist Sampling Theorem

I have not had time to write this section but I have to mention it because it's possibly the most important and profound thing to know in all of detector physics...

For now - the sample rate $f_s$ required to accurately reconstruct an analog signal from a digital signal is $f_s = 2f_{\text{sig}}$, where $f_{\text{sig}}$ is the **highest** frequency that is present in the input signal. Otherwise you get **aliasing**. Aliasing is in the general case impossible to detect, and impossible to correct. In practice, due to noise and quantisation of voltages, you need $f_s > 2f_{\text{sig}}$. For example, CD digital audio is sampled at a rate of 44.1kHz. The highest sound that a human being can hear is about 20kHz.

## 4.13 Other important "real world" parameters

In a real dessign, you may have to care also about these (look at the TL081, OPA851 and LM7171 datasheets to compare these!)

- **input bias current** - the amount of current that actually does go in the input (it's not zero!)
- **input & output offset voltage** - the actual differences between $V_+$ and $V_-$ that "counts as" zero, and induces a DC offset at the output for a zero input.
- **Slew Rate** - this is really complicated. The maximum rate at which the output voltage can change for large signals. Note this can contradict the available bandwidth (and often high power or high voltage amplifier circuits are limited by slew rate not bandwidth)
- **Total Harmonic Distortion** - put in a pure sine wave, you get out a small amount of higher harmonics as well due to non-linearity. How much of that is there?
- **Power Supply Rejection Ratio** - If you inject a frequency into the power supply line, how much of it appears at the output?
- **Common Mode Rejection Ratio** - an ideal op-amp does $V_{\text{out}} = A(V_+ - V_-)$ (purely differential). A real one does $V_{\text{out}} = A(V_+ - V_-) + \frac{A_{\text{CM}}}{2}(V_+ + V_-)$. i.e. if you have a common mode signal at both inputs some amount of it will get erroneously amplified. CMRR is the ratio between $A$ and $A_{\text{CM}}$. Can be a real problem for both DC accuracy and noise performance. Inverting configurations all but eliminate common mode signals so they're usually "better" to use in this sense
- **Rail to Rail (R2R) capability** - some op-amps cannot produce signals at their outputs all the way to the power supply rails. Some can. Check, if this is important to your circuit!
- **Input referred noise** - usually specified as some amount of integrated 1/f + some amount of white noise per $\sqrt{\omega}$

and there are many others, but we are just covering the basics here!