Physics Beyond the Standard Model

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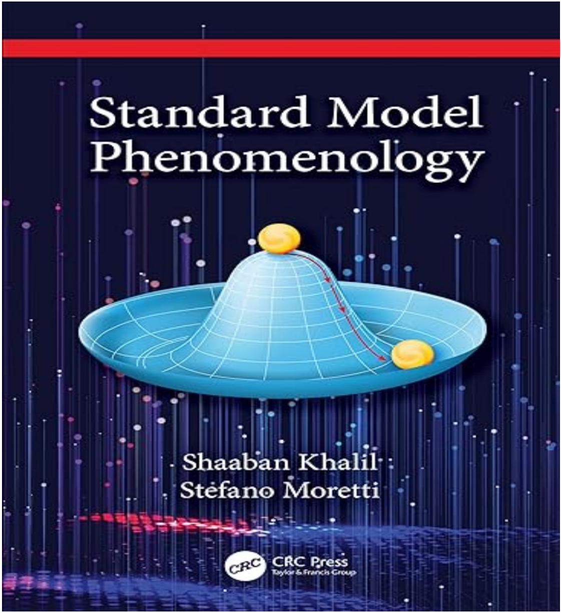
Course Outline

1. SM Overview
2. Evidence for BSM
3. Directions of BSM
4. Simple SM Extensions: B-L Extension of the SMI
5. $SU(5)$ Grand Unified Field Theory
6. Supersymmetry
7. Extra Dimensions

SM Overview

- **Standard Model is defined by**
  - **4-dimension QFT (Invariant under Poincare group).**
  - **Symmetry: Local** $SU(3)_C \times SU(2)_L \times U(1)_Y$.
  - **Particle content (Point particles):**
  - **3 fermion (quark and Lepton) Generations.**
  - **No Right-handed neutrinos: Massless Neutrinos.**
- **Symmetry breaking: one Higgs doublet.**
- **No candidate for Dark Matter.**
- **SM does not include gravity.**


Standard Model
Phenomenology
Shaaban Khalil
Stefano Moretti
CRC Press
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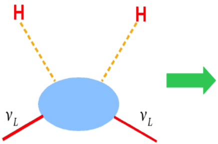
1- Neutrino Mass:

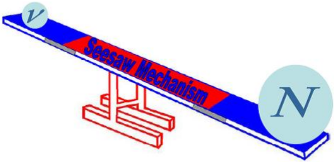
- **In the SM, quarks and electrons acquire masses through Yukawa couplings :** $\mathcal{L}_{Yuk} \sim \bar{Q}_L \phi u_R$.
- **Neutrinos remain massless because there are no RH ν in the SM.**
- **However, it has proven experimentally (from neurino oscillations) that** $m_\nu \neq 0$ **.**
- **Needs a mechanism to give ν masses...**

If lepton number is violated



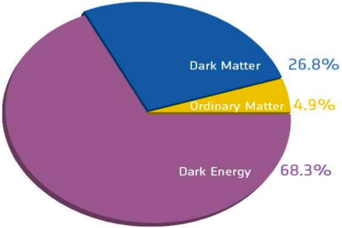
$LH\,LH / \mathcal{M}$

Dim-5 operator, Weinberg (80)



2- Dark Matter:

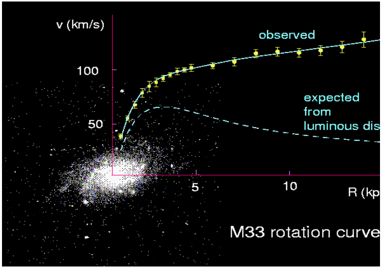
- **Most astronomers, cosmologists and particle physicists are convinced that 90% of the mass of the Universe is due to some non-luminous matter, called 'Dark Matter/Energy'.**



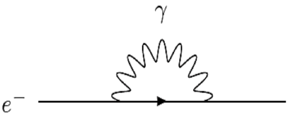
- **The velocity of rotating objects**

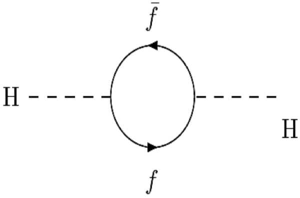
$$v(r) = \sqrt{\frac{G\ M(r)}{r}}$$

- **The observation of 1000 spiral galaxies showed that away from the centre of galaxies the rotation velocities do not drop off with distance.**
- **The explanation for these is to assume that disk galaxies are immersed in extended DM halos.**
- **Dark Matter must be non-baryonic. No such candidate in the Standard Model**



3- Higgs Mass Hierarchy:





$$\delta m_e = 2\frac{\alpha_{em}}{\pi} m_e \log\frac{\Lambda}{m_e} + \ \ldots$$

$$\delta m_H^2(f) = -2N_f\frac{|\lambda_f|^2}{16\pi^2}[\Lambda^2 - 2m_f^2 \ln\frac{\Lambda}{m_f} \ + \ldots]$$

$m_e \to 0$, Chiral Symmetry

m_H^2 no symmetry, Quadratic div.!!

$$\Lambda = 10^{19}\ \text{GeV}$$

$$\delta m_e \ = \ 0.24\ m_e$$

$$\delta m_H^2 \ \simeq \ 10^{30}\ \text{GeV}$$

In addition, there are a number of questions we hope will be answered:

- **Electroweak symmetry breaking, which is not explained within the SM.**
- **Why is the symmetry group is** $SU(3) \times SU(2) \times U(1)$?
- **Can forces be unified?**
- **Why are there three families of quarks and leptons?**
- **Why do the quarks and leptons have the masses they do?**
- **Can we have a quantum theory of gravity?**
- **Why is the cosmological constant much smaller than simple estimates would suggest?**

Directions of BSM

- **Extension of gauge symmetry.**
- **Extension of Higgs Sector.**
- **Extension of Matter Content.**
- **Extension with Flavor Symmetry.**
- **Extension of Space-time dimenstions (Extra-dimensions).**
- **Extension of Lorentz Symmetry (Supersymmetry).**
- **Incorporate Gravity (Supergravity).**
- **One dimension object (Superstring).**

Simple SM Extensions: Incorporating B-L into the SM

- **The $B-L$ extension of the SM is based on the gauge group**

$$SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \qquad S.K(2007)$$

- **Under $U(1)_{B-L}$, we require the transformation**

$$\psi_L \to e^{iY_{B-L}\theta(x)}\psi_L, \qquad \psi_R \to e^{iY_{B-L}\theta(x)}\psi_R.$$

In this model:

1. **Three right-handed neutrinos, N_R^i, $i = 1, 2, 3$; with $B-L$ charge $= -1$.**
2. **An extra gauge boson corresponding to $B-L$ gauge symmetry, Z'.**
3. **An extra SM singlet scalar, χ with $B-L$ charge $= +2$, are introduced.**

Particle	ℓ_L	e_R	N_R	ϕ	χ
Y_{B-L}	-1	-1	-1	**0**	$+2$

- **Lagrangian: fermion and gauge kinetic terms**

$$\mathcal{L}_{B-L} = -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}C_{\mu\nu}C^{\mu\nu} + i\,\bar{l}D_\mu\gamma^\mu l + i\,\bar{e}_R D_\mu\gamma^\mu e_R + i\,\bar{\nu}_R D_\mu\gamma^\mu \nu_R$$

- $C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ **is the field strength of the $U(1)_{B-L}$.**

$U(1)_{B-L}$ Symmetry Breaking

- These new particles have significant impact on the SM phenomenology & interesting LHC signatures.
- $U(1)_{B-L}$ gauge symmetry can be spontaneously broken by a SM singlet complex scalar field χ:

$$|\langle\chi\rangle| = \frac{v'}{\sqrt{2}}$$

- $SU(2)_L \times U(1)_Y$ gauge symmetry is broken by a complex $SU(2)$ doublet of scalar field ϕ:

$$|\langle\phi\rangle| = \frac{v}{\sqrt{2}}$$

- The most general Higgs potential:

$$V(\phi,\chi) = m_1^2 \phi^\dagger \phi + m_2^2 \chi^\dagger \chi + \lambda_1 (\phi^\dagger \phi)^2 + \lambda_2 (\chi^\dagger \chi)^2 + \lambda_3 (\chi^\dagger \chi)(\phi^\dagger \phi)$$

- For $V(\phi,\chi)$ bounded from below, we require: $\lambda_3 > -2\sqrt{\lambda_1 \lambda_2}$ and $\lambda_1, \lambda_2 \geq 0$.
- For non-zero local minimum, we require $\lambda_3^2 < 4\lambda_1\lambda_2$.
- After the B-L gauge symmetry breaking, the gauge field C_μ acquires mass: $M_{Z'}^2 = 4g"v'^2$.
- Strongest Limit on $M_{Z'}/g"$ comes from LEP II: $M_{Z'}/g" \simeq \mathcal{O}(\mathrm{TeV}), g" \sim \mathcal{O}1 \Longrightarrow v' > \mathcal{O}(\mathrm{TeV})$

Neutrino masses and mixing in BLSM

- The lepton Yukawa interaction is given by

$$\mathcal{L}_{\rm Yukawa} = \lambda_e \bar{l}\phi e_R + \lambda_\nu \bar{l}\tilde{\phi}\nu_R + \frac{1}{2}\lambda_{\nu_R}\bar{\nu^c}_R \chi \nu_R + h.c.$$

- After $U(1)_{B-L}$ symmetry breaking, the Yukawa interaction: $\lambda_{\nu_R}\chi\nu_R\nu_R$ leads to right handed neutrino mass: $M_R = \frac{1}{\sqrt{2}}\lambda_{\nu_R} v'$.

- Also the electroweak symmetry breaking implies Dirac neutrino mass term : $m_D = \frac{1}{\sqrt{2}}\lambda_\nu v$.

- Therefore, the mass matrix of the left and right-handed neutrino is given by

$$\left(\begin{array}{cc} 0 & m_D \\ m_D & M_R \end{array} \right).$$

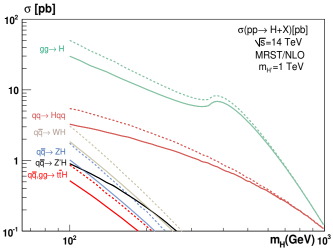
- Since $M_R \gg m_D$, the light and heavy neutrino masses are:

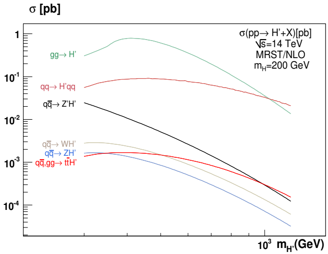
$$\begin{aligned} m_{\nu_L} &= -m_D M_R^{-1} m_D^T, \\ m_{\nu_H} &= M_R. \end{aligned}$$

- Thus $m_{\nu_L} \sim$ eV, with a TeV scale M_R, if $m_D \simeq 10^{-4}$ GeV, ı.e $\lambda_\nu \simeq \lambda_e$.

Higgs Bosons in BLSM

- BLSM Higgs sector consists of one complex $SU(2)_L$ doublet and one complex scalar singlet.
  - Six scalar degrees of freedom
  - Four are eaten by $Z', Z, W^{\pm}$ after symmetry breaking
  - Two physical degrees of freedom: ϕ, χ
- Neutral Higgs boson mass matrix: $\frac{1}{2}M^2(\phi,\chi) = \begin{pmatrix} \lambda_1 v^2 & \frac{\lambda_3}{2} v v' \\ \frac{\lambda_3}{2} v v' & \lambda_2 v'^2 \end{pmatrix}.$
- The masses of eigenstates: H and H' are $\quad m^2_{H,H'} = \lambda_1 v^2 + \lambda_2 v'^2 \mp \sqrt{(\lambda_1 v^2 - \lambda_2 v'^2)^2 + \lambda_3^2 v^2 v'^2}.$
- Mixing is controlled by $\lambda_3 = 0$: $\quad \lambda_3 = 0 \Longrightarrow m_\phi = \sqrt{2\lambda_1} v, \quad m_\chi = \sqrt{2\lambda_2} v'$





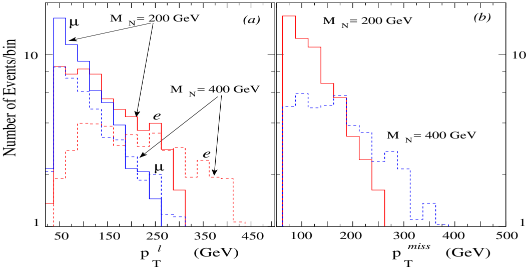
- The cross sections of light and heavy Higgs production as function of $m_H/m_{H'}$, for $m_{Z'} = 600$ GeV.

- The interactions of the Z' boson with the SM fermions are described by $\mathcal{L}^{Z'}_{\rm int} = \sum_f Y^f_{B-L}\, g''\, Z'_\mu\, f\gamma^\mu f$.
- The decay widths of $Z' \to f\bar{f}$ are then given by

$$\Gamma(Z' \to l^+l^-) \approx \frac{(g'' Y^l_{B-L})^2}{24\pi} m_{Z'} \quad \Gamma(Z' \to q\bar{q}) \approx \frac{(g'' Y^q_{B-L})^2}{8\pi} m_{Z'} \left(1 + \frac{\alpha_s}{\pi}\right), \quad q \equiv b, c, s$$

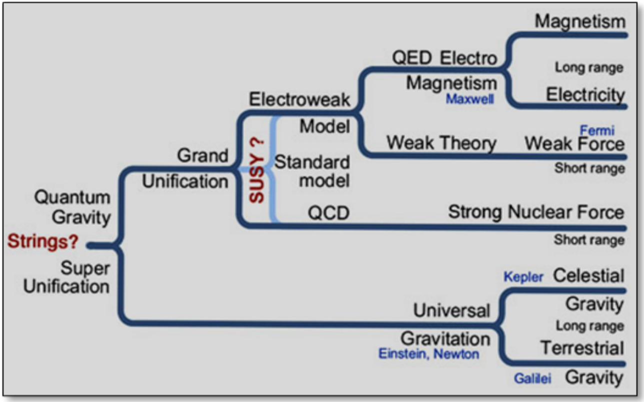
$$\Gamma(Z' \to t\bar{t}) \approx \frac{(g'' Y^q_{B-L})^2}{8\pi} m_{Z'} \left(1 - \frac{m_t^2}{m_{Z'}^2}\right) \left(1 - \frac{4m_t^2}{m_{Z'}^2}\right)^{1/2} \left(1 + \frac{\alpha_s}{\pi} + O\left(\frac{\alpha_s m_t^2}{m_{Z'}^2}\right)\right)$$

- The rate for the pair production of the heavy neutrinos depends on $M_{Z'}$ and the $B-L$ coupling g''.
- These decays are very clean with four hard leptons in the final states and large missing energy due to the associated neutrinos.



$SU(5)$ Grand Unified Field Theory

- The trials of unifying forces started since the 19th century when Farady and Maxwell combined electricity and magnetism, and ever since the trials to describe the whole universe by a single elegant law are continued.



Gauge bosons

- The gauge fields A_μ unify in the 24-dimensional adjoint representation. It decomposes under the SM subgroup as following:

$$24 = (8,1)_0 \oplus (3,2)_{-5/3} \oplus (\bar{3},2)_{5/3} \oplus (1,3)_0 \oplus (1,1)_0$$

- 8 glouns of $SU(3)_C$ $\quad$ $(W^\pm, Z^0)$ of $SU(2)$ $\quad$ $U(1)_Y$ singlet $\quad$ 12 New gauge bosons (X^i, Y^i)

- $\mathbf{A}_\mu = A^a_\mu T_a$,where T_a is the 24 generators of $SU(5)$. $G^8_\mu = A^{1\ldots8}_\mu T_{1\ldots8}$, $W^\pm_\mu = \frac{1}{\sqrt{2}}(\mathbf{A}^9_\mu \mp \mathbf{A}^{10})$, $W^3_\mu = A^{11}_\mu T_{11}$ and $B_\mu = A^{12}_\mu T_{12}$, is the hypercharge.

  The 12 new gauge bosons is defined by linear combinations like

$$\bar{X}^r_\mu = \frac{1}{\sqrt{2}}(\mathbf{A}^{13}_\mu - i\mathbf{A}^{14}_\mu), \qquad X^r_\mu = \frac{1}{\sqrt{2}}(\mathbf{A}^{13}_\mu + i\mathbf{A}^{14}_\mu)$$

- X and Y bosons are charged under both $SU(3)$ and $SU(2)$ groups, that means in $SU(5)$ there are new vertices between quarks and leptons. The strengthest experimental bounds on the (lepto-quark) interactions come from the bounds on the baryon number violation processes (proton decay) .

- For completeness, we explicitly write A_μ matrix with physical fields as follows:

$$A_\mu = \begin{pmatrix} G_1^1 - \frac{2}{\sqrt{30}}B & G_2^1 & G_3^1 & \bar{X}^1 & \bar{Y}^1 \\ G_1^2 & G_2^2 - \frac{2}{\sqrt{30}}B & G_3^2 & \bar{X}^2 & \bar{Y}^2 \\ G_1^3 & G_2^3 & G_3^3 - \frac{2}{\sqrt{30}}B & \bar{X}^3 & \bar{Y}^3 \\ X^1 & X^2 & X^3 & \frac{1}{\sqrt{2}}W^3 + \frac{3}{\sqrt{30}}B & W^+ \\ Y^1 & Y^2 & Y^3 & W^- & -\frac{1}{\sqrt{2}}W^3 + \frac{3}{\sqrt{30}}B \end{pmatrix}.$$

- SM has $8+3+1=12$ gauge bosons. It is clear that $SU(5)$ has an extra 12 gauge bosons $X^i, \bar{X}^i, Y^i, \bar{Y}^i$, which should be super-heavy.

- The covariant derivative $D_\mu = \partial_\mu - igA_\mu$ acts on the $SU(5)$ spinors as follows:

$$D_\mu \psi_p = \partial_\mu \psi_p - ig(A_\mu)_{pq}\psi_q$$
$$D_\mu \chi_{pq} = \partial_\mu \chi_{pq} - ig(A_\mu)^{pr}\chi_{rq} - ig(A_\mu)^{qs}\chi_{ps}$$

- Fundamental representation of $SU(5)$ is 5, which has the following composition:

$$5 = (3,1)_{-1/3} \oplus (1,2)_{1/2}, \qquad 5^* = (3^*,1)_{1/3} \oplus (1,2)_{-1/2}.$$

- The next $SU(5)$ representation is 10, which is defined as $10 = (5 \bigotimes 5)_A$. It has the following composition:

$$10 = (3^*,1)_{-4/3} \oplus (3,2)_{1/3} \oplus (1,1)_2$$

Unification of matter

- The SM left-handed fermions transform under $5^* \oplus 10$ irreducible representation of SU(5).

$$5 \;\equiv\; (\psi_i)_R = \begin{pmatrix} d_r \\ d_g \\ d_b \\ e^+ \\ -\nu_e^c \end{pmatrix}_R \;, \quad 5^* \;\equiv\; (\psi^i)_L \equiv\; \psi_L^c = \begin{pmatrix} d_r^c \\ d_g^c \\ d_b^c \\ e^- \\ -\nu_e \end{pmatrix}_L .$$

Charge quantization

Arises naturally here, because the tracelessness of the SU(5) generators guarantees that the charge of the quark should be (1/3) the charge of the electron.

$$10 \equiv (\chi_{ij})_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L .$$

Higgs Sector and the Hierarchy Nightmare

- We require the Higgs sector to provide for:
  1. The breaking patter: $SU(5) \xrightarrow{M_X} SU(3) \times SU(2) \times U(1)_Y \xrightarrow{M_W} SU(3) \times U(1)_{em}$
  2. The fermion masses are generated at the M_W scale.
- The 24 adjoint Higgs, (Φ), contains a SM singlet that could break the SU(5) down to the SM.
- The most general Φ Higgs potential is :

$$V_\Phi = m_1^2 Tr\Phi^2 + \lambda_1 [Tr\Phi^2]^2 + \lambda_2 Tr\Phi^4,$$

- Ons can show that $\langle\Phi\rangle$ is given by

$$\langle\Phi\rangle = \begin{pmatrix} v & & & \\ & v & & \\ & & -\frac{3}{2}v & \\ & & & -\frac{3}{2}v \end{pmatrix}$$

with $v \simeq 10^{15}$ GeV so that lepto-quark bosons X and Y gets masses of order v: $m_X^2 = m_Y^2 = \frac{20}{3} g_G^2 v^2$.

- Then the SM electroweak symmetry breaking happens by the 5 fundamental representation of Higgs scalars (H) whose vacuum expectation value is of order (100) GeV.

- The potential of 5 representation of Higgs scalar, H is

$$V(H) = -\frac{\mu_5^2}{2} H^+ H + \frac{\lambda}{4} (H^+ H_5)^2$$

Assuming color unbroken, the VEV of H is:

$$\langle H \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_H \end{pmatrix}$$

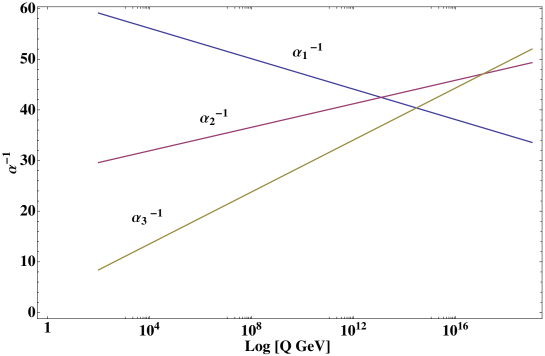
where $v_H^2 = \frac{2\mu_5^2}{\lambda}$ and $M_W^2 = g^2 \frac{v_H^2}{4}$.

- The color triplet scalar H^α can not be light and one consider a full potential $V = V(\Phi) + V(H) + V(\Phi, H)$ with serious fine-tuning to generate a splitting between the masses of the triplet and doublet in H_5.

Unification of gauge couplings

- In the SM there are three gauge coupling constants g_i $(i = 1, 2, 3)$ correspond to the three gauge groups $U(1)_Y$, $SU(2)_L$, $SU(3)_c$, respectively. $\alpha_i = g_i^2/4\pi$.

- The running of the gauge couplings constants with the energy scale Q in the Standered model.



- In a GUT theory, all the gauge couplings at high energy scale say M_X are described by a single gauge coupling.
- At energy scale $Q < M_X$ the evolution of the SU(n) gauge coupling constants with energy is controlled by the renormalization group equation

$$\frac{d\alpha_i^{-1}}{dt} = \frac{b_i}{2\pi}$$

$t = lnQ$, where Q is the running energy scale. b_i determined by the gauge group and the matter multiplets to which the gauge bosons couples:

$$b_1 = \frac{41}{10} \;\; U_Y(1); \qquad b_2 = -\frac{19}{6} \;\; SU_L(2); \qquad b_3 = -7 \;\; SU_c(3).$$

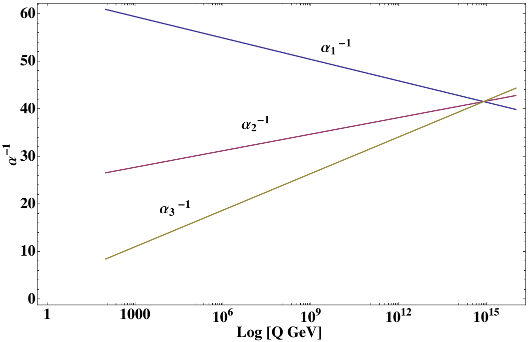
$$\alpha_i^{-1}(Q_0) \; = \alpha_i^{-1}(Q) \; + \frac{b_i}{2\pi} \; ln\frac{Q}{Q_0} \tag{1}$$

The energy scale running from Q_0 to Q. At the unification scale $Q = M_X : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_5$.

- Take $Q_0 = 100$ GeV (the electroweak scale), and solve Equ's (1) to get:

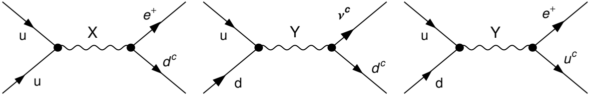
$$M_X = 7.99 \times 10^{14} \ \text{GeV}$$

- The predicted unification scale by minimal-SU(5) theory for the SM gauge coupling constants.



- This means the unification scheme doesn't realize in the prototype or the minimal $SU(5)$ because from the proton decay constraints $M_X > 10^{15}$ GeV. Also from cosmological constraints M_X should be less than M_P by around two orders of magnitude. In non-minimal $SU(5)$ new Higgs scalars are added to realize the unification with keeping the low energy scale constrains.

- Baryon number violating processes via X and Y boson exchange



$$\Gamma(p \to \pi^0 e^+) \approx \frac{m_p^5}{M_X^4}$$

- The experimental proton life time:$\tau_p(p \to e^+\pi^0) \simeq 10^{34}$, gives a lower bound on M_X: $M_X \geq 10^{16}$ GeV

- This is very close to the predicted unification scale from supersymmetric GUT, which makes seraches for

Yukawa sector and Fermion Masses

- For constructing the SU(5) invariant Yukawa Lagrangian, one uses:

$$5^* \times 10 = 5 + 45^*, \quad 10 \times 10 = 5^* + 45 + 50, \quad 5^* \times 5^* = 10^* + 15^*$$

- We have only two invariant Yukawa mass terms:

$$5^* \otimes 10 \otimes 5^*_H, \quad 10 \otimes 10 \otimes 5_H,$$

- The Yukawa Lagrangian at GUT scale is

$$\mathcal{L}_Y^G = (\Gamma_1)\ (5_L^{C\alpha})^T\ C\ 10_L^{\alpha\beta}\ 5_H^{*\beta}\ + \Gamma_2\ \epsilon_{\alpha\beta\gamma\delta r}\ (10_L^{\alpha\beta})^T\ C\ 10_L^{\gamma\delta}\ 5^r + h.c,$$

- After electroweak symmetry breaking

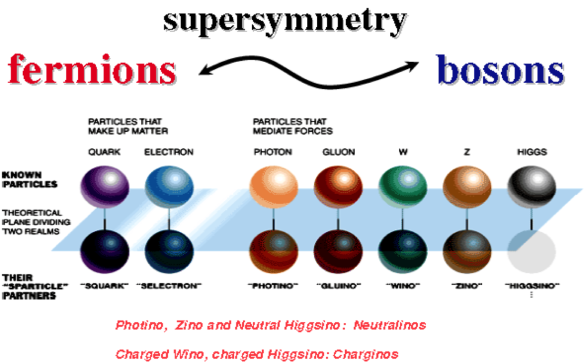
$$L_m = (Y_1\ \bar{d}_R\ d_L\ - Y_1^T\ \bar{e}_R\ e_L)\ \upsilon_5^*/\sqrt{2} + 4\ (Y_2\ \bar{u}_R u_L - Y_2^T \bar{u}_R u_L)\ \upsilon_5/\sqrt{2},$$

- The following mass relations are obtained:

$$m_e = m_d, \quad m_s = m_\mu \quad m_b = m_\tau$$

- Running the fermion masses down to the low scale, one gets $m_b \simeq 3m_\tau$, in a good agreement with the experimental results.
- The mass relation for light generations: $m_s \simeq 3m_\mu$, $\frac{m_\mu}{m_e} \simeq \frac{m_s}{m_d}$ are violated experimentally.

Supersymmetric Extension of the SM



- SUSY is an extension of the space time symmetry. Super Poincare algebra: P_μ (translation), $M_{\mu\nu}$ (rotation and Lorentz transformation), Q_α (SUSY transformation).
- SUSY ensures the stability of hierarchy between the week and the Planck scales.
- Supersymmetric theories are promising candidates for unified theory beyond the SM.
- With SUSY, the mechanism of the electroweak symmetry breaking is natural .

Hierarchy problem and SUSY

- String and GUT unification $\rightarrow$ A cutoff scale Planck scale (10^{19} GeV).
- SUSY is a symmetry to avoid the fine tuning in the renormalization of the Higgs boson mass at the level of $\mathcal{O}(10^{34})$.

$$M_h^2 = M_{h,tree}^2 + c\frac{g^2}{4\pi^2}M_{pl}^2 \qquad (w/o\ SUSY)$$

$$M_h^2 = M_{h,tree}^2(1 + c'\frac{g^2}{4\pi^2}\ln(M_{pl}/M_W)) \quad (with\ SUSY)$$

- In SUSY, the loop diagrams that are quadratically divergent cancel, term by term against the equivalent diagrams involving superpartners.
- If $m_H \sim O(100)$ GeV, the masses of superpartners should be $\lesssim O(1)$ TeV.
- Thus, some of the superpartners will be detected at the LHC.

Gauge Coupling Unification

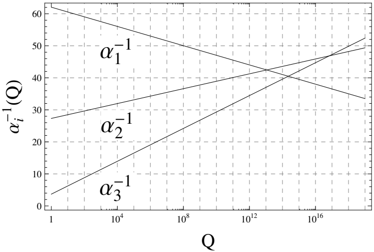
- Additional support for low scale (~ 1 TeV) SUSY follows from gauge coupling unification. At EW scale, $\alpha_s \gg \alpha$, yet quantum corrections introduce energy dependence:

$$\frac{d\alpha_i(t)}{dt} = \frac{b_i}{2\pi}\alpha_i^2(t)$$

- SM couplings evolve with μ according to

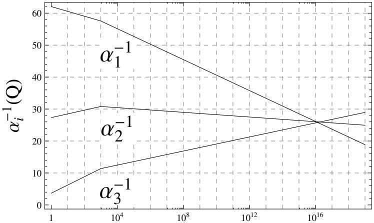
$$SU(3) \quad : \quad b_3 = -11 + \frac{4}{3}n_g = -7$$
$$SU(2) \quad : \quad b_2 = -\frac{22}{3} + \frac{4}{3}n_g + \frac{n_H}{6} = -\frac{19}{16},$$
$$U(1) \quad : \quad b_1 = \frac{4}{3}n_g + \frac{n_H}{10}, = \frac{41}{10}$$

- Within the SM, couplings do not come to a common value at any scale.



- In MSSM

$$SU(3) \quad : \quad b_3 = -9 + 2N_g = -3$$
$$SU(2) \quad : \quad b_2 = -6 + 2N_g + \frac{N_H}{2} = 1$$
$$U(1) \quad : \quad b_1 = 2n_g + \frac{3}{10}n_H = \frac{33}{5}$$



What is Supersymmetry

- Supersymmetry (SUSY): a symmetry between bosons and fermions.

$$Q_\alpha |\text{Bosnon}\rangle = \text{Fermion}\rangle \quad Q_\alpha |\text{Fermion}\rangle = \text{Bosnon}\rangle$$

  Q and Hermitan conjugate $\bar{Q}$ are fermonic operators, carry spin $1/2 \rightarrow$ SUSY is a space-time symmetry.

- SUSY introduced in 1973 as an extension of the Poincare group (an extension of the special relativity).

- In QFT, there are two type of symmetries:
  - External (Space-time) symmteries: translation: $x_\mu \rightarrow x_\mu + a_\mu$, Lorentz transformations: $x_\mu \rightarrow \Lambda_\mu^{\ \nu} x_\nu$
  - Internal symmetries: transformations on the fields: $\phi^a(x) \rightarrow M^a_{\ b}\phi^b(x)$, *e.g.*, EM $U(1)$ & flavour $SU(3)$ gauges. If $M^a_{\ b}$ is independent of x_μ, it is *global* symmetry & if it is x_μ dependent it is *local*.

- In particle physics, symmetries play crucial roles. According to Noether's theorem, every conserved physical quantity: mass, spin, electric charge, colour, etc., corresponds to a space-time or internal symmetry.

- There were several attempts to combine internal with external symmetries in a bigger symmetry group. However, in 1967, Sidney Coleman and Jeffrey Mandula showed that it is impossible to achieve a non-trivial combination of internal and external symmetries.

- SUSY is considered as a possible loophole of this theorem, since it contains additional generators that are not scalars but rather spinors.

Chiral Superfields

- A chiral superfield is defined through the conditions

$$\begin{aligned} \overline{D}_{\dot\alpha}\Phi(x,\theta,\overline{\theta}) &= 0 \quad \text{(left-chiral)} \\ D_{\alpha}\Phi(x,\theta,\overline{\theta}) &= 0 \quad \text{(right-chiral)} \end{aligned}$$

- Let us define new bosonic coordinates $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. Thus, we have

$$\bar{D}_{\dot\alpha} y^\mu = \left(-\bar{\partial}_{\dot\alpha} - i\theta^\alpha \sigma^\nu_{\alpha\dot\alpha}\partial_\nu\right)\left(x^\mu + i\theta\sigma^\mu\bar{\theta}\right) = -i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha} = 0.$$

- Also, $\bar{D}_{\dot\alpha}\theta^\alpha = 0$ and $D_\alpha\bar{\theta}^{\dot\alpha} = 0 \Longrightarrow \bar{D}_{\dot\alpha}\Phi = 0 \Longrightarrow$ it is a chiral superfield.

- The most general chiral superfield can be written as

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$$

where $\phi(y)$ is a complex scalar field, $\psi(y)$ is a spinor field and $F(y)$ is an auxiliary field.

- The expression should be read as a Taylor expansion:

$$\begin{aligned} \varphi(y) &= \varphi(x) - i\theta\sigma^\mu\overline{\theta}\partial_\mu\varphi(x) + \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu\varphi, \\ \psi(y)\theta &= \psi(x)\theta - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\theta^\beta\partial_\mu\psi_\beta, \\ F(y)\theta\theta &= F(x)\theta\theta, \end{aligned}$$

Vector Superfields

- The chiral superfields, introduced above, can describe spin 0-bosons and spin-1/2 fermions.

- For describing the spin-1 gauge bosons of the SM, one introduces vector superfields V, defined from the general superfield by imposing a covariant reality constraint:

$$V(x, \theta, \bar{\theta}) = V^{\dagger}(x, \theta, \bar{\theta}).$$

- The vector superfield has the following expansion:

$$\begin{aligned} V(x, \theta, \bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \theta\sigma^{\mu}\bar{\theta}v_{\mu} + \frac{i}{2}\theta\theta\,[M(x) + iN(x)] \\ &- \frac{i}{2}\bar{\theta}\bar{\theta}\,[M(x) - iN(x)] + \theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)\right] \\ &+ \bar{\theta}\bar{\theta}\theta\left[\lambda(x) - \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)\right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) - \frac{1}{2}\partial_{\mu}\partial^{\mu}C(x)\right], \end{aligned}$$

- The presence of a real vector field in the vector superfield suggests that we use vector superfields to construct SUSY gauge theories.

- Examples of vector superfields:

$$\Phi^{\dagger}\Phi \quad \& \quad \Phi + \Phi^{\dagger}$$

- A general vector superfield has eight bosonic and eight fermionic components, which are far too many to describe a single supermultiplet. To reduce their number, we introduce a generalisation of the usual concept of gauge transformations:

$$V \to V + \Lambda + \Lambda^\dagger,$$

where Λ is a chiral superfield.

- Under this transformation, the real vector field v_μ transforms as

$$v_\mu(x) \to v_\mu + i\partial_\mu \left[\alpha(x) - \alpha^*(x)\right]$$

- The SUSY gauge transformations allow for the possibility of gauging away the unphysical fields. In this physical gauge, the WZ gauge, the vector supermultiplet V is given by

$$V(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta}\; v_\mu(x) + \theta^2\bar{\theta}_{\dot{\alpha}}\; \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2\theta^\alpha\; \lambda_\alpha(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x)$$

where $D(x)$ is a non-propagating auxiliary field as it was F in the chiral superfield.

- Notice that, analogously to F, D also transforms under a SUSY transformation into a total derivative.

Supersymmetry Breaking

- SUSY cannot be an exact symmetry of Nature.

- If it were, it would imply the existence of, *e.g.*, a selectron with the same mass as the electron, ~ 0.5 MeV, and squarks with the same mass of quarks, for which there is no experimental evidence.

- In order for SUSY to play a role in particle physics, it must be a broken symmetry at energies at least of the order of the EW scale.

- As any other symmetry, SUSY can be broken either spontaneously, dynamically or explicitly.

- From SUSY algebra, we have
$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu .$$

- Thus, by multiplying from the right-hand side by $(\bar{\sigma}^\nu)^{\dot{\beta}\alpha}$, we find
$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} = 2\sigma^\mu_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} P_\mu = 2\mathrm{Tr}[\sigma^\mu \bar{\sigma}^\nu] P_\mu = 4\eta^{\mu\nu} P_\mu .$$

- For $\nu = 0$ with $\bar{\sigma}^0 = \mathbb{I}_{2\times 2}$, one finds that the Hamiltonian H can be written as
$$H = \frac{1}{4}(\bar{Q}_1 Q_1 + Q_1 \bar{Q}_1 + \bar{Q}_2 Q_2 + Q_2 \bar{Q}_2) \geq 0,$$

- The Hamiltonian is semi-positive definite. If a vacuum state $|0\rangle$ is supersymmetric, *i.e.*, $Q_\alpha |0\rangle = \bar{Q}_{\dot{\alpha}} |0\rangle = 0$, then a zero vacuum energy is obtained: $E_{\rm vacuum} = \langle 0|H|0\rangle = 0$.

Soft SUSY Breaking

- Spontaneous SUSY breaking does not lead to a phenomenological viable model.
- The other possibility is to break SUSY explicitly by adding to the SUSY invariant Lagrangian a set of terms that violate SUSY and do not introduce quadratic divergences.
- These terms are called 'soft SUSY breaking terms'.
- In order not to introduce quadratic divergences and consequently spoil the SUSY solution to the gauge hierarchy problem, the set of soft SUSY breaking terms contains only mass terms and couplings with positive mass dimension.
- This type of restricted terms have been catalogued by Girardello and Grisaru as follows

  1. Masses for the scalars: $\tilde{m}^2_{ij}\phi^*_i\phi_j$.
  2. Masses for the gauginos: $\frac{1}{2}M_a\lambda^a\lambda^a$.
  3. Bilinear scalar interactions: $\frac{1}{2}B_{ij}\phi_i\phi_j + \mathrm{h.c.}$
  4. Trilinear scalar interactions: $\frac{1}{3!}\phi_i\phi_j\phi_k + \mathrm{h.c.}$

- The soft terms are very important since they determine the SUSY spectrum and contribute to the Higgs potential generating the radiative breakdown of the EW symmetry.

  The number of soft SUSY breaking terms is enormous. They parametrize our ignorance of the SUSY breaking mechanism.

Minimal Supersymmetric Standard Model

- The MSSM is a straightforward supersymmetrisation of the SM with the minimal number of possible new parameters.

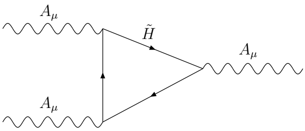
- The MSSM i is based on the SM gauge group: $SU(3)_C \times SU(2)_L \times U(1)_Y$, with the following particle content:

Supermultiplet	SM	SUSY	$SU(3)_C \times SU(2)_L \times U(1)_Y$
Q_L	quarks $q = (u_L, d_L)^T$ (spin $\frac{1}{2}$)	squarks $\tilde{q} = (\tilde{u}_L, \tilde{d}_L)^T$ (spin 0)	$(3, 2, 1/6)$
U_L^c	quarks u_L^c (spin $\frac{1}{2}$)	squarks $\tilde{u}_L^c$ (spin 0)	$(\bar{3}, 1, -2/3)$
D_L^c	quarks d_L^c (spin $\frac{1}{2}$)	squark $\tilde{d}_L^c$ (spin 0)	$(\bar{3}, 1, 1/3)$
L_L	leptons $l = (\nu_L, e_L)^T$ (spin $\frac{1}{2}$)	sleptons $(\tilde{l}) = (\tilde{\nu}_L, \tilde{e}_L)^T$ (spin 0)	$(1, 2, 1/2)$
E_L^c	leptons e_L^c (spin $\frac{1}{2}$)	sleptons $\tilde{e}_L^c$ (spin 0)	$(1, 1, 1)$
H_u	Higgs $H_u = (H_u^0, H_u^+)^T$ (spin 0)	Higgsino $\tilde{H}_u = (\tilde{H}_u^0, \tilde{H}_u^+)$ (spin $\frac{1}{2}$)	$(1, 2, 1/2)$
H_d	Higgs $H_d = (H_d^-, H_d^0)^T$ (spin 0)	Higgsino $\tilde{H}_d = (\tilde{H}_d^-, \tilde{H}_d^0)^T$ (spin $\frac{1}{2}$)	$(1, 2, 1/2)$

- The hypercharge Y is defined by $Y = Q - I_3$.

- In the SM one Higgs doublet, H, is used only to generate masses for both up and down quarks after EWSB, through the Yukawa interactions $Y_d q_L H d_L^c + Y_u q_L \tilde{H} u_L^c + \mathrm{h.c.}$, where $\tilde{H} = i\sigma_2 H^*$.

- In the MSSM, where the superpotential is an analytic function of only chiral superfields, the anti-chiral H^* cannot be included in the superpotential and a new Higgs doublet with opposite hypercharge should instead be introduced.



- Another reason for the necessity of adding another Higgs doublet in the MSSM is to cancel the triangle anomaly generated by the fermionic partner of the Higgs superfield.

  This anomaly is cancelled in the SM due to the vanishing of Tr$[Y^3]$ and Tr$[T_3^2 Y]$, where T_3 stands for the $SU(2)_L$ third generator.

  In a SUSY model with just one Higgs doublet, the fermionic partner of this Higgs (Higgsino) contributes to the triangle anomaly.

  This contribution would remain not cancelled. Therefore, a second Higgs doublet superfield, with opposite hypercharge, must be added in order to cancel this contribution.

- Three gauge (vector) superfields corresponding to the SM gauge groups $(B_\mu, \tilde{B})$ for $U(1)_Y$, $(W_\mu^a, \tilde{W}^a)$ for $SU(2)_L$ with $a = 1, 2, 3$ and $(G_\mu^a, \tilde{g}^a)$ for $SU(3)_C$ with $a = 1, \ldots 8$.

- With these superfields, the MSSM Lagrangian can be written as

$$\mathcal{L}_{\rm MSSM} = \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm matter} + W + \mathcal{L}_{\rm soft}.$$

where the gauge Lagrangian $\mathcal{L}_{\rm gauge}$ includes all the gauge interactions in the MSSM.

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4} F_G^{a\ \mu\nu} F^a_{G\ \mu\nu} + i \bar{\lambda}^a_G \bar{\sigma}^\mu D_\mu \lambda^a_G + \frac{1}{2} D^a D_a$$

- The terms obtained from the kinetic Lagrangian of chiral superfields are given by

$$\mathcal{L}_{\rm matter} = (D^\mu \phi_i)^\dagger (D_\mu \phi_i) + i \bar{\psi}_i \gamma^\mu D_\mu \psi_i + F_i^* F_i + i g_a \sqrt{2} (\phi^* T^a \lambda^a \psi + {\rm h.c.}) - \frac{1}{2} g_a^2 (\phi_i^* T^a \phi_i)^2.$$

$D_\mu = \partial_\mu + i g_1 Y B_\mu + i g_2 \frac{\sigma^a}{2} W_\mu^a + i g_3 \frac{\lambda^a}{2} G_\mu^a$ and ϕ, ψ refer to the MSSM scalars and fermions.

- The SM Yukawa interactions are included in the MSSM superpotential, which describes the interactions between Higgs bosons and matter superfields. This superpotential can be written as

$$W = Y_u Q U^c H_u + Y_d Q D^c H_d + Y_e L E^c H_d + \mu H_d H_u.$$

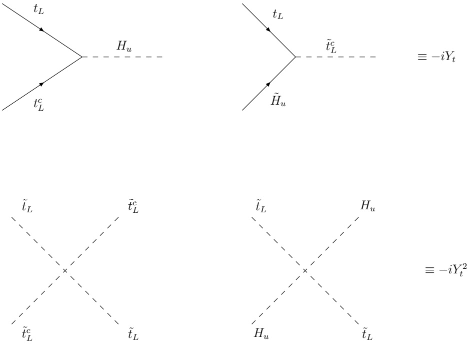
The parameter μ has mass dimension one and gives supersymmetric masses to both fermionic and bosonic components of the chiral superfields H_u and H_d.

- As an explicit example, let us consider the superpotential of the top superfield: $W = Y_t Q_t H_u t_L^c$.

- The interaction Lagrangian associated with this superpotential is given by

$$
\begin{aligned}
\mathcal{L}_{\rm int} &= -\frac{1}{2} W^{ij} \psi_i \psi_j - \frac{1}{2} W^{ij\,*} \bar{\psi}_i \bar{\psi}_j - W^i W_i^{\,*} \\
&= -\frac{Y_t}{2}\Big(t_L H_u t_L^c + \tilde{t}_L \tilde{H}_u t_L^c + t_L \tilde{H}_u \tilde{t}_L^c + {\rm h.c.}\Big) - Y_t^2 \left(|H_u \tilde{t}_L^c|^2 + |H_u \tilde{t}_L|^2 + |\tilde{t}_L \tilde{t}_L^c|^2 \right).
\end{aligned}
$$

This term in the superpotential leads to the following four Feynman rules:



- Due to the fact that Higgs and lepton doublet superfields have the same $SU(3)_C \times SU(2)_L \times U(1)_Y$ quantum numbers, we have the following additional terms;

$$W' = \lambda_{ijk} L_i L_j E^c_k + \lambda'_{ijk} L_i Q_j D^c_k + \lambda''_{ijk} D^c_i D^c_j U^c_k + \mu'_i L_i H_u.$$

- These terms violate baryon and lepton numbers explicitly and lead to proton decay at unacceptable rates.

- To avoid this problem of too rapid a proton decay, a new symmetry (called R-parity is commonly introduced in order to remove these terms: $R_P = (-1)^{3B+L+2S}$, where B and L are baryon and lepton number and S is the spin.

- Two remarkable phenomenological implications of the presence of R-parity:

  1. SUSY particles are produced or destroyed only in pairs;
  2. the LSP LSP is absolutely stable and, hence, it might constitute a possible candidate for DM.

- In addition to the above interactions, one should add the soft SUSY breaking terms to the Lagrangian. Following the general classification of the soft SUSY breaking terms:

$$\begin{aligned}
\mathcal{L}_{\rm soft} &= -\frac{1}{2} M_a \lambda^a \lambda^a - m^2_{\tilde{q}_{ij}} \tilde{q}^*_i \tilde{q}_j - m^2_{\tilde{u}_{ij}} \tilde{u}^*_i \tilde{u}_j - m^2_{\tilde{d}_{ij}} \tilde{d}^*_i \tilde{d}_j - m^2_{\tilde{\ell}_{ij}} \tilde{\ell}^*_i \tilde{\ell}_j \\
&- m^2_{\tilde{e}_{ij}} \tilde{e}^*_i \tilde{e}_j - m^2_{H_u} |H_u|^2 - m^2_{H_d} |H_d|^2 - \Big[Y^A_{uij} \tilde{q}_i \tilde{u}_j H_u \\
&+ Y^A_{dij} \tilde{q}_i \tilde{d}_j H_d + Y^A_{eij} \tilde{\ell}_i \tilde{e}_j H_d - B\mu H_u H_d + {\rm h.c.} \Big].
\end{aligned}$$

- The soft terms $m^2_{\tilde{q}}$, $m^2_{\tilde{u}}$, $m^2_{\tilde{d}}$, $m^2_{\tilde{\ell}}$ and $m^2_{\tilde{e}}$ are Hermitian 3×3 matrices in flavour space, while the trilinear couplings in most cases are given by $Y^A_{fij} \equiv (A_f)_{ij}(Y_f)_{ij}$, with $f = u, d, e$, as complex 3×3 matrices.

- Also the gaugino masses M_a and the bilinear coupling B of mass dimension one are generally complex numbers.

- The soft SUSY terms induce about 100 free parameters which reduce the predictivity of the MSSM.

- In what is called cMSSM, a kind of universality among the soft SUSY breaking terms at the GUT scale $M_{\rm GUT} = 3 \times 10^{16}$ GeV is assumed. In this case, this large number of soft SUSY breaking terms is reduced to the following four parameters:

$$\begin{aligned} m^2_{\tilde{q}_{ij}} &= m^2_{\tilde{u}_{ij}} = m^2_{\tilde{d}_{ij}} = m^2_{\tilde{\ell}_{ij}} = m^2_{\tilde{e}_{ij}} = m_0^2 \delta_{ij}, \\ m^2_{H_u} &= m^2_{H_d} = m_0^2, \\ A^{ij}_u &= A^{ij}_d = A^{ij}_e = A_0 \delta^{ij}, \\ M_1 &= M_2 = M_3 = m_{1/2}. \end{aligned}$$

- The parameter m_0 is called 'universal scalar mass', A_0 is called 'universal trilinear coupling' and $m_{1/2}$ is called 'universal gaugino mass'.

- This class of models is motivated by mSUGRA where SUSY breaking is mediated by gravity interactions with minimal Kähler potential and minimal gauge kinetic function.

Higgs Bosons in the MSSM

- As discussed last time, successful mass generation needs two doublets

$$H_u = \begin{pmatrix} H_u^+ \\ \frac{1}{\sqrt{2}}(v_u + h_u + ia_u) \end{pmatrix} \quad H_d = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_d + h_d + ia_d) \\ H_d^- \end{pmatrix}$$

- β is a mixing angle: $v_u = v \sin\beta$, $v_d = v\cos\beta$, where $v = 246$ GeV is the SM Higgs VEV.

- Two doublets have eight degrees of freedom: three form the longitudinal polarisation states of EW gauge bosons, three neutral and a charged pair should remain.

- The Higgs potential of the MSSM gets following contributions
  - the superpotential gives terms $|\mu|^2(|H_u|^2 + |H_d|^2)$
  - the soft SUSY breaking Lagrangian gives terms $m_{H_u}^2|H_u|^2 + m_{H_d}^2|H_d|^2 + B\mu H_u H_d$
  - the U(1) gauge group gives the contribution $\frac{1}{8}g'^2|H_u^\dagger H_u - H_d^\dagger H_d|^2$ (the extra factor $1/4$ comes from the generator being $Y/2$)
  - the SU(2) gauge group gives the contribution $\frac{1}{8}g^2|H_u^\dagger\sigma^i H_u + H_d^\dagger\sigma^i H_d|^2$

$$\begin{aligned} V &= (m_{H_u}^2 + |\mu|^2)H_u^\dagger H_u + (m_{H_d}^2 + |\mu|^2)H_d^\dagger H_d + B\mu(H_u^T \cdot i\sigma_2 \cdot H_d + \mathrm{h.c.}) \\ &+ \frac{1}{8}\left(g^2 + (g')^2\right)\left|H_u^\dagger H_u - H_d^\dagger H_d\right|^2 + \frac{1}{2}g^2|H_u^\dagger H_d|^2. \end{aligned}$$

CP-even Neutral Higgs

- In the CP-even Higgs sector the mass matrix is (basis h_d, h_u)

$$m_H^2 = \begin{pmatrix} B\mu \tan\beta + m_Z^2 \cos^2\beta & B\mu - m_Z^2 \sin\beta\cos\beta \\ B\mu - m_Z^2 \sin\beta\cos\beta & B\mu \cot\beta + m_Z^2 \sin^2\beta \end{pmatrix}.$$

- The eigenvalues are

$$m_{h,H}^2 = \frac{m_A^2 + m_Z^2 \pm \sqrt{(m_A^2 + m_Z^2)^2 - 4m_A^2 m_Z^2 \cos^2 2\beta}}{2},$$

where the smaller eigenvalue has a limit $m_h \leq m_Z |\cos 2\beta|$ (if $m_A > m_Z$). The heavier one has an mass $\mathcal{O}(m_A)$.

- The one-loop corrections to the scalar potential can be given in the form

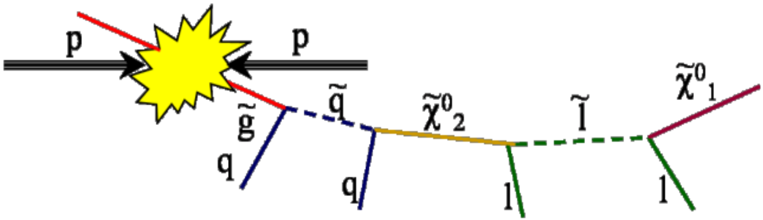
$$V^{(1)} = \frac{1}{64\pi^2} \mathrm{Str} \left\{ M^4(\varphi_i) \left[\ln\left(\frac{M^2(\varphi_i)}{Q^2} \right) - C_s \right] \right\},$$

where C_s is a spin-dependent constant, which gets values $C_0 = 3/2$, $C_{1/2} = 3/2$, $C_1 = 5/6$.

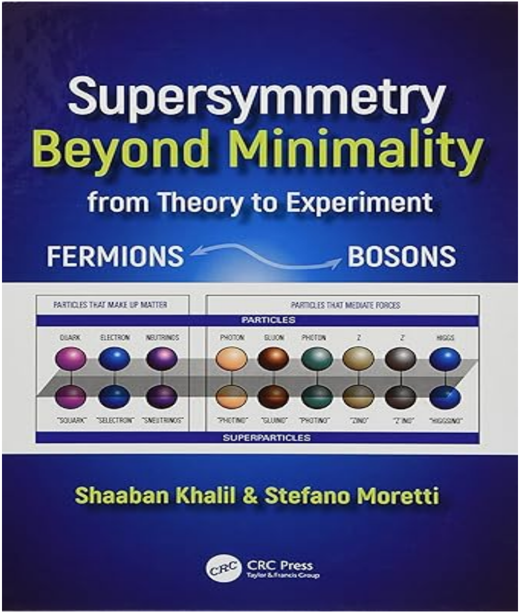
- In MSSM the light CP even Higgs mass was predicted to be less than 130 GeV, consistently with the experimentally observed 125 GeV Higgs boson.

SUSY searches at the LHC

- At LHC, the total SUSY particle production cross section is largely dominated by strongly interacting sparticles.
- A typical high mass SUSY signal has squarks and gluinos which decay through a number of steps to quarks, gluons, charginos, neutralinos, W, Z, Higgses and, finally, to a stable $\tilde{\chi}_1^0$



- The search for the production and decay of SUSY particles by the multi-purpose LHC experiments (CMS and ATLAS) is described by events with two or more energetic jets and significant missing transverse energy.


Supersymmetry
Beyond Minimality
from Theory to Experiment
FERMIONS
BOSONS
PARTICLES THAT MAKE UP MATTER
PARTICLES THAT MEDIATE FORCES
PARTICLES
QUARK
ELECTRON
NEUTRINOS
PHOTON
GLUON
PHOTON
HIGGS
"SQUARK"
"SELECTRON"
"SNEUTRINOS"
"PHOTINO"
"GLUINO"
"PHOTINO"
"ZINO"
"HIGGSINO"
SUPERPARTICLES
Shaaban Khalil & Stefano Moretti
CRC Press
Taylor & Francis Group

Extra Dimensions

- General Relativity: why 4 dimensions?
- Possible existence of new spatial dimensions beyond the four we see have been under consideration for about eighty years already.
- The first ideas date back to the early works of Kaluza and Klein around the 1920?s, who tried to unify electromagnetism with Einstein gravity.
- Extra dimensions aim to unify the fundamental forces of the universe.
- Extra dimensions are fundamental ingredient for String Theory, since all versions of the theory are naturally and consistently formulated only in a space-time of more than four dimensions (actually 10, or 11 if there is M-theory).
- Extra dimensions offer a potential solution to the hierarchy problem.
- Extra dimensions can potentially explain cosmological inflation and the nature of dark matter and dark energy.

General relativity in 5D spacetime

- One of the first attempts to formulate a unified field theory. Introduced by Theodor Kaluza in 1921.
- The five dimensional line element is given by

$$d\hat{s}^2 = g_{MN} dx^M dx^N$$

- The five dimensional metric is assumed as,

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} - \kappa^2\phi^2 A_\mu A_\nu & -\kappa\phi^2 A_\mu \\ -\kappa\phi^2 A_\nu & -\phi^2 \end{pmatrix}$$

- $\hat{g}_{MN}$ becomes the gravitational tensor potential framed by the electromagnetic four-potential A_μ and scalar field ϕ.
- It was assumed that the metric is independent of the extra dimensional coordinate y. This assumption is known as the *cylindrical condition*: $\partial g_{MN}/\partial x_4 = 0$
- Along with the identifications $g_{44} = -\phi^2 = -1$, $\kappa = \sqrt{\frac{16\pi G}{c^4}}$,the resulting field equations $G_{MN} = 0$ are

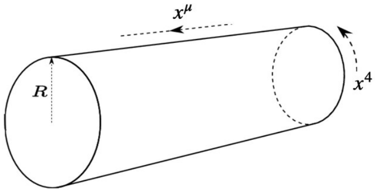
$$\begin{aligned} \tilde{G}_{\mu\nu} &= \frac{8\pi G}{c^4} T_{\mu\nu} \\ \nabla^\mu F_{\mu\nu} &= 0 \end{aligned}$$

- with $T_{\mu\nu} \equiv \frac{1}{2}(g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_\mu{}^\alpha F_{\nu\alpha})$. This situation is known as "Kaluza miracle".

Compactified extra dimension

- To justify the cylinder condition, Oskar Klein assumed a microscopic, curled-up dimension. Compactified in toroidal fashion.

$$X(x^\mu, x^4) = X(x^\mu, x^4 + 2\pi R)$$



- All fields are periodic in $y = x_4$ and may be expanded in a Fourier series:

$$\begin{aligned} g_{\mu\nu}(x,y) &= \sum_{n=-\infty}^{+\infty} g_{\mu\nu n}(x) e^{in\cdot y/R} \\ A_\mu(x,y) &= \sum_{n=-\infty}^{+\infty} A_{\mu n}(x) e^{in\cdot y/R} \\ \phi(x,y) &= \sum_{n=-\infty}^{+\infty} \phi_n(x) e^{in\cdot y/R} \end{aligned}$$

with $\quad g^*_{\mu\nu n}(x) = g_{\mu\nu -n}(x), \quad A^*_{\mu n}(x) = A_{\mu -n}(x), \quad \phi^*_n(x) = \phi_{-n}(x)$

- So, the Kaluza-Klein theory describes an infinite number of four-dimensional fields.

- The equations of motion corresponding to the above action are,

$$
\begin{aligned}
(\partial^\mu \partial_\mu - \partial^y \partial_y) g_{\mu\nu}(x,y) &= (\partial^\mu \partial_\mu + \frac{n^2}{R^2}) g_{\mu\nu n}(x) = 0 \\
(\partial^\mu \partial_\mu - \partial^y \partial_y) A_\mu(x,y) &= (\partial^\mu \partial_\mu + \frac{n^2}{R^2}) A_{\mu n}(x) = 0 \\
(\partial^\mu \partial_\mu - \partial^y \partial_y) \phi(x,y) &= (\partial^\mu \partial_\mu + \frac{n^2}{R^2}) \phi_n(x) \;\; = 0
\end{aligned}
$$

- Comparing these with the standard Klein-Gordon equation, we get 'mass' corresponding to these fields as,

$$m_n \sim \frac{n}{R}$$

  where n is the mode of excitation.

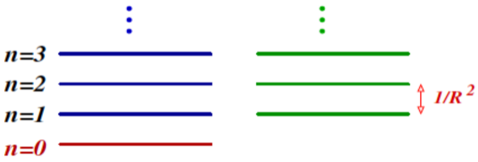
- In four dimensions we see all these excited states with mass or momentum $\sim O(n/R)$. Since we want to unify the electromagnetic interactions with gravity, the natural radius of compactification will be the Planck length: $R = \dfrac{1}{M_p}$, where the Planck mass $M_p \sim 10^{18}\, GeV$.

- The resulting action of the scalar field (called dilaton) is given by

$$S = \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)} + \sum_{n=1}^{\infty} \left[\partial_\mu \phi^{(n)\dagger} \partial^\mu \phi^{(n)} - \frac{n^2}{R^2} \phi^{(n)\dagger} \phi^{(n)} \right] \right\}.$$

- From the 4D point of view that the action describes an (infinite) series of particles (Kaluza-Klein tower) with masses $m_{(n)} = n/R$.

- If the field $\Phi(x^\mu, y)$ has a 5D mass m_0, then the 4D Kaluza-Klein particles will have masses, $m_{(n)}^2 = m_0^2 + n^2/R^2$.



KK mass spectrum for a field on the circle.

- In $5D$, the gauge field $A_M(x^\mu, y)$ has the following Fourier decomposition along the compact dimension,

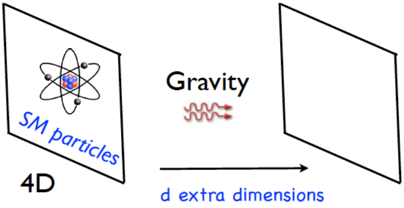
$$A_M(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_n A_M^{(n)}(x^\mu) e^{i\frac{n}{R}y}.$$

- The action of $5D$ gauge field becomes

$$\begin{aligned} S &= \int d^4x\, dy \left[-\frac{1}{4} F_{MN} F^{MN} \right] \\ &= \int d^4x \left\{ \left(-\frac{1}{4} F_{\mu\nu}^{(0)} F^{(0)\mu\nu} + \frac{1}{2} \partial_\mu A_5^{(0)} \partial^\mu A_5^{(0)} \right) + \sum_{n\geq 1} 2 \left(-\frac{1}{4} F_{\mu\nu}^{(-n)} F^{(n)\mu\nu} + \frac{1}{2} \frac{n^2}{R^2} A_\mu^{(-n)} A_\mu^{(n)} \right) \right\}. \end{aligned}$$

- We can see that zero modes contain a 4D gauge field and a real scalar.

Brane-world models



- If SM fields are localized to a four-dimensional brane. The only restriction on the radius would be

$$R \sim \frac{{M_{Pl}}^{\frac{2}{n}}}{{M_*}^{1+\frac{2}{n}}}$$

- In 1998 Arkani-Hamed, Dimopoulos and Dvali (ADD) realize that extra dimensions could explain the weakness of gravity: $G_N \ll G_F$.

- For m_{EW} is the fundamental Planck scale and choose R such that the observed mass scale is M_{pl}

$$R \sim 10^{\frac{30}{n}-17} cm \times (\frac{1\,TeV}{m_{EW}})^{1+\frac{2}{n}}$$

- Two extra dimensions ($n = 2$) $\rightarrow R \sim 100\mu m$. Deviation from Newton's law would be accommodated by the experimental limit on gravity.

Warped extra dimensions

- Warped space-times, the metric warps exponentially along the extra dimension

$$ds^2 = f(y) g_{\mu\nu}(x) dx^\mu dx^\nu + g_{ab}(y) dy^a dy^b$$

- Assuming the following
  - An S^1 symmetry:

$$y \to y + 2\pi R$$

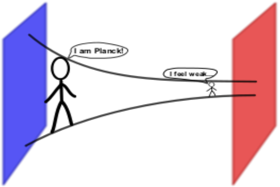
  - A $\mathbb{Z}_2$ symmetry

$$y \to -y$$

- A warp factor satisfying the field equations and the previous assumptions is $f(y) = e^{-2k|y|}$
- An important feature of this model is that it can only admit an AdS space: $k = \sqrt{-\Lambda\kappa}$, where κ is the 5D Einstein constant related to the Planck mass as $\kappa^2 \sim \frac{1}{M^3}$ and Λ is the cosmological constant.

The metric eventually takes the form

$$ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$



Hierarchy problem

- This model solves the hierarchy problem by connecting the four-dimensional Planck scale and mass parameters to the five-dimensional scales.

- The law energy limit is approached by a weak-field background perturbation $h_{\mu\nu}(x^\mu) \ll 1$:

$$ds^2 = e^{-2kT(x^\mu)|\phi|} dx^\mu dx^\nu [\eta_{\mu\nu} + h_{\mu\nu}(x^\mu)] + T^2(x^\mu) dy^2$$

  Where the modulus field $T(x)$ is stabilized at r_c being the vacuum expectation value $\langle T(x) \rangle \equiv r_c$

$$ds^2 = e^{-2kr_c|\phi|} dx^\mu dx^\nu [\eta_{\mu\nu} + h_{\mu\nu}(x^\mu)] + {r_c}^2 dy^2$$

- The effective action implies:

$$S = -M^3 \int d^5x \sqrt{g^{(5)}} (R^{(5)} + \kappa^2) \supset -M^3 \int dy e^{-2k|y|} \int d^4x \sqrt{g^{(4)}} (R^{(4)} + \kappa^2)$$

- We now compare this to the four-dimensional action $S = -M_{Pl}^2 \int d^4x \sqrt{g^{(4)}} (R^{(4)} + \kappa^2)$ to get

$$M_{pl}^2 = M^3 \int_{y=-b}^{y=b} dy e^{-2k|y|} = \frac{M^3}{k} [1 - e^{-2kb}]$$

- It is evident by now that this scenario provides a quite different approach than ADD model. The effect of the warp factor is negligible $M_{Pl} \approx M_*$

Hierarchy problem

- The Electroweak mass scale however will be modified. The matter field which, unlike gravity, is localized to one of the branes. The action for the Higgs scalar

$$S^H = \int d^4x \sqrt{g^{ind}} [g_{\mu\nu} D^\mu H D^\nu H - \lambda((H^\dagger H) - v^2)^2]$$

- The induced metric at the brane at $y = b$ is $g^{ind}_{\mu\nu} = e^{2kR} \eta_{\mu\nu}$

$$S^H = \int d^4x e^{-4kb} [e^{2kb} \eta_{\mu\nu} D^\mu H D^\nu H - \lambda((H^\dagger H) - v^2)^2]$$

- The field can be redefined as $\tilde{H} = e^{-kb} H$ to get a canonically normalized field. The action is then

$$S^H = \int d^4x [\eta_{\mu\nu} \partial^\mu \tilde{H} \partial^\nu \tilde{H} - \lambda((\tilde{H}^\dagger \tilde{H}) - (e^{-kb} v)^2)^2]$$

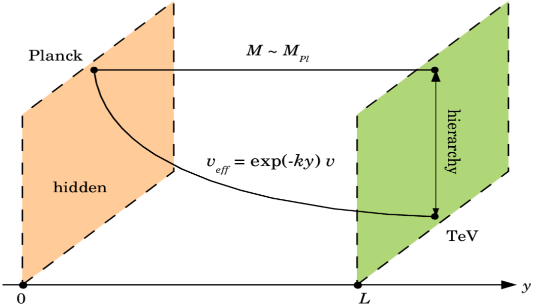
- Thus:

$$\tilde{v} = e^{-kR} v.$$

- If v is regarded as the fundamental mass scale, the warp factor can be used to generate the TeV scale of the weak scale.

- The brane at $y = R$ is referred to as the TeV brane and that at $y = 0$ is referred to as the Planck brane where the warp factor would have no effect and the mass scale parameters are of the Planck mass order.

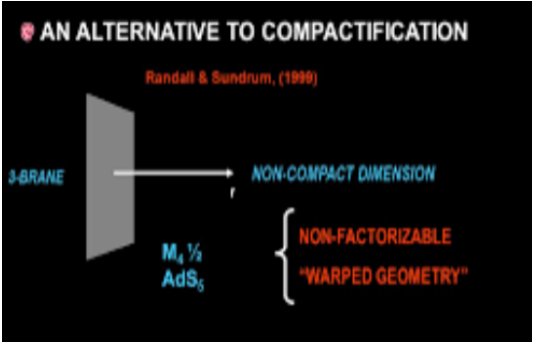
RS1 Model



- In this model, no large dimension necessary to explain weakness of gravity
- Graviton's interaction is exponentially suppressed away from Gravity brane
- Gravity is weak everywhere except Gravity brane
- Mass hierarchy natural on Weak brane!

RS2 model

- This model is known as an Alternative to Compactification.
- The RS2 model uses the same geometry as RS1, but there is no TeV brane.
- The particles of the standard model are presumed to be on the Planck brane.
- This model was originally of interest because it represented an infinite 5-dimensional model, which, in many respects, behaved as a 4-dimensional model.

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