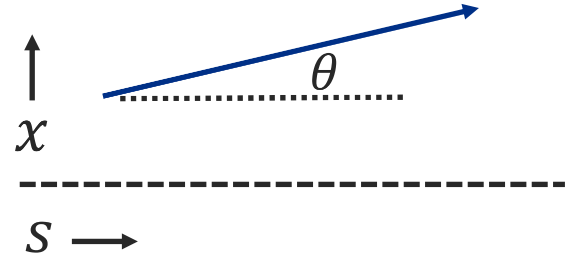
Fundamentals of Particle Accelerators II

Dr. Karie Badgley
ASP 2024- Morocco

Overview

- Emittance
- Phase space
- Acceleration
  - Cyclotrons
  - RF cavities
- Phase stability

Last Time



$$x(s) = x_0 cos\left(\sqrt{K}s\right) + \frac{x'_0}{\sqrt{K}} sin(\sqrt{K}s)$$

$$x'(s) = -x_0\sqrt{K}sin\left(\sqrt{K}s\right) + x'_0 cos(\sqrt{K}s)$$

$$M = \begin{pmatrix} cos\left(\sqrt{K}s\right) & \dfrac{sin(\sqrt{K}s)}{\sqrt{K}} \\ -\sqrt{K}sin\left(\sqrt{K}s\right) & cos(\sqrt{K}s) \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$x(s) = A\sqrt{\beta(s)}cos(\psi(s) + \delta)$$

$$x(s) = A\sqrt{\beta(s)}cos(\psi(s) + \delta)$$

$$\binom{x}{x'}_{s_{0+C}} = \begin{pmatrix} cos\Delta\psi + \alpha sin\Delta\psi & \beta sin\Delta\psi \\ -\gamma sin\Delta\psi & cos\Delta\psi - \alpha sin\Delta\psi \end{pmatrix} \binom{x}{x'}_{s_0}$$

α, β, γ are the Twiss parameters

Emittance

$$x(s) = A\sqrt{\beta(s)}cos(\psi(s) + \delta)$$

Constants of integration

We can write the constant A as $\sqrt{\epsilon}$ and find this constant, emittance, is a useful value in describing the beam

$$emittance = \epsilon = constant$$

From the amplitude of the orbit equation, we can find the maximum and minimum particle displacement

$$cos(\psi(s) + \delta) = \pm 1 \qquad Max\ displacement\ x_{max} = \pm\sqrt{\epsilon\beta(s)}$$

This sets a limit on the minimum beam aperture a machine can have to prevent particle loss

-We haven’t included other effects which influence the beam such as resonance, space-charge, …

Emittance

We can also express emittance in terms of the Twiss parameters by eliminating the trigonometric terms in the betatron oscillation

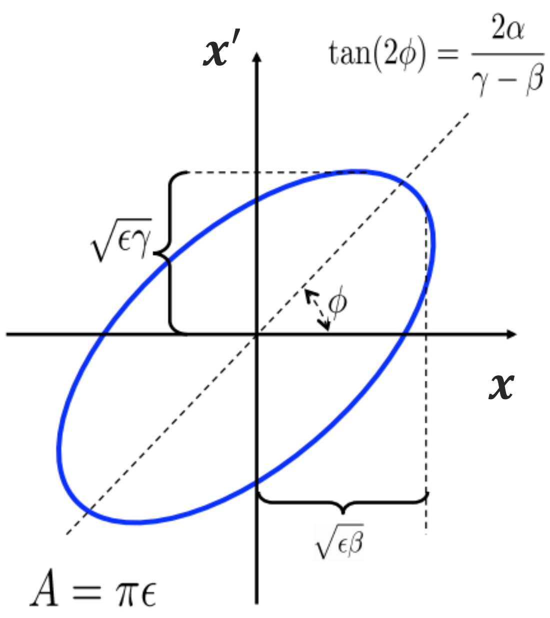
$$x(s) = \sqrt{\epsilon\beta(s)}cos(\psi(s) + \delta)$$

If we multiply x and x' by αx and $\beta x'$:

$$\alpha(s)x(s) + \beta(s)x'(s) = -\sqrt{\epsilon\beta(s)}sin(\psi(s) + \delta)$$

Squaring and summing the above equations yields:

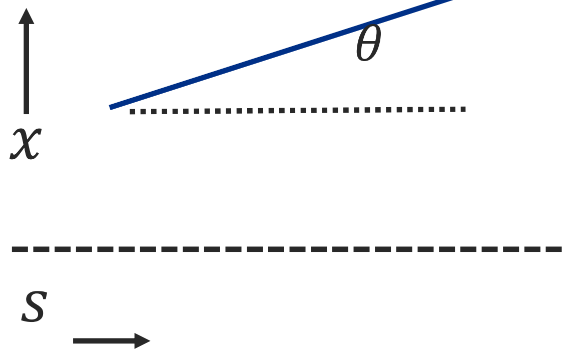
$$\epsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



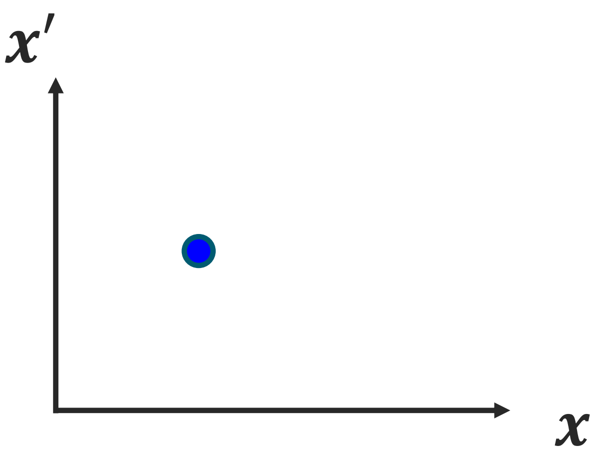
This Courant-Snyder invariant will be constant* for all locations through the lattice. Represents the area in phase space, measure of accelerator performance

Shape and orientation of the ellipse will change

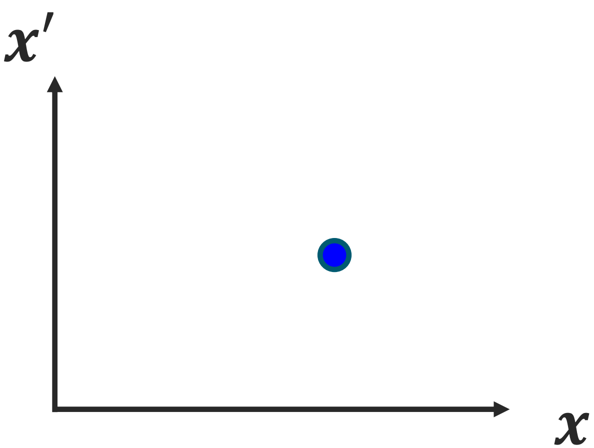
Phase Space



$$\frac{dx}{ds} = x' \sim \theta$$

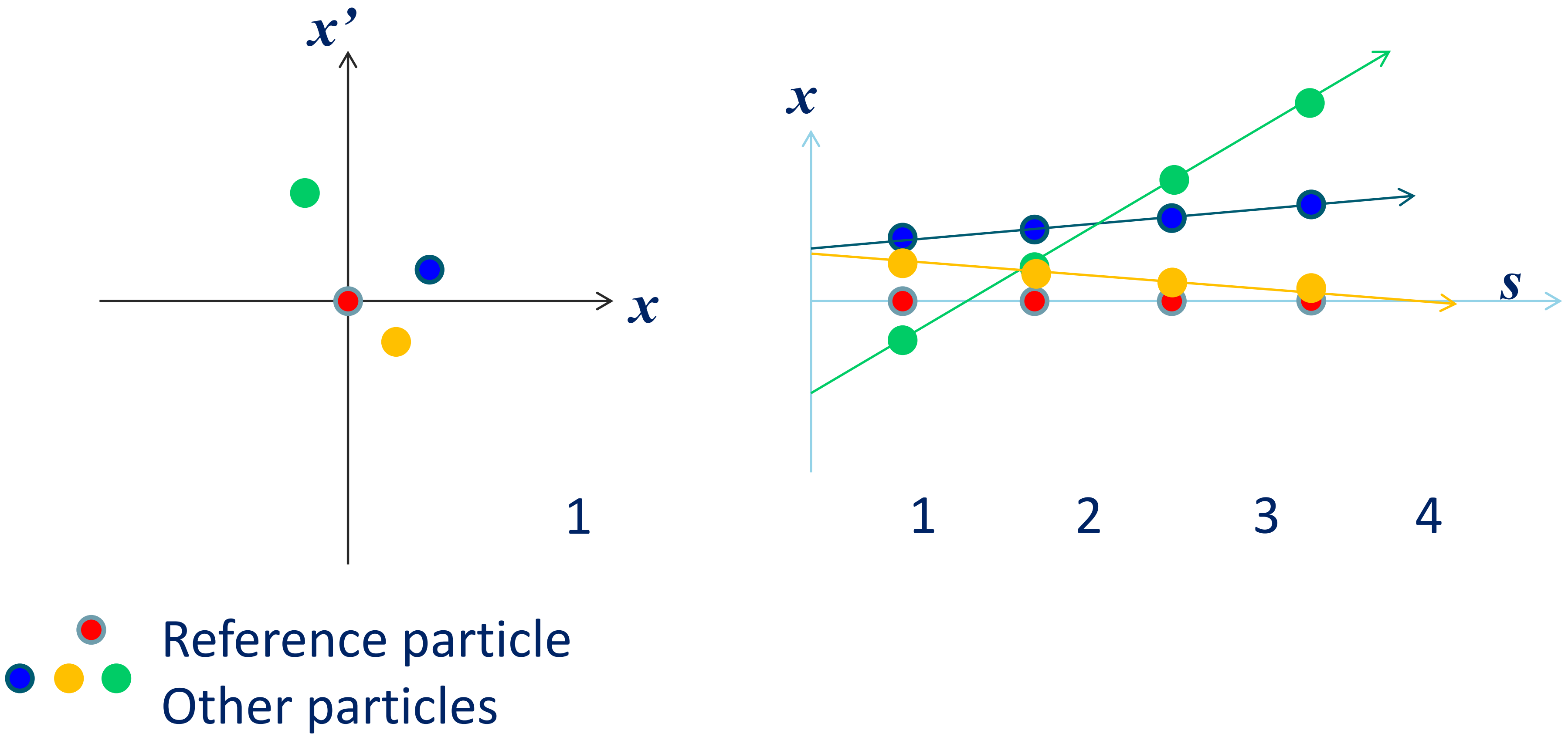


Initial s(0)



Final s

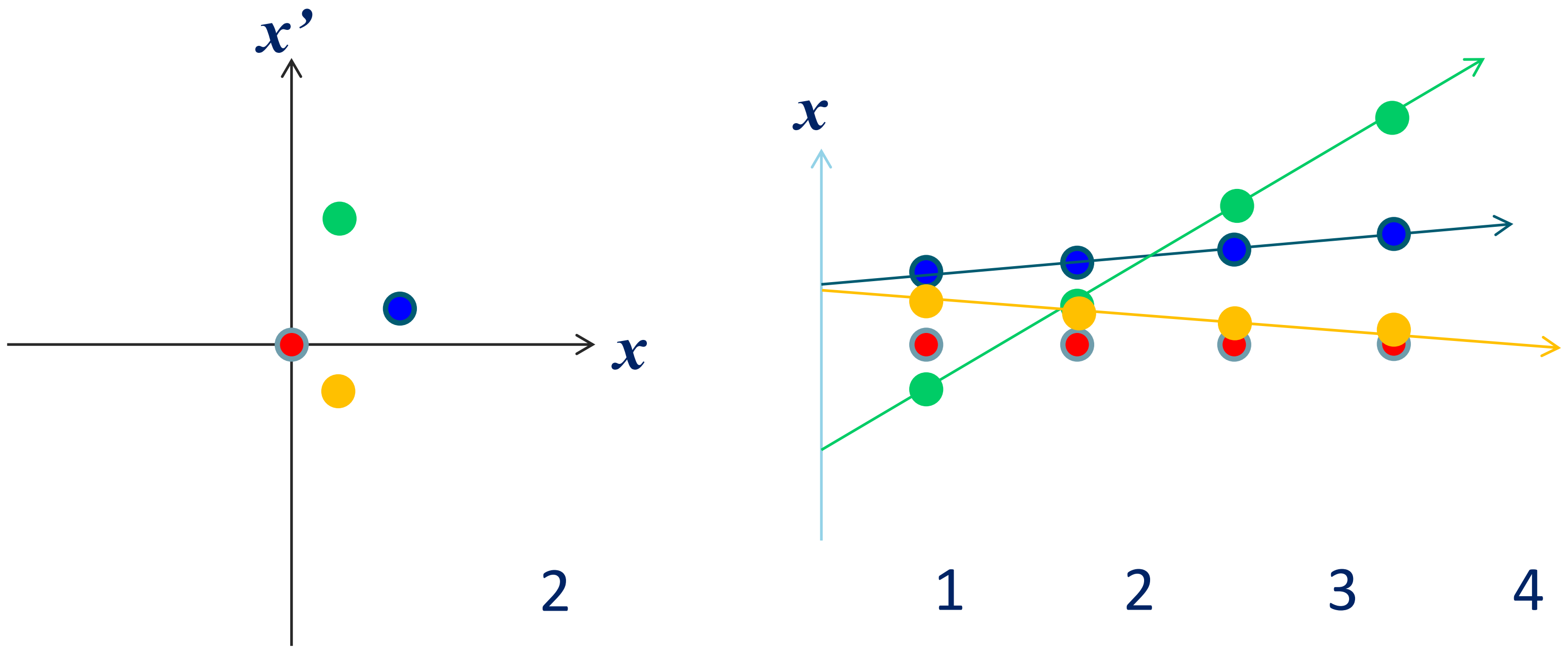
Single Particle to Beam Ellipse

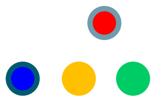


(No magnetic field from 1 to 4)

Slide by C. Biscari

Single Particle to Beam Ellipse

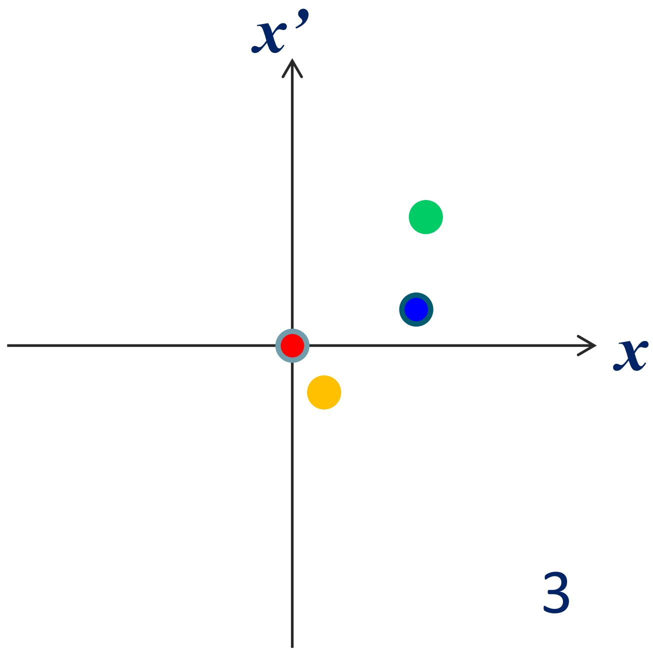


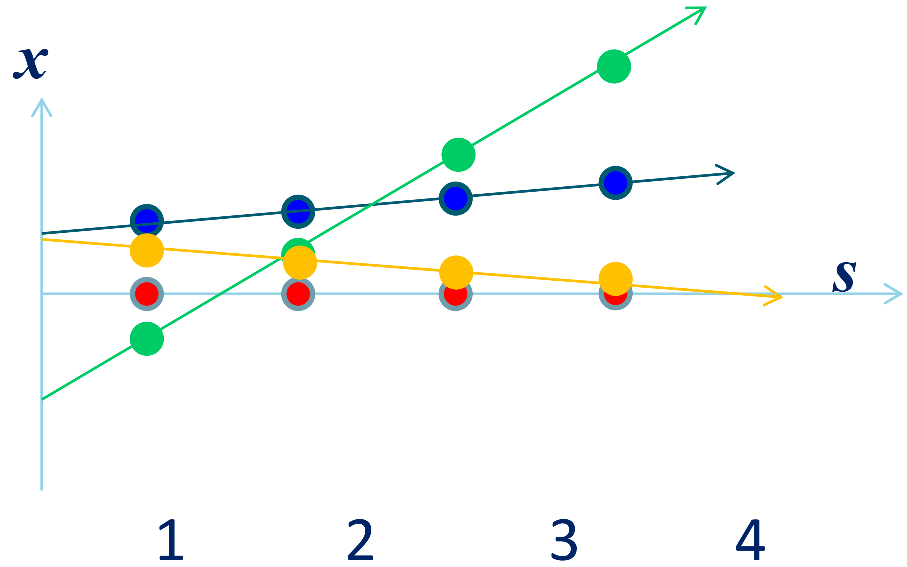


(No magnetic field from 1 to 4)

Slide by C. Biscari

Single Particle to Beam Ellipse





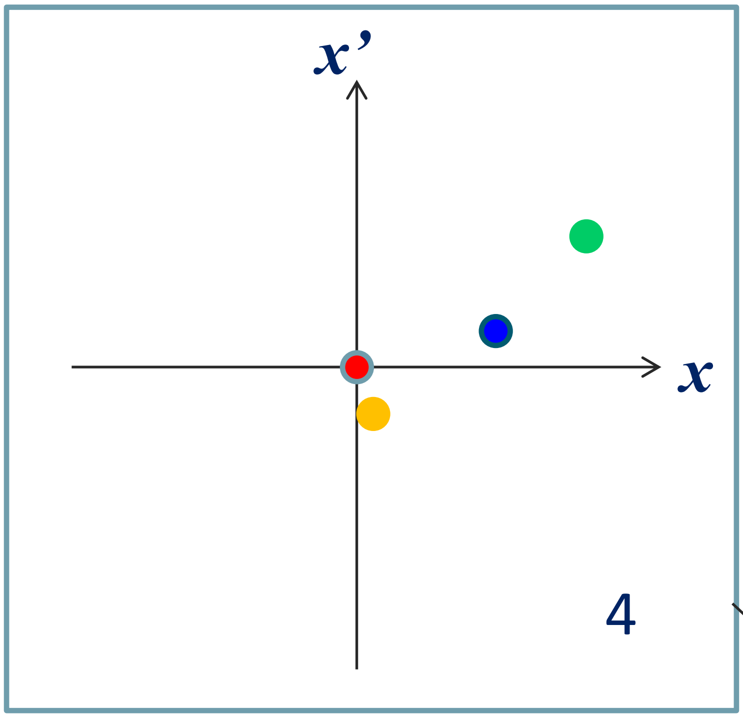
Reference particle

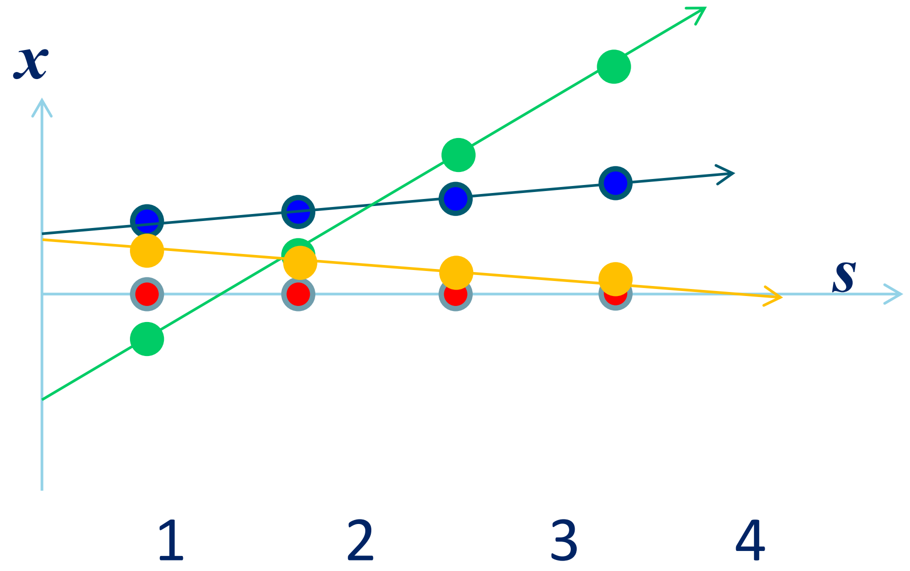
Other particles

(No magnetic field from 1 to 4)

Slide by C. Biscari

Single Particle to Beam Ellipse





Reference particle

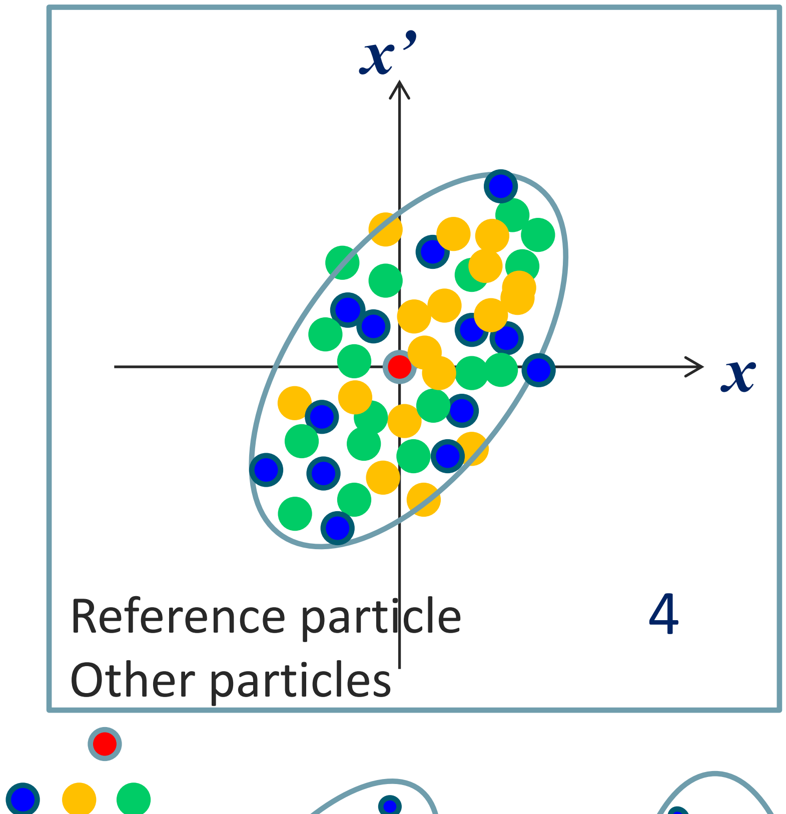
Other particles

Phase space

(No magnetic field from 1 to 4)

Slide by C. Biscari

Single Particle to Beam Ellipse



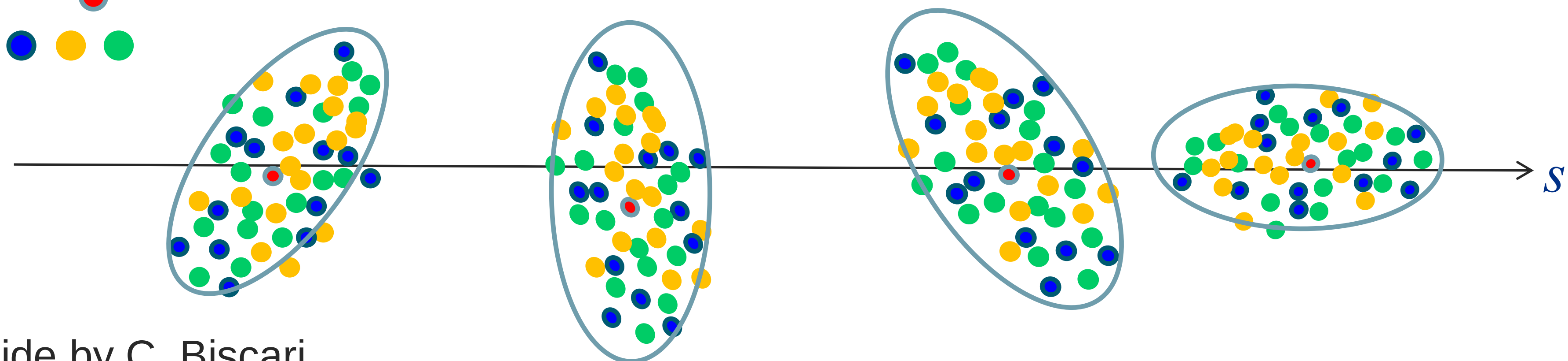
Emittance ~ Area of phase space

Beam will have emittance in each plane

- Horizontal (x, x')
- Vertical (y,y')
- Longitudinal (Time-Energy)

For unaccelerated particles, the area of the ellipse will remain constant, but the ellipse orientation and shape will change along s

Particles will not be evenly distributed in phase space



Slide by C. Biscari

Beam Size

If you know the emittance and the Twiss parameters at a point in the accelerator, the beam dimensions σ_x and σ'_x can be obtained

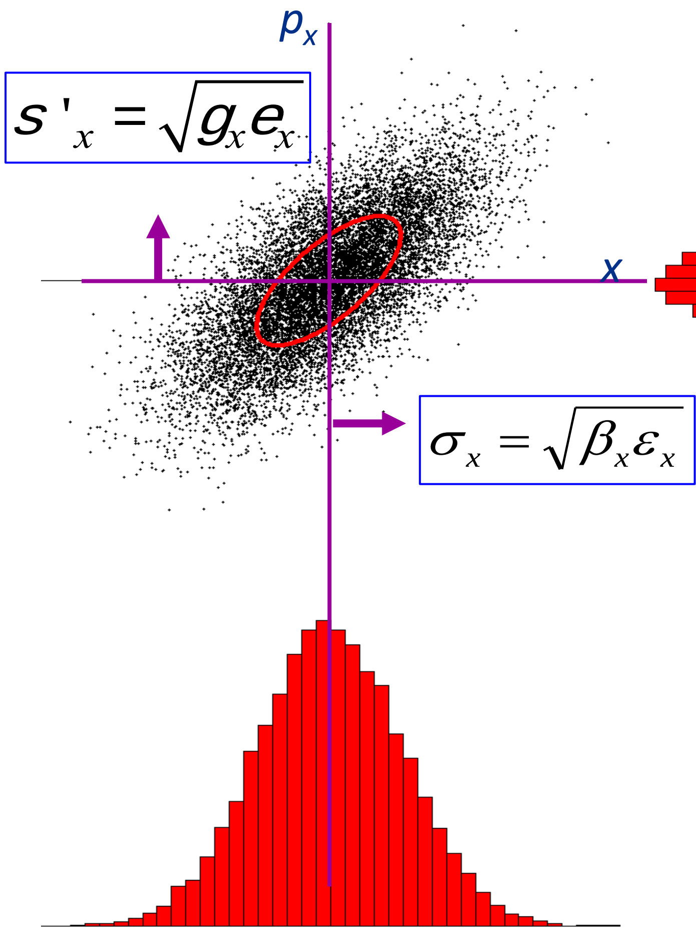
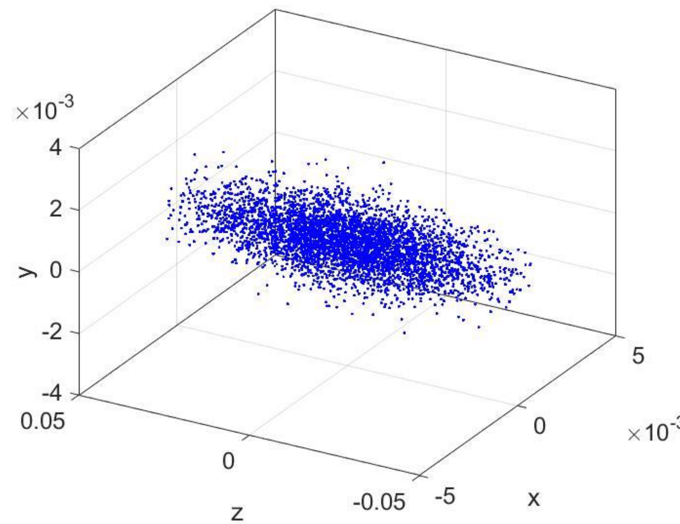


Image from C. Biscari



Gaussian

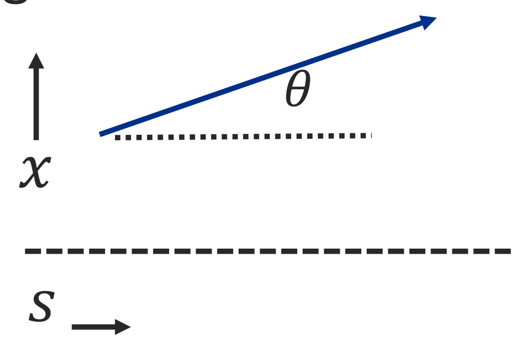
Adiabatic Damping

According to **Liouville's theorem,** the phase space area is constant if there are only conservative forces acting on the beam

– Magnetic fields of dipoles and quadrupoles are conservative

When there is acceleration, the emittance decreases proportional to increase in momentum

$$x' = \frac{dx}{ds} = \frac{dp_x}{dp}$$



We can also define a **normalized emittance**

$$\varepsilon_n = \varepsilon\beta\gamma$$

β γ are relativistic, not Twiss!

With acceleration, the area in the $x - x'$ plane is no longer constant, but in the $x - p_x$ plane will remain constant

RMS Emittance

If the beam doesn't have an elliptical or gaussian distribution, a more general form of the emittance can be defined

$$\epsilon_{RMS} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x \cdot x' \rangle^2}$$

Here $\langle \ \rangle$ is the variance

$$\langle x \rangle = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

average

Luminosity

An important performance measure of a collider is the luminosity, number of particles passing through a cross section per second

For colliding beams[cm^{-2}s^{-1}]:

events per second

production cross section

$$\frac{dN_{events}}{dt} = L\sigma$$

For head on collisions of a bunch of N particles:

number of bunches

particles per bunch

revolution frequency

$$L = \frac{N^2 n f_{rev}}{4\pi\sigma_x\sigma_y}$$

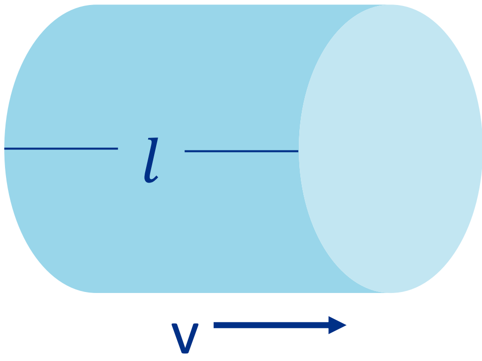
beam size

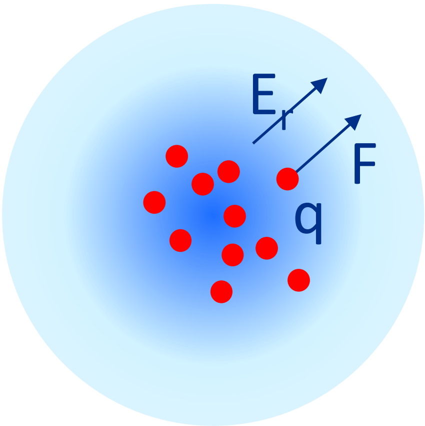
For round beams:

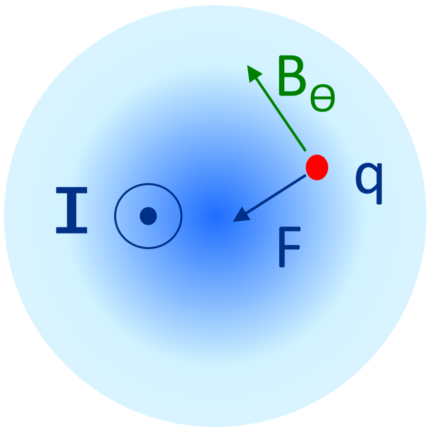
$$L = \frac{N^2 n f_{rev}}{4\pi\epsilon\beta}$$

emittance and betatron function

Space Charge







$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \Rightarrow \quad F_r = q(E_r - vB_\theta)$$

Gaussian distribution:

$$n(r) = \frac{N}{2\pi l \sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

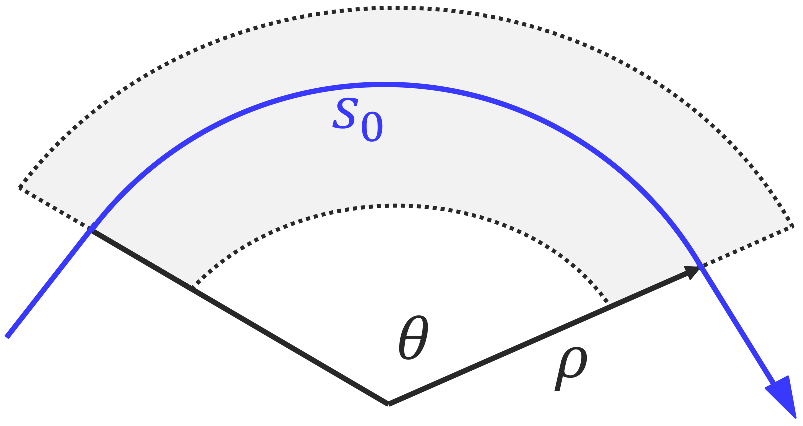
$$F_r = \frac{Nq^2}{2\pi\epsilon_0 l}(1 - \beta^2)\frac{1 - e^{-\frac{r^2}{2\sigma^2}}}{r}$$

Larger issue for lower energy

Longitudinal Motion

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Dipole Bend



$$s_0 = \rho\theta$$

$$qB = \frac{p}{\rho}$$

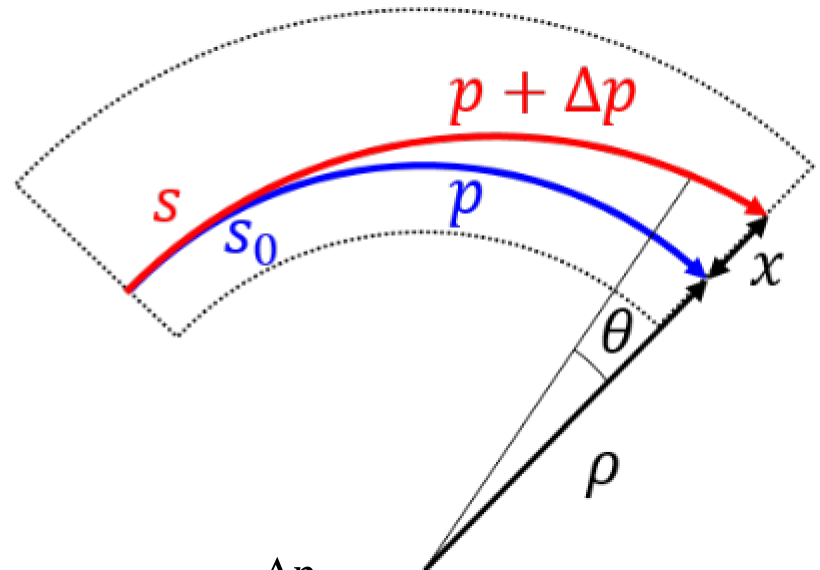
$$\theta = \frac{s_0}{\rho} = \frac{qBs_0}{p}$$

- Bend angle depends on momentum
- Similar to optics where index of refraction depends on frequency

Off-momentum particle

- If particle is off from design momentum (which it will be), it will have a slightly different orbit
- Radius off by x, path length:

$$ds_0 = \rho d\theta \;\Rightarrow\; ds_0 = (\rho + x)d\theta$$



- Relative difference in path length:

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

$$\Delta x(s) = D_x(s)\frac{\Delta p}{p_0}$$

- D_x is dispersion
  – Change in closed orbit (position) as function of momentum

Rob Williamson

Momentum Compaction

- Integrate to get total path length change

$$\Delta C = \oint \mathrm{d}l = \oint \frac{x}{\rho(s_0)} \mathrm{d}s_0 = \oint \frac{D_x(s_0)}{\rho(s_0)} \frac{\mathrm{d}p}{p} \mathrm{d}s_0$$

- Momentum compaction, α_c, is the change in closed orbit length as a function of momentum

$$\alpha_c \equiv \frac{\mathrm{d}L/L}{\mathrm{d}p/p} = \frac{1}{L} \oint \frac{D_x(s_0)}{\rho(s_0)} \mathrm{d}s_0 \approx \frac{1}{C} \sum_i \langle D_x \rangle_i \, \theta_i$$

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p}$$

Updating equations of motion

- Our equations of motion now have an extra term:

$$x'' + K(s)x = 0 \quad \Rightarrow \quad x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0} = \frac{\delta}{\rho}$$

- We can use a sum of solutions to the previous homogenous equations with an additional term: $x = x_{Hom} + D(s)\delta$

$$x(s) = x_0\,C(s) + x'_0\,S(s) + \delta D(s)$$

$$x'(s) = x_0\,C'(s) + x'_0\,S'(s) + \delta D'(s)$$

Previous solutions had this form

New term

Matrix Form

- We can add this to our matrix

Previous transfer matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & d(s) \\ m_{21} & m_{22} & d'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix}$$

$$\frac{\Delta p}{p_0} = \delta$$

Velocity and Kinetic Energy

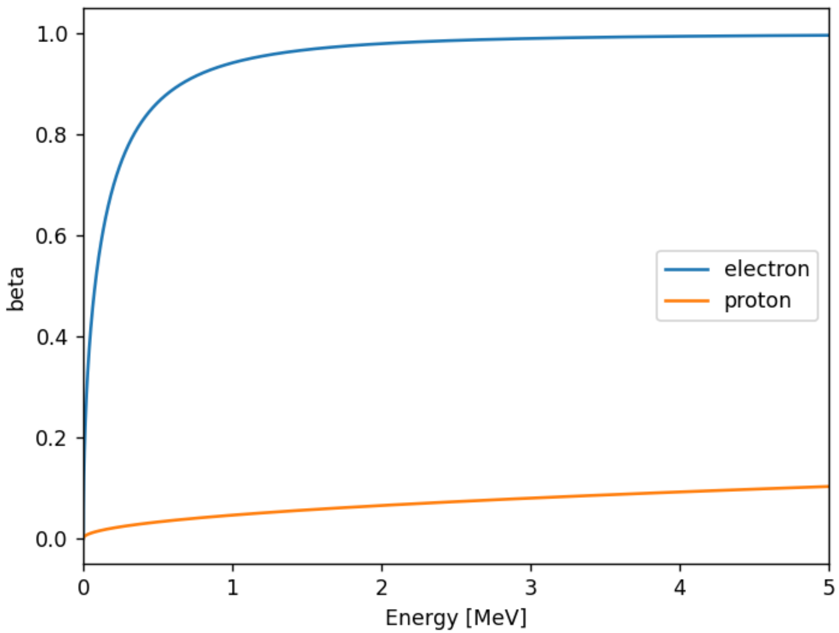
Particle	Rest mass, eV/c²
Electron, e^-	0.511×10^6
Proton, e^+	938×10^6

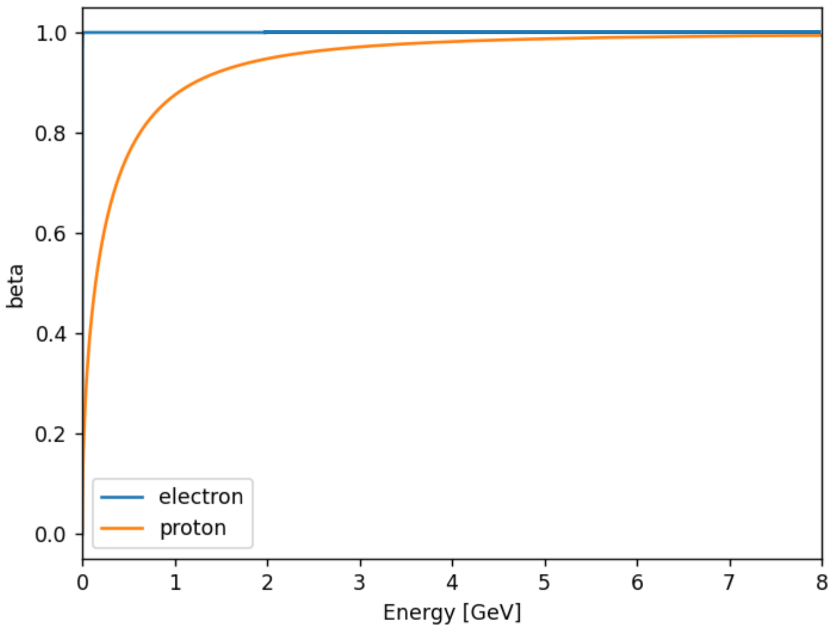
$$\beta = \frac{v}{c}$$

$$U = \gamma mc^2$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$K = U - mc^2$$





Electrons are relativistic at few MeV, protons at GeV

Electrostatic Fields-DC

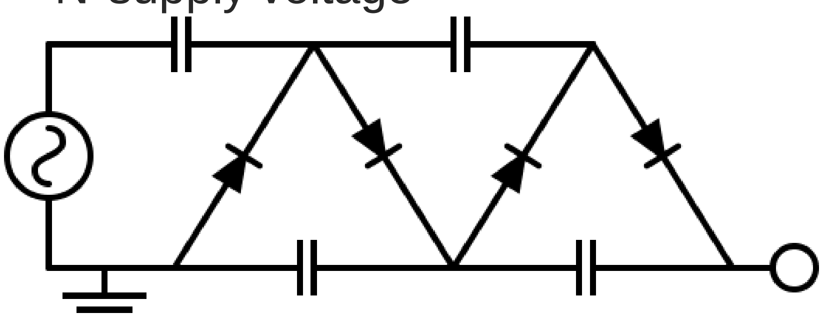
- If we set B=0, we can only get static electric fields
  - Limited energy gain ~60 MeV/q

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

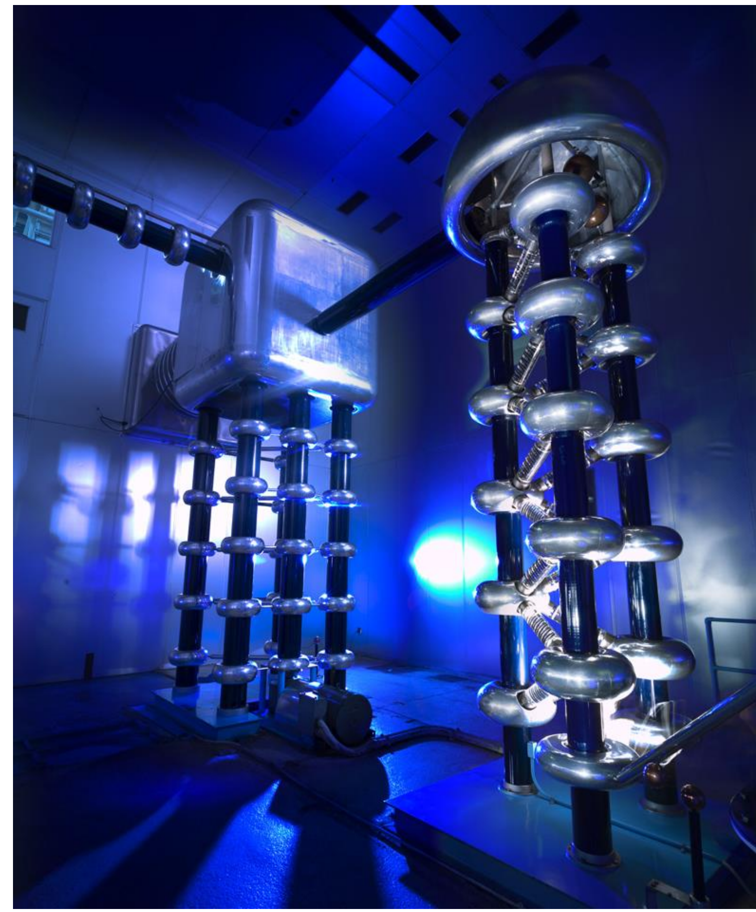
0

- 1929 Robert Van de Graaff
  - Up to ~ 5MV

- 1932 Cockroft-Walton
  - For N number of stages, able to get N*supply voltage



A two-stage Cockcroft–Walton multiplier
Wikipedia.org

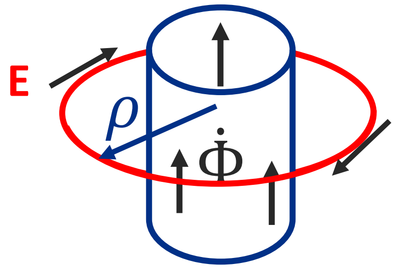


The Need for AC

From Faraday's law, a changing magnetic flux will produce a tangentially directed electric field

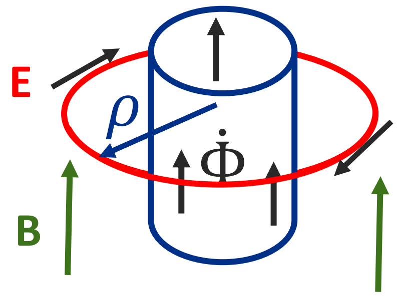
$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$

If there were a cylindrical region of changing magnetic flux, it would produce an E field around the cylinder

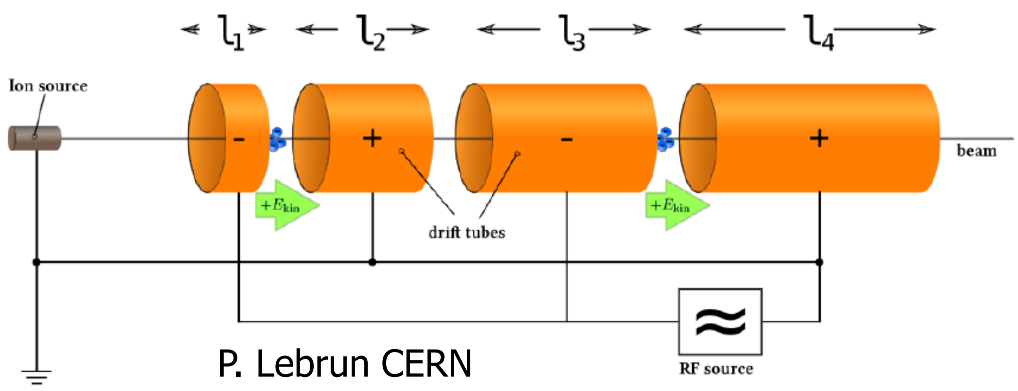


If now there is a B field perpendicular to this E field, we could have a particle travel around the circle at radius ρ

This is the idea behind the Betatron, the first circular accelerator to operate at a constant orbit radius



Alvarez Linac - Drift Tube



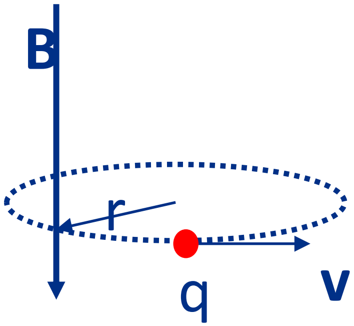
Acceleration occurs in the gaps between the drift tubes, length of tubes grows with velocity

Synchronism condition: $$L = \frac{v}{2f_{RF}}$$

First practical linac (200 MHz, 32 MeV) built by L. Alvarez at Berkeley in 1946

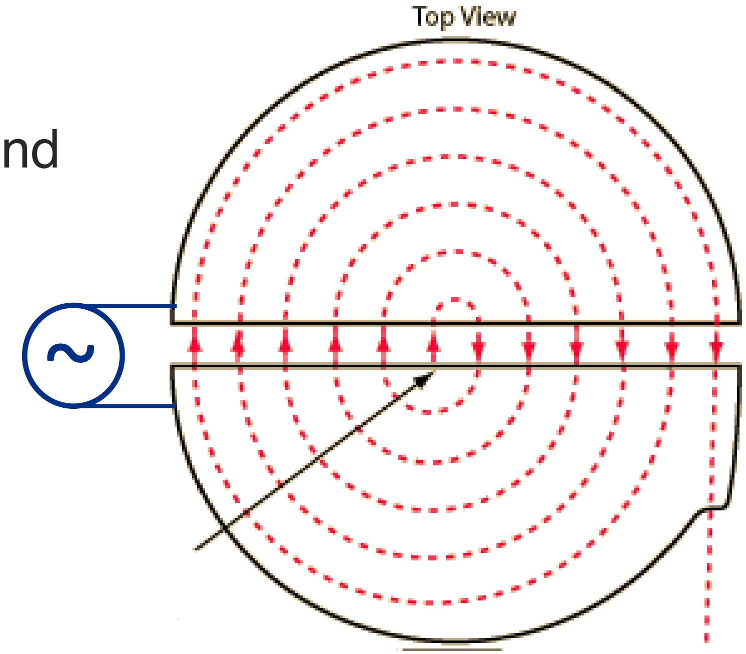
Cyclotron

$$F = q(v \times B) = \frac{Mv^2}{r}$$



Another method is to accelerate particles in a circular path between two D shaped pole pieces and apply an alternating voltage across the gap

Cyclotron frequency: $\omega = \frac{qB}{M}$



Particles must be isochronous "same time" and arrive at the gap at the same time to be accelerated-constant ω

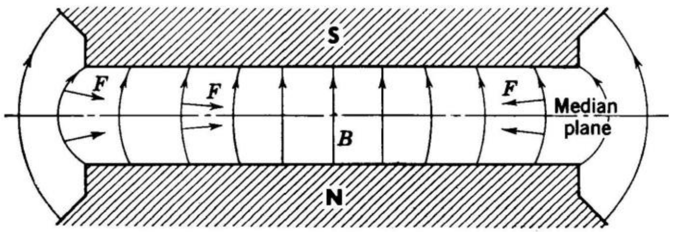
Now add relativity:

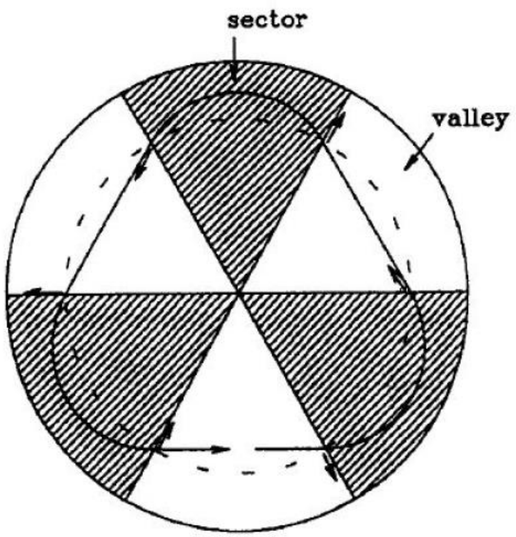
$$\beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \mathrm{M} = \gamma M_0$$

B must increase as γB_0 to maintain isochronicity

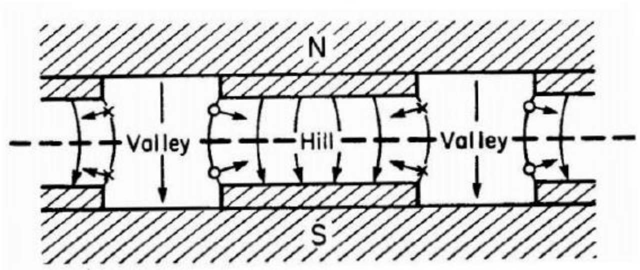
Cyclotrons

- If we radially increase B to maintain isochronicity, we destroy the weak focusing, limited for protons to about ~12 MeV





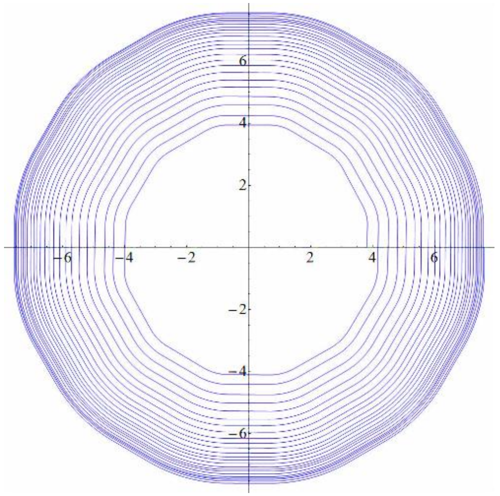
- L.H. Thomas proposed a separated sector cyclotron which allowed the radial field to increase, and gained focusing between the sectors

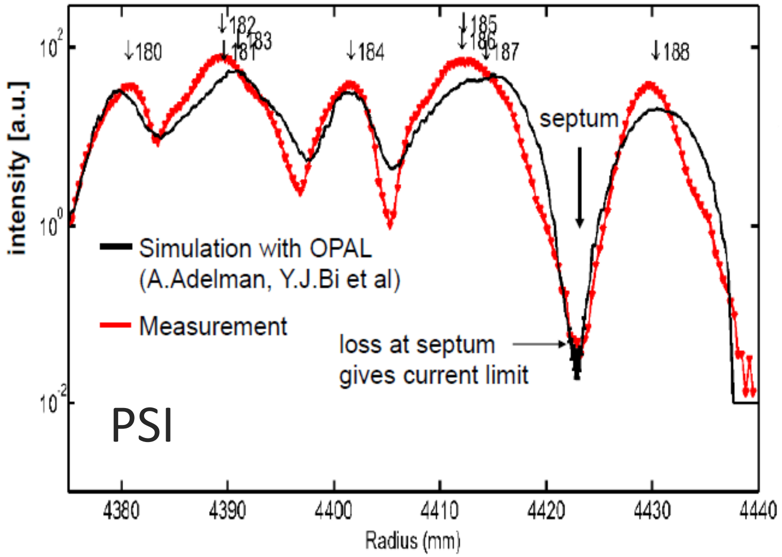


Fermilab

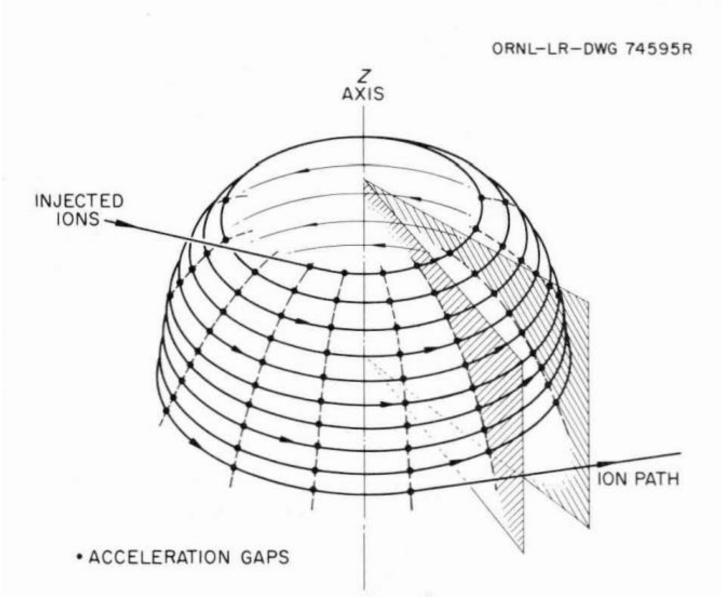
Cyclotrons

- Another issue arises if you don't have enough energy gain per turn, the turns can overlap



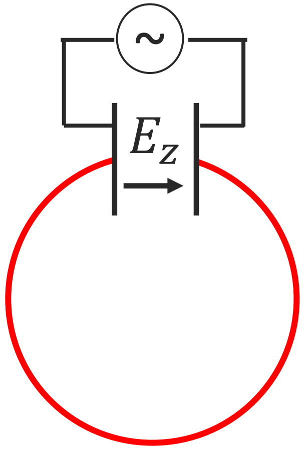


- To overcome this, in 1963 F.M. Russell proposed a "beehive" separated orbit cyclotron (never built)



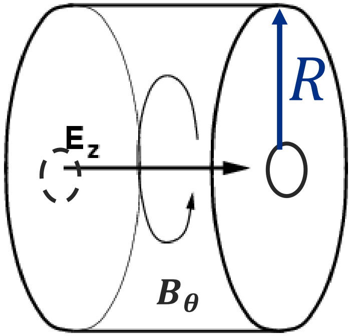
Fermilab

Resonant Cavity-Pillbox



Another setup would be to have an oscillating field in a region(or multiple regions) only when the particles are passing through

We also only want to produce an electric field in E_z, the direction of particle motion, and a magnetic field B_θ



Maxwells' equations reduce to:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \frac{1}{c^2}\frac{\partial E_z}{\partial t} \qquad \frac{\partial E_z}{\partial r} = \frac{\partial B_\theta}{\partial t}$$

Take the derivative w.r.t r

+

plug in to eliminate B_θ

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

Resonant Cavity

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

A solution with frequency ω will have the form:

$$E_z = E(r)e^{i\omega t} \qquad E'' + \frac{E'}{r} + \left(\frac{\omega}{c}\right)^2 E = 0$$

This has the form of Bessel's equation of zero order, with know solutions:

$$E(r) = E_0 J_0\left(\frac{\omega}{c} r\right)$$

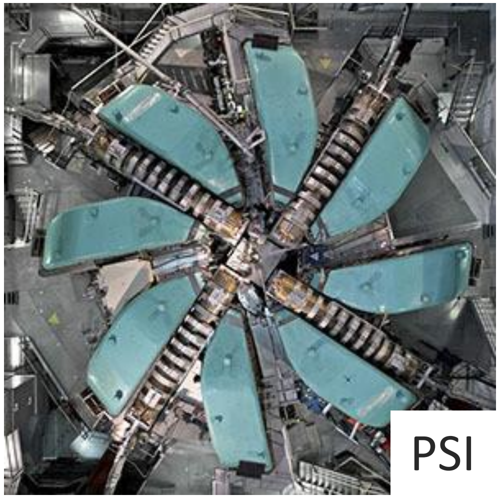
The surface of the pillbox is conducting, so at r=R, E=0 and the lowest frequency mode will be:

$$\frac{2\pi f}{c} R = 2.405$$

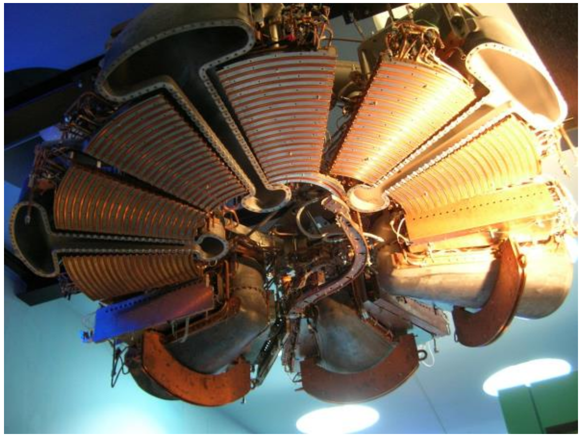
For a reasonable R ~30 cm, the frequency will be in the 400 MHz range- RF range

Cyclotrons

- D. W. Kerst proposed increasing the focusing by increasing the angle the particles make with the sectors
  - TRIUMF, Texas A&M, Michigan State, PSI
  - Problem is it creates an odd shape gap to put an accelerating structure



- TRITRON, was able to fully separate orbits through a combination of edge focusing, individual gradient windings along each sector
  - RF cavities were superconducting



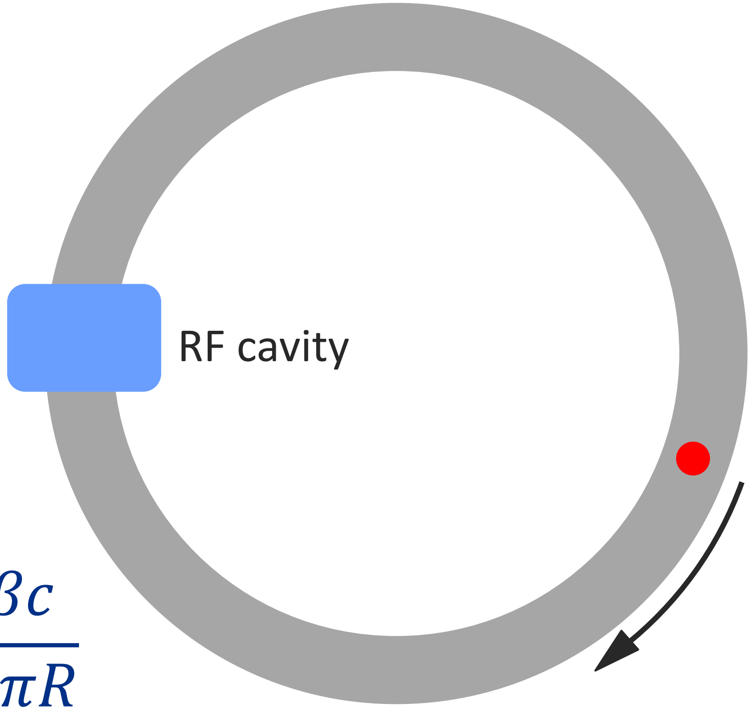
Fermilab

Synchrotron

“Particles should be constrained to move in a circle of constant radius thus enabling the use of an annular ring of magnetic field...which would be varied in such a way that the radius of curvature remains constant as the particle gains energy through successive accelerations by an alternating electric field applied between coaxial hollow electrodes.” - Mark Oliphant

- B increases synchronously with rising E
- Cavity has field oscillating with $f_{RF} = hf_{rev}$
- Synchronous particle
  - Energy gain per turn:

$$\Delta E = \sim qV sin\varphi_s$$



$$f_{rev} = \frac{\beta c}{2\pi R}$$

Talk on Light Sources and Applications Monday 9:30

RF Cavity

There is a limit to the effective longitudinal length of the cavity

- If too long, the particle would be in the cavity when the field flipped and would decelerate the particle

The change in energy of a particle crossing a gap is given by:

max voltage

particle's initial velocity

$$\Delta E = \int_{t_0}^{t_0+T} \frac{qV_0}{g} \cos(\omega t)\, v dt$$

RF frequency, not revolution ω

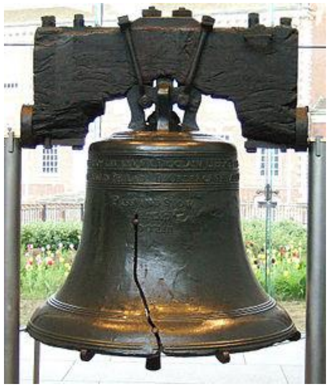
gap length

The transit time T through the cavity must satisfy:

$$g = \int_{t_0}^{t_0+T} v dt$$

Quality Factor

- The quality factor, Q, is a figure of merit for a cavity
  - The higher the Q the better
  - High quality EM resonators: Typical $Q_0 > 10^{10}$



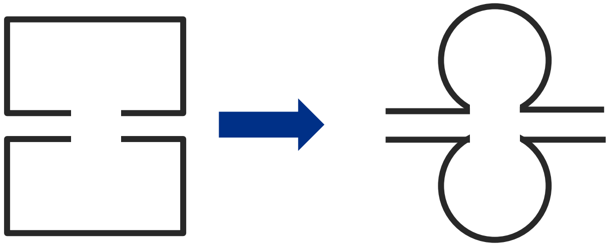
Ring for ~1 year

- Q is a ratio of the total stored energy to power lost

$$Q = \frac{\omega U}{P}$$

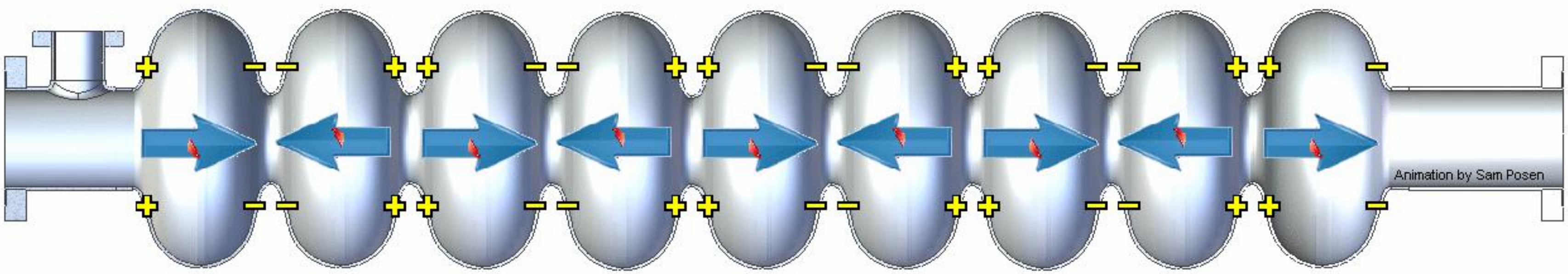
- Q is a measure of the power loss in the walls of the cavity due to current flowing through resistive walls

- The power loss can be reduced by
  - Shaping the pillbox surface
  - Making the walls out of superconducting material



A single cavity is good, but multiple cavities much better!

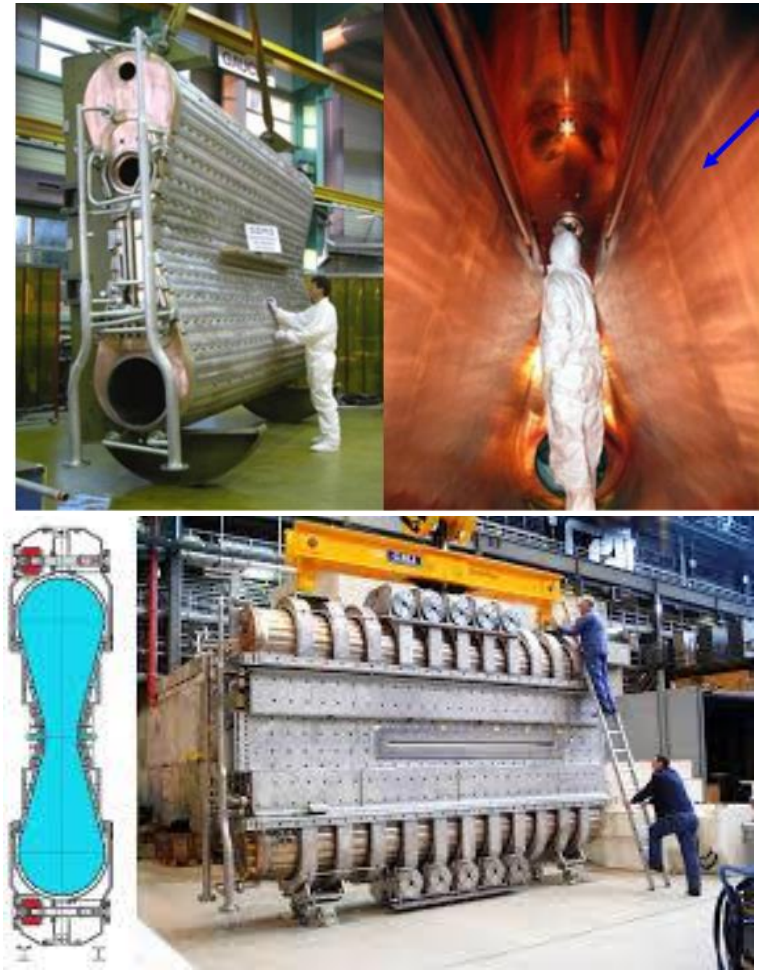
Often multicell cavities are grouped together and run from a single source

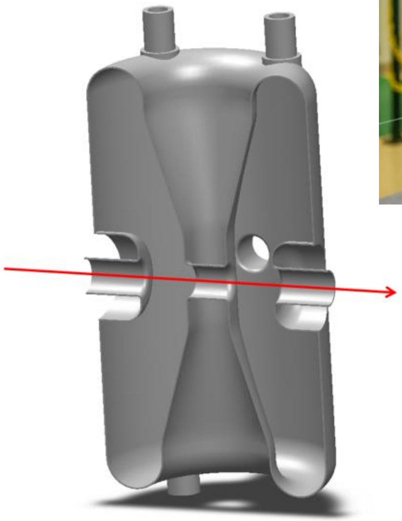


GIF produced by Sam Posen

Other Cavity Shapes

Photo: Ryan Postel, Fermilab





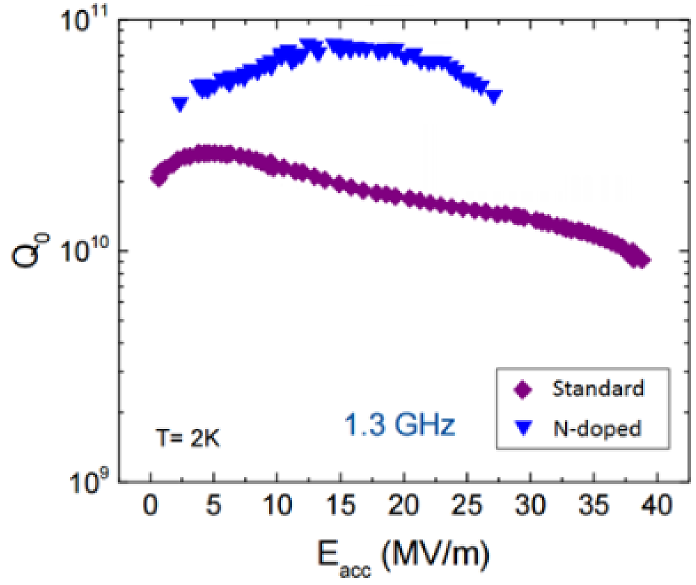
J. Holzbauer Fermilab

M. Seidel PSI

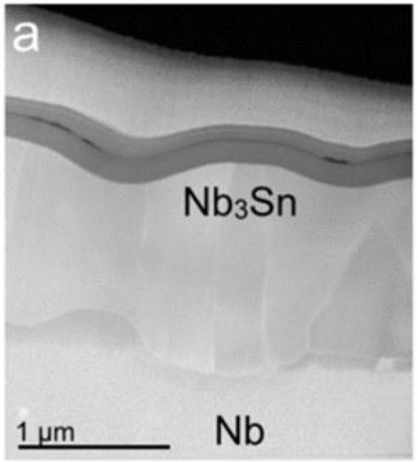
R. Leuxe CERN

Superconducting Cavities

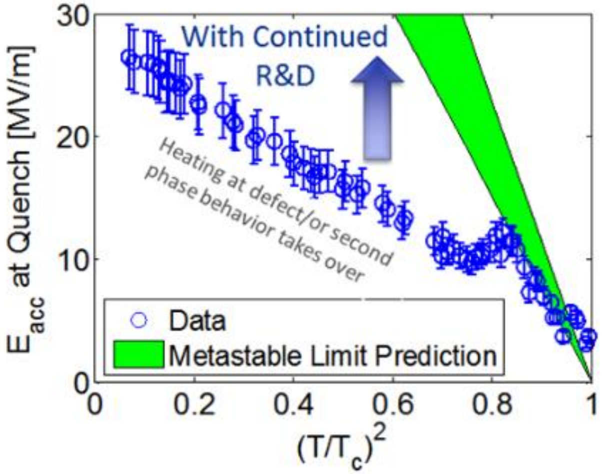
- Fermi has a large superconducting cavity research group
- Working to improve Q through
  - doping (Nitrogen)
  - efficient Meissner expulsion
  - coating cavities with superconducting Nb_3-Sn



https://td.fnal.gov/srf-rd/



TEM cross sectional image of a Nb_3Sn layer on a niobium substrate

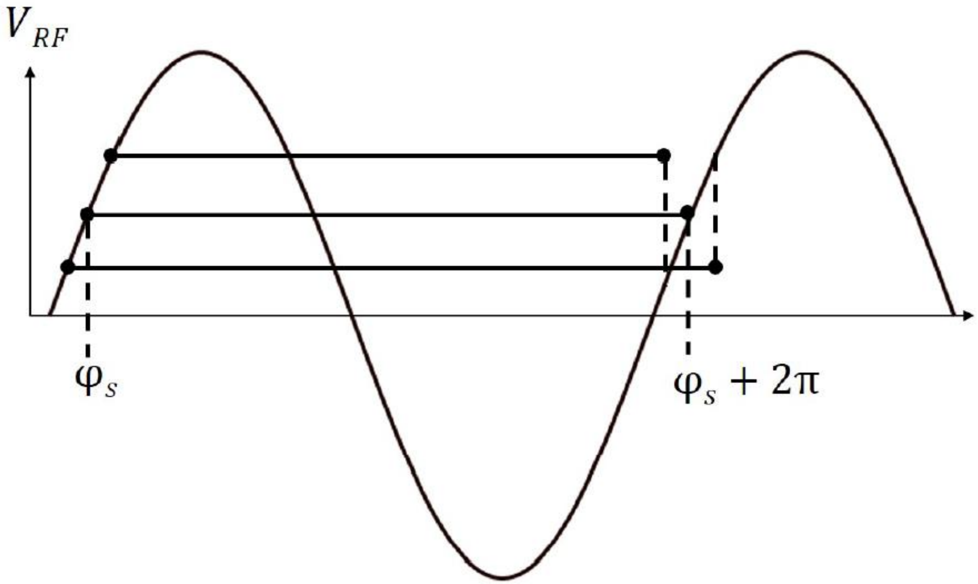


Voltage in RF Cavity

- A cyclotron or synchrotron is designed so the reference particle hits the RF wave at a desired phase φ_s

$$V_{RF}(t) = V sin(\omega_{RF} t + \varphi_s)$$

- A synchronous particle would return to the same location on the voltage curve after one period (revolution)



Frequency change with changing momentum

- Revolution frequency change:

$$f_{rev} = \frac{\beta c}{2\pi R}$$

$$\frac{df}{f} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

Change in orbit length

Change in velocity

- In terms of momentum compaction

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p}$$

$$\frac{df}{f} = \frac{d\beta}{\beta} - \alpha_c \frac{\Delta p}{p} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{\Delta p}{p}$$

Transition Energy

- Relative change in revolution frequency:

$$\frac{df}{f} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{\Delta p}{p} = \eta\frac{\Delta p}{p}$$

- η is the slip factor

$$\eta = \frac{1}{\gamma^2} - \alpha_c$$

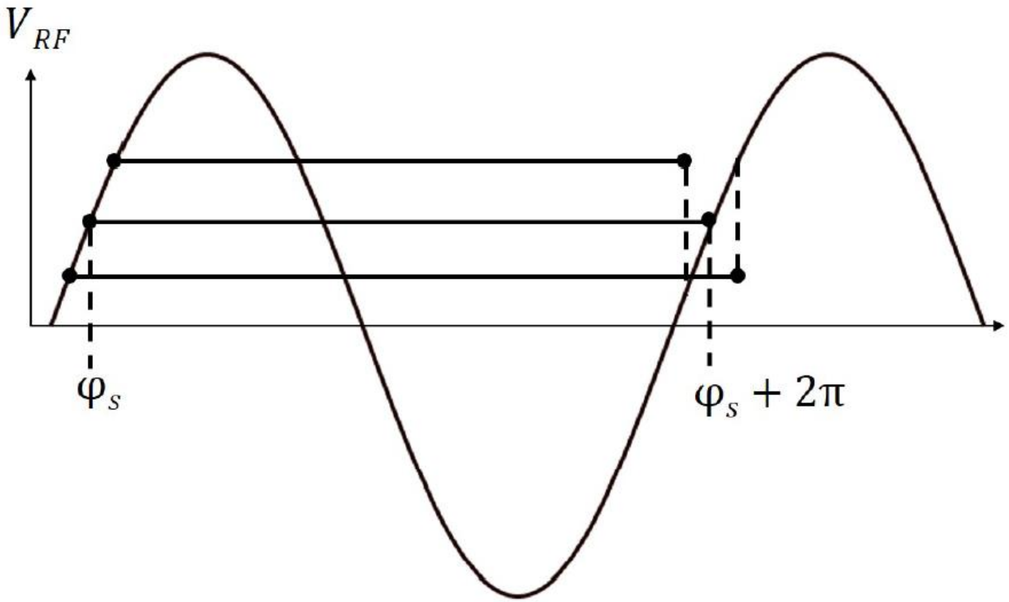
- Transition energy when $\eta = 0$

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha_c}}$$

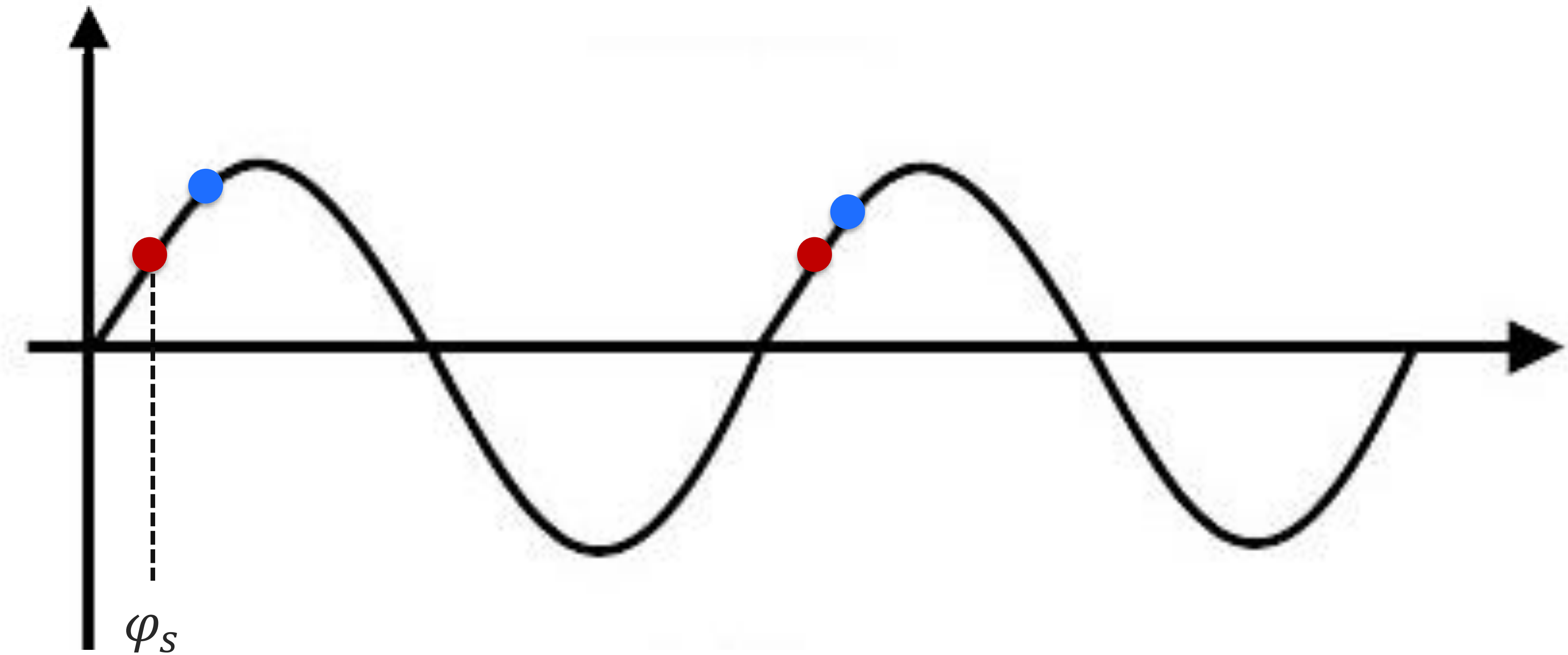
- Below transition, frequency is dominated by $\frac{d\beta}{\beta}$ term
  - Particles behave ~non-relativistically
- Above transition, $\frac{\Delta L}{L}$ term dominates
  - Particles behave relativistically

Phase stability below transition, η >0

- Particles with higher energy hitting the RF wave earlier in its ramp up cycle and receiving a smaller energy gain.
- The slower particles hit the RF wave after the reference particle where the RF wave has risen higher and thus receive a larger energy gain.
- By the next RF cavity, or on the next RF cycle, the particles have been adjusted toward the timing of the reference particle, and oscillate about its timing.

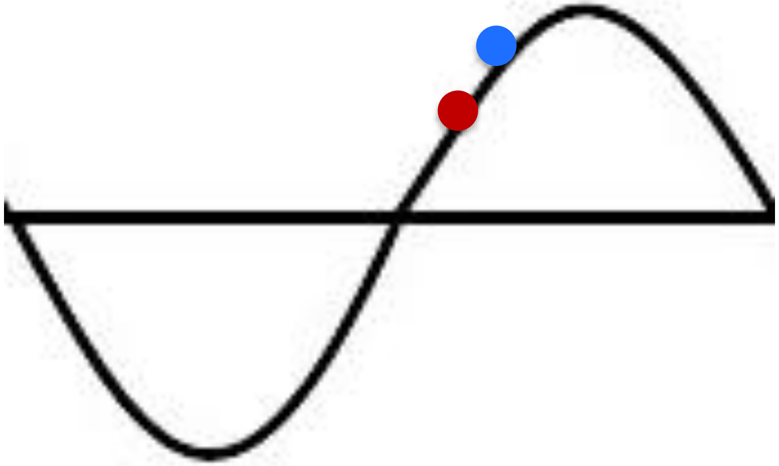


Below Transition



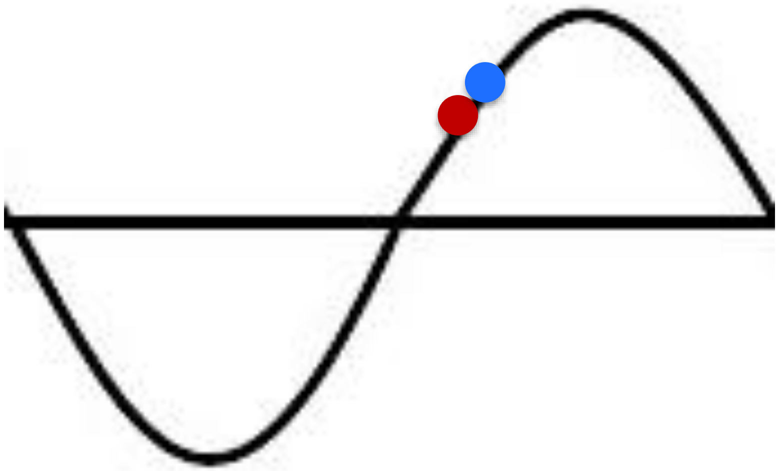
- Red particle revolution equals RF frequency
- Blue particle is later in time, sees a higher voltage, gains more energy, less late to the next cycle
- …

Subsequent turns

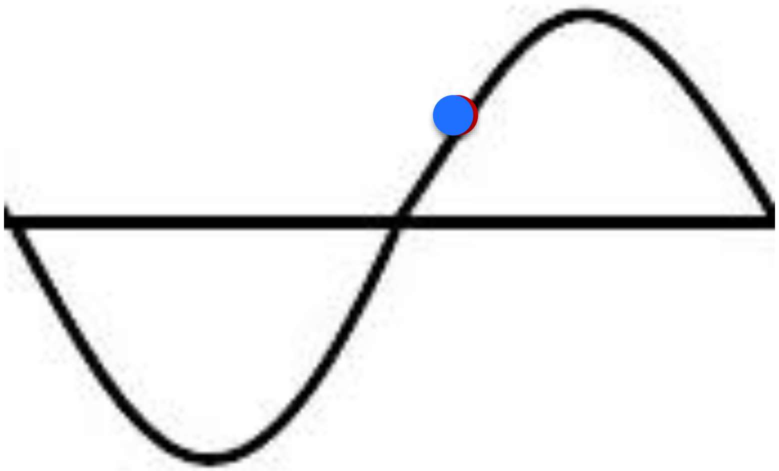


Fermilab

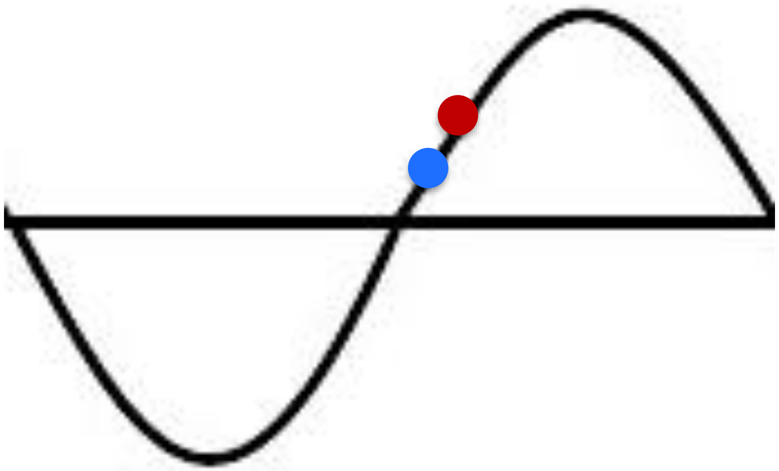
Subsequent turns



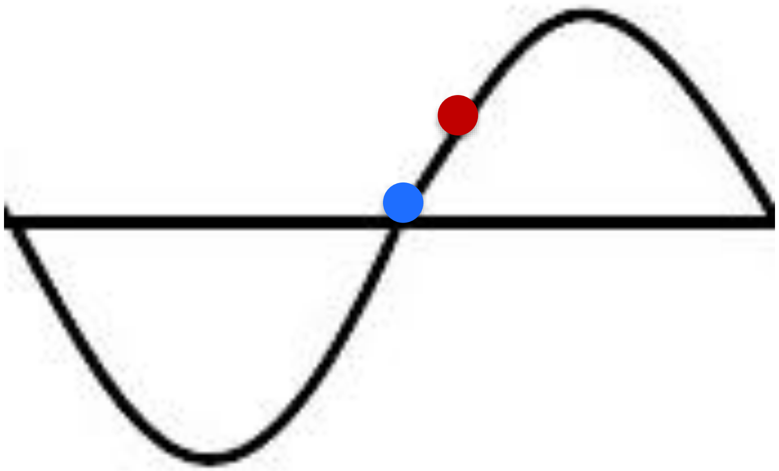
Subsequent turns



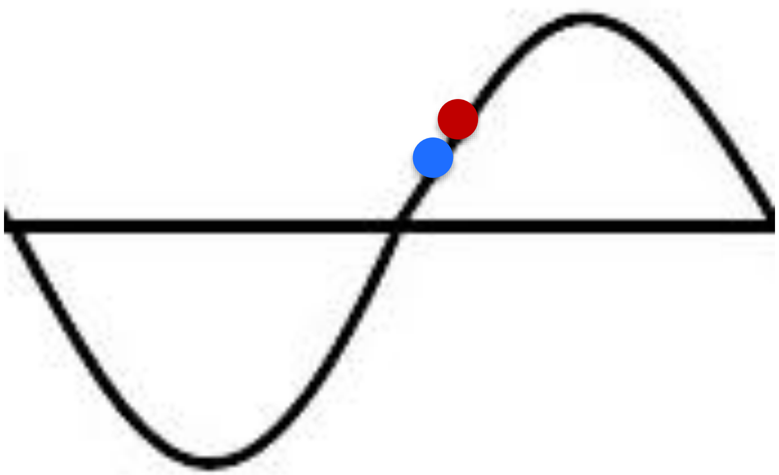
Subsequent turns



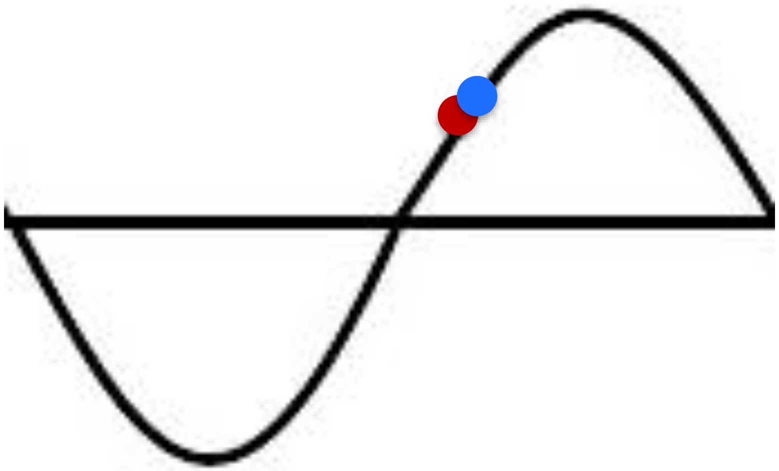
Subsequent turns



Subsequent turns

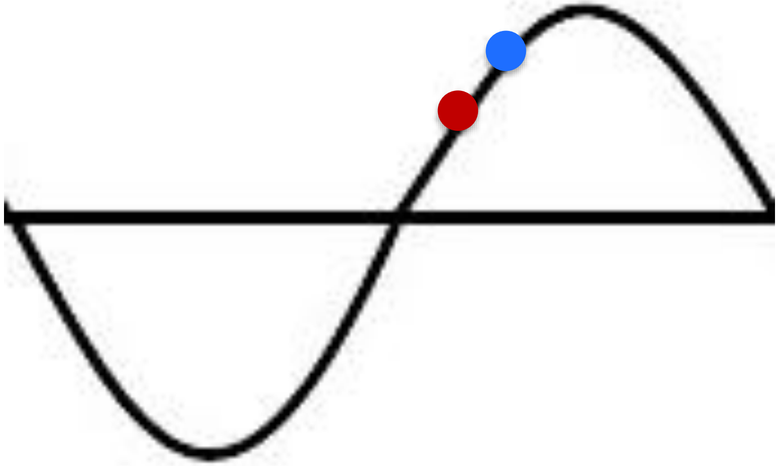


Subsequent turns



Fermilab

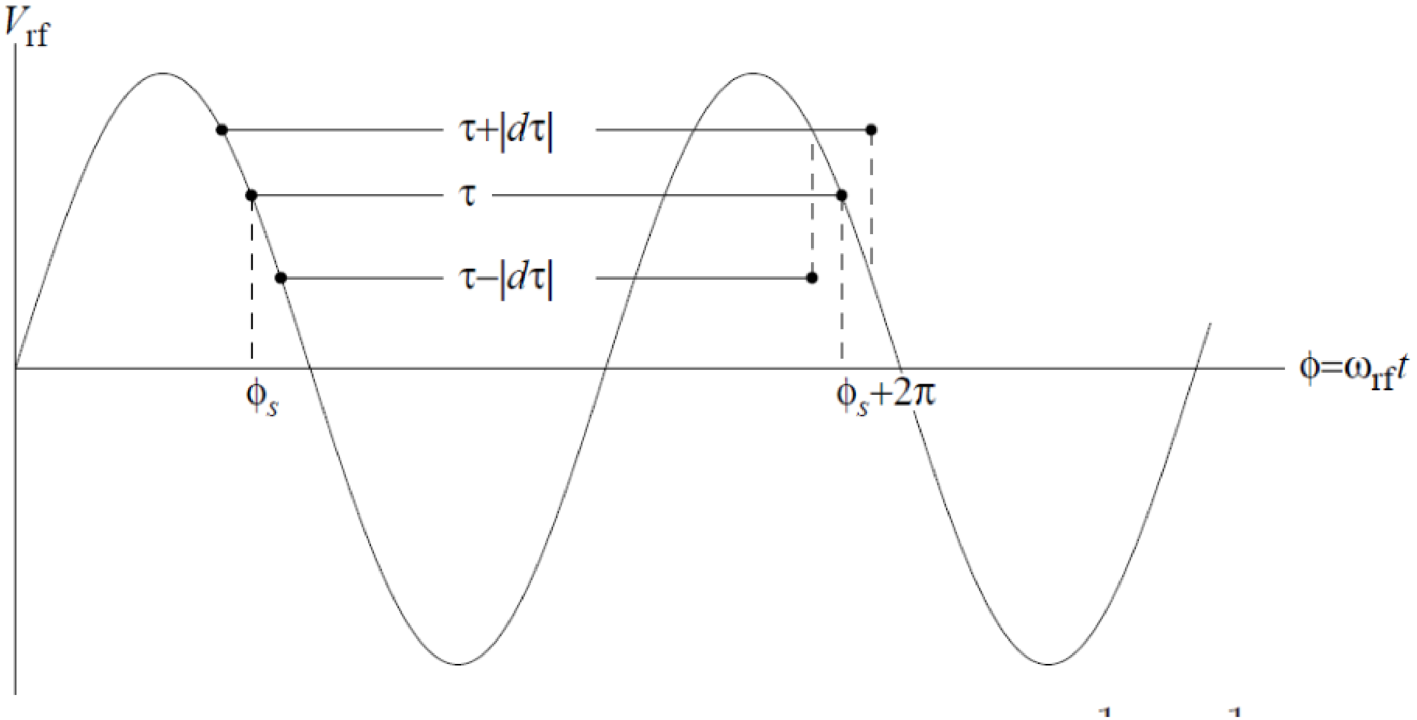
Subsequent turns



Synchrotron Oscillation

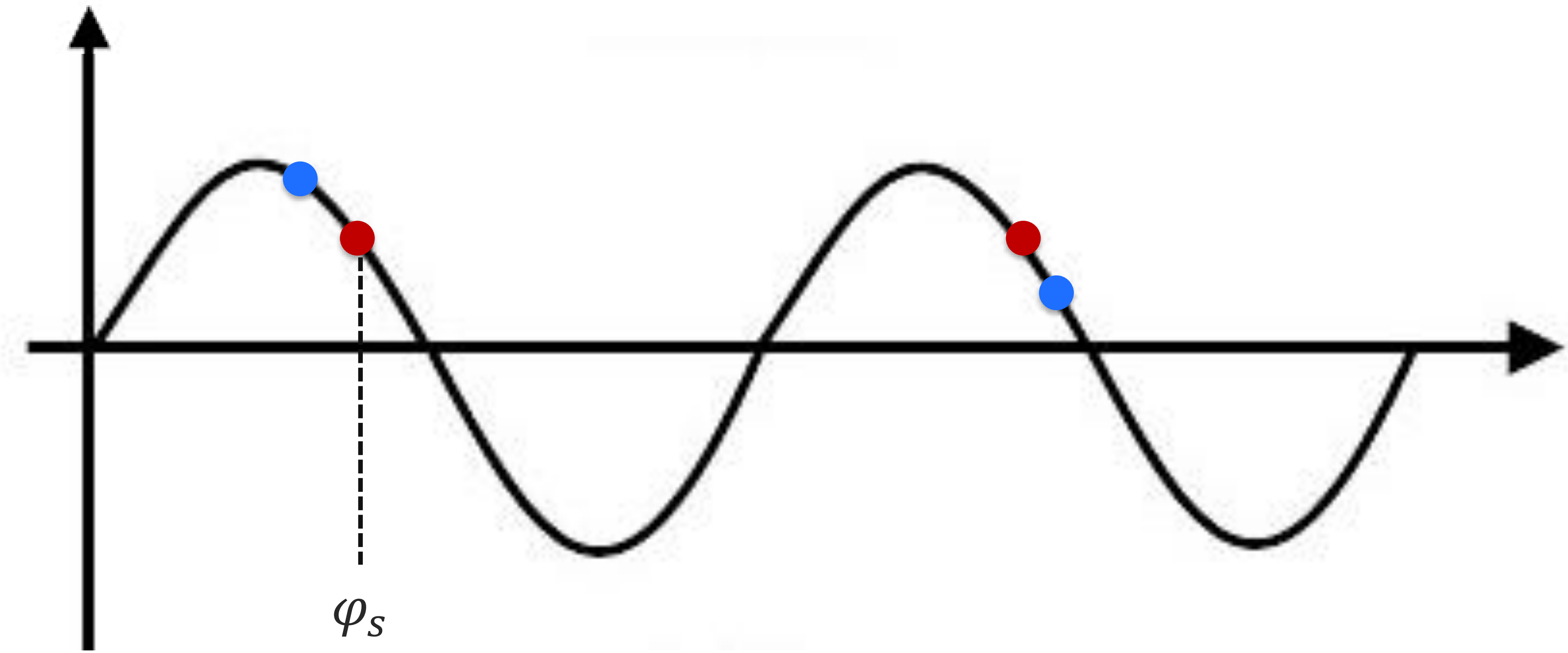
- The blue particle has made one oscillation around the red particle.
- This motion is similar to a pendulum with the RF forming a potential well
- This stable region is called the RF bucket
- For particles below transition, we sat on the rising edge of the sine wave
- For particles above transition, we shift to the falling edge of the sine wave

Above Transition , η <0

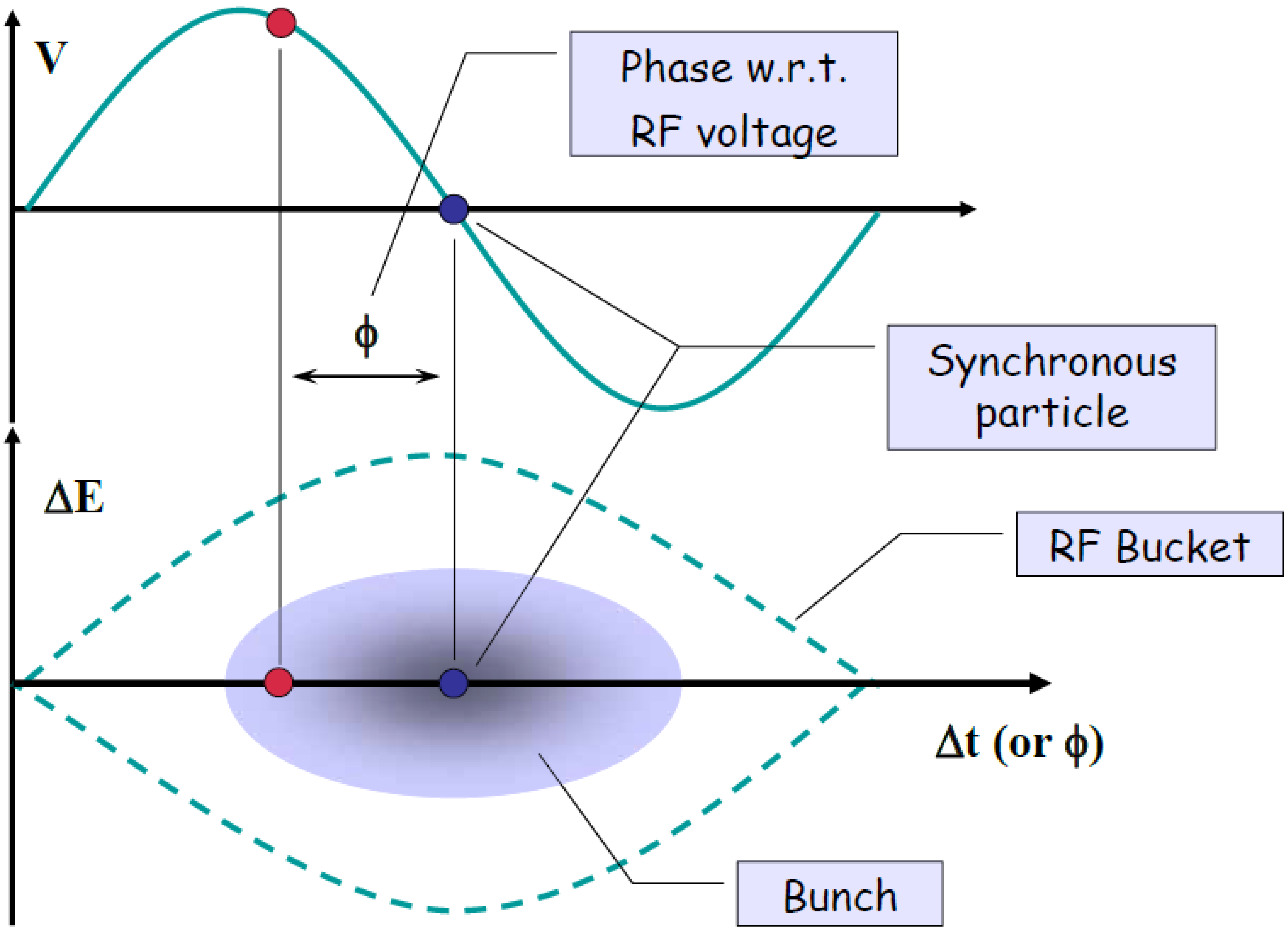


Fermilab

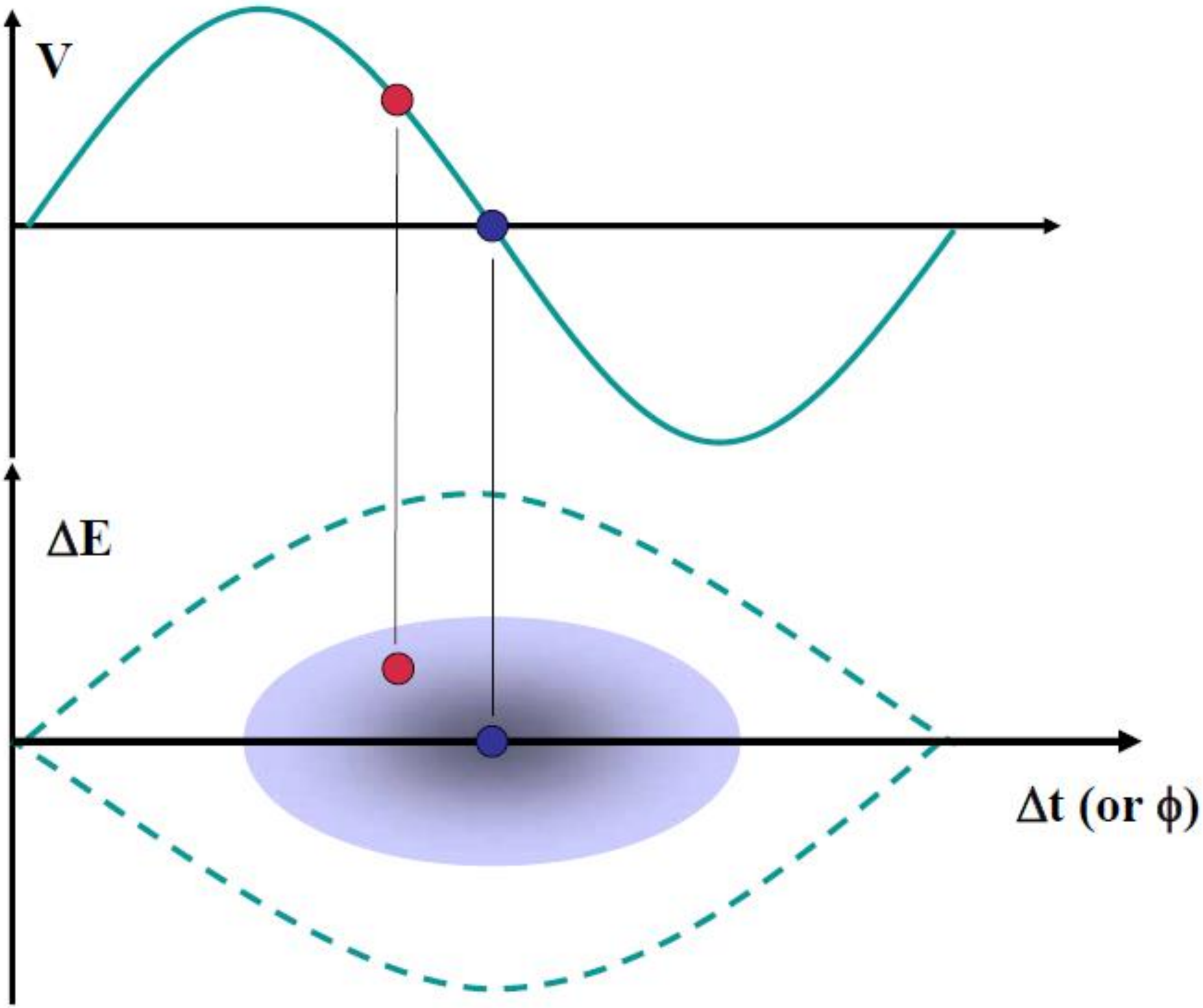
Above Transition

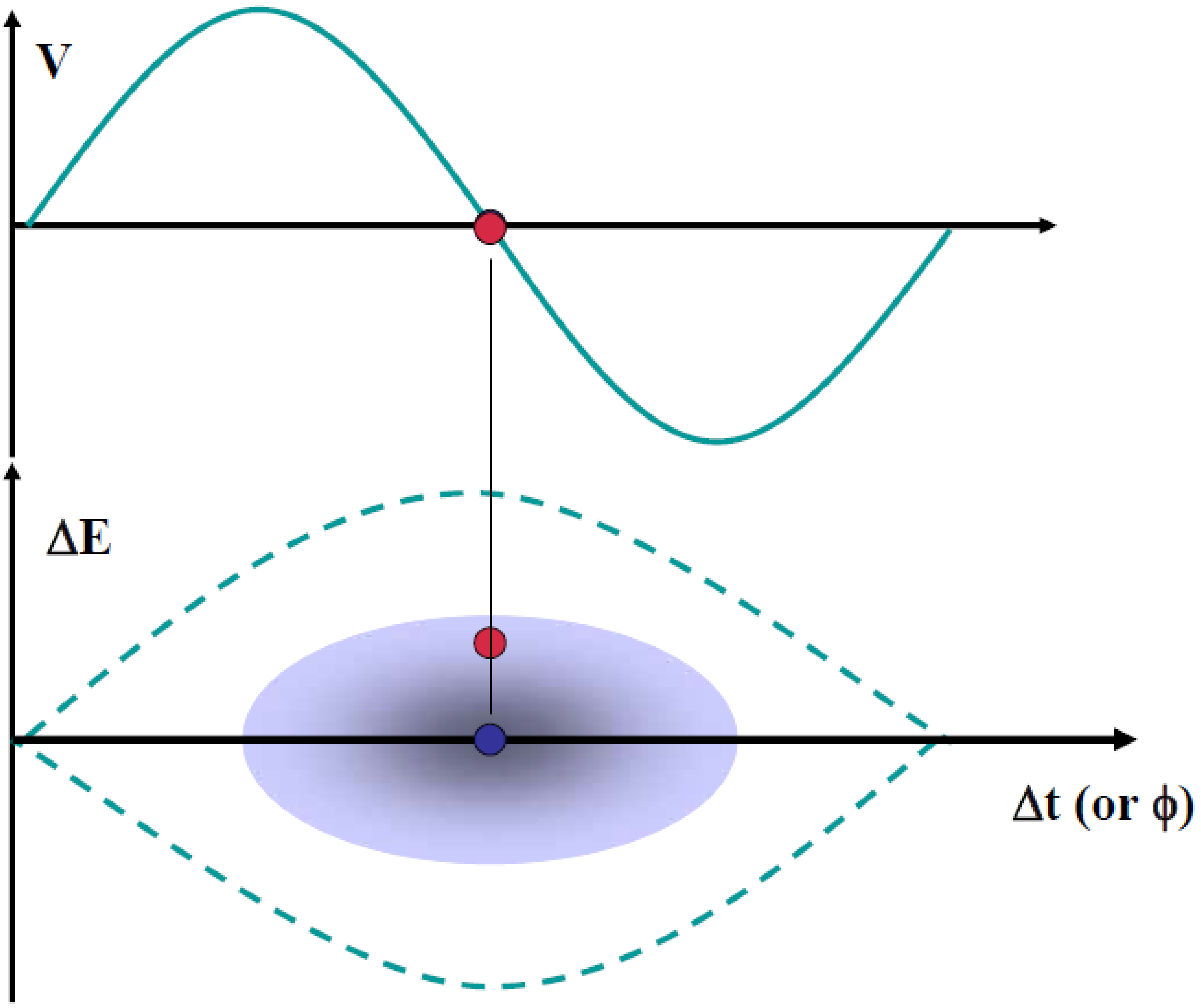


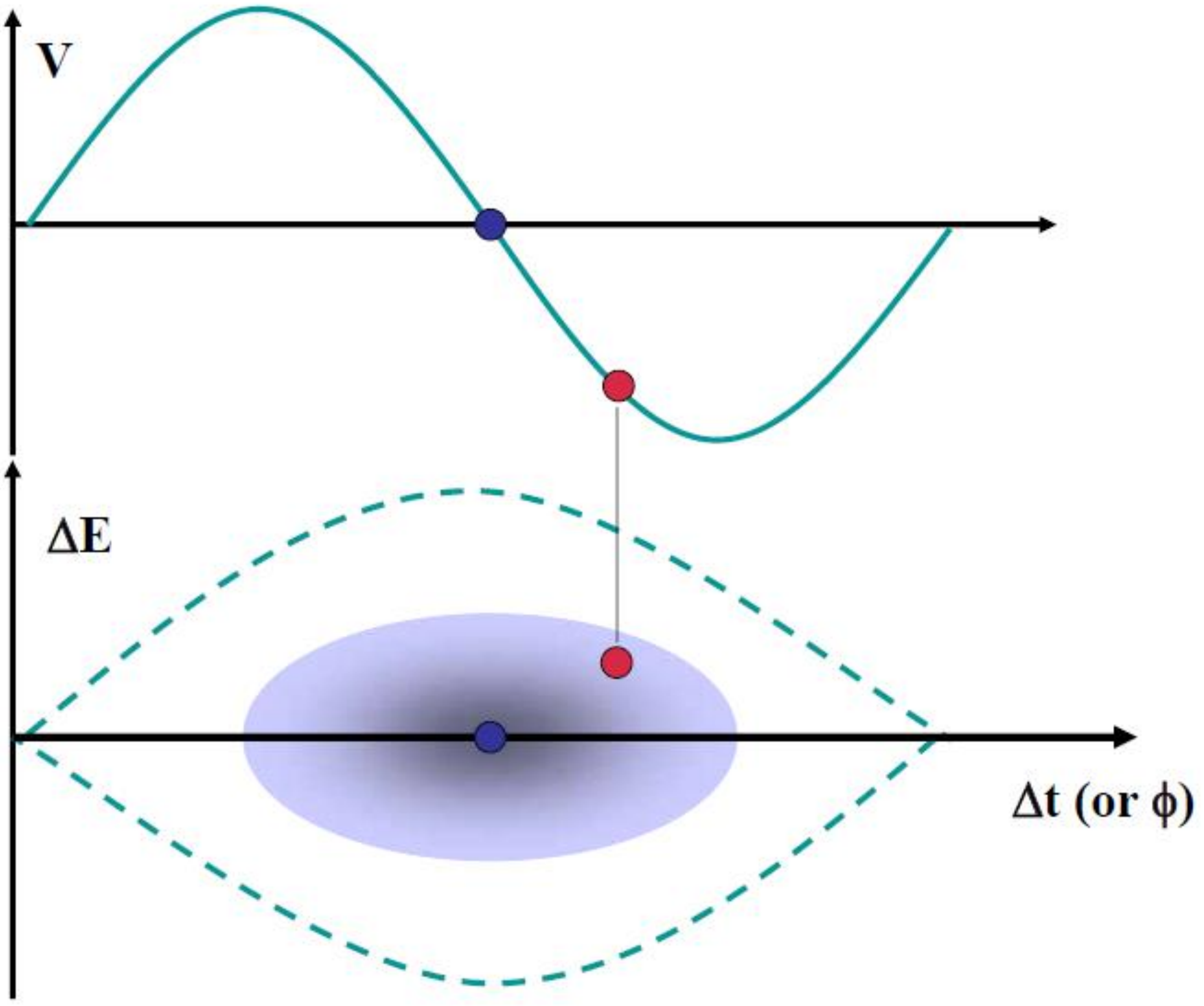
- Higher momentum particle has a lower f than the synchronous particle

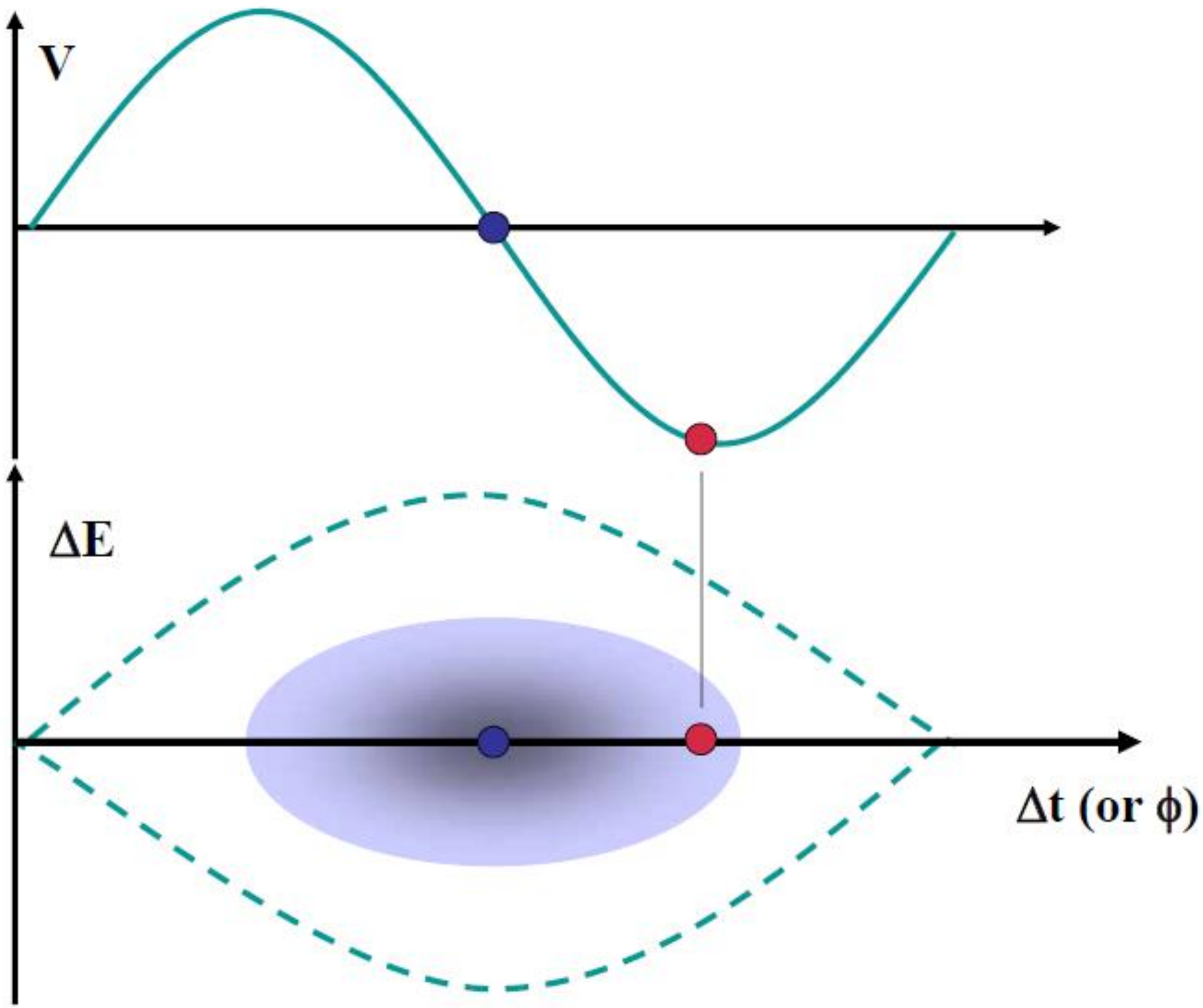


Slides by E. Wildner

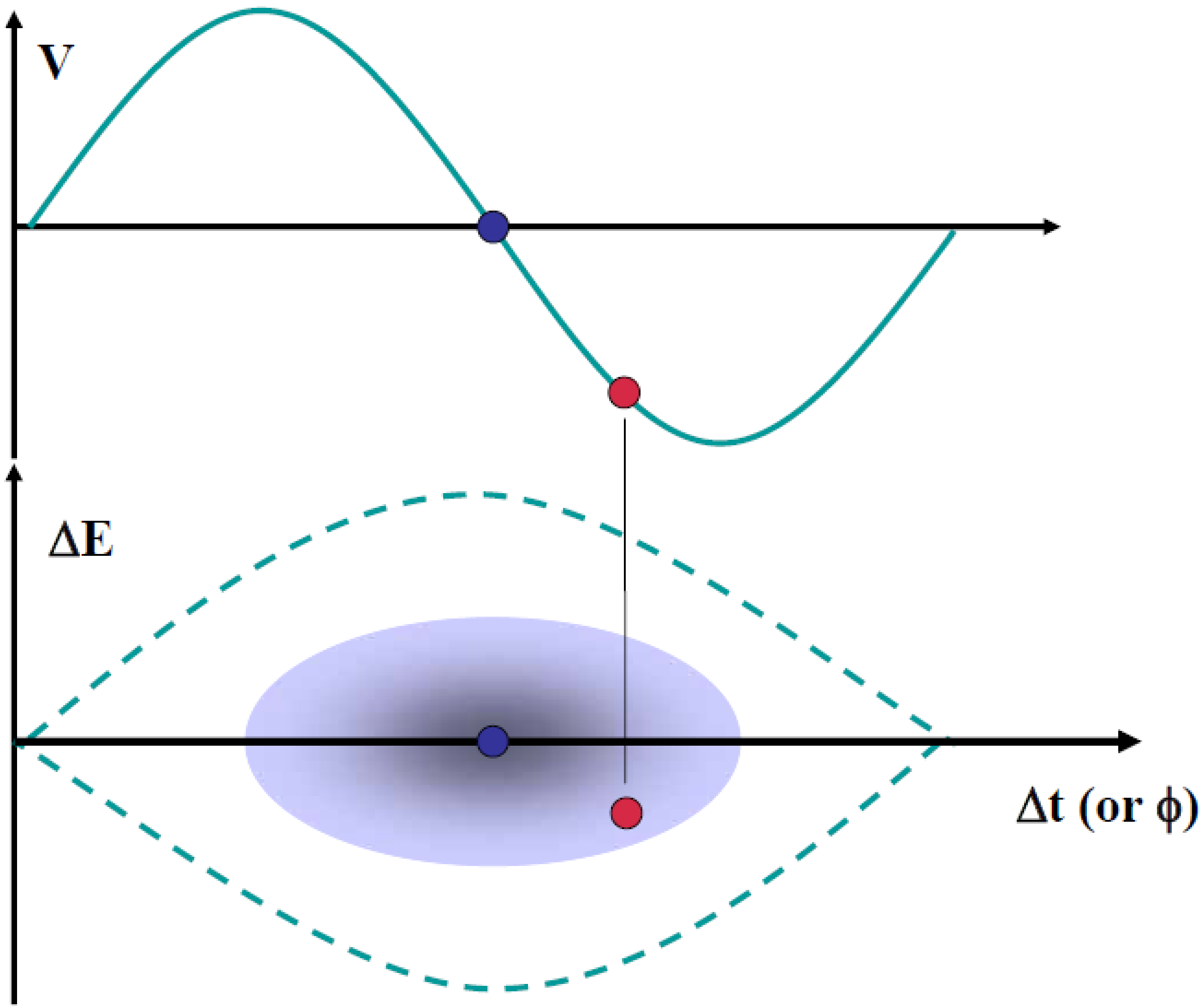

V
ΔE
Δt (or ϕ)

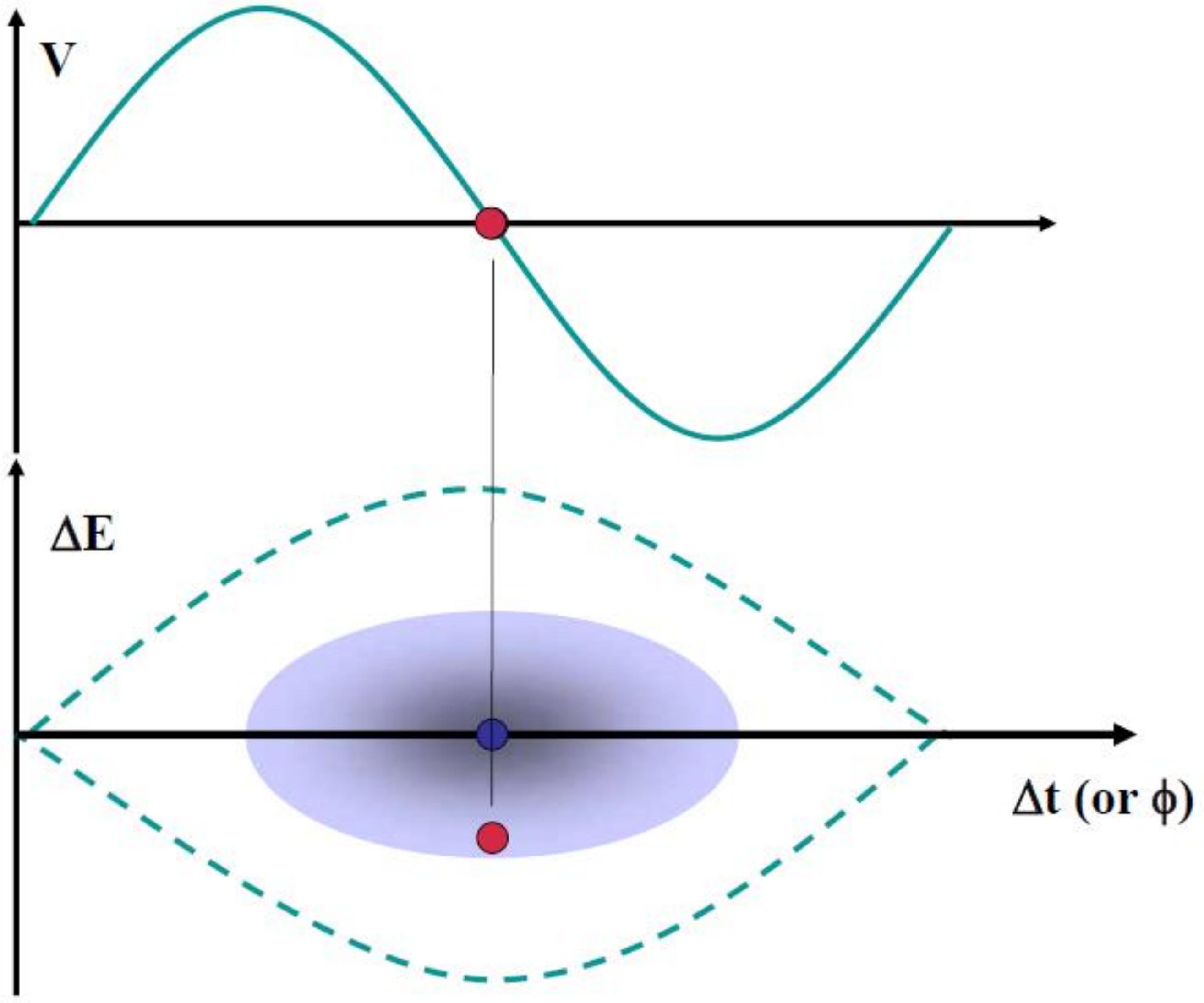

V
ΔE
Δt (or ϕ)


V
ΔE
Δt (or ϕ)



Fermilab


V
ΔE
Δt (or ϕ)



γ_t

So many other topics!

- We have just barely touched on the basics of accelerators and equations of motion with simple assumptions (perfect magnetic fields, alignment, …)
- Hopefully you gained a sense of the various research topics and areas in accelerators (Material science, beam dynamics, ..)
- The really exiting stuff is in the details, some of which will be covered later this week
- If you have any questions about these lectures or Fermilab, please reach out

Kbadgley@fnal.gov

References

Lectures:

- African School of Physics (ASP)
  - Previous lectures (https://www.africanschoolofphysics.org/)
- US Particle Accelerator School (USPAS)
  - (https://uspas.fnal.gov/)
- Cern Accelerator School (CAS)
  - (https://cas.web.cern.ch/)

Books:

- Particle Accelerator Physics- Helmut Wiedemann
- Introduction to the Physics of Particle Accelerators- Mario Conte and William MacKay
- An introduction to the Physics of High Energy Accelerators- D.A. Edwards and M.J. Syphers

This is a second order differential inhomogeneous differential equation, so the solution is

$$x(s) = x_0 C(s) + x_0' S(s) + \delta d(s)$$

$$x'(s) = x_0 C'(s) + x_0' S'(s) + \delta d'(s)$$

Where d(s) is the solution particular solution of the differential equation

$$d'' + Kd = \frac{1}{\rho}$$

We solve this piecewise, for K constant and find

$$K > 0: \quad d(s) = \frac{1}{\rho K}\left(1 - \cos\sqrt{K}s\right)$$

$$d'(s) = \frac{1}{\rho\sqrt{K}} \sin\sqrt{K}s$$

$$K < 0: \quad d(s) = -\frac{1}{\rho K}\left(1 - \cosh\sqrt{K}s\right)$$

$$d'(s) = \frac{1}{\rho\sqrt{K}} \sinh\sqrt{K}s$$

Transition crossing

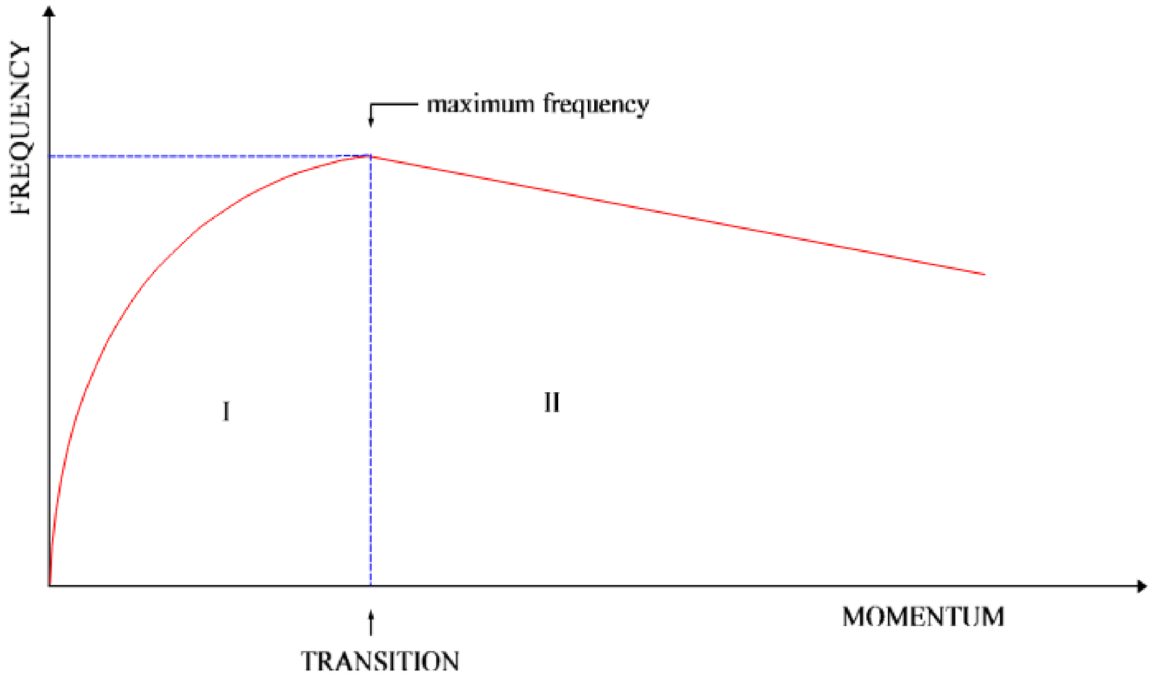


Figure 1: Revolution frequency .v. particle momentum

https://intranet.cells.es/Intranet/Labs/Elec/chap6.pdf