Fundamentals of Particle Accelerators I

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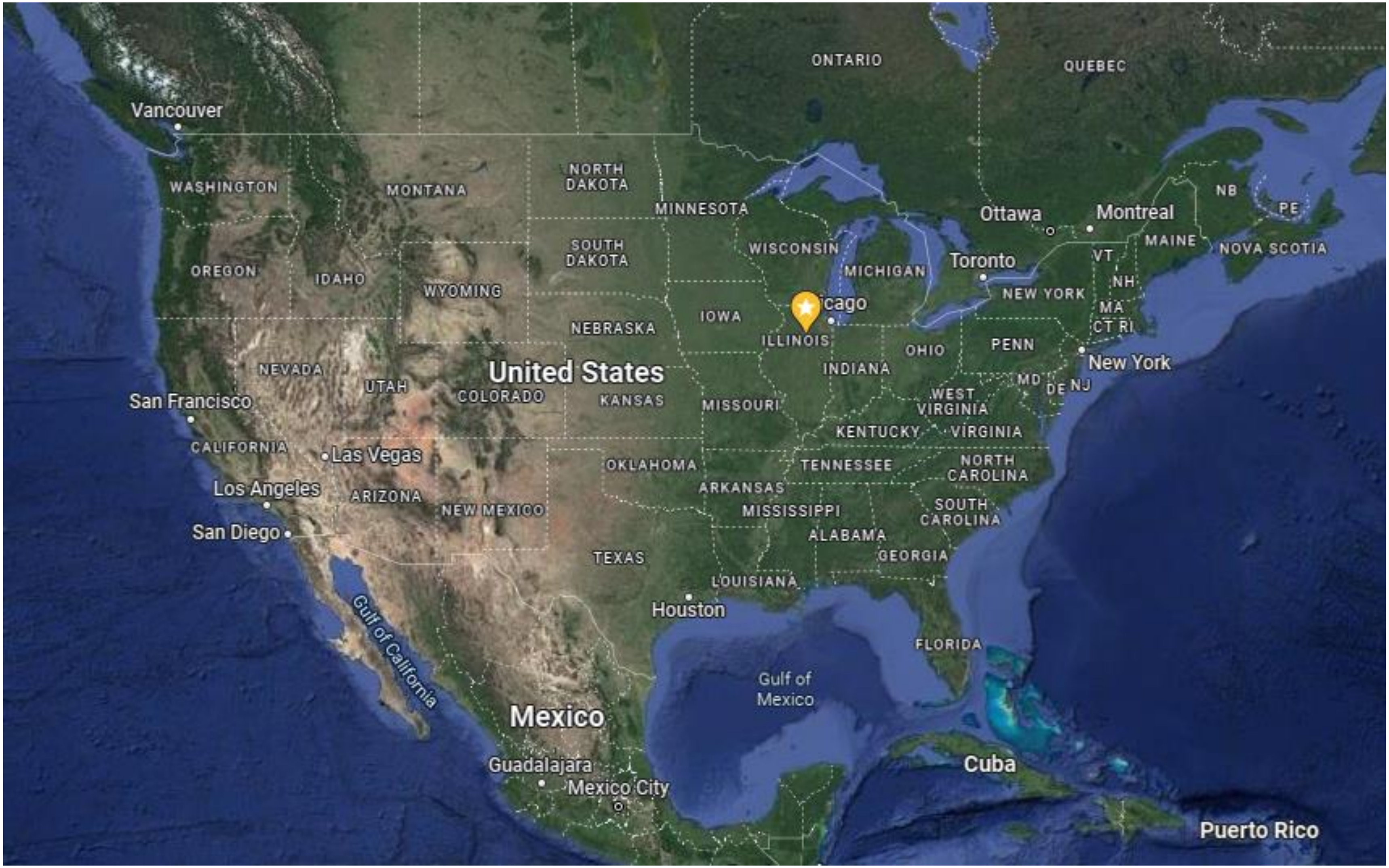
ASP 2024- Morocco

Fermilab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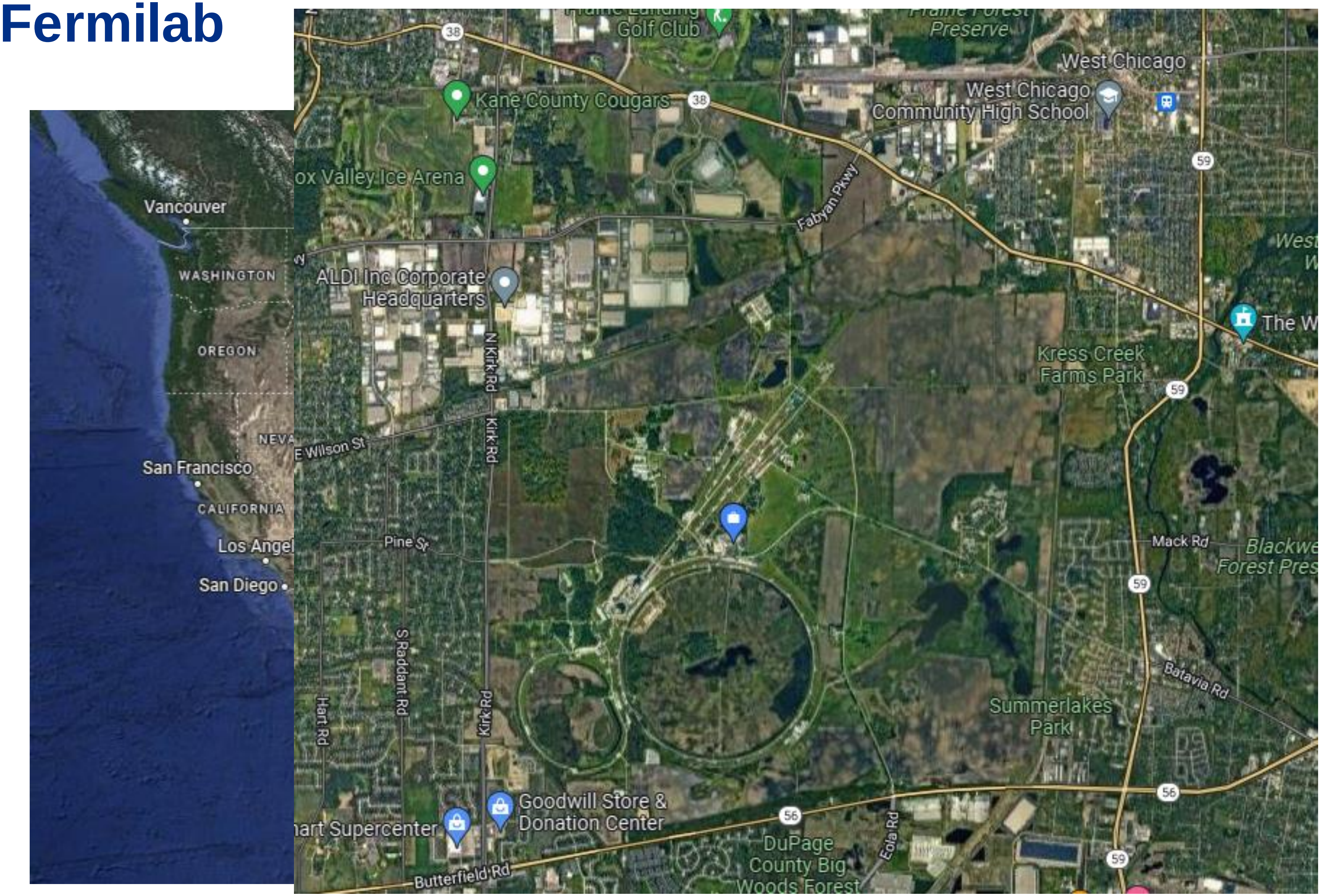
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N Kirk Rd
E Wilson St
Kirk Rd
Pine St
Mack Rd
Blackwe Forest Pres
Batavia Rd
S Raddant Rd
Hart Rd
Summerlakes Park
56
Goodwill Store & Donation Center
art Supercenter
DuPage County Big Woods Forest
Eola Rd
Butterfield Rd

Fermilab


Golf Club
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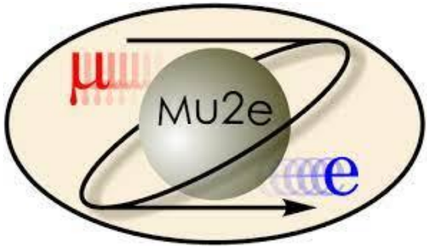
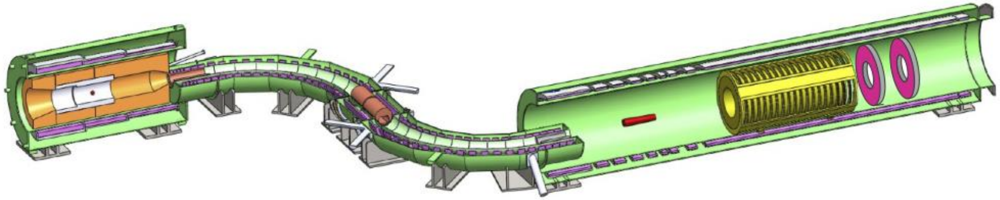
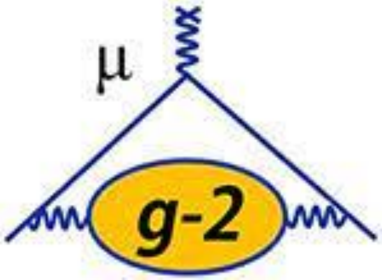
Muon Experiments







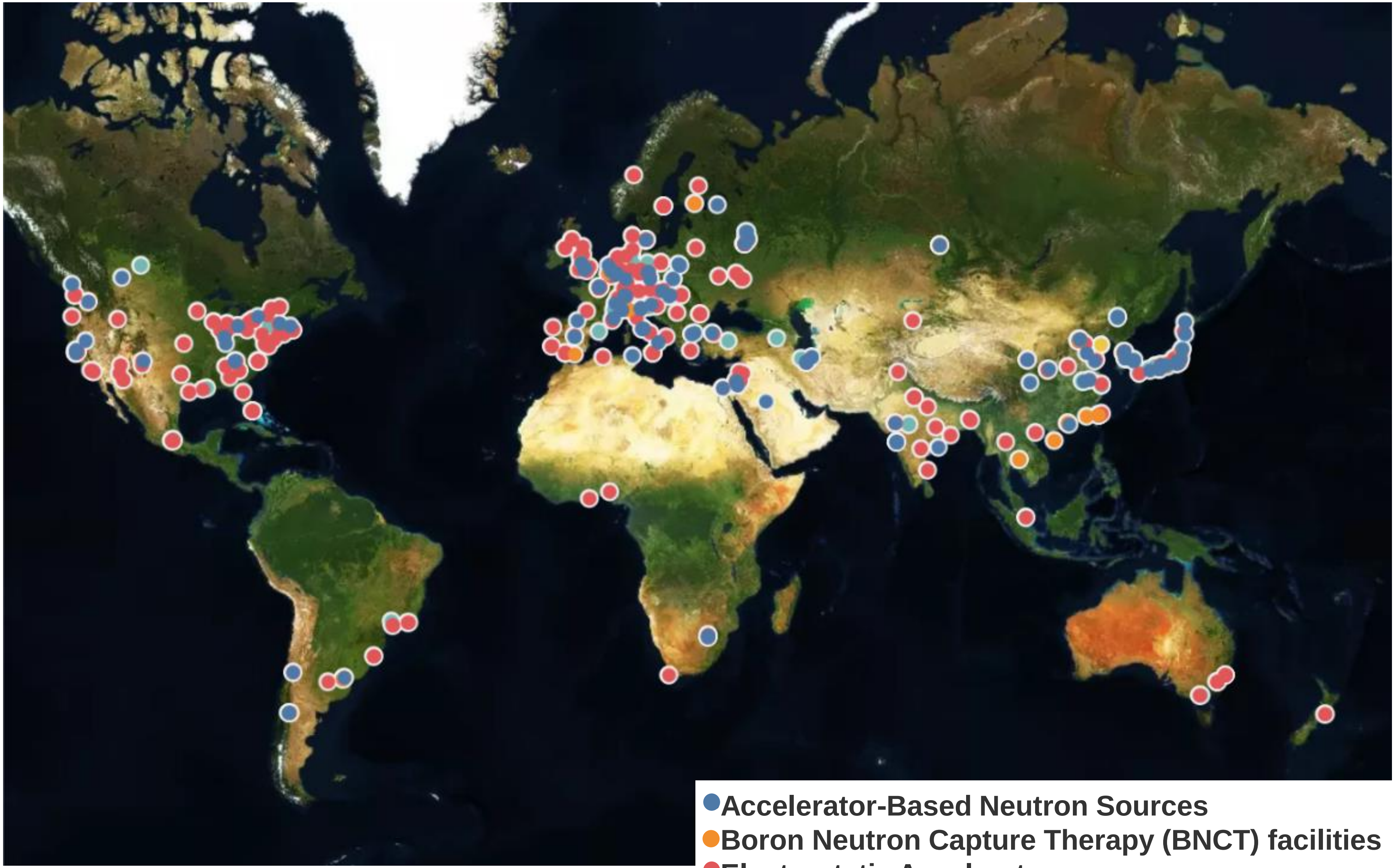
PIP-II

Upgrade to accelerator to enable experiments such as DUNE

Overview

- Why accelerate particles
- A bit of accelerator history
- Components of an accelerator
- Magnets
- Equations of Transvers Motion
  - Weak Focusing
- Matrix Representation
- Strong Focusing
- Betatron motion

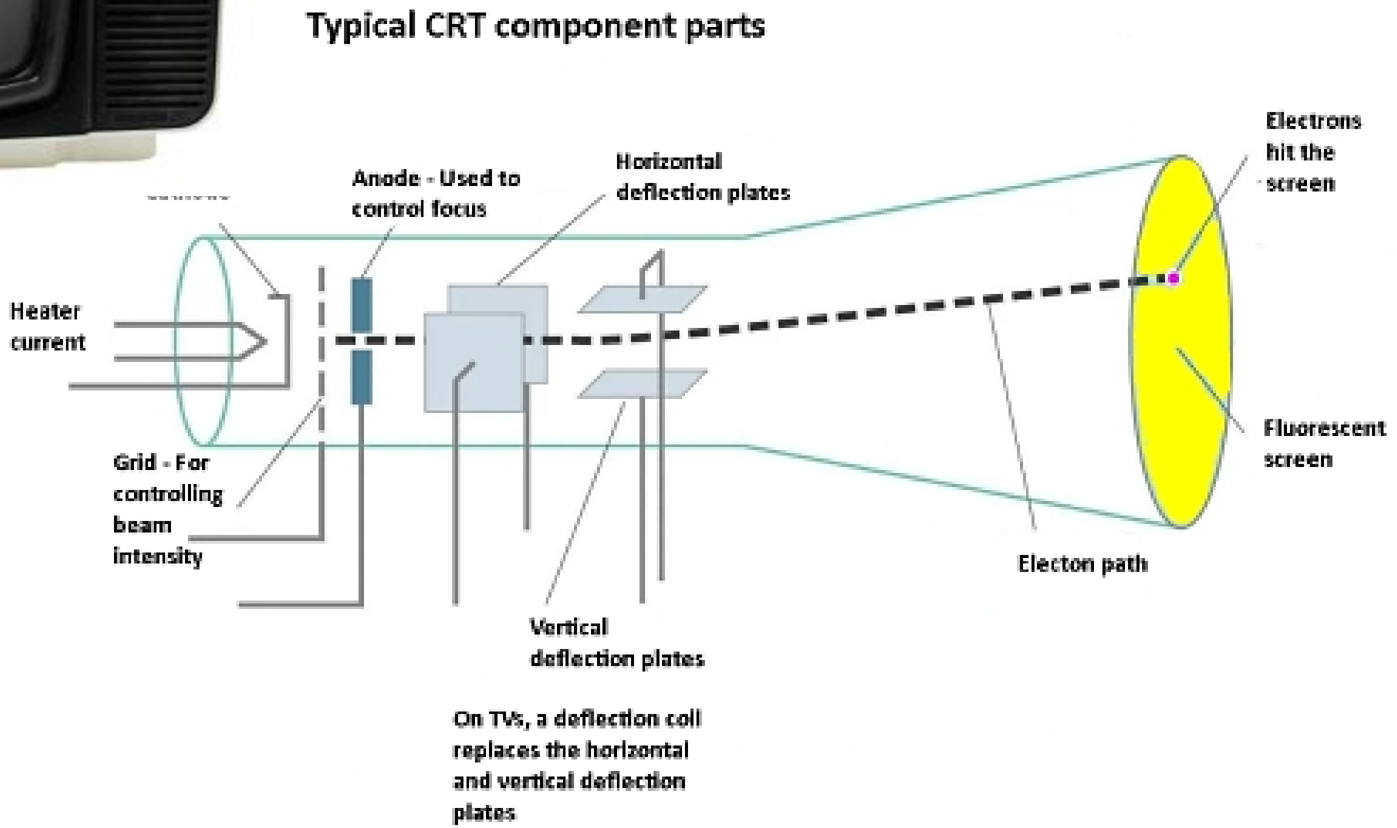
Accelerators Worldwide



● Accelerator-Based Neutron Sources
● Boron Neutron Capture Therapy (BNCT) facilities
● Electrostatic Accelerators
● Synchrotron Light Sources
● X-ray Free Electron Laser Sources

Accelerator Applications

Cathode ray tube TVs



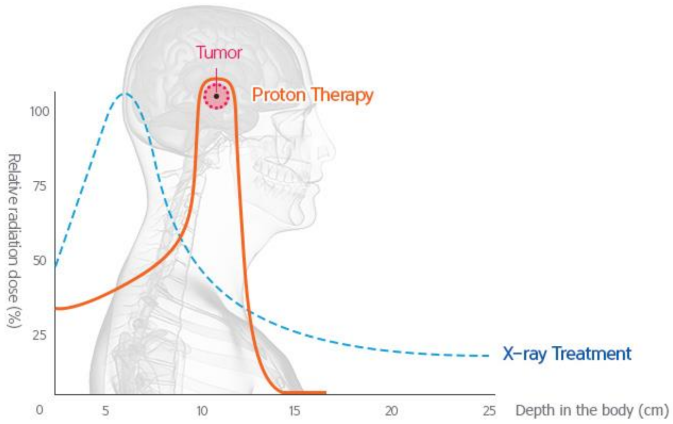
On TVs, a deflection coil replaces the horizontal and vertical deflection plates

Accelerator Applications- Medicine

Of the ~35,000 accelerators worldwide, roughly half are medical

Proton Therapy

Reduce dose to surrounding healthy tissue

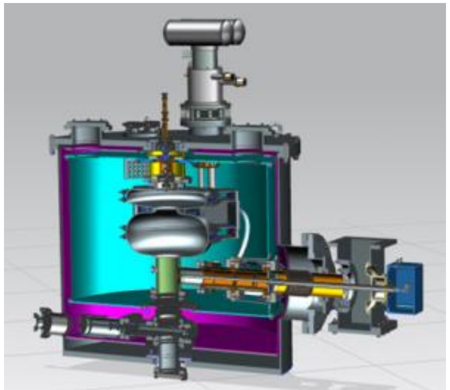


samsunghospital.com

Imaging

Device Sterilization

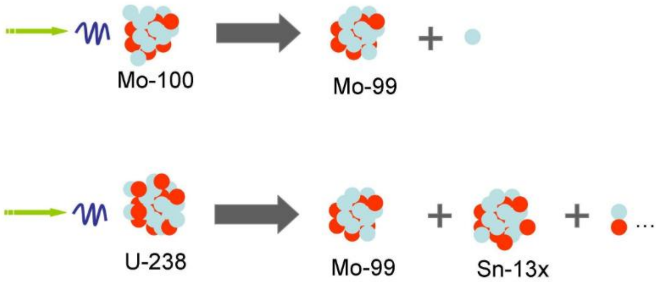
Looking to replace ethlylene oxide and cobalt-60 with x-rays from electron beams



Fnal.gov

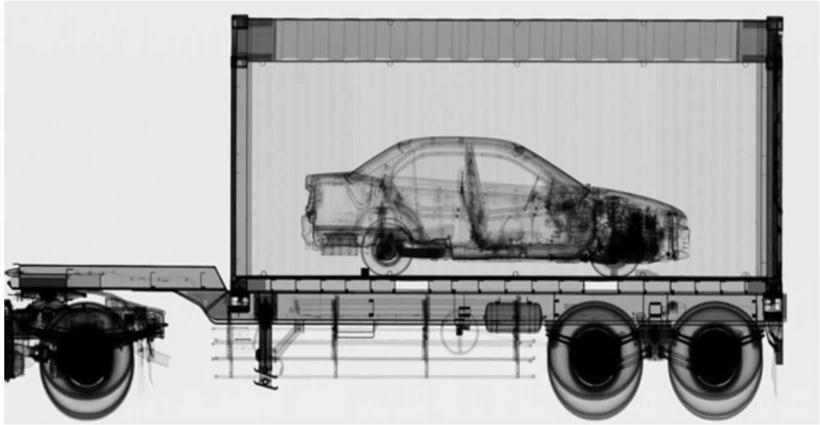
Isotope Production

Mo-99 to Tc-99m



www.triumf.ca

Accelerator Applications- Security



Cargo containers scanned at ports and border crossings

Accelerator-based sources of X-Rays can be far more penetrating (6MV) than Co-60 sources.

Container must be scanned in 30 seconds.

Image source: Varian medical systems

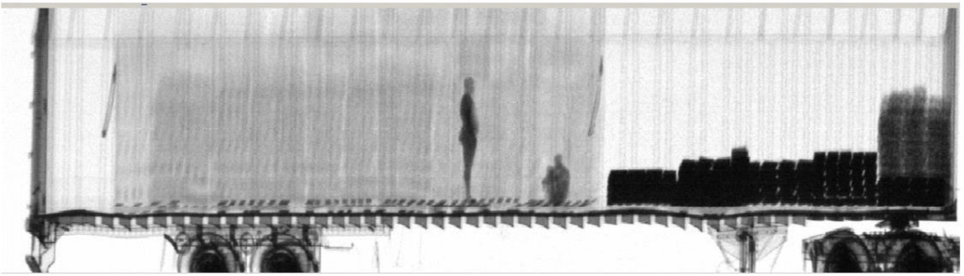
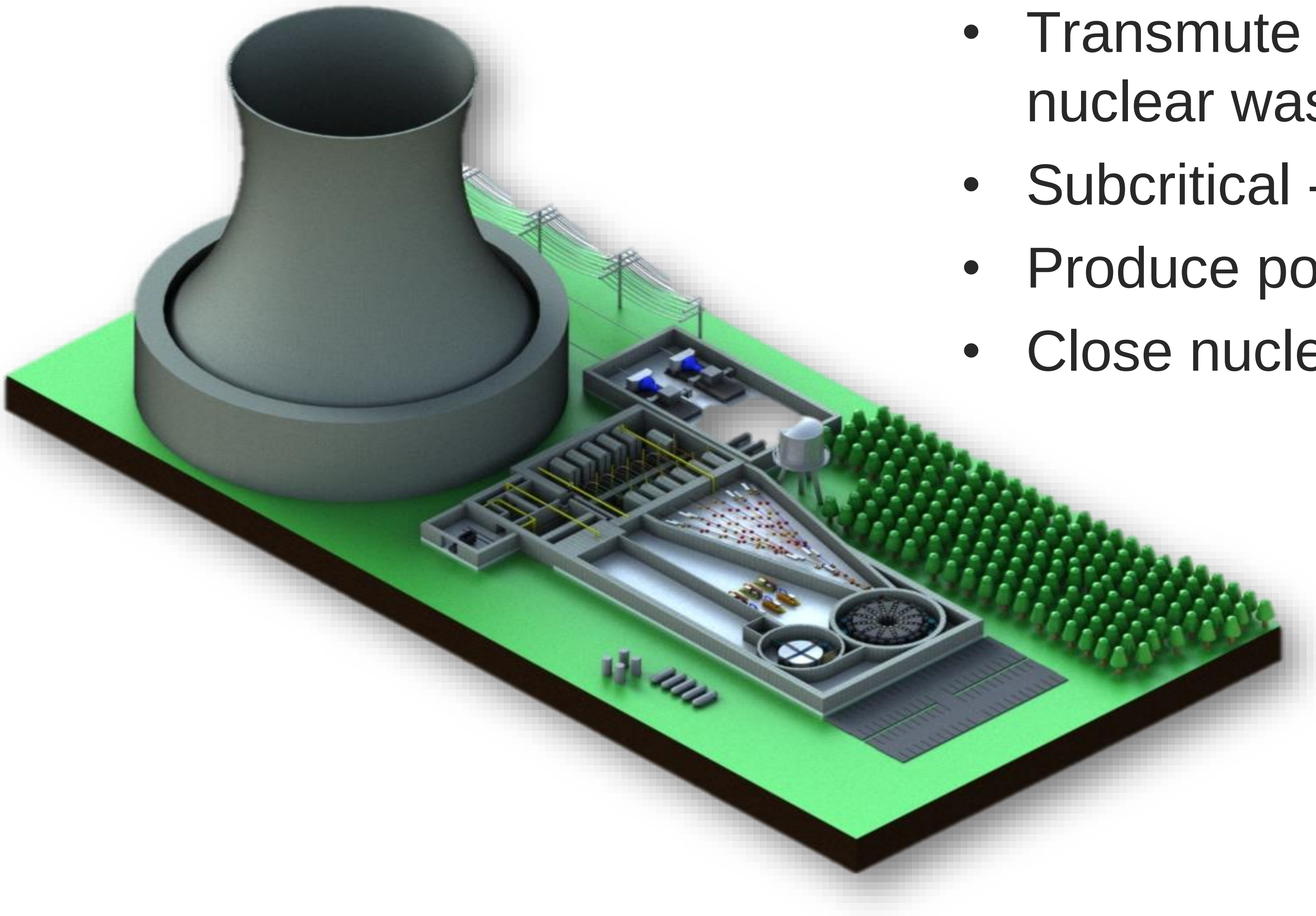


Image: dutch.euro

Accelerator Applications- Energy/Environment

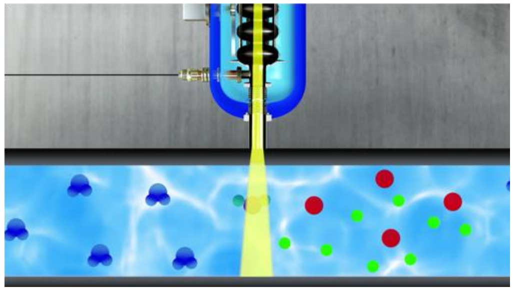


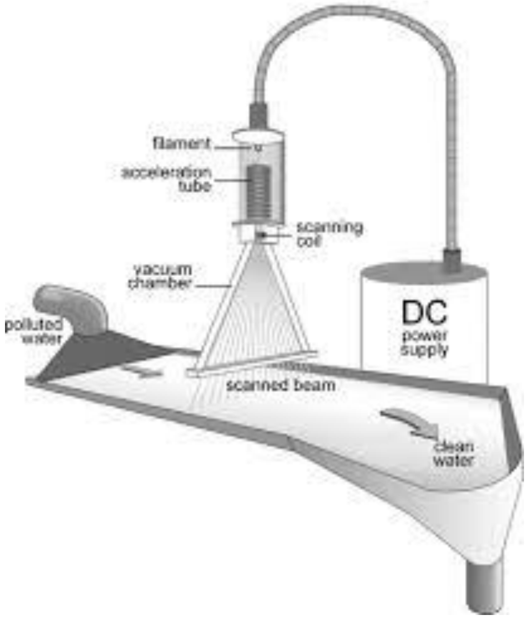
- Transmute long lived nuclear waste
- Subcritical - Safe
- Produce power
- Close nuclear fuel cycle

Accelerator Applications- Energy/Environment

Wastewater treatment

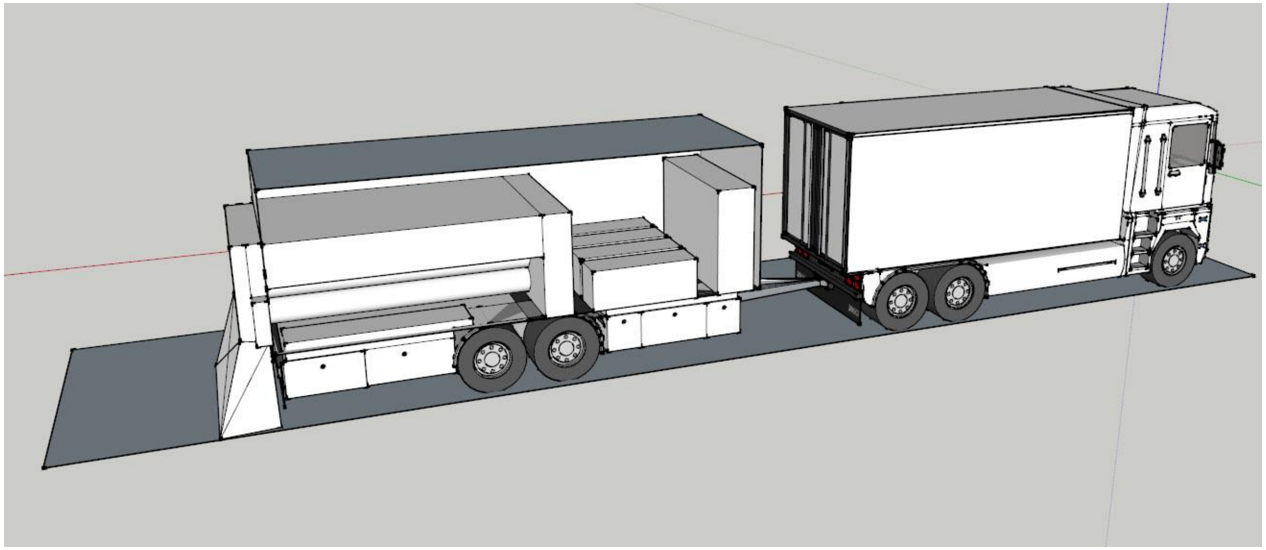
High energy electron to break down pollutants





Accelerator on a truck

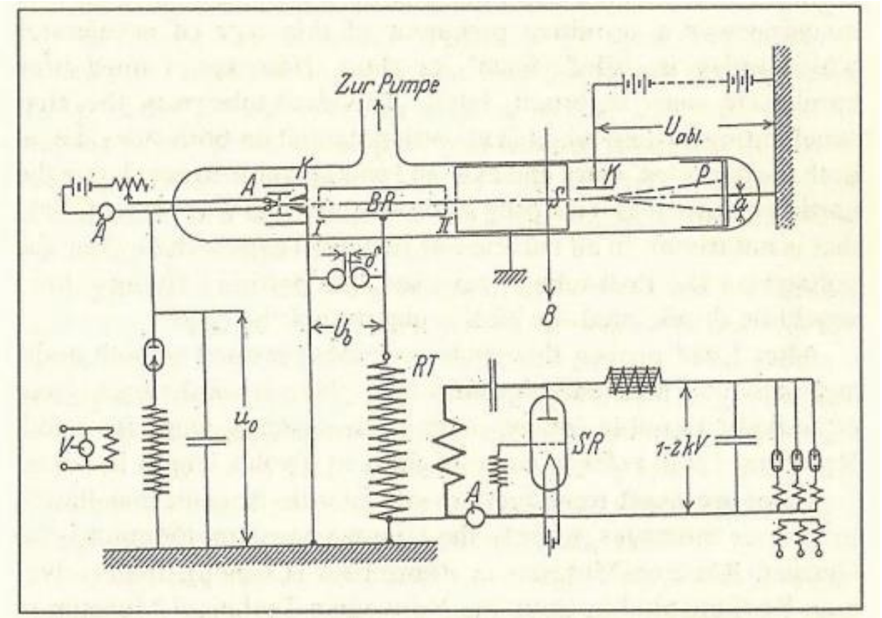
Use electron beam to resurface road



...and so many more!

The Accelerator is Born

- 1919 Ernst Rutherford called for “copious supply” of particles more energetic than produced by natural radioactive sources
- 1924 Gustav Ising developed the concept of a linear particle accelerator (Linac)
- 1928 Rolf Wideröe builds the first linac in Aachen, Germany
  - He first tried to build a betatron, but when that was unsuccessful, switched to a linac for his thesis

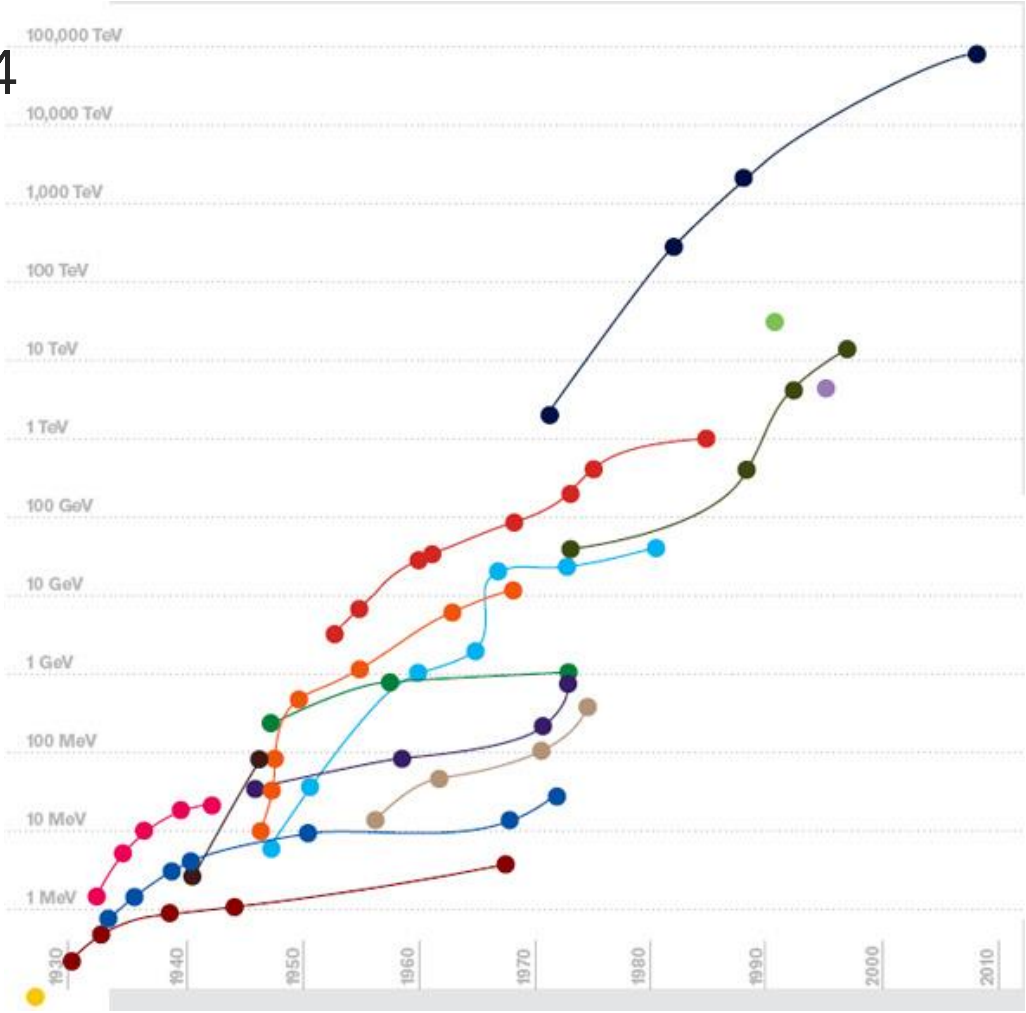


Wideröe, Über ein neues Prinzip zur Herstellung hoher Spannungen, Archiv für Elektrotechnik 21, 387 (1928)

Livingston Plot

- It was estimated in a 2014 Symmetry article that there were over 30,000 operating particle accelerators
- In his 1954 book, Stanley Livingston noted that advances in accelerator technology allowed a factor of 10 increase in energy every 6-7 years

Laboratory energy of particles colliding with a proton at rest to reach the same center of mass energy



https://www.symmetrymagazine.org/article/october-2009/deconstruction-livingston-plot

Units

1 eV= energy of a particle $q = e$ when accelerated across a 1 V potential

$$e = 1.6 \times 10^{-19}\ C$$

$$1\ eV = 1.6 \times 10^{-19}\ J$$

Through the relationship between mass and energy, the rest mass can also be expressed in terms of eV

$$U = mc^2$$

Particle	Rest Mass, kg	Rest mass, eV/c^2
Electron, e^-	9.11×10^{-31}	0.511×10^6
Proton, e^+	1.67×10^{-27}	938×10^6

Relativity Review

$$c = 2.99792 \times 10^8 m/s$$

$$\beta = \frac{v}{c}$$

Particle velocity

Speed of light

$\gamma \approx 1$ non-relativistic

$\gamma > 1$ relativistic

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Rest mass

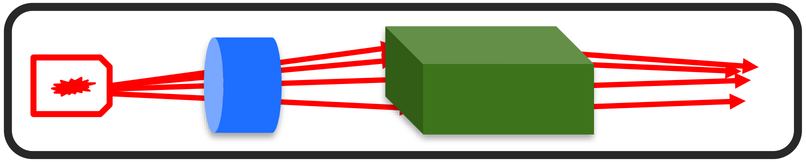
Momentum $p = \gamma m v = \beta \gamma m c$

Total energy $U = \gamma m c^2$

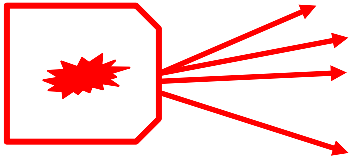
Kinetic energy $K = U - mc^2$

When we refer to the energy of a particle, it is the kinetic energy

Anatomy of an Accelerator



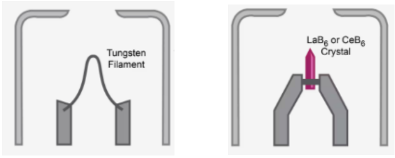
Charged particles



Source

- Electrons
- Protons
- Ions

Thermionic- heated cathode



www.thermofisher.com

Field emission - strong E field to induce emission

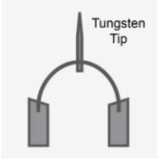
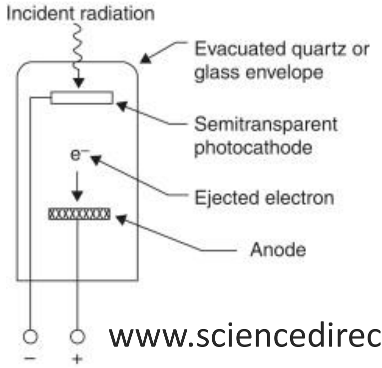
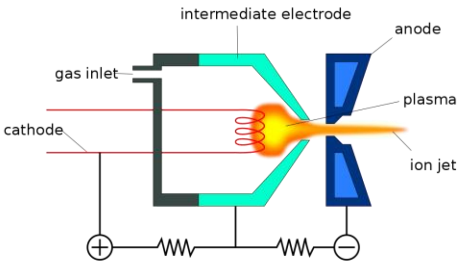


Photo emission - light to produce electrons through photoelectric effect



www.sciencedirect.com

Ion source – electron ionization, plasma, …

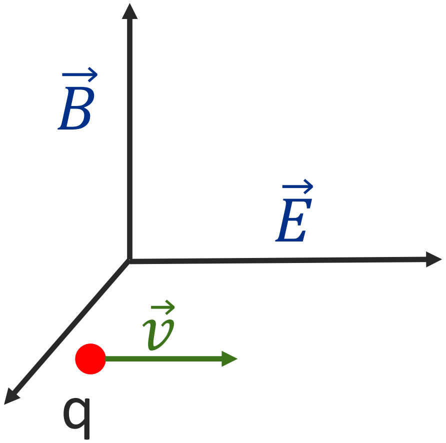


wikipedia.org

Electromagnetic force on a charged particle

Lorentz Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$



$$\Delta K = Work = \vec{F} \cdot \vec{d} = q\vec{E} \cdot \vec{d} + q\underbrace{(\vec{v} \times \vec{B}) \cdot \vec{d}}_{0}$$

- Force from the Electric field is the direction of particle velocity
  - Used to accelerate the particle in the direction of the E field
- Force from the magnetic field is perpendicular to particle velocity
  - Used to bend and focus the particle

Electromagnetic force on a charged particle

Loretnz Force:

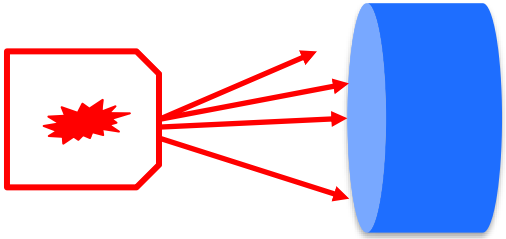
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- Magnitude of Force
  - Force from magnetic field scales with velocity
  - Velocity of high energy particle ~3x10^8 m/s
  - Using a high E field of 1 MV/m and medium B field of 1 T

  the force from the B field will be ~300 times stronger

$$\frac{\vec{v} \times \vec{B}}{\vec{E}} \approx \frac{3 \times 10^8}{1 \times 10^6} \approx 300$$

Anatomy of an Accelerator



$$\vec{E} = \underbrace{-\vec{\nabla}V}_{DC} - \underbrace{\frac{\partial \vec{A}}{\partial t}}_{\mathrm{A}C}$$

Source

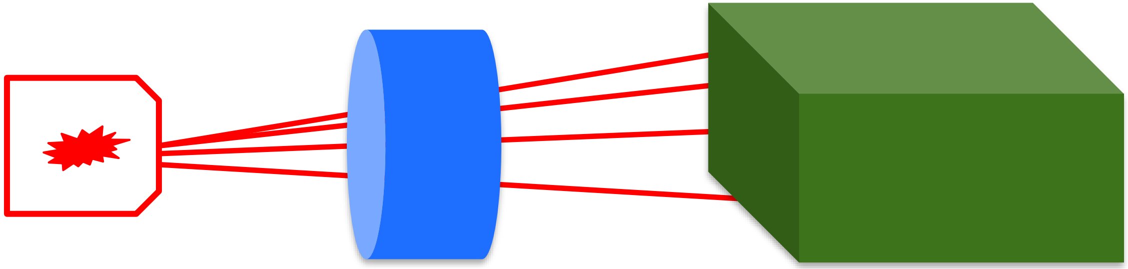
- Electrons
- Protons
- Ions

Electric Field

- Electrostatic(DC)
- Time-varying(AC)

More on this when we get to longitudinal motion next lecture

Anatomy of an Accelerator

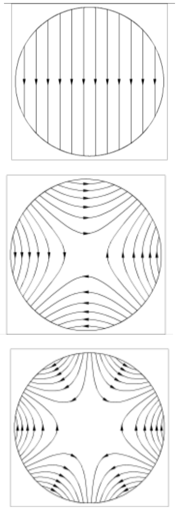


Source
  - Electrons
  - Protons
  - Ions

Electric Field
  - DC
  - AC

Magnets
  - Dipole
  - Quadrupole
  - Sextupole
  - …

Types of magnets



- Dipoles – bending (transport, energy selection…)
- Quadrupoles - focusing
- Sextupoles - correction
- Combined function

- Correctors
- Septa
- Kickers
- Solenoids

Less common/specialty magnets

Magnetic Field Harmonics

The magnetic field can be found from the expansion*:

More slides in the backup if anyone is curious about this

$$B_y + iB_x = n \sum_{n=1} C_n \, \mathbb{z}^{n-1} = n \sum_{n=1} C_n \, (x + iy)^{n-1}$$

Plugging in C_n, it takes the form:

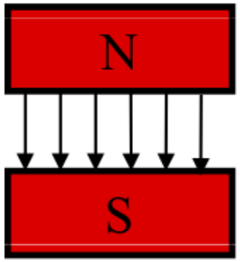
$$C_n = (B_n + iA_n)$$

$$B_y + iB_x = \sum_n (B_n + iA_n)\,(x + iy)^{n-1} = B_0 \sum_n (b_n + ia_n)\,(x + iy)^{n-1}$$

where B_0 is the reference field, the coefficients b_n and a_n correspond to normal and skew terms, and n gives the order of the pole

n=1 corresponding to a dipole, n=2 a quadrupole, n=3 a sextupole…

Dipole (two pole, n=1)



$$B_y + iB_x = \sum_n (B_n + iA_n)\,(x+iy)^{n-1}$$

$$B_y + iB_x = (B_1 + iA_1)(x+iy)^0 = \underbrace{B_1 + iA_1}_{C}$$

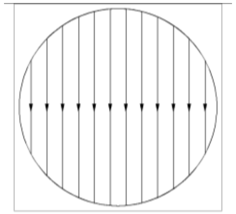
Equate real and imaginary parts:

$B_y = B_1$ $\quad$ $iB_x = iA_1$

"Normal": C=real, A_1=0

$$B_y = B_1$$

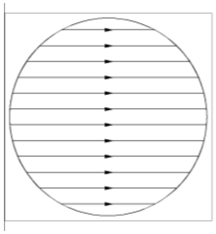
$$B_x = 0$$



"Skew": C=imaginary, B_1=0

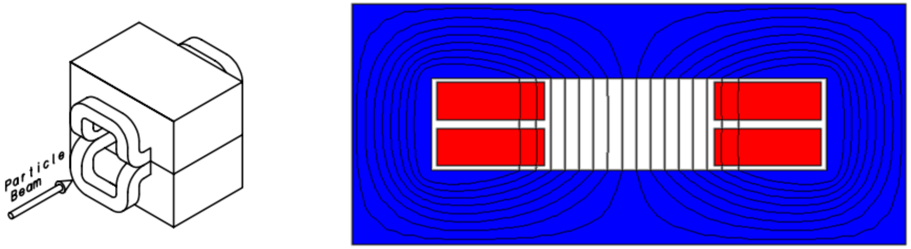
$$B_x = A_1$$

$$B_y = 0$$

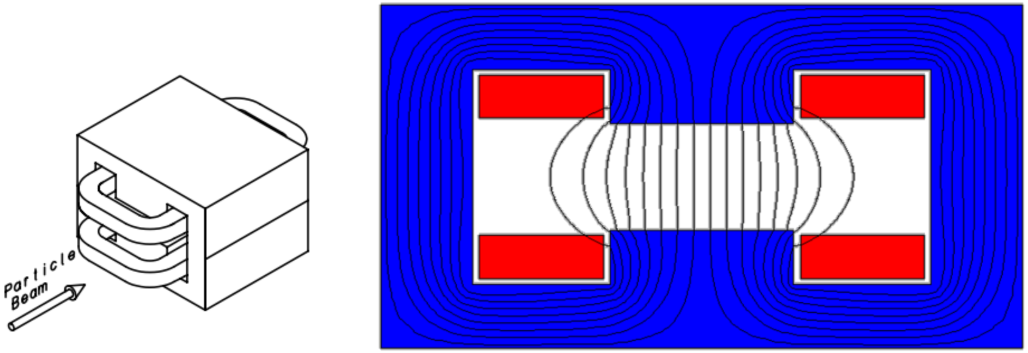


Force?

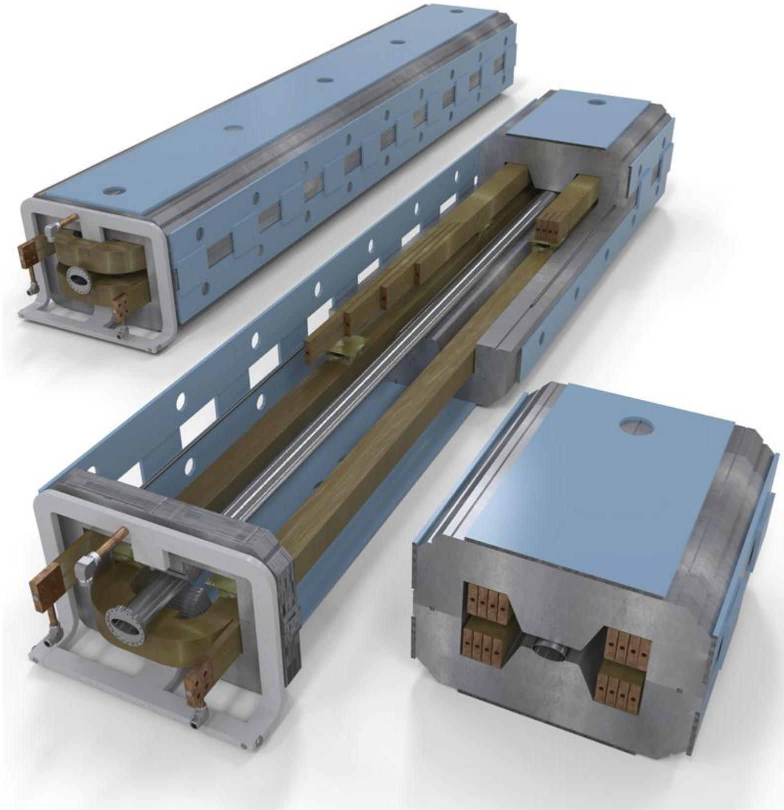
Dipoles



Window frame dipole

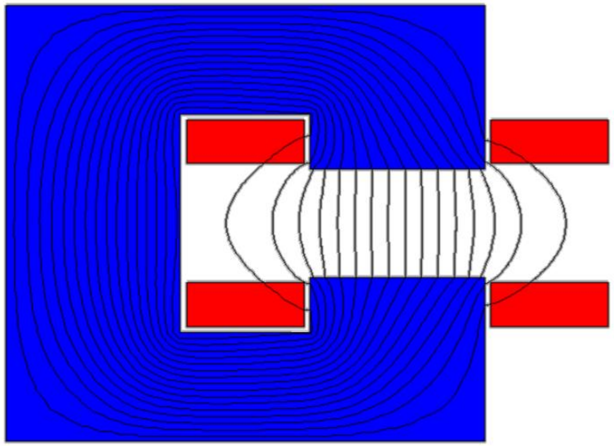


H dipole

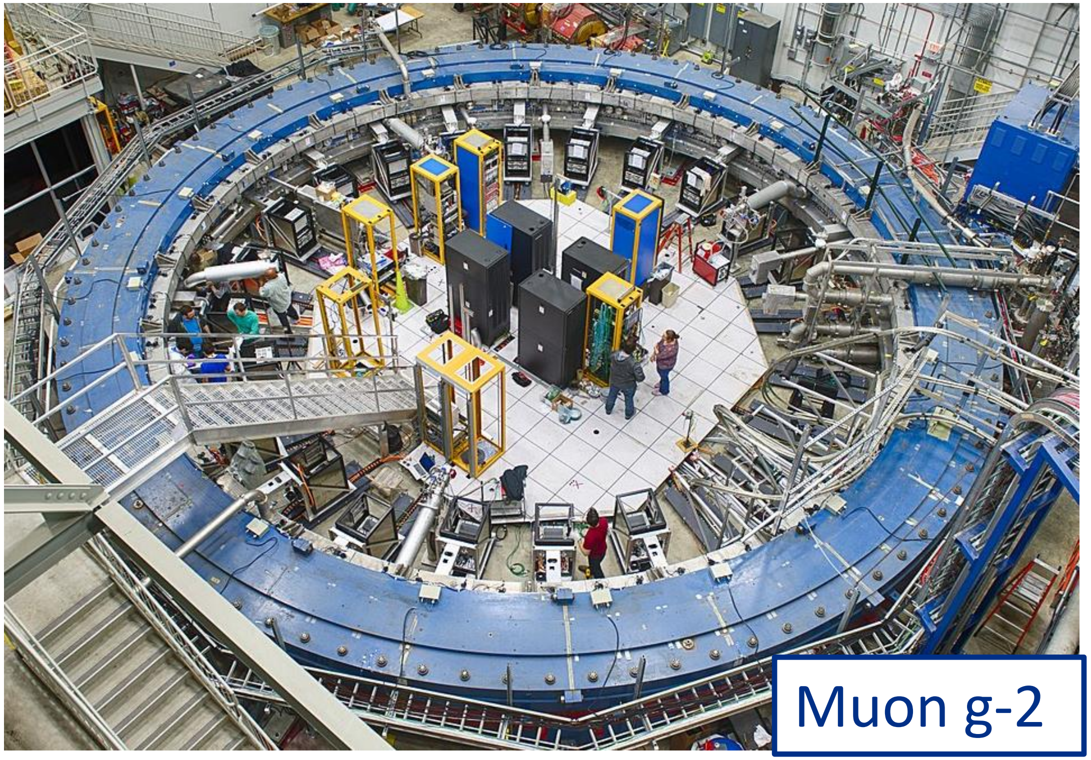


Dipole (FNAL)

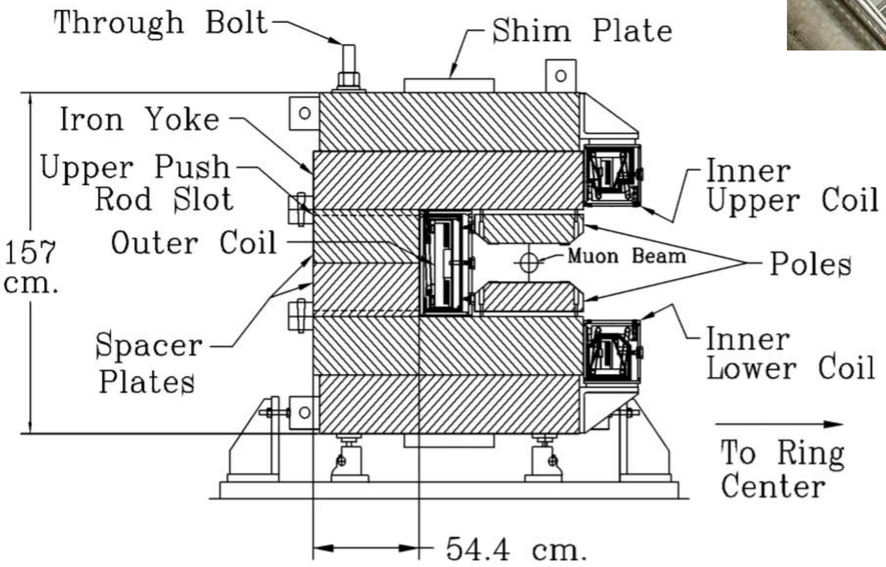
Dipoles



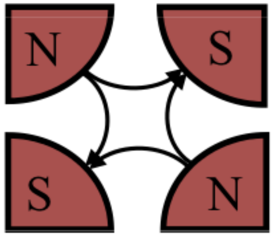
C dipole



Muon g-2



Quadrupole (four pole, n=2)



$$B_y + iB_x = n\sum_{n=1} C_n\, \mathbb{Z}^{n-1} = n\sum_{n=1} C_n\, (x+iy)^{n-1}$$

$$B_y + iB_x = 2(C_2)(x+iy)^1 = 2C_2x + i2C_2y$$

Normal, C is real:

$$B_y = 2C_2x \qquad B_x = 2C_2y \qquad \frac{\partial B_y}{dx} = 2c_2 = g \qquad \frac{\partial B_x}{dy} = 2c_2 = g$$

Skew, C is imaginary:

$$B_x = 2C_2x \qquad B_y = -2C_2y$$

$$\boxed{B = \underbrace{g}_{}y\hat{x} + \underbrace{g}_{}x\hat{y}}$$

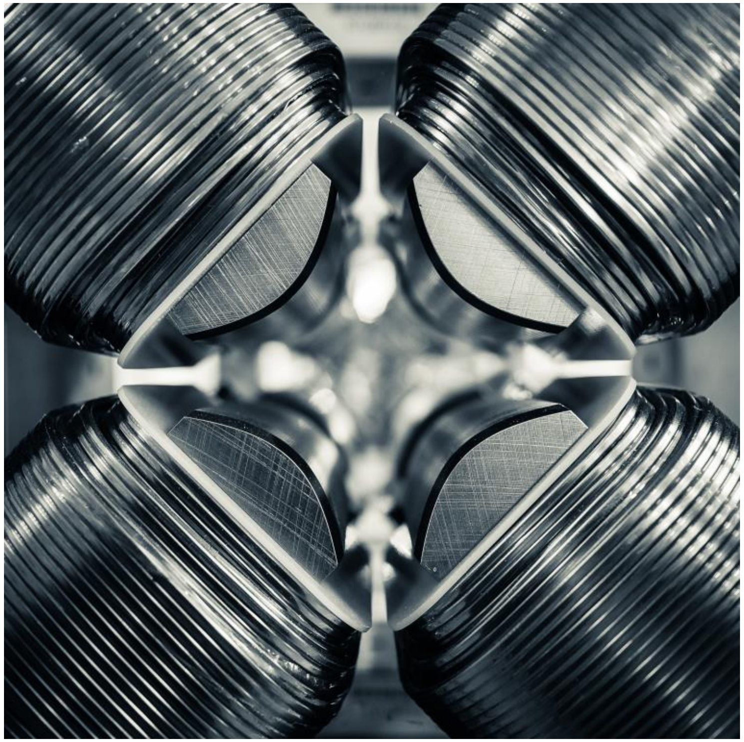
Gradient (T/m)

The quadrupole field varies linearly with the distance from the magnet center. It **focuses** the beam in one direction and **defocuses** in the other. An F or focusing quadrupole focuses the particle beam along the horizontal plane.

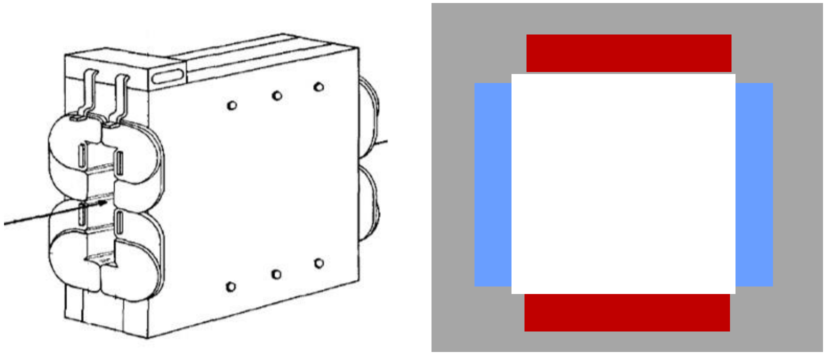
Force?

Quadrupole

ALBA SR Quadrupole

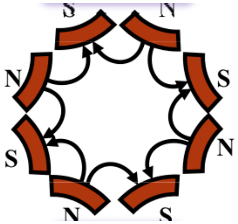


Fermilab Quadrupole



Panofsky Quadrupole

Sextupole (six pole, n=3)



$$B_y + iB_x = -n \sum_{n=1} C_n \, (x + iy)^{n-1}$$

$$B_y + iB_x = -3C_3 \, (x + iy)^2 \quad \Longrightarrow \quad B_y + iB_x = -3C_3(x^2 \; - y^2) - i6C_3xy$$

Normal, C is real:

$$B_x = -6C_3xy \qquad B_y = -3C_3(x^2 \; - y^2)$$

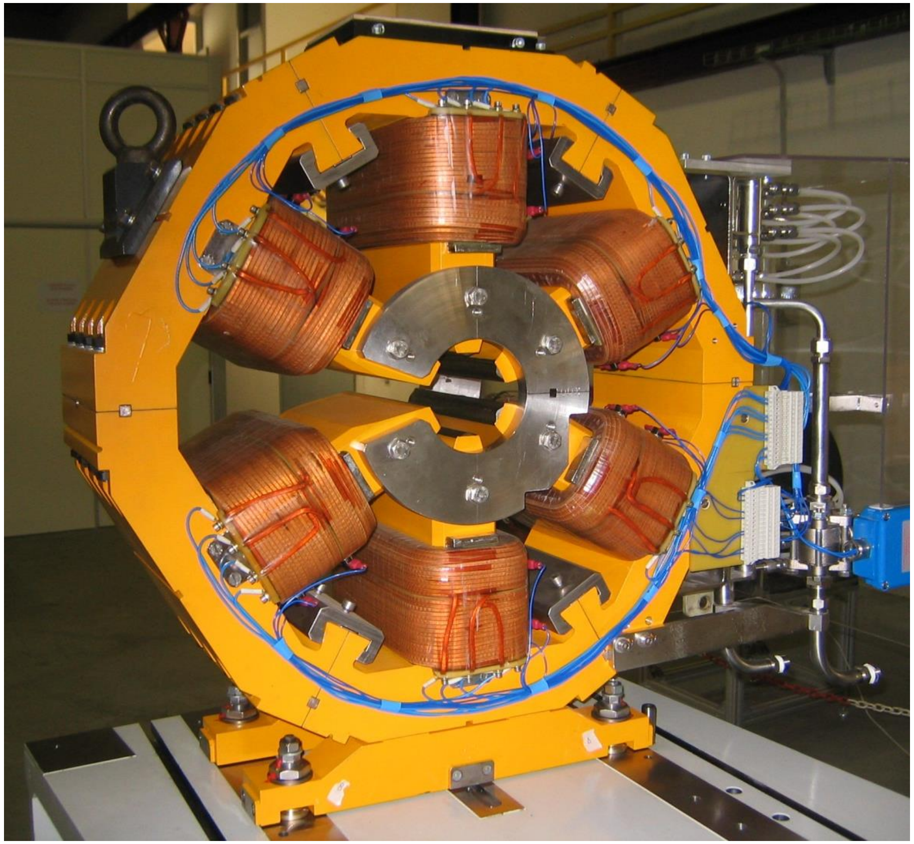
$$\Downarrow$$

$$\frac{\partial^2 B_y}{\partial x^2} = B'' = -6C_3$$

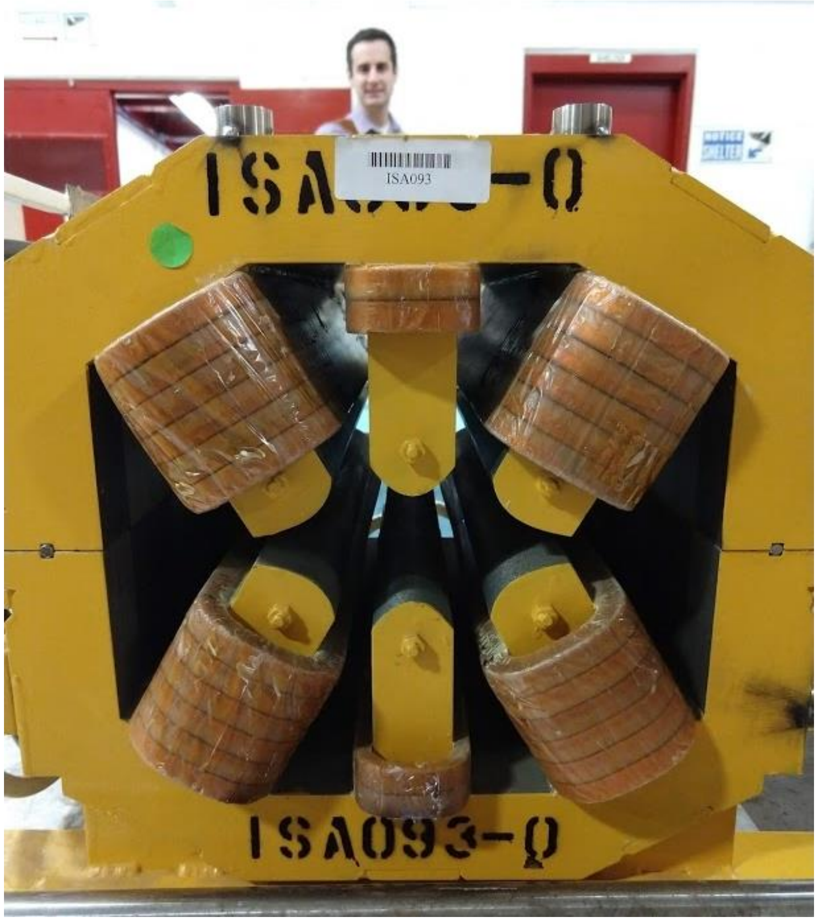
$$B_x = B''xy \qquad B_y = \frac{B''}{2}(x^2 \; - y^2)$$

The sextupole field varies *quadratically* with the distance from the magnet center. It’s purpose is to effect the beam at the edges. An *F* sextupole will steer the particle beam toward the center of the ring. Note that the sextupole also steers along the 60 and 120 degree lines.

Sextupole

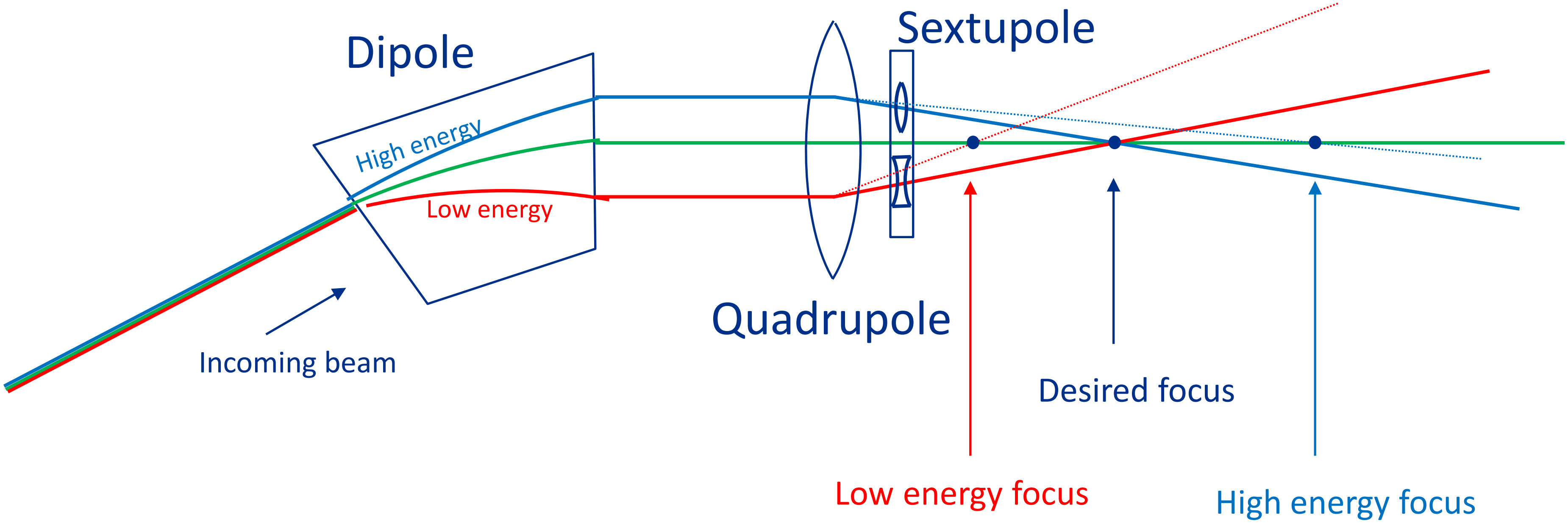


ALBA SR Sextupole

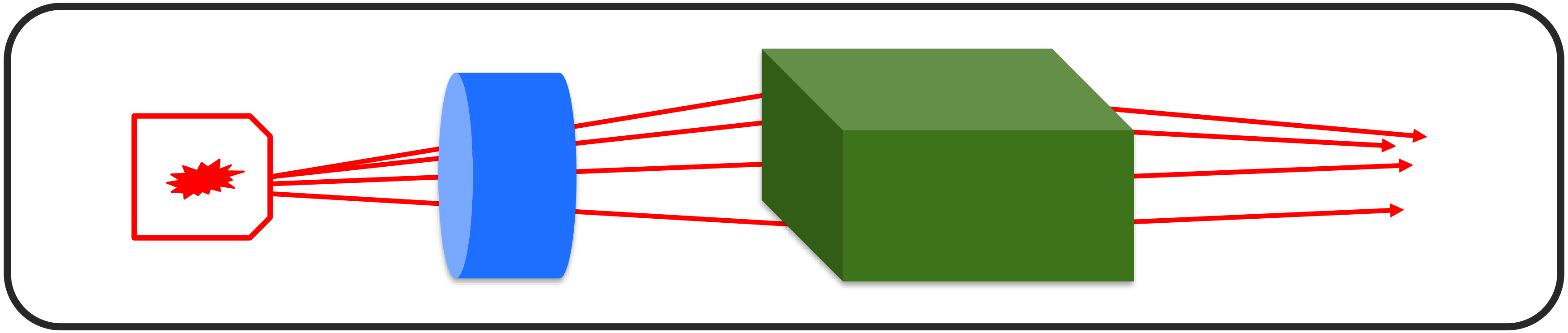


Sextupole (FNAL)

Optics Analogy



Anatomy of an Accelerator



Source
- Electrons
- Protons
- Ions

Electric Field
- DC
- AC

Magnets
- Dipole
- Quadrupole
- Sextupole

Vacuum system

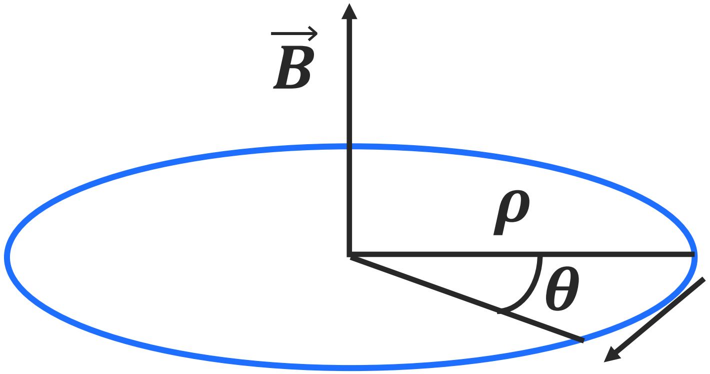
- Power Supplies
- Cryogenics
- Beam diagnostics
- Control system

Equations of Transverse Motion

$$\vec{F} = q(\cancelto{0}{\vec{E}} + \vec{v} \times \vec{B})$$

Motion in a uniform B field

$$\vec{F} = q(\cancelto{0}{\vec{E}} + \vec{v} \times \vec{B})$$



$$\vec{F} = q(\vec{v} \times \vec{B}) = m\frac{v^2}{\rho}$$

$$qvB = \frac{\gamma m_0 v^2}{\rho}$$

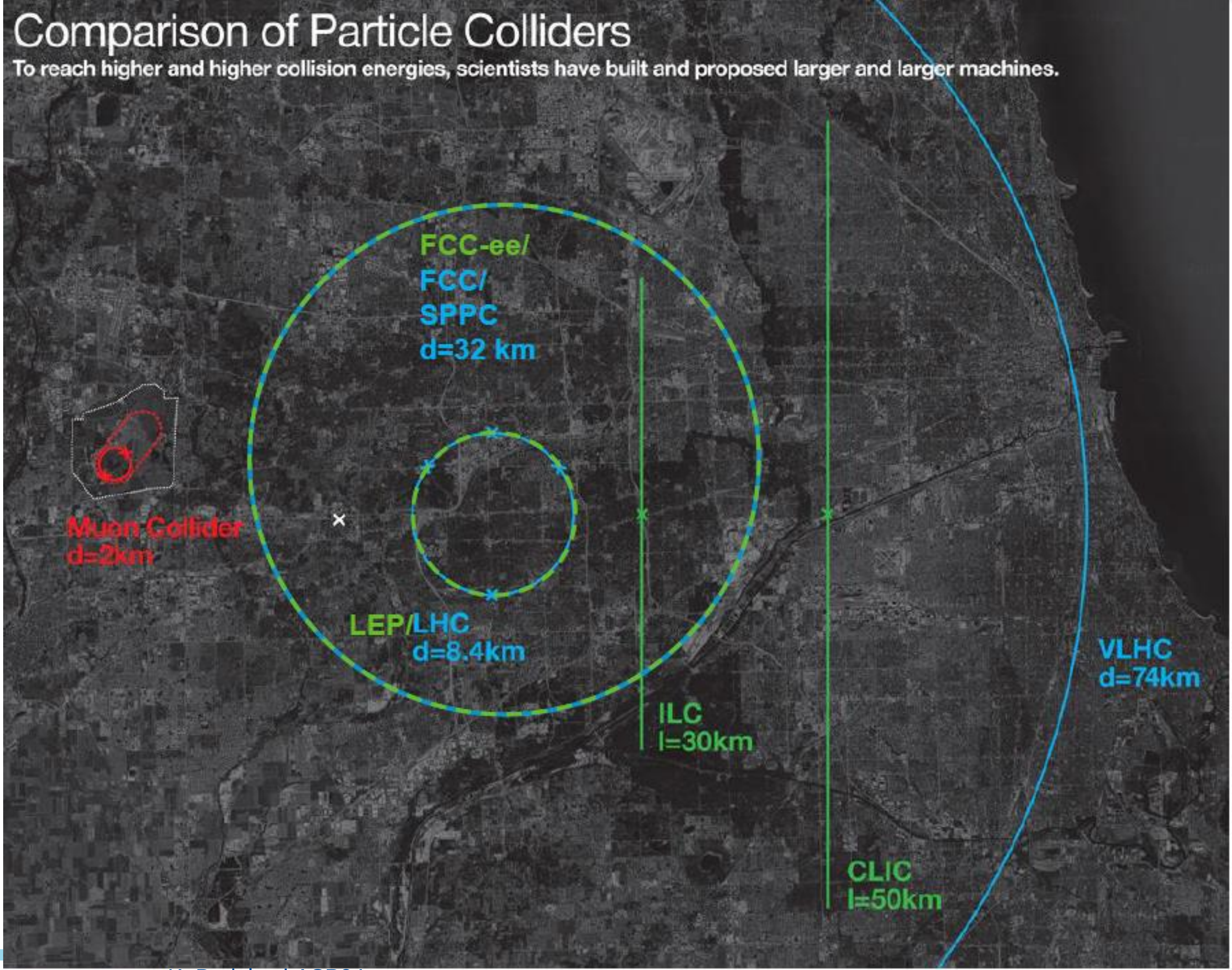
In terms of momentum, $p = \gamma m_0 v$

$$qB = \frac{p}{\rho}$$

$$\boxed{\frac{\boldsymbol{p}}{\boldsymbol{q}} = \boldsymbol{B\rho}}$$

Magnetic rigidity

How hard a particle is to deflect [T m]



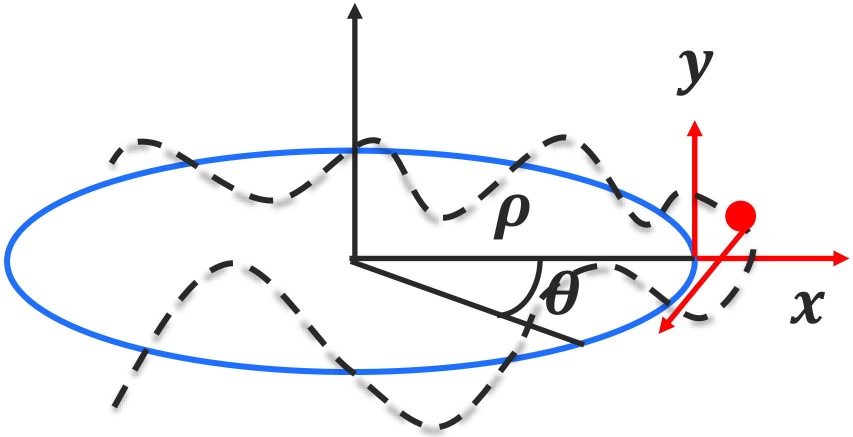
Comparison of Particle Colliders

To reach higher and higher collision energies, scientists have built and proposed larger and larger machines.

Source: Fermilab

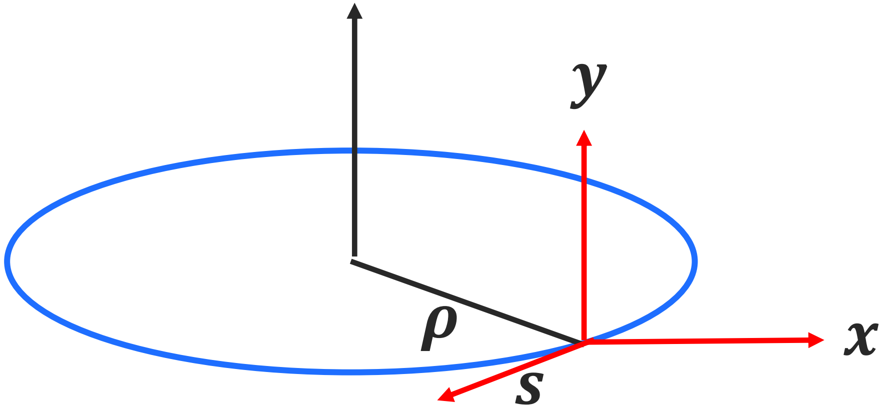
Equations of motion

Particle motion will be expanded about the ideal or design trajectory ($\boldsymbol{x}$, $\boldsymbol{y}$)



$$\vec{B} = (B_x, B_y, 0)$$

Reference frame:



$\boldsymbol{x}$: horizontal

$\boldsymbol{y}$: vertical

$\boldsymbol{s}$: longitudinal-along the ideal trajectory(x=y=0)

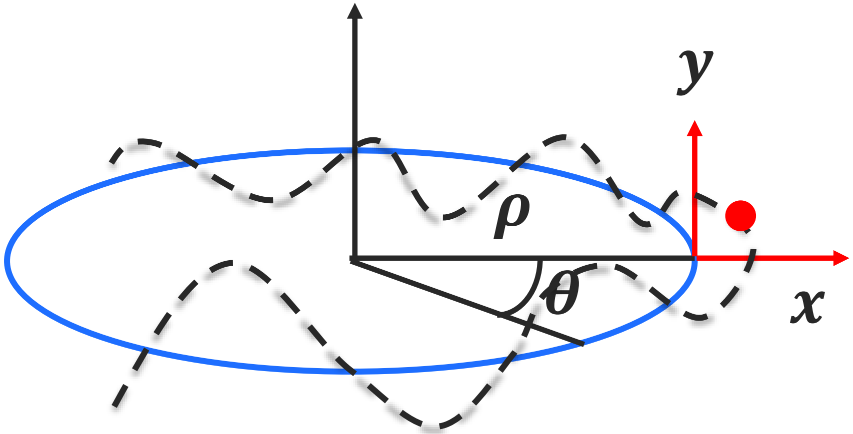
Equations of motion

$$\vec{B} = (B_x, B_y, 0)$$

Particle motion will be expanded about the ideal or design trajectory ($\boldsymbol{x}$, $\boldsymbol{y}$)

$$F_x = -qvB_y = ma_{rad}$$

$$F_y = qvB_x$$



Radial acceleration:

$$a_{rad} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$r \rightarrow \rho + x$$

$$F_x = m\frac{d^2(\rho + x)}{dt^2} - m(\rho + x)\left(\frac{d\theta}{dt}\right)^2 = -qvB_y$$

Equations of motion- Horizontal

$$F_x = m\frac{d^2(\rho + x)}{dt^2} - m(\rho + x)\left(\frac{d\theta}{dt}\right)^2 = -qvB_y$$

$$\frac{d\theta}{dt} = \omega = \frac{v}{\rho + x}$$

$$F_x = m\frac{d^2(\rho + x)}{dt^2} - \frac{mv^2}{\rho + x} = -qvB_y$$

ρ is the radius and is constant

$$F_x = m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho + x} = -qvB_y$$

Equations of motion- Horizontal

$$F_x = m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho + x} = -qvB_y$$

$x \ll \rho$ so we can do a Taylor expansion

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!}f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + ..$$

$$\frac{1}{\rho + x} \approx \frac{1}{\rho} - \frac{x}{\rho^2} + \cdots \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

$$F_x = m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -qvB_y$$

Equations of motion- Horizontal

$$F_x = m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -qvB_y$$

Divide by m and multiply the r.h.s. by v/v

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{-qv^2B_y}{\underbrace{mv}_{p}}$$

$$\frac{q}{p} = \frac{1}{B\rho}$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{-v^2B_y}{B\rho}$$

Equations of motion

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{-v^2 B_y}{B\rho}$$

We can also do a Taylor expansion of the B_y field about the reference orbit if we assume $\frac{dB_y}{dx}$ is small

$$B_y(x) = B_0 + \frac{dB_y}{dx}x + \cdots$$

Define the gradient $g = \frac{dB_y}{dx}$

$$B_y(x) = B_0 + gx + \cdots$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{-v^2(B + gx\,)}{B\rho} = \frac{-v^2}{\rho} - \frac{v^2gx}{B\rho}$$

Equations of motion - Horizontal

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{-v^2}{\rho} - \frac{v^2 gx}{B\rho} \quad\Longrightarrow\quad \frac{d^2x}{dt^2} + \left(\frac{v^2x}{\rho^2}\right) = -\frac{v^2 gx}{B\rho}$$

Convert from t to s

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} \qquad \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''v^2$$

$$\frac{d^2xv^2}{ds^2} + \left(\frac{v^2x}{\rho^2}\right) = -\frac{v^2 gx}{B\rho} \quad\Longrightarrow\quad \frac{d^2x}{ds^2} + \frac{x}{\rho^2} + \frac{gx}{B\rho} = 0$$

Equations of motion - Horizontal

$$\frac{d^2x}{ds^2} + \frac{x}{\rho^2} + \frac{gx}{B\rho} = 0$$

$$x'' + \left(\frac{1}{\rho^2} + \frac{g}{B\rho}\right)x = 0$$

We can define $k = \frac{g}{B\rho}$

$$x'' + \left(\frac{1}{\rho^2} + k\right)x = 0$$

and $\mathrm{K} = \frac{1}{\rho^2} + k$

$$\boxed{x'' + Kx = 0}$$

Equations of Motion - Vertical

A similar treatment of the vertical motion yields

$$\frac{d^2y}{ds^2} - \frac{dB_x}{dy}\frac{y}{B\rho} = 0$$

$$B_x(y) = \underbrace{\frac{dB_x}{dy}}_{g} y + \cdots$$

$$\frac{d^2y}{ds^2} - \frac{gy}{B\rho} = 0$$

We can define $k = \frac{g}{B\rho}$

$$\boxed{y'' - ky = 0}$$

Quick Aside on Springs

The form of this equation should look familiar $\boxed{x'' + Kx = 0}$

Recall Hooke's law for a mass, m, on a spring, k $\vec{F} = -k\vec{x}$

$$F = ma \qquad -kx = ma$$

$$-kx = m\frac{d^2x}{dt^2} \qquad -kx = mx''$$

$$-\frac{k}{m}x = x'' \qquad x'' + \frac{k}{m}x = 0 \qquad \omega = \sqrt{\frac{k}{m}}$$

Solutions to the Equations of Motion

Horizontal: $K = \frac{1}{\rho^2} + k$

Vertical: $K = -k$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{l} x'' + Kx = 0 \\ y'' + Ky = 0 \end{array}$$

These look like our familiar harmonic motion equations with known solutions of the form:

$$x(s) = Acos(\omega s) + Bsin(\omega s)$$

$$x'(s) = -A\omega sin(\omega s) + B\omega cos(\omega s)$$

$$x''(s) = -A\omega^2 cos(\omega s) - B\omega^2 sin(\omega s) = -\omega^2 x(s)$$

$$\omega = \sqrt{K}$$

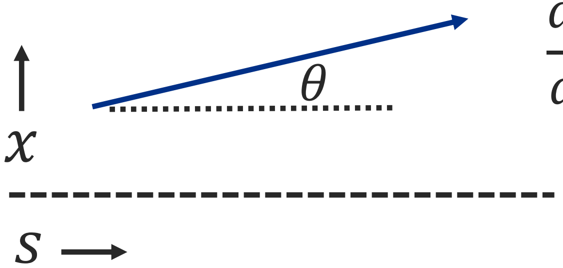
Matrix Representation

$$x(s) = A\cos\left(\sqrt{K}s\right) + B\sin(\sqrt{K}s)$$

$$x'(s) = -A\sqrt{K}\sin\left(\sqrt{K}s\right) + B\sqrt{K}\cos(\sqrt{K}s)$$

The constants A and B can be found from initial conditions,

$$x(0) = x_0 \qquad x'(0) = x'_0 \quad \Rightarrow \quad A = x_0 \qquad B = \frac{x'_0}{\sqrt{K}}$$



$$\frac{dx}{ds} = x' \sim \theta$$

$$\boxed{\begin{aligned} x(s) &= x_0\cos\left(\sqrt{K}s\right) + \frac{x'_0}{\sqrt{K}}\sin(\sqrt{K}s) \\ x'(s) &= -x_0\sqrt{K}\sin\left(\sqrt{K}s\right) + x'_0\cos(\sqrt{K}s) \end{aligned}}$$

Matrix Reminder

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} aV_1 + bV_2 \\ cV_1 + dV_2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Matrix Representation

$$x(s) = x_0 cos\left(\sqrt{K}s\right) + \frac{x'_0}{\sqrt{K}} sin(\sqrt{K}s)$$

$$x'(s) = -x_0\sqrt{K}sin\left(\sqrt{K}s\right) + x'_0 cos(\sqrt{K}s)$$

These equation can now be expressed in matrix form

$$\binom{x}{x'}_s = M\binom{x}{x'}_{s0} \qquad M = \begin{pmatrix} cos\left(\sqrt{K}s\right) & \dfrac{sin(\sqrt{K}s)}{\sqrt{K}} \\ -\sqrt{K}sin\left(\sqrt{K}s\right) & cos(\sqrt{K}s) \end{pmatrix}$$

Horizontal Focusing

For $K > 0$, this is focusing

$$x'' + Kx = 0$$

$$M = \begin{pmatrix} cos(\sqrt{K}s) & \dfrac{sin(\sqrt{K}s)}{\sqrt{K}} \\ -\sqrt{K}sin(\sqrt{K}s) & cos(\sqrt{K}s) \end{pmatrix}$$

For $K < 0$, this is defocusing

$$x'' - Kx = 0$$

$$x(s) = Acos(\sqrt{K}s) + Bsin(\sqrt{K}s)$$

$$\Downarrow$$

$$cos(ix) = \cosh(x)$$

$$-isin(ix) = \sinh(x)$$

$$M = \begin{pmatrix} cosh(\sqrt{K}s) & \dfrac{sinh(\sqrt{K}s)}{\sqrt{K}} \\ \sqrt{K}sinh(\sqrt{K}s) & cosh(\sqrt{K}s) \end{pmatrix}$$

Weak Focusing

Define a field index

$$B_y(x) = B_0 + \frac{dB_y}{dx}x \qquad B_x(y) = \frac{dB_x}{dy}y$$

$$n = -\frac{\rho}{B_0}g$$

Fields of this shape lead to focusing when $0 < n < 1$

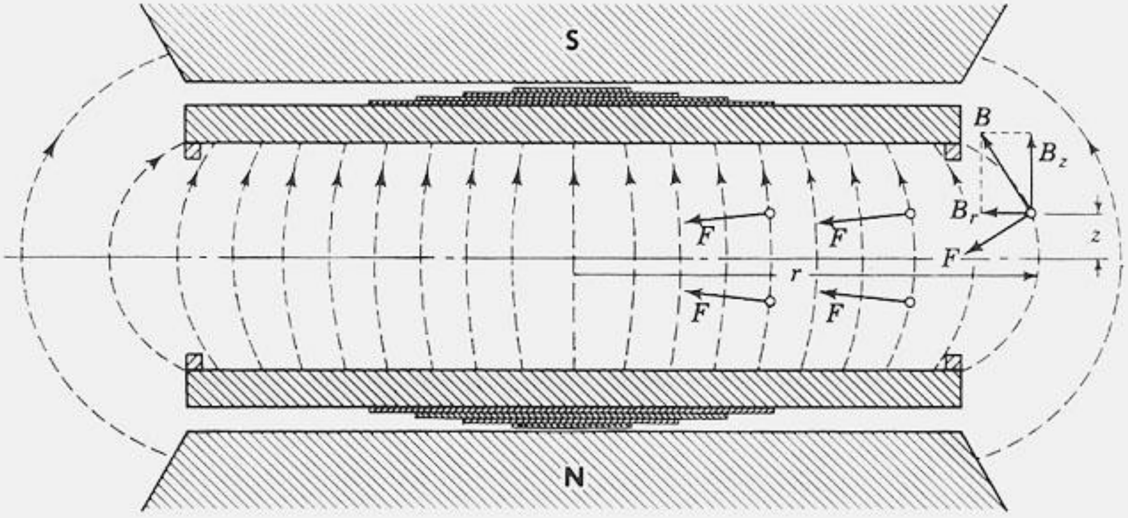


Fig. 6-7. Radially decreasing magnetic field between poles of a cyclotron magnet, showing shims for field correction.

Weak Focusing

Several early machines relied on weak focusing

- Cyclotrons relied on the uneven field between poles
  - First cyclotron built by
    Ernest Lawrence in 1930, 4” diam.

- The Betatron, first built by Donald Kerst in 1940, uses this field shape

- In 1943, Marcus Oliphant develops the idea for the synchrotron
  - The most famous weak focusing was the Bevatron built at Berkely in 1954, led to the discovery of the antiproton(Nobel Prize)

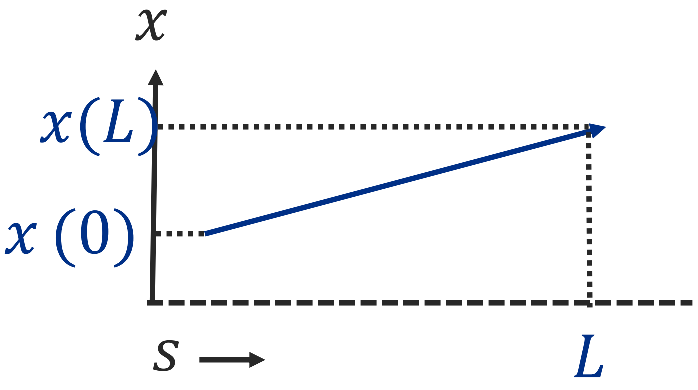
Drift Space

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

For $K = 0$, this is just a drift space of length L

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$



$$x(s) = x(0) + Lx'(0)$$

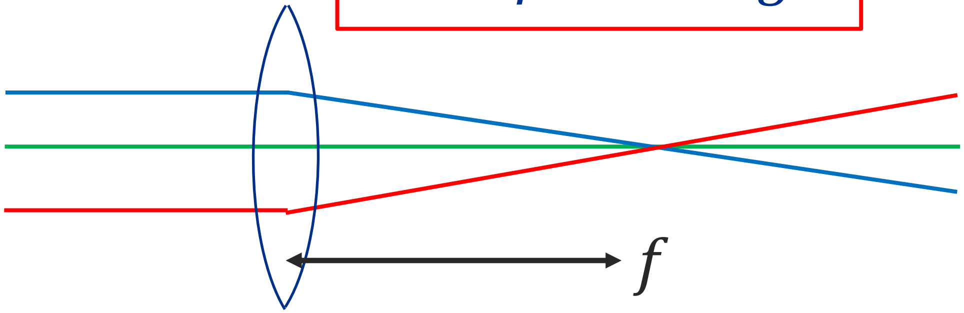
$x'(0) \ = x'(L)$ Slope hasn't changed

Thin Lens Approximation

$$\boxed{k = \frac{g}{B\rho} \qquad f = \frac{B\rho}{gL}}$$

For a focusing quadrupole of length L

$$M = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{\sin(\sqrt{K}L)}{\sqrt{K}} \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$



If the focal length is much longer than the length of the quadrupole

$$f = \frac{1}{kL} \gg L$$

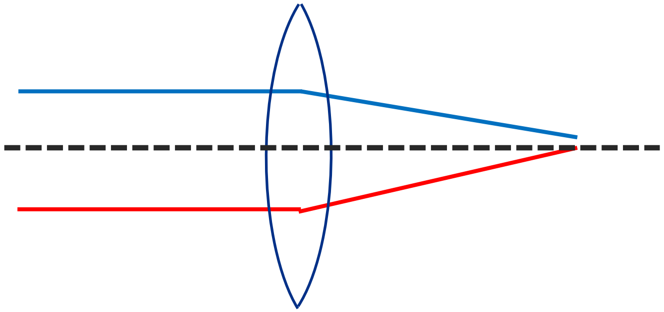
We can rewrite the focusing and defocusing matrices as:

$$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Focusing Thin Lens

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$



$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$

$x(s) = x(0)$ Initial position hasn’t changed

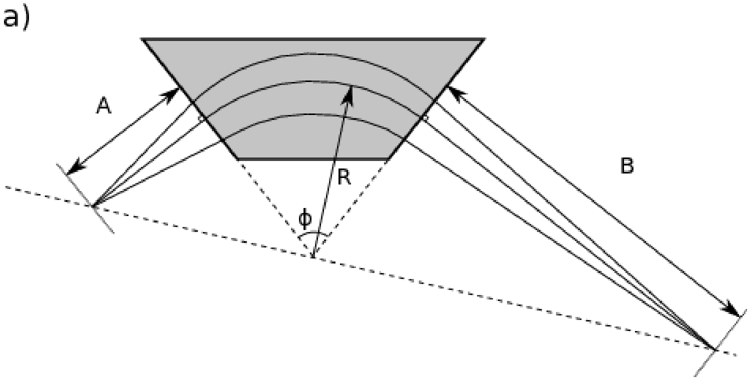
$x'(0) = x'(0) - \frac{1}{f} x(0)$ Slope changed

Sector Dipole Bend

Particle trajectory is perpendicular to the dipole edge

Horizontal plane: $K = {}^{1}/_{\rho^2} - k$

Vertical plane: $K = k$



If $k = 0, L = \rho\theta$

$$M_H = \begin{pmatrix} cos\theta & \rho sin\theta \\ -\frac{1}{\rho}sin\theta & cos\theta \end{pmatrix}$$

ρ = bending radius

θ = bending angle

$$M_V = \begin{pmatrix} 1 & \rho\theta \\ 0 & 1 \end{pmatrix}$$

Looks like drift

Transfer Matrices

A simple beam line can now be constructed by combining these elements as a product of the matrices

$$M = M_N \cdot \cdots \cdot M_4 \cdot M_3 \cdot M_2 \cdot M_1$$

From $\begin{pmatrix} x_0 \\ {x'}_0 \end{pmatrix}$, the final position and divergence of the particle are $\begin{pmatrix} x_1 \\ {x'}_1 \end{pmatrix}$

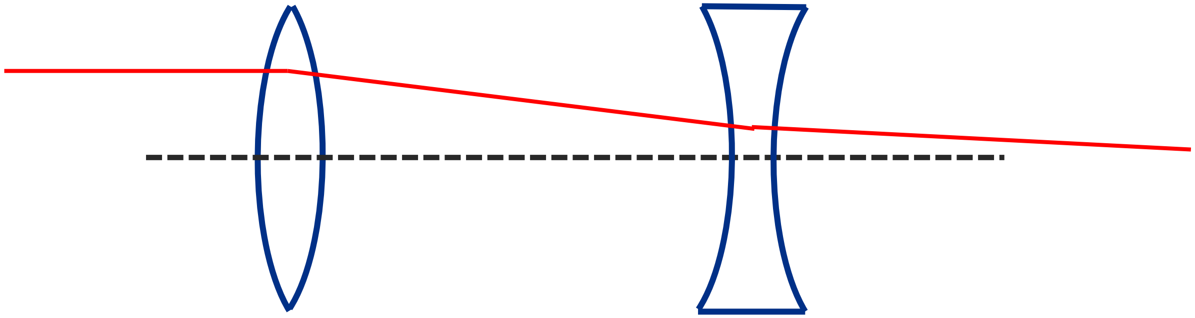
$$\begin{pmatrix} x_1 \\ {x'}_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ {x'}_0 \end{pmatrix}$$

The elements of the transfer matrix can be referenced with the following notation:

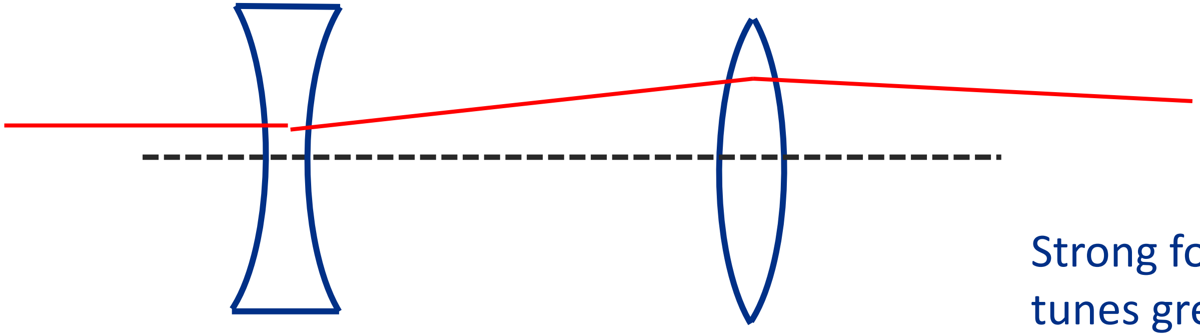
$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

Strong Focusing

A common combination is a focusing(F) quadrupole followed by a drift, then a defocusing(D) quadrupole, and another drift. Often referred to as FODO or doublet



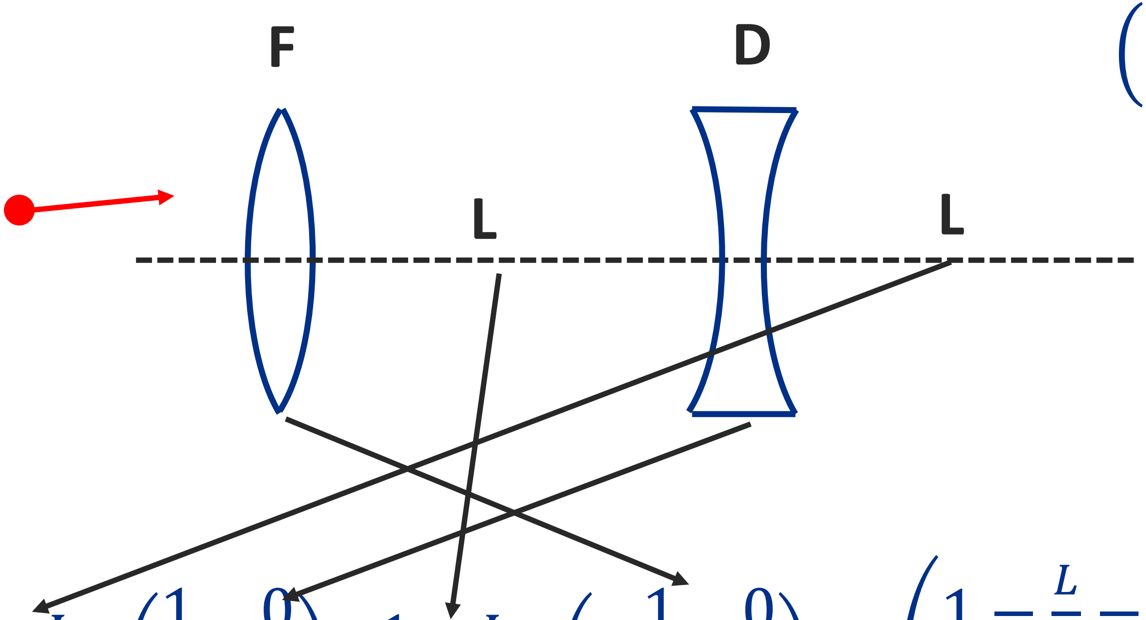
The result of this doublet, no matter the order FODO or DOFO, results in a net focusing in the horizontal and vertical direction



Strong focusing also has tunes greater than 1

FODO

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 & 2L + \frac{L^2}{f} \\ \frac{-L}{f^2} & 1 + \frac{L}{f} \end{pmatrix}$$

The particle moves from left to right, first encountering the F quadrupole, so we apply that matrix first, and so on

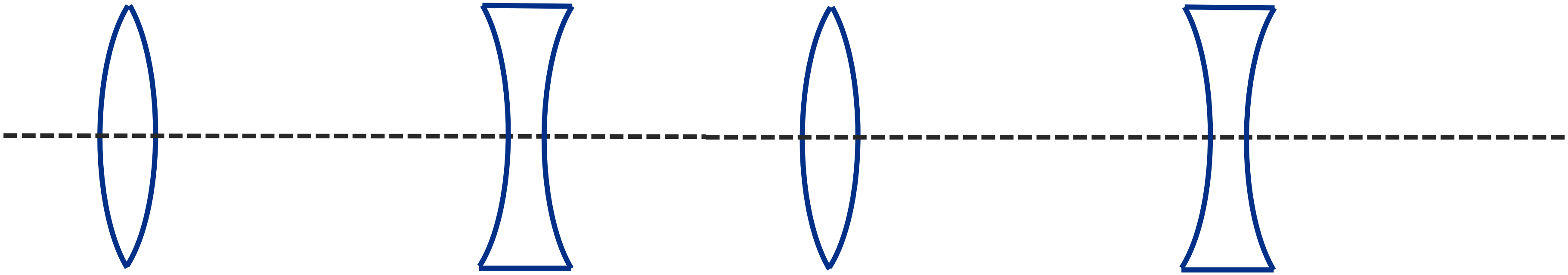
This is only for the x or y, the sign of the quadrupole will need to change for the other plane

Periodic

We will build our accelerator out of cells which are periodic such that:

$$\vec{B}(x,y,s+C)=\vec{B}(x,y,s)$$

C is the length of a cell, could be circumference of a circular machine or the length of a FODO cell



Twiss Parameters

The equations of motion found previously:

$x'' + Kx = 0$ If K = constant => motion of harmonic oscillator

$x'' + K(s)x = 0$ If K varies with s: Hill's equation (well studied D. E.)

The solution of the Hill equation is given by:

$$x(s) = Aw(s)\, cos(\psi(s) + \delta)$$

Constants of integration

The constants can be distributed and the solution written:

$$x(s) = w(s)\ (A_1 \cos\psi(s) + A_2 \sin\psi(s))$$

$$x'(s) = \left(A_1 w' + \frac{A_2 k}{w}\right) cos\psi(s) + \left(A_2 w' - \frac{A_1 k}{w}\right) \sin\psi(s)$$

Twiss Parameters

As before, solving for initial conditions of x, x' at $s = s_0$

$$A_1 = \frac{x_0}{w(s)} \qquad A_2 = \frac{x'_0 w(s) - x_0 w'(s)}{k}$$

Matrix for propagation over one period, $s_0 \; to \; s_0 + C$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_{0+C}} = \begin{pmatrix} cos\Delta\psi - \frac{ww'}{k} sin\Delta\psi & \frac{w^2}{k} sin\Delta\psi \\ -\frac{1 + (ww'/k)^2}{w^2/k} sin\Delta\psi & cos\Delta\psi + \frac{ww'}{k} sin\Delta\psi \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Phase of particle’s oscillation advances by

$$\Delta\psi = \int_{s_0}^{s_0+C} \frac{kds}{w^2(s)}$$

Twiss Parameters

$$\binom{x}{x'}_{s_{0+C}} = \begin{pmatrix} cos\Delta\psi - \frac{ww'}{k} sin\Delta\psi & \frac{w^2}{k} sin\Delta\psi \\ -\frac{1+(ww'/k)^2}{w^2/k} sin\Delta\psi & cos\Delta\psi + \frac{ww'}{k} sin\Delta\psi \end{pmatrix} \binom{x}{x'}_{s_0}$$

We can define new variables

$$\beta(s) = \frac{w^2(s)}{k}$$

$$\alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds} = -\frac{1}{2}\frac{d}{ds}\left(\frac{w^2(s)}{k}\right)$$

$$\gamma(s) = \frac{1+\alpha^2}{\beta}$$

The phase advance becomes:

$$\Delta\psi = \int_{s_0}^{s_0+C} \frac{ds}{\beta\,(s)}$$

α, β, γ are the Twiss parameters

Twiss Parameters

The matrix simplifies to:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_{0+C}} = \begin{pmatrix} cos\Delta\psi + \alpha sin\Delta\psi & \beta sin\Delta\psi \\ -\gamma sin\Delta\psi & cos\Delta\psi - \alpha sin\Delta\psi \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

or even more succinctly to:

$$M = cos\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + sin\mu \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\mu = \Delta\psi$$

phase advance over C

The α, β, γ functions can also be transformed using the elements of the transport matrix

$$\begin{pmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{pmatrix}_f = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & 1+2M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{pmatrix}_i$$

Betatron Motion

We can now describe the particle motion or oscillation

$$x(s) = A\sqrt{\beta(s)}cos(\psi(s) + \delta)$$

Deviation from nominal in one plane

Betatron function defines the beam envelope, similar to wavenumber

Small β –lots of oscillations
Large β –few oscillations

Phase advance

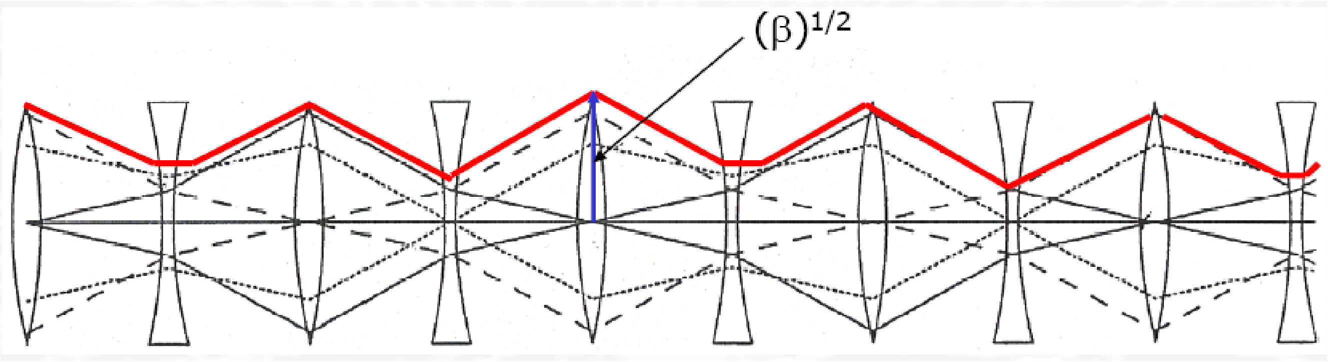
$$\Delta\psi = \int_{s_0}^{s_0+C} \frac{ds}{\beta\,(s)}$$

Phase advance in one turn
"**Betatron Tune**"

$$\mathrm{Q}_{\mathrm{x,y}} = \frac{1}{2\pi}\oint \frac{\mathrm{ds}}{\beta_{\mathrm{x,y}}\,(\mathrm{s})}$$

Betatron Oscillation

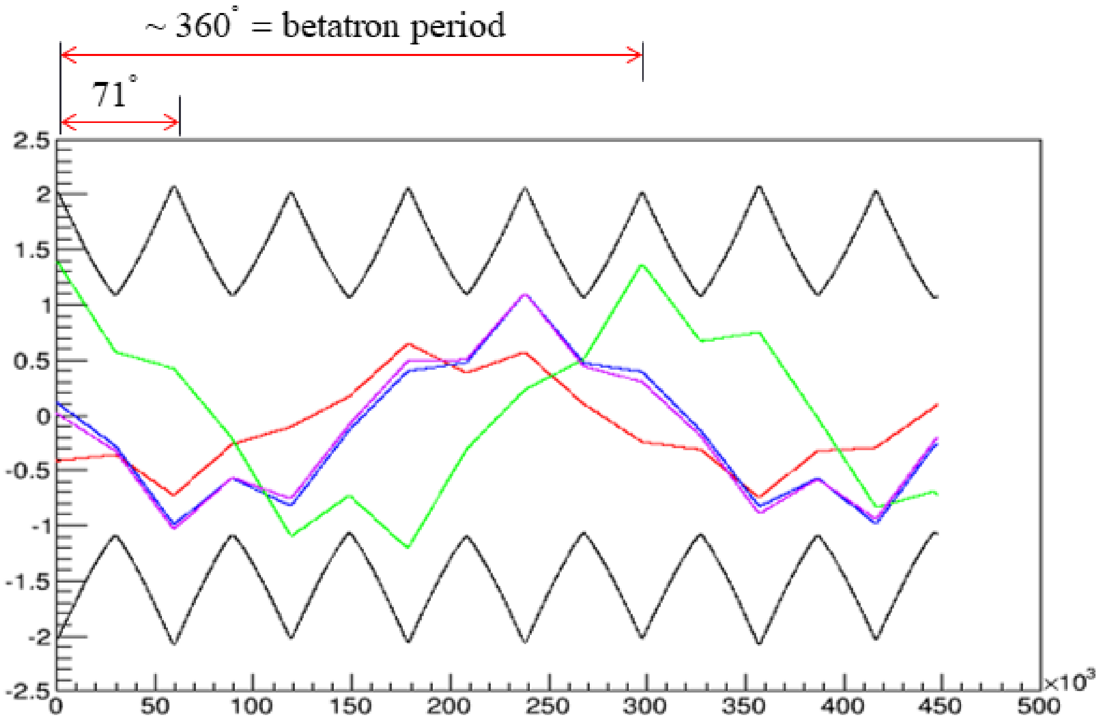
- Sinusoidal motion in vertical and horizontal are known as betatron oscillations
- The betatron function represents a bounding envelope to the beam motion, not the beam motion itself
- Particles oscillate around the closed orbit, a number of times which is determined by the betatron tune



Computer Codes for Accelerator Design

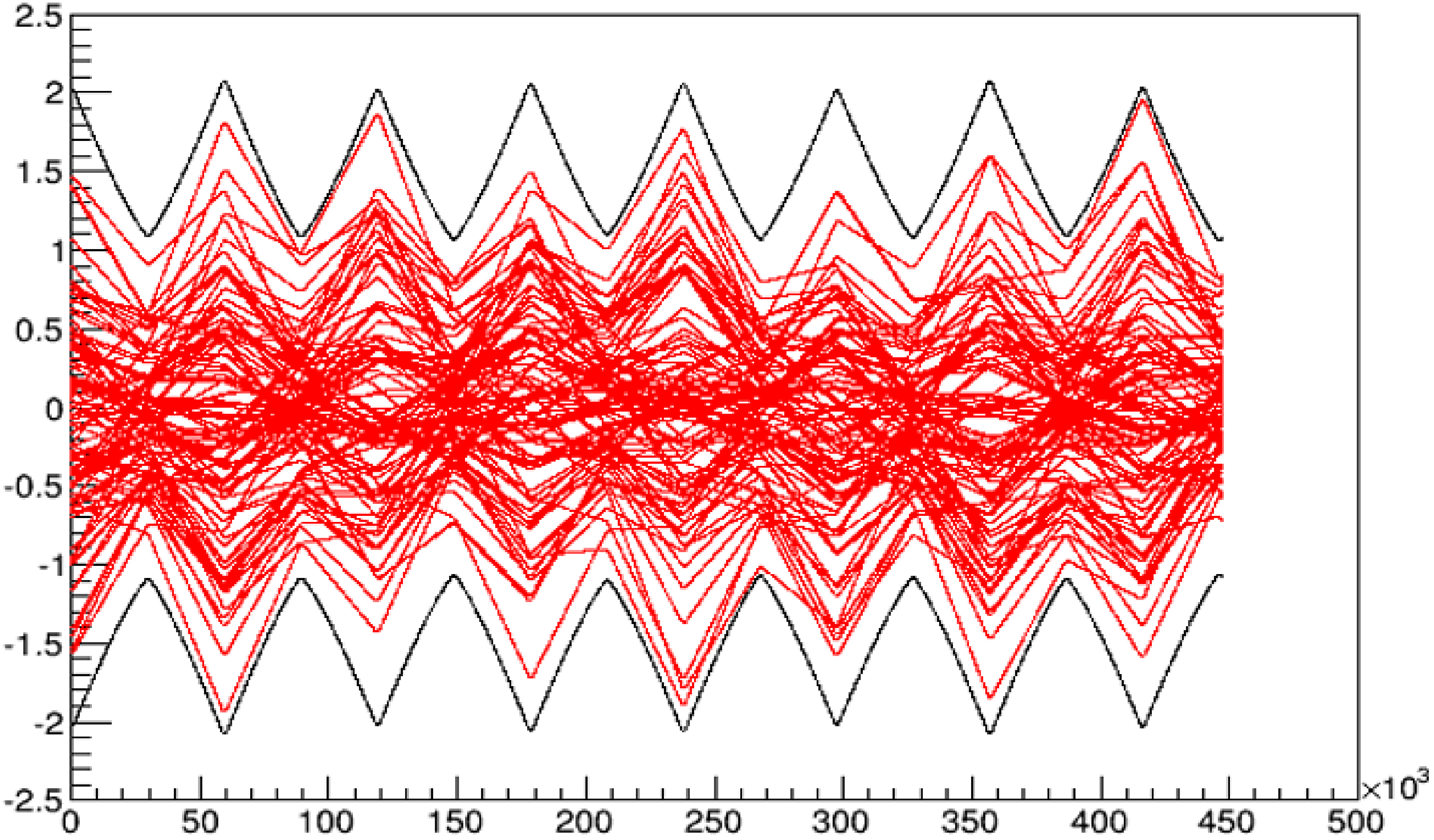
- The calculations with multiple elements can get complex quickly, so we can turn to computer codes
- MAD-X is one of the standard codes, but there are many others

MAD - Methodical Accelerator Design
CERN - BE/ABP Accelerator Beam Physics Group



E. Prebys using g4beamline

Beam Envelope



E. Prebys using g4beamline

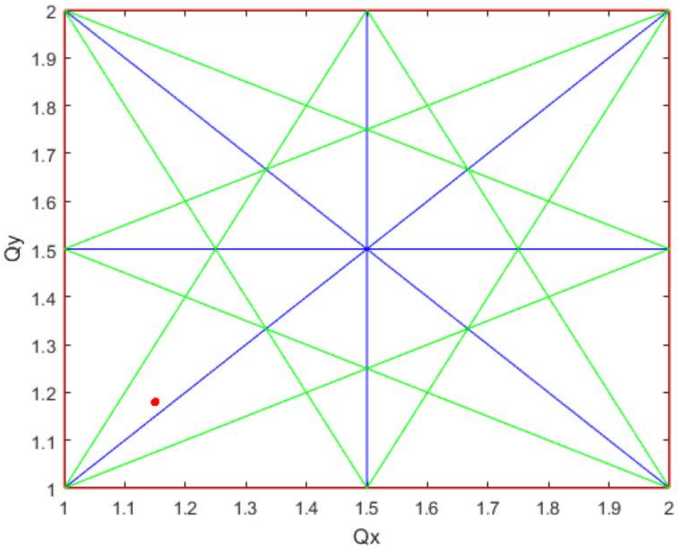
Tune

Why is the tune so important?

- If not carefully chosen, it can lead to harmful resonances which in turn can lead to beam blow-up
- Integer values should be avoided $Q_x, Q_y = m$
- Coupling between the x and y motion can also result from magnet or alignment errors

Coupling tunes to avoid:

- Integer sum
  - $Q_x + Q_y = m$
- Half integer tunes
  - $2Q_x = \pm m, 2Q_y = \pm m$
- Walknsaw resonance
  - $Q_x - 2Q_y = m, \pm 3Q_x = m$
- Other higher order



Tune diagram showing the first(red), second(blue), and third(green) order resonances

To be continued…

Bonus Slides

Laplace’s Equation

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

In a region free of currents and permeable materials, two dimensional magnetic fields can be derived from Laplace’s equation

Any analytic function of a complex variable satisfies Laplace’s equation

Complex Functions

$$\mathbb{z} = x + iy \qquad (x, y) \in D \qquad F(\mathbb{z}) = \boldsymbol{A} + i\boldsymbol{V} = \sum_{n=1}^{\infty} C_n \mathbb{z}^n$$

$$F(x + iy) = F_x(x, y) + iF_y(x, y) = \sum_{n=1}^{\infty} C_n (x + iy)^n$$

A complex function is analytic if it converges with its power series in a domain D. To be analytic, the real and imaginary parts of the function must obey the Cauchy-Riemann equations.

$$\boxed{\begin{gathered} \frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} = 0 \\[2em] \frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x} = 0 \end{gathered}}$$

Augustin Louis Cauchy
French 1789-1857

Bernhard Riemann
German 1826-1866

Analytic Complex Function

$$F(x+iy) = F_x(x,y) + iF_y(x,y)$$

$$F(\mathbb{z}) = (\boldsymbol{A} + iV)$$

$Cauchy - Riemann:$

$$\boxed{\begin{gathered}\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} = 0 \\ \frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x} = 0\end{gathered}} \Rightarrow \boxed{\begin{gathered}\frac{\partial \boldsymbol{A}}{\partial x} - \frac{\partial V}{\partial y} = 0 \\ \frac{\partial \boldsymbol{A}}{\partial y} + \frac{\partial V}{\partial x} = 0\end{gathered}} \Rightarrow \boxed{\begin{gathered}\frac{\partial \boldsymbol{A}}{\partial x} = \frac{\partial V}{\partial y} \\ \frac{\partial \boldsymbol{A}}{\partial y} = -\frac{\partial V}{\partial x}\end{gathered}}$$

$F(\mathbb{Z}) = (\boldsymbol{A} + iV)$

Vector potential

- Using $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, we can define a vector potential A such that

$$\boxed{\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}}$$

- Adding a gradient to this potential ($\boldsymbol{A}' = \boldsymbol{A} + \nabla f$) still satisfies

$$\boldsymbol{\nabla} \times \boldsymbol{A}' = \boldsymbol{\nabla} \times \boldsymbol{A} + \underbrace{\cancel{\boldsymbol{\nabla} \times \boldsymbol{\nabla} f}}_{0} = \boldsymbol{B}$$

Scalar potential

- For charge and magnetic material free regions, $\boldsymbol{\nabla} \times \boldsymbol{B} = 0$ and we can define a scalar potential

$$\boxed{B = -\boldsymbol{\nabla} V}$$

The function of a complex variable

$\boldsymbol{A}$: Vector potential
V : Scalar potential

$$\boldsymbol{F} = \boldsymbol{A} + iV$$

$$\boxed{\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad \boxed{\boldsymbol{B} = -\boldsymbol{\nabla} V} = -\left(\boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}\right)$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\underbrace{\cancel{\boldsymbol{\nabla} \cdot} \boldsymbol{A}}_{0 \text{ (Coulomb gauge)}}) - \boldsymbol{\nabla}^2 \boldsymbol{A} = 0 \longrightarrow \boldsymbol{\nabla}^2 \boldsymbol{A} = 0$$

$\boldsymbol{A}$ satisfies the Laplace equation!

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = -\boldsymbol{\nabla}^2 V = 0 \longrightarrow \boldsymbol{\nabla}^2 V = \boldsymbol{0}$$

V also satisfies the Laplace equation!

The complex function $\boldsymbol{F} = \boldsymbol{A} + iV$ must also satisfy the Laplace equation
$\boldsymbol{\nabla}^2 \boldsymbol{F} = 0$

Fields from the 2D function of a complex variable

$Cauchy - Riemann:$

$$\boxed{\begin{gathered}\frac{\partial \boldsymbol{A}}{\partial x} = \frac{\partial V}{\partial y} \\ \frac{\partial \boldsymbol{A}}{\partial y} = -\frac{\partial V}{\partial x}\end{gathered}}$$

$$\boxed{\boldsymbol{B} = -\boldsymbol{\nabla} V} = -\left(\boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}\right)$$

$$B_x = -\frac{\partial V}{\partial x} \qquad B_y = -\frac{\partial V}{\partial y}$$

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A}{\partial y} \qquad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A}{\partial x}$$

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\partial F(\mathbb{z})}{\partial \mathbb{z}} = \frac{\partial \boldsymbol{A} + i\partial V}{\partial x + i\partial y}$$

Fields from the 2D function of a complex variable

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\partial F(\mathbb{z})}{\partial \mathbb{z}} = \frac{\partial \boldsymbol{A} + i\partial V}{\partial x + i\partial y}$$

$$\boldsymbol{F}(\mathbb{z}) = \boldsymbol{A} + iV, \quad \mathbb{z} = x + iy$$

∂x

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\frac{\partial \boldsymbol{A}}{\partial x} + i\frac{\partial V}{\partial x}}{\frac{\partial x}{\partial x} + i\frac{\partial y}{\partial x}}$$

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\partial \boldsymbol{A}}{\partial x} + i\frac{\partial V}{\partial x}$$

∂y

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\frac{\partial \boldsymbol{A}}{\partial y} + i\frac{\partial V}{\partial y}}{\frac{\partial x}{\partial y} + i\frac{\partial y}{\partial y}}$$

$$\boldsymbol{F}'(\mathbb{z}) = -i\frac{\partial \boldsymbol{A}}{\partial y} + \frac{\partial V}{\partial y}$$

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A}{\partial y}$$

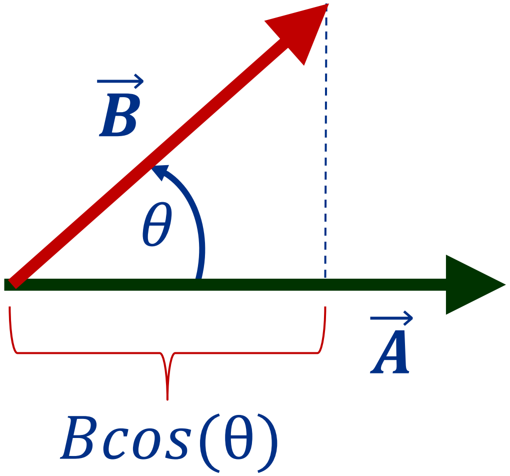
$$B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A}{\partial x}$$

$$\boldsymbol{F}'(\mathbb{z}) = -B_y - iB_x \qquad \boldsymbol{F}'(\mathbb{z}) = -iB_x - B_y$$

$$\boxed{\boldsymbol{B}^* = B_x - iB_y = i\boldsymbol{F}'(\mathbb{z})} \qquad \boxed{B_y + iB_x = -\boldsymbol{F}'(\mathbb{z})}$$

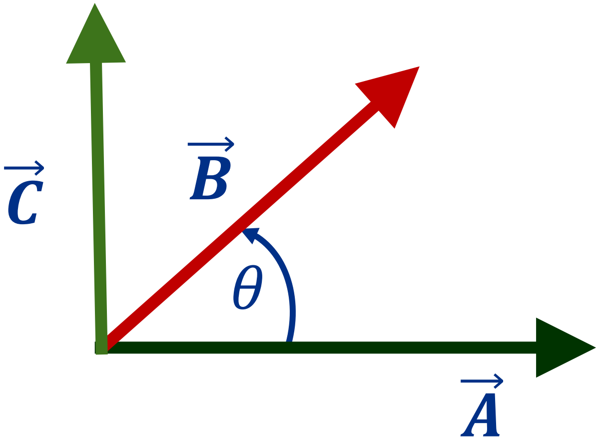
Vector Operations

- Scalar “dot” product



$$\boxed{\vec{A}\cdot\vec{B} = ABcos(\theta) = |\vec{A}||\vec{B}|\cos(\theta)}$$

- Vector “cross” product



$$\boxed{\vec{C} = \vec{A}\times\vec{B} = ABsin(\theta)}$$

Resulting vector perpendicular to the plane formed by A and B

Differential Operators

- Grad operator

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right)$$

- Gradient

$$\vec{\nabla}\phi \equiv \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)$$

- Divergence

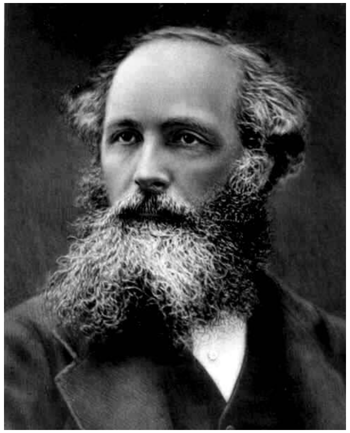
$$\vec{\nabla} \cdot \vec{A} \equiv \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

- Curl

$$\vec{\nabla} \times \vec{A} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{j} + \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \hat{k}$$

Maxwell's Equations

(in vacuum)



James Clerk Maxwell
Scottish 1831-1879

Gauss's law

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_o}$$

$$\oiint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\varepsilon_o}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\oiint \boldsymbol{B} \cdot d\boldsymbol{A} \ = 0$$

Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$

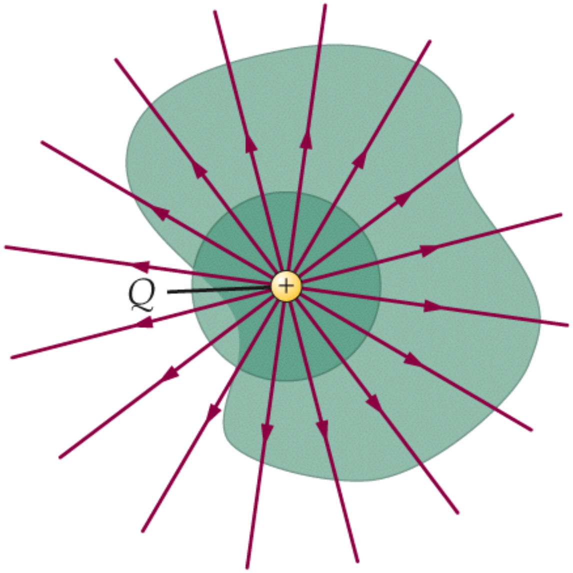
Ampere's law

$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_o \boldsymbol{I} + \mu_o \varepsilon_o \iint \frac{\partial \boldsymbol{E}}{\partial t} \cdot d\boldsymbol{A}$$

Gauss's Law

The total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ϵ_0



$$\boxed{\Phi_E = \oiint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{encl}}{\epsilon_0}}$$

$$\boxed{\oiint \boldsymbol{B} \cdot d\boldsymbol{A} \; = 0}$$

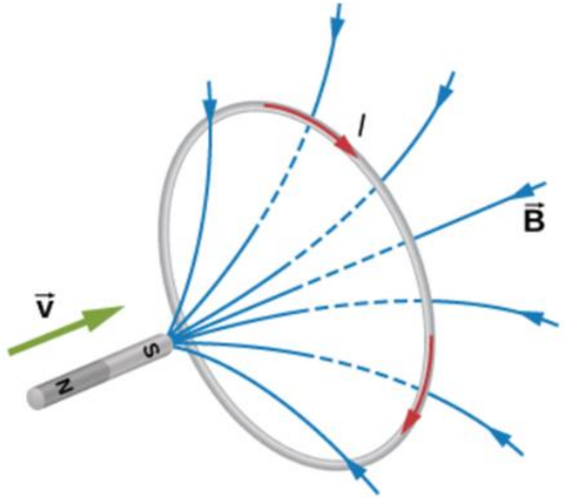
No known magnetic monopoles

ϵ_o *is electric constant* $= 8.85418781762\times 10^{-12}\ \mathrm{A^2{\cdot}s^4{\cdot}kg^{-1}{\cdot}m^{-3}}$

Faraday's Law

The integrated electric field around any closed loop is proportional to the rate of change of the magnetic flux passing through the loop

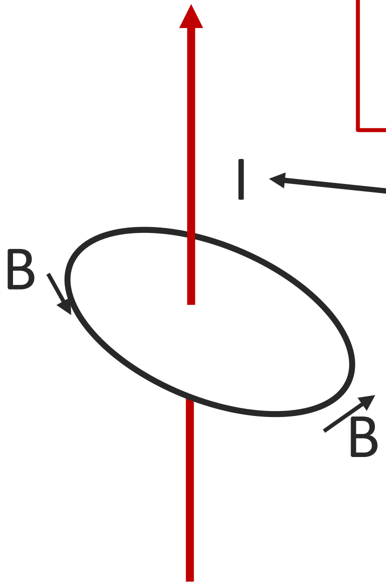
$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$



Ampere's Law

The current passing through a surface is equal to the line integral of the B field around that closed surface

$$\boxed{\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_o \boldsymbol{I} + \mu_o \varepsilon_o \iint \frac{\partial \boldsymbol{E}}{\partial t}\cdot d\boldsymbol{A}}$$



Enclosed current

"Displacement current" in charging capacitor for example