Factorisation and Resummation for Jet Cross Sections

Based on work with Thomas Becher (2309.17355)

Jürg Haag, University of Zürich, QCD-Seminar @ CERN, 3 Juni 2024

Introduction

Exclusive jet cross-sections

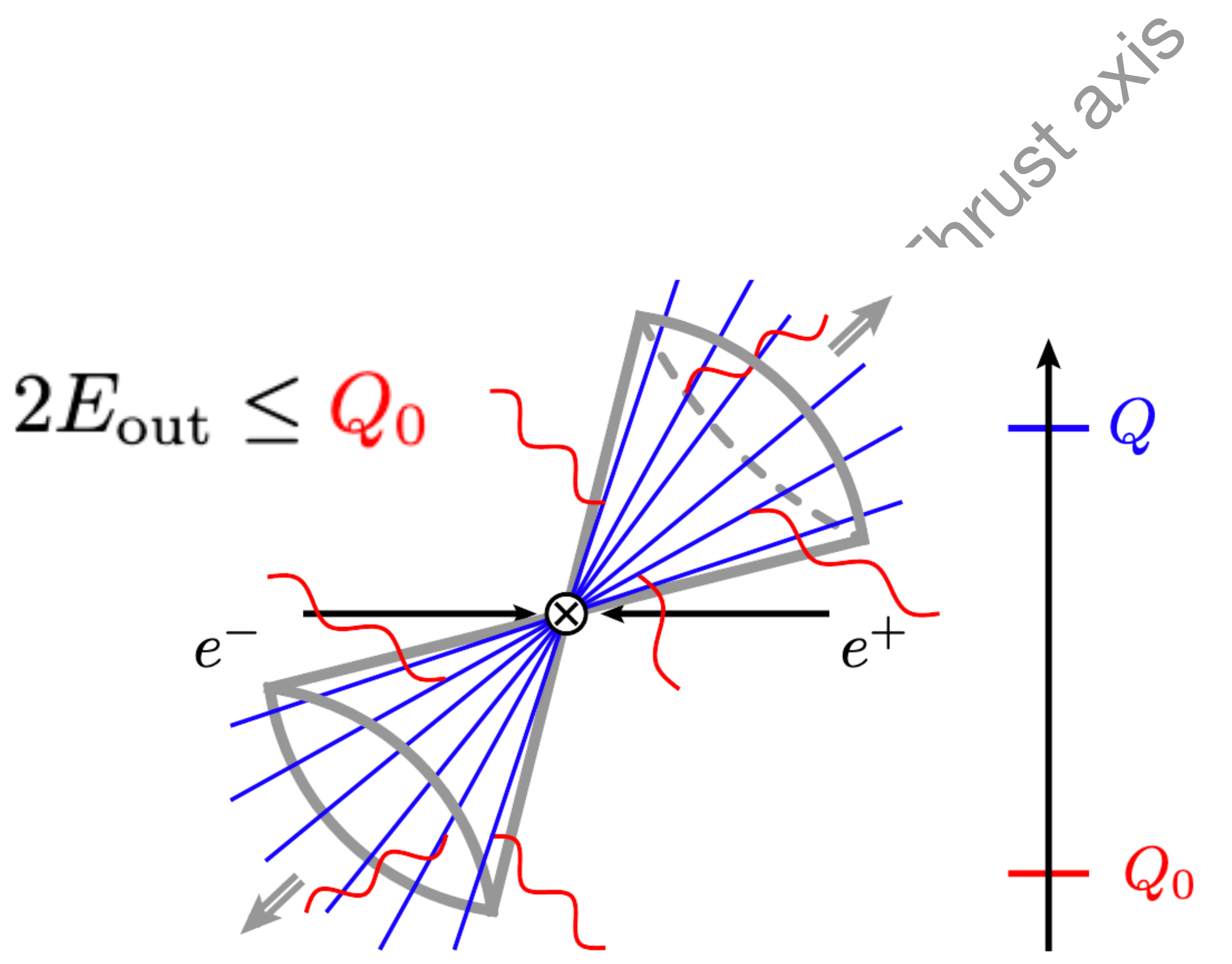
Gaps between jets

- We are interested in cross sections of the form

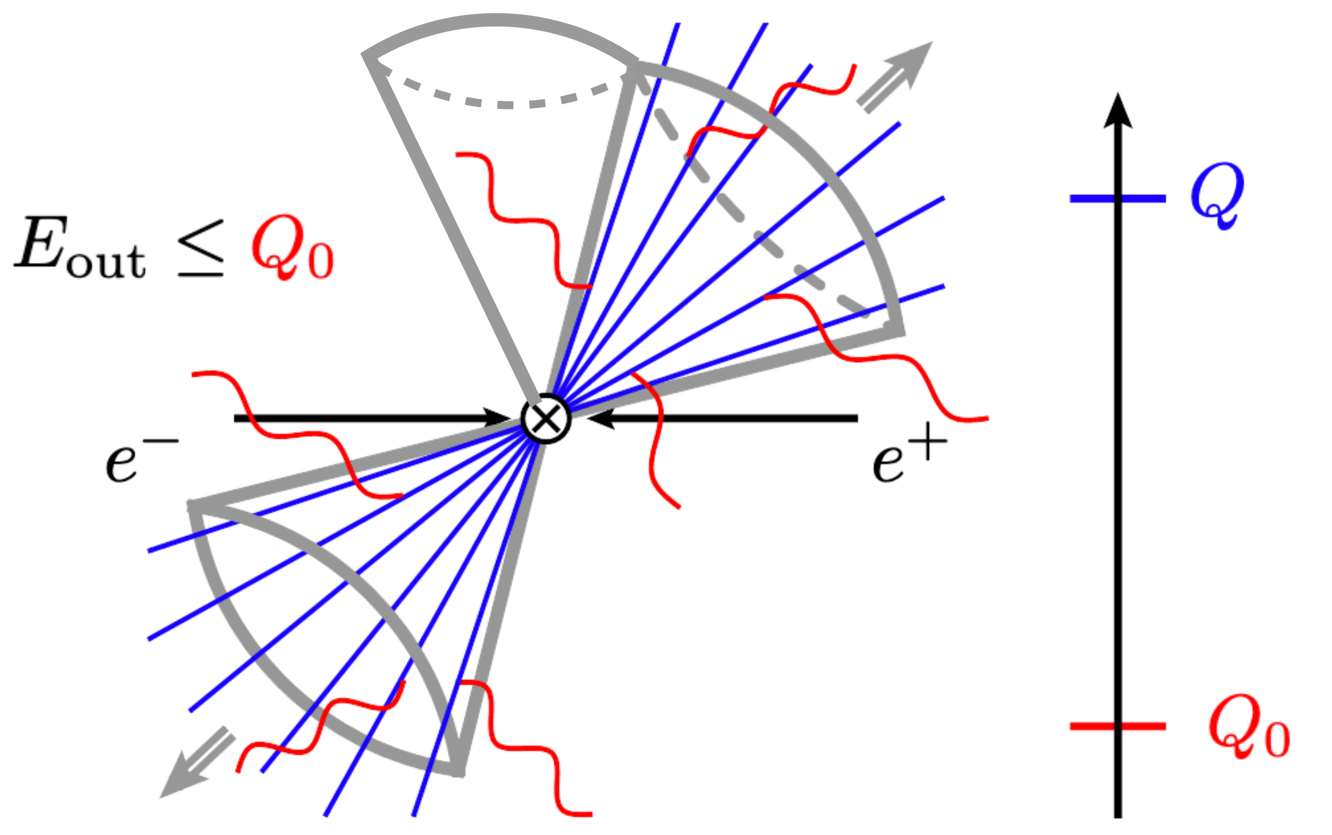
$$\sigma\left(Q_0\right)=\frac{1}{2Q^2}\sum_{m=M}^{\infty}\prod_{i=1}^{m}\int\left[dp_i\right]\left|\mathcal{M}_m(\{\underline{p}\})\right|^2\delta\left(Q-E_{\text{tot}}\right)\delta^{(d-1)}\left(\vec{p}_{\text{tot}}\right)\Theta\left(Q_0-2E_{\text{out}}\right)$$

- The energy veto $\Theta\left(Q_0-2E_{\text{out}}\right)$ introduces non-global logarithms $\alpha_S\log\left(\frac{Q}{Q_0}\right)\sim 1$

- What does "out" mean?
  - Fixed cone cross section: "out" depends only on the hard scale dynamics
  - Sequential clustering: "out" also depends on the soft scale dynamics.



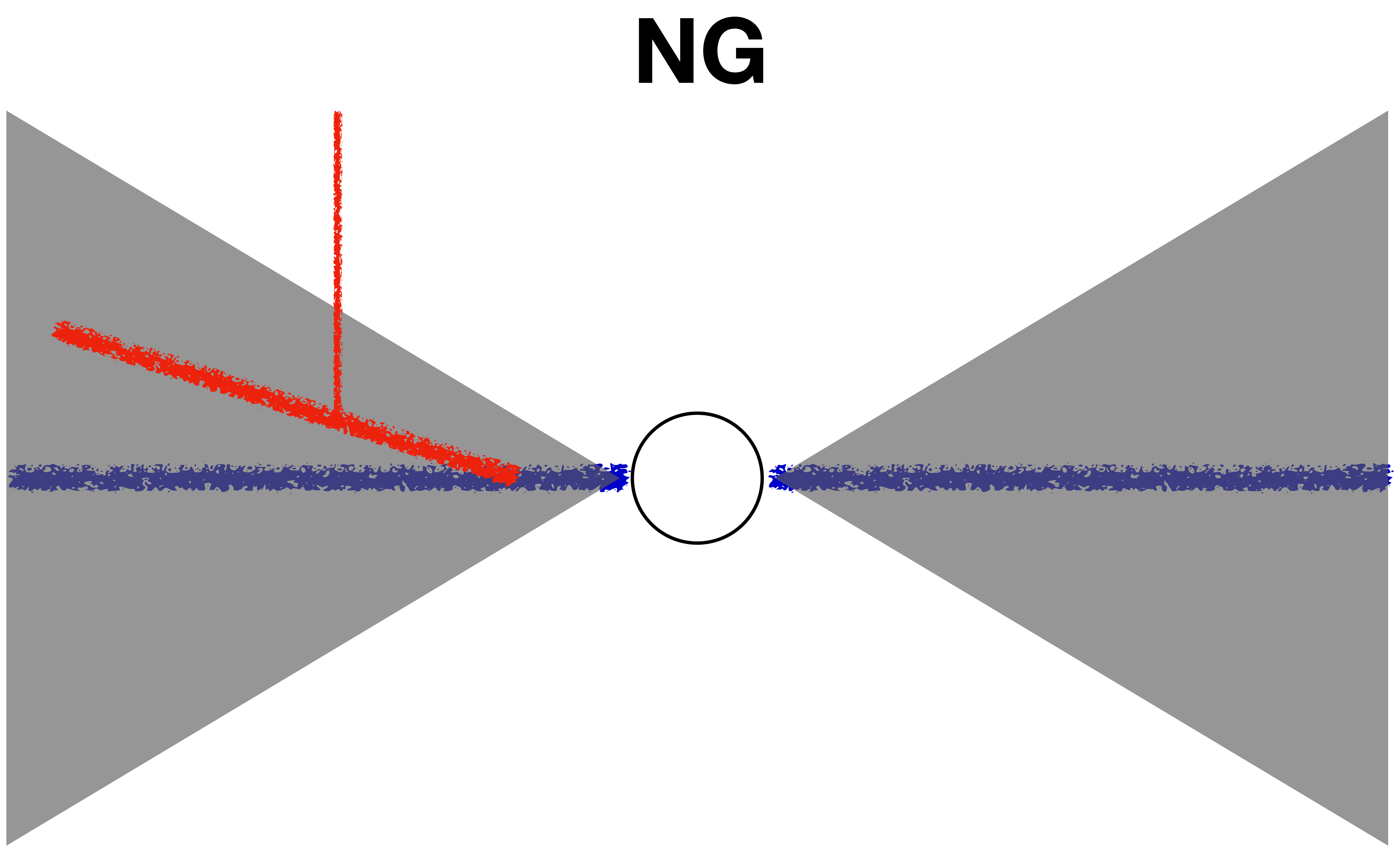
Fixed cone



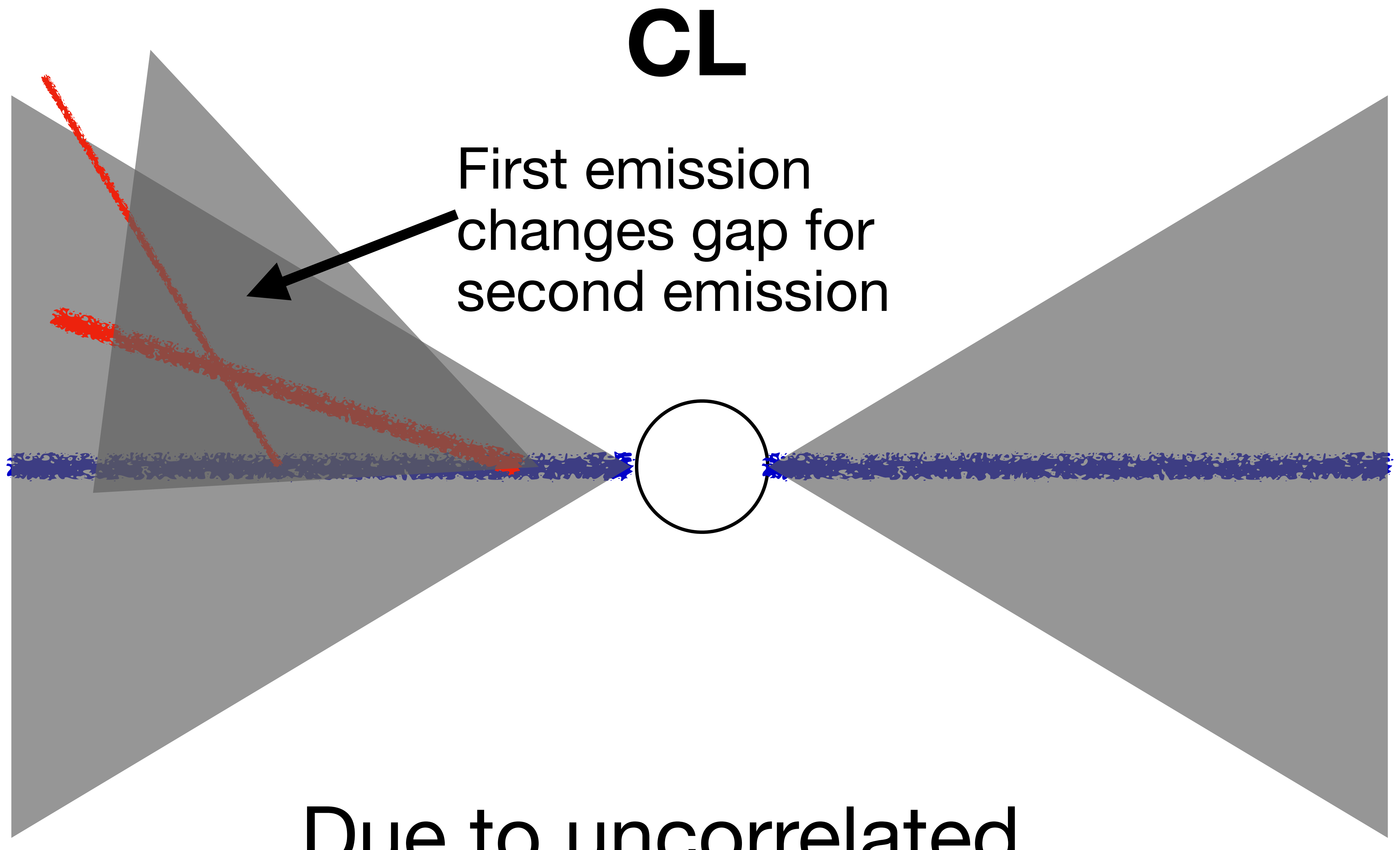
Clustering Constraint

Figures adjusted from 1605.02737

Non-Global v.s. Clustering Logs



Due to correlated emissions



Due to uncorrelated emissions. Even exist in QED.

Non-Global and Clustering Logarithms (an incomplete history)

- NGLs first discovered by Dasgupta and Salam in 2001 (hep-ph/0104277)
- NLL NG resummation was recently achieved. (GNOLE: 2111.02413, SCET: 2307.02283,PANSCALES: 2307.11142)
- LL Beyond leading color:

  (Weigert: hep-ph/0312050, Hatta, Ueda: 2011.04154, Plätzer et.al.: 1312.2448, 1802.08531,1905.08686)
- Super Leading Logs (SLL) discovered 2006: (Forshaw, Kyrieleis, Seymour: hep-ph/0604094; Resummation: Becher et al 2107.01212)
- Clustering Logarithms (CL): Found shortly after NGLs (Appleby, Seymour: hep-ph/0211426])
- Analyzed in SCET: (R. Kelley, J.R. Walsh and S. Zuber: 1202.2361,1203.2923) A factorization theorem for CL in SCET as a product of a hard and soft function was thought to be impossible

What does “out” mean?

- We run an inclusive k_T-type jet clustering on the partons $\{p_1, \ldots, p_n\}$ which yields the jet momenta $\{P_1, \ldots, P_{n_J}\}$.
- For each jet, decide whether it is “in” or “out”, e.g.,
    - only the M hardest jets are “in” for M jet cross sections or
    - only the jets which are in a cone with (half)-opening angle α around the thrust axis are “in”
- Then define $E_{\text{out}} = \sum_{j=1}^{n_{\mathrm{J}}} P_j^0 \Theta_{\text{out}}\left(P_j\right)$

k_T-type clustering algorithm

1. For a list of partons with momenta $\{p_1, \ldots, p_n\}$, determine the distances

$$d_{ij} = \min(E_i^{2p}, E_j^{2p}) \frac{\Delta R_{ij}^2}{R^2}, \quad i \neq j \in \{1, \ldots, n\}$$

$$d_i = E_i^{2p}, \quad i \in \{1, \ldots, n\},$$

$p = 1$: k_T $p = 0$: C/A $p = -1$: Anti-k_T

Determines the clustering distance

Gives an approximate order in which things cluster

2. Find the minimum of the d_{ij} and d_i.

3. If it is a d_{ij}, combine the two partons into a single one with combined momentum $p_{ij} = p_i + p_j$ and return to step 1.

4. Otherwise, if the minimum is a d_i, declare the corresponding particle to be a jet, remove it from the list of particles, and return to 1.

5. Stop when no particles remain.

Factorisation and Resummation

Factorisation Theorem

Factorisation for clustering:

$$\sigma(Q,Q_0) = \sum_{m=M}^{\infty} \langle \boldsymbol{\mathcal{H}}_m(\{\underline{n}\},\{\underline{z}\},Q,\mu) \otimes_z \boldsymbol{\mathcal{S}}_m(\{\underline{n}\},\{\underline{z}\},Q_0,\mu)\rangle$$

It is convenient to have the hard particles ordered by energy

$$\boldsymbol{\mathcal{H}}_m(\{\underline{n}\},\{\underline{z}\},Q) = \frac{1}{2Q^2}\left(\prod_{i=1}^{m}\int \frac{dE_i E_i^{d-3}}{\tilde{c}^{\epsilon}(2\pi)^2}\right)\widetilde{\boldsymbol{\mathcal{H}}}_m(\{\underline{p}\})\times$$
$$(2\pi)^d\delta(Q-E_{\text{tot}})\delta^{d-1}(\vec{p}_{\text{tot}})\Theta_{\text{in}}(\{\underline{p}\})\prod_{j=1}^{m}\delta\left(z_j-\frac{E_j}{E_{j-1}}\right)$$

$$\boldsymbol{\mathcal{S}}_m(\{\underline{n}\},Q_0) = \sum\!\!\!\!\!\!\!\int_X \langle 0|\boldsymbol{\mathcal{S}}_1^\dagger(n_1)\ldots\boldsymbol{\mathcal{S}}_m^\dagger(n_m)|X\rangle\langle X|\boldsymbol{\mathcal{S}}_1(n_1)\ldots\boldsymbol{\mathcal{S}}_m(n_m)|0\rangle\theta(Q_0-2E_{\text{out}})$$

We need the energy fractions $z_i = \frac{E_i}{E_{i-1}}$ to solve Θ_{in}. Θ_{in} is not a simple product anymore

RG Evolution and Resummation

Anomalous dimensions

- The renormalised hard function satisfies an RGE:

$$\frac{d}{d\ln\mu}\boldsymbol{\mathcal{H}}_m(\{\underline{n}\},\{\underline{z}\},Q,\mu) = -\sum_{l=M}^{m}\boldsymbol{\mathcal{H}}_l(\{\underline{n}\},\{\underline{z}\},Q,\mu)\,\boldsymbol{\Gamma}^H_{lm}(\{\underline{n}\},\{\underline{z}\},Q,\mu)$$

- The anomalous dimension is a matrix in multiplicity space

$$\boldsymbol{\Gamma}^H = \frac{\alpha_s}{4\pi}\begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \cdots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \cdots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \cdots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \ldots .$$

- $\boldsymbol{R}_m$ is obtained from the soft emission limit of $\mathscr{H}^{(0)}_{m+1}$

- $\boldsymbol{V}_m$ is obtained from the soft gluon loop limit of $\mathscr{H}^{(1)}_{m}$

Deriving the one-loop Anomalous Dimension

Real Anomalous Dimension

Consider the hard function:

$$\mathcal{H}_{m+1}\left(\{\underline{n}, n_q\}, \{\underline{z}, z_q\}, Q, \epsilon\right) = \frac{1}{2Q^2}\prod_{i=1}^{m+1}\int \frac{dE_i E_i^{d-3}}{\tilde{c}^{\epsilon}(2\pi)^2}\delta(z_m - \frac{E_i}{E_{i-1}})$$

$$\times\, \widetilde{\boldsymbol{\mathcal{H}}}_{m+1}(\{\underline{p}, q\})(2\pi)^d \delta\big(Q - \sum_{i=1}^{m+1} E_i\big)\delta^{(d-1)}\left(\vec{p}_{\rm tot} + \vec{q}\right)\Theta_{\rm in}(\{\underline{p}, q\})$$

Take the soft limit:

$$\widetilde{\boldsymbol{\mathcal{H}}}_{m+1}(\{\underline{p}, q\}) = |\mathcal{M}_{m+1}(\{\underline{p}, q\})\rangle\langle\mathcal{M}_{m+1}(\{\underline{p}, q\})| \xrightarrow{q\to 0} \frac{1}{E_q^2} W_{ij}^q\, \boldsymbol{T}_i^a\, \widetilde{\boldsymbol{\mathcal{H}}}_m(\{\underline{p}\})\, \boldsymbol{T}_i^{\tilde{a}}$$

Perform the E_q integral:

$$\boldsymbol{\mathcal{H}}_{m+1}\left(\{\underline{n}, n_q\}, \{\underline{z}, z_q\}, Q\right) = \frac{\alpha_s}{4\pi}\frac{1}{2\epsilon}\underbrace{4\delta(z_q)\Theta_{\rm in}\left(n_q\right)\sum_{(ij)} W_{ij}^q\, \boldsymbol{T}_i^a\, \boldsymbol{\mathcal{H}}_m\left(\{\underline{n}\}, \{\underline{z}\}, Q\right)\, \boldsymbol{T}_j^{\tilde{a}}}_{-\boldsymbol{R}_m}$$

RG Evolution and Resummation

Anomalous dimensions

- At NLO, the anomalous dimensions are given by

$$\boldsymbol{V}_m = 2\sum_{(ij)} (\boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}) \int [d\Omega_q]\, W_{ij}^q$$

$$\boldsymbol{R}_m = -4\,\delta(z_q)\sum_{(ij)} \boldsymbol{T}^a_{i,L}\boldsymbol{T}^{\tilde{a}}_{j,R} W_{ij}^q \Theta_{\text{in}}(n_q)$$

Strong energy ordering!

- At NNLO the structure of the anomalous dimension gets more complicated:

$$\boldsymbol{v}_m = \boldsymbol{v}_m^{\text{fc}} \quad \boldsymbol{r}_m = \delta(z_q)\boldsymbol{r}_m^{\text{fc}} \quad \boldsymbol{d}_m = \delta(z_{qr})F(z_r)$$

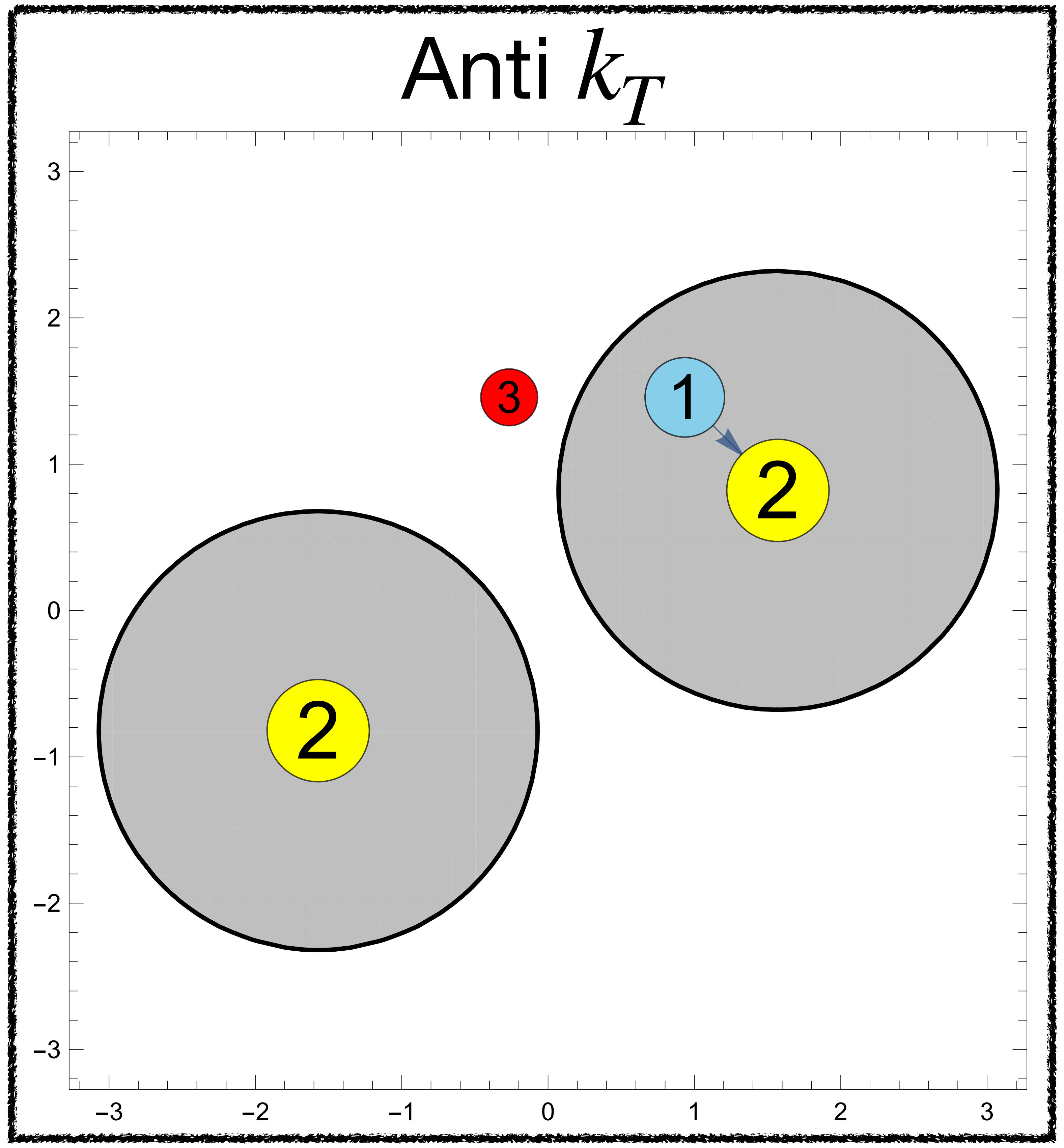
- New and interesting
- Contains 4-parton correlators
- Depends on details of clustering, e.g. WTA or E scheme

Parton Shower

- Shower time $t = \frac{\alpha_S}{4\pi} \log(\frac{Q}{Q_0})$
- Shower generates real emissions by randomly choosing dipoles and emission times according to $\boldsymbol{V}_m$.
- Stops, once a new emission does not satisfy the “in”-condition.
- Angular integrals are done with the MC-sampling
- Energy integrals are trivial at LL due to the $\delta(z_i)$ term in $\boldsymbol{R}_m$
- The only additional difficulty related to the clustering is to determine the “in” condition for each new emission

Strongly Ordered Clustering

Example Situation for diet production

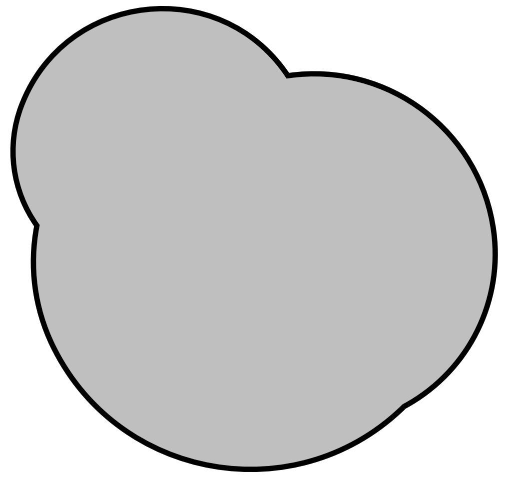


Primary particles

First emission “in”

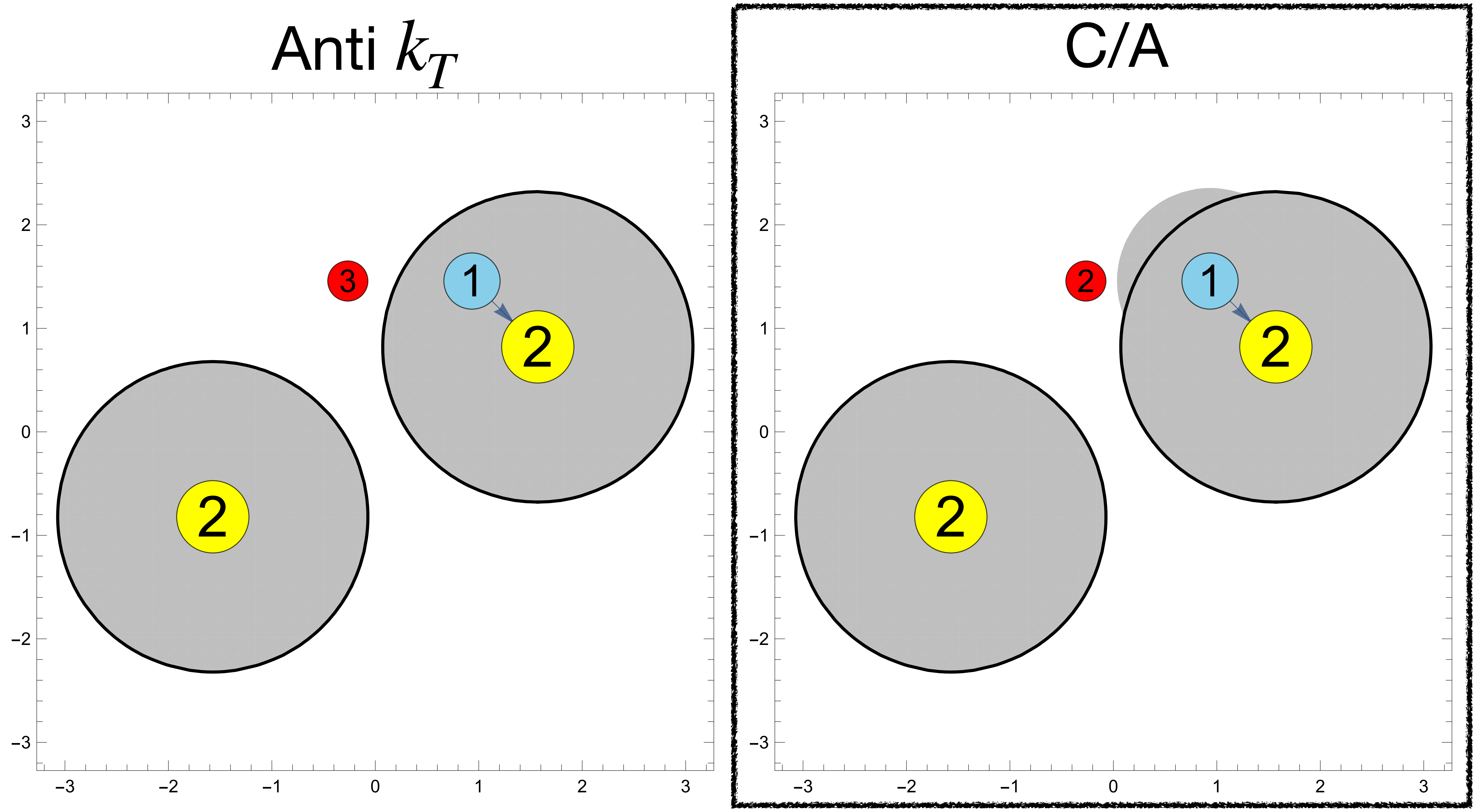
Second emission “in”

Second emission “out”



Phase space, where the second emission would be “in”

Example Situation for diet production

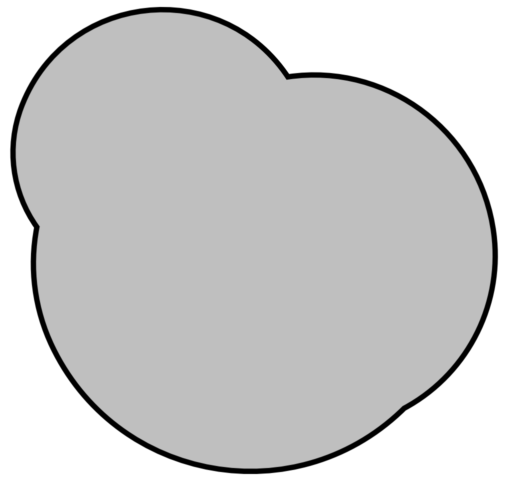


Primary particles

First emission “in”

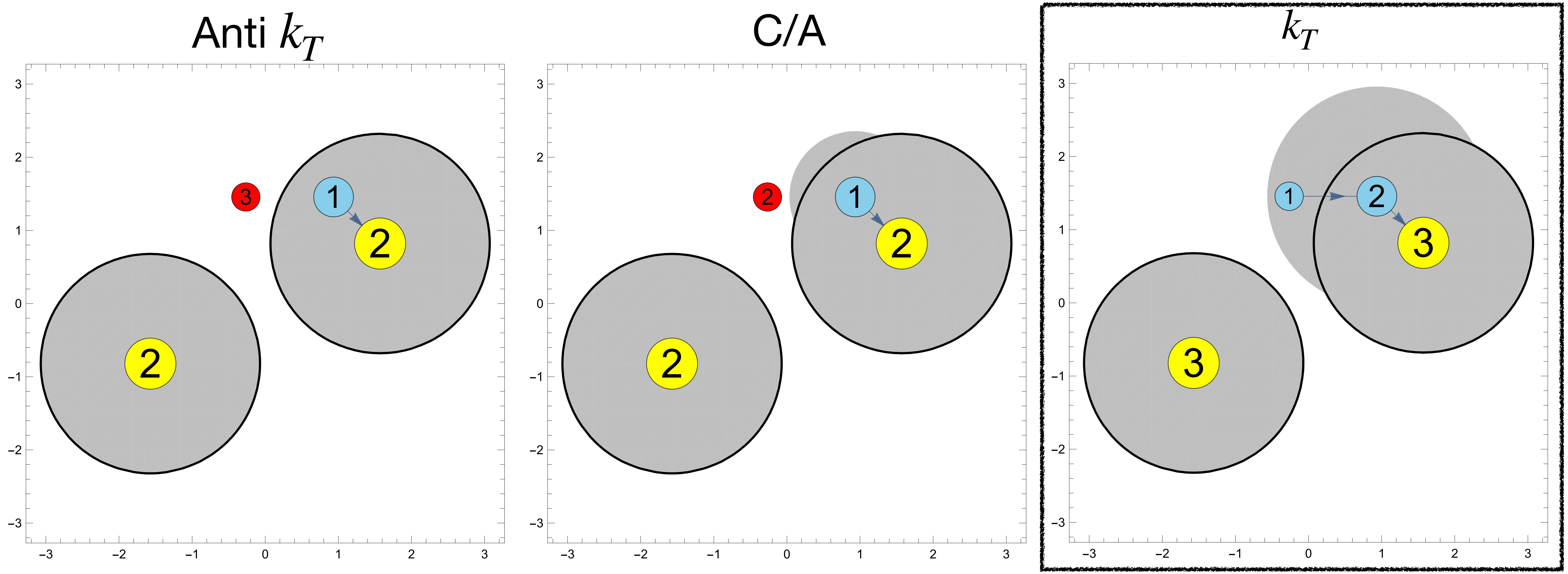
Second emission “in”

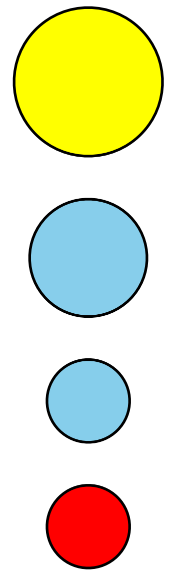
Second emission “out”



Phase space, where the second emission would be in

Example Situation for diet production



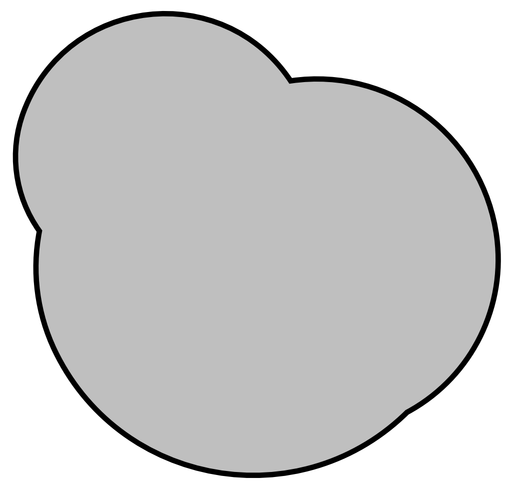


Primary particles

First emission “in”

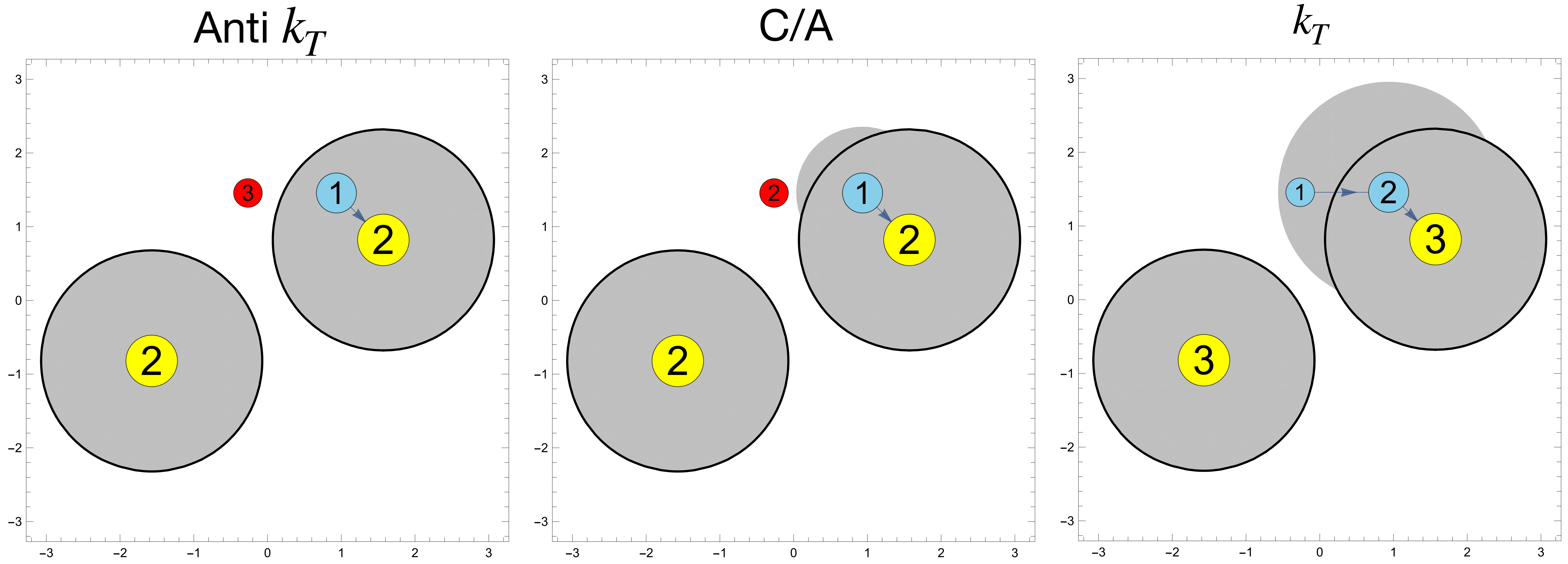
Second emission “in”

Second emission “out”



Phase space, where the second emission would be in

Example Situation for diet production



Primary particles

First emission “in”

Second emission “in”

Second emission “out”

Phase space, where the second emission would be in

Take Away

The “in” region for p_{m+1} is obtained by

Anti $\mathbf{k_T}$

Putting a “circle” of radius R around every jet $\{P_1, \ldots, P_{n_J}\}$

No growth! (Like fixed cones)

C/A

Putting a “circle” of radius $\delta_i \leq R$ around every particle $\{p_1, \ldots, p_m\}$, where δ_i is the distance with which p_i became a jet or was clustered with a harder parton

Steady growth!

$\mathbf{k_T}$

Putting a “circle” of radius R around every particle $\{p_1, \ldots, p_m\}$:

Fast growth!

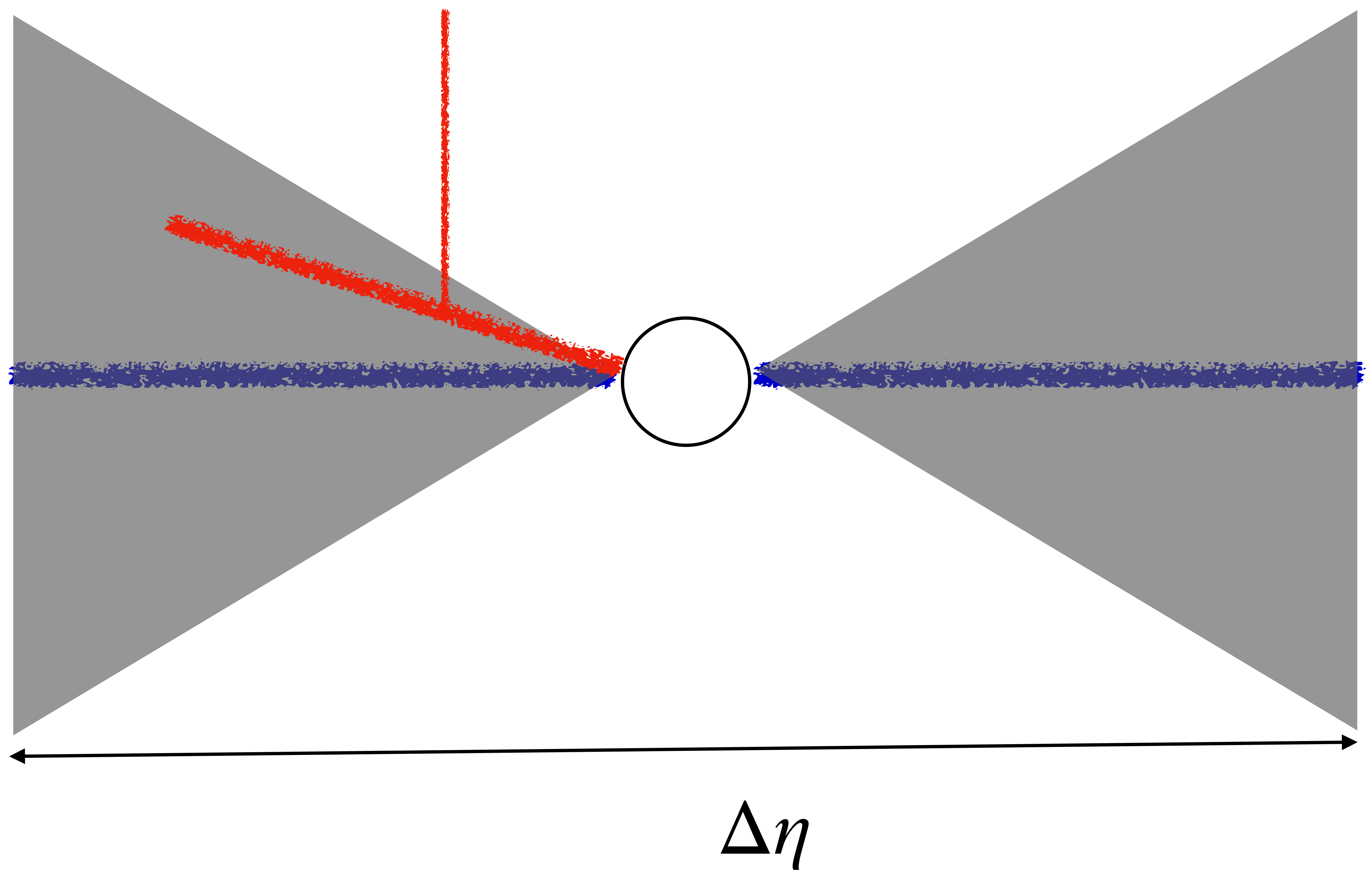
LL Features

Set-up 1 (Central rapidity gap)

$$\frac{\Delta R_{ij}^2}{R^2} = \frac{\Delta\eta_{ij}^2 + \Delta\phi_{ij}^2}{R^2}$$

WRT thrust axis

(Extra) jets are "out" if they land in the gap



Hard particles:

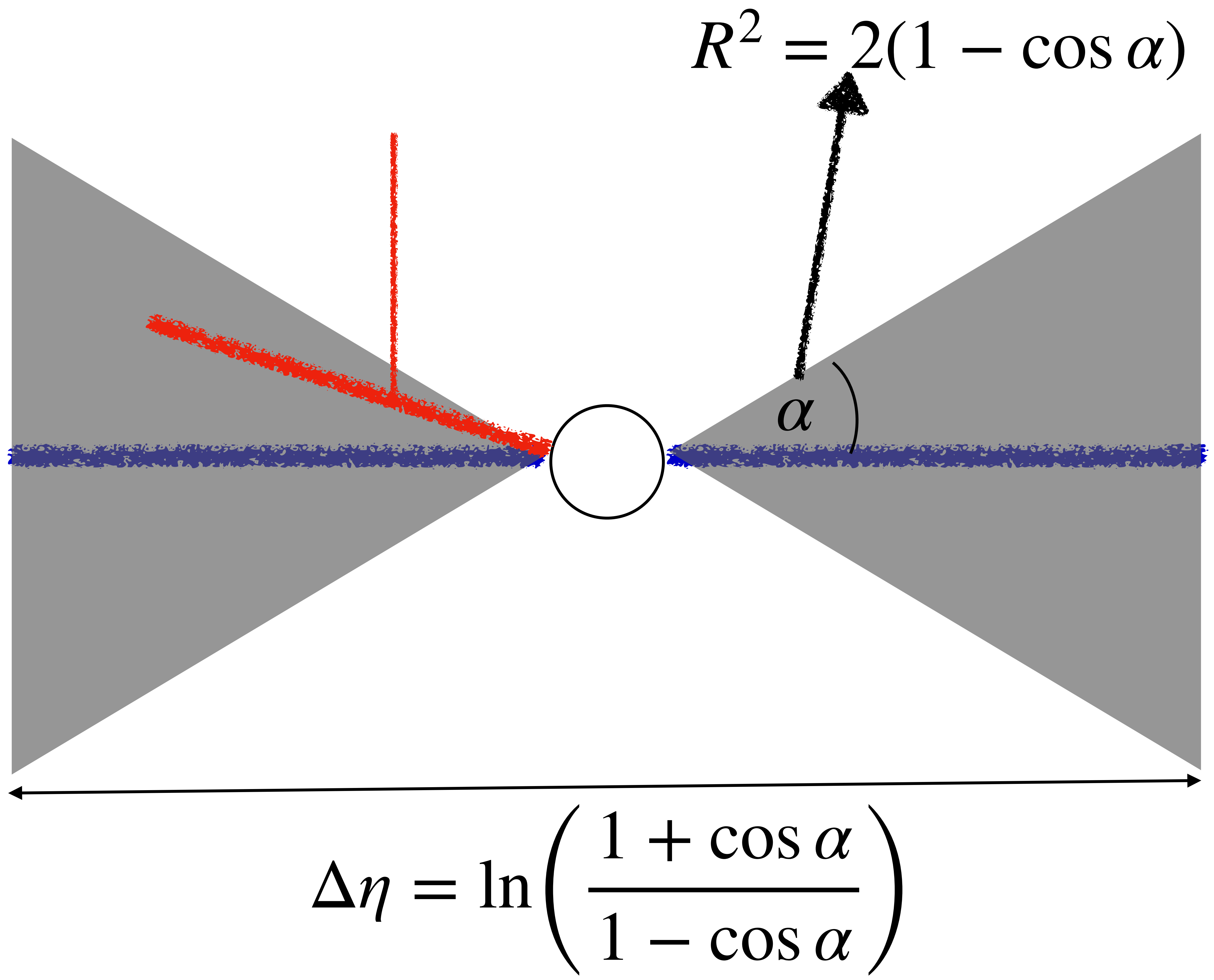
Soft emisssions:

- Set-up allows for analytic resummation of CL up to power-correction in R (Delenda, Appleby, Dasgupta, Banfi hep-ph/0610242)
- Blue and red particles cannot cluster

Set-up 2 (Veto on extra jets)

$$\frac{\Delta R_{ij}^2}{R^2} = \frac{2(1-\cos\theta_{ij})}{R^2}$$

Only the two hardest jets are in.



Hard particles:

Soft emisssions:

- CLs have not been resummed analytically
- Blue and red particles <u>can</u> cluster

Common Misconceptions

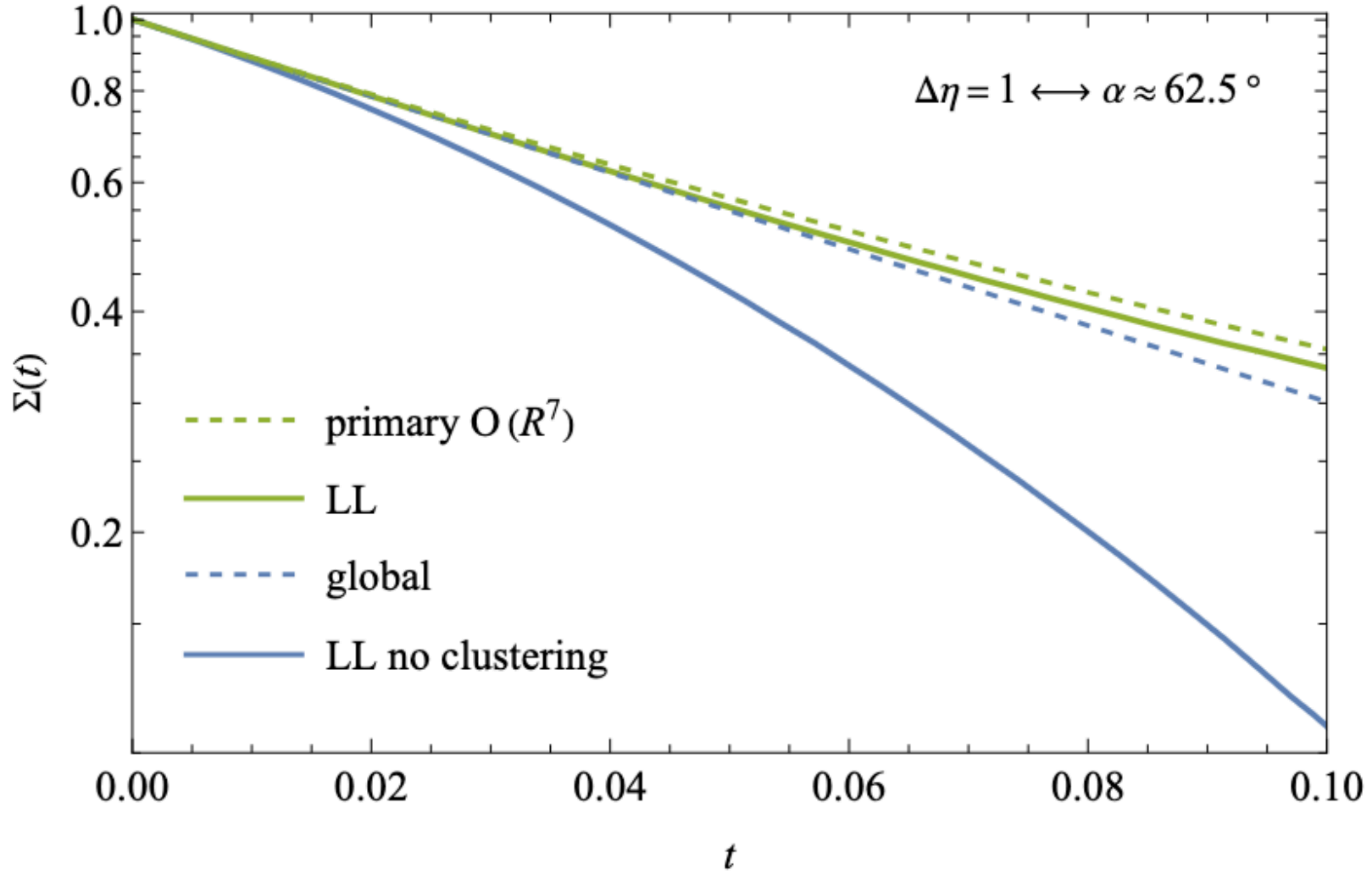
- Anti k_T clustering does not produce CL: This statement depends on the exact definition of the jets and the “in”-condition. For Setup 2 this is true at LL but not necessarily at NLL. For Setup 1 it is false.
- All k_T-type clustering algorithms give the same CL at order α_S^2: This is true for Setup 1, not for Setup 2. I. e., it is true when there is only one possible pair that can be clustered.

Setup 1: k_T-Clustering With Central Rapidity Gap

As a check, we compared with
(Delenda, Appleby, Dasgupta, Banfi hep-ph/0610242)
and reproduced their result.

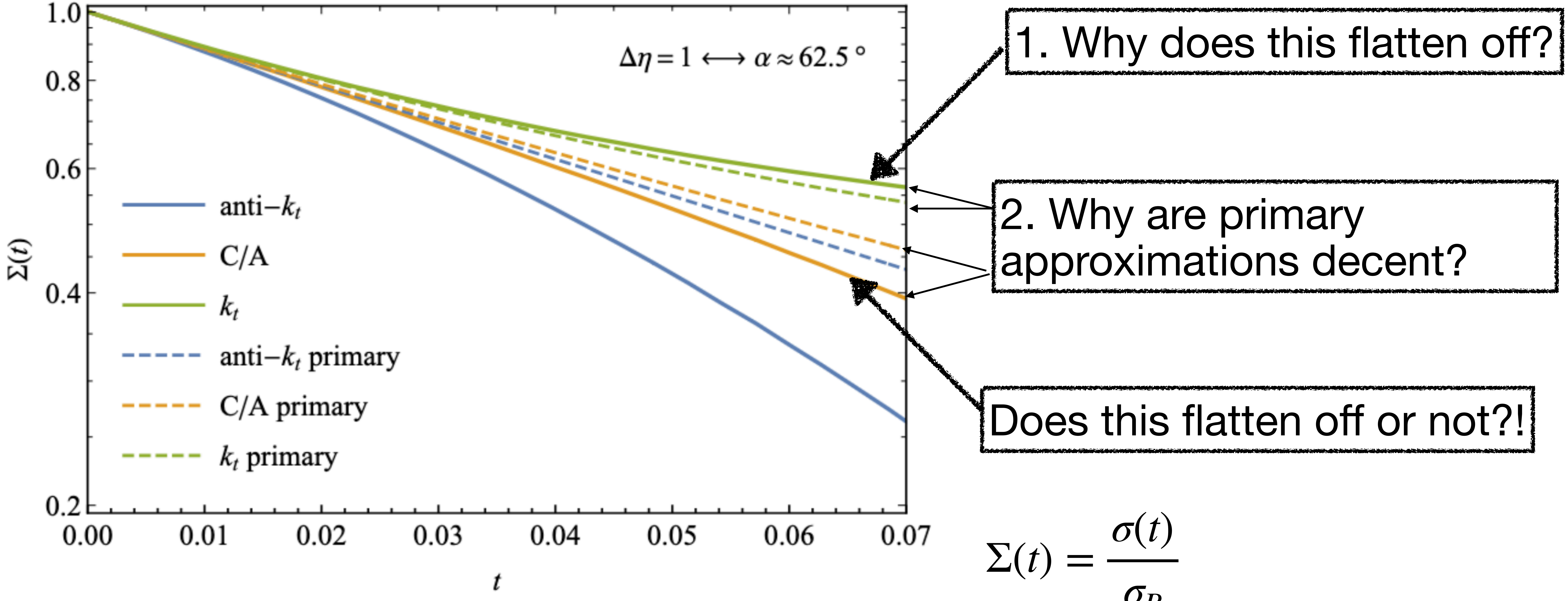
Here and in the following, we always plot the gap fraction for diet production:

$$\Sigma(t) = \frac{\sigma(t)}{\sigma_B}$$



Effect of k_t jet-clustering on the gap fraction for a fixed central rapidity gap of $\Delta\eta = 1$. In this set-up, emissions can never cluster with the primary jets.

Setup 2: Clustering Effects With Jet Veto On Extra Jets

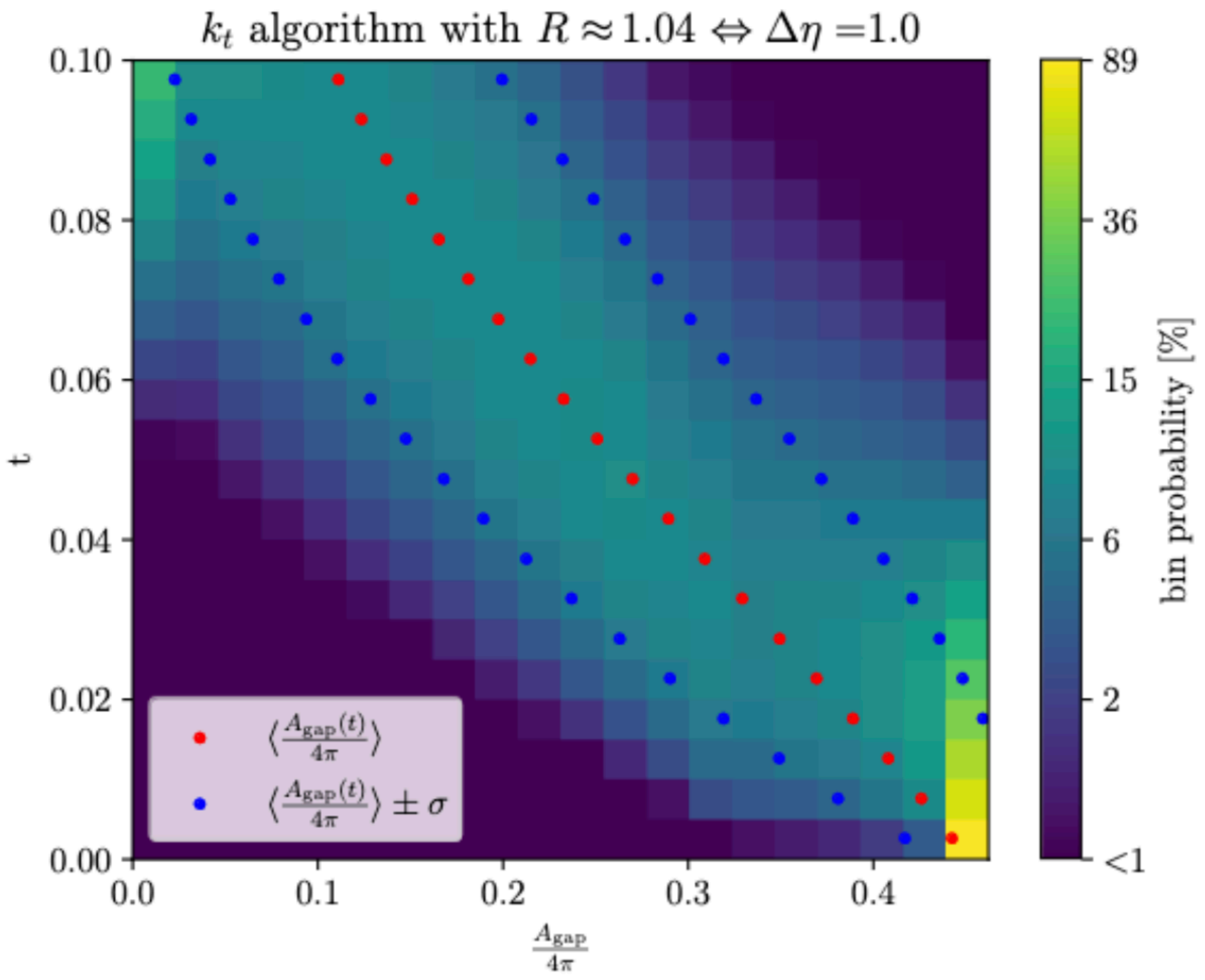


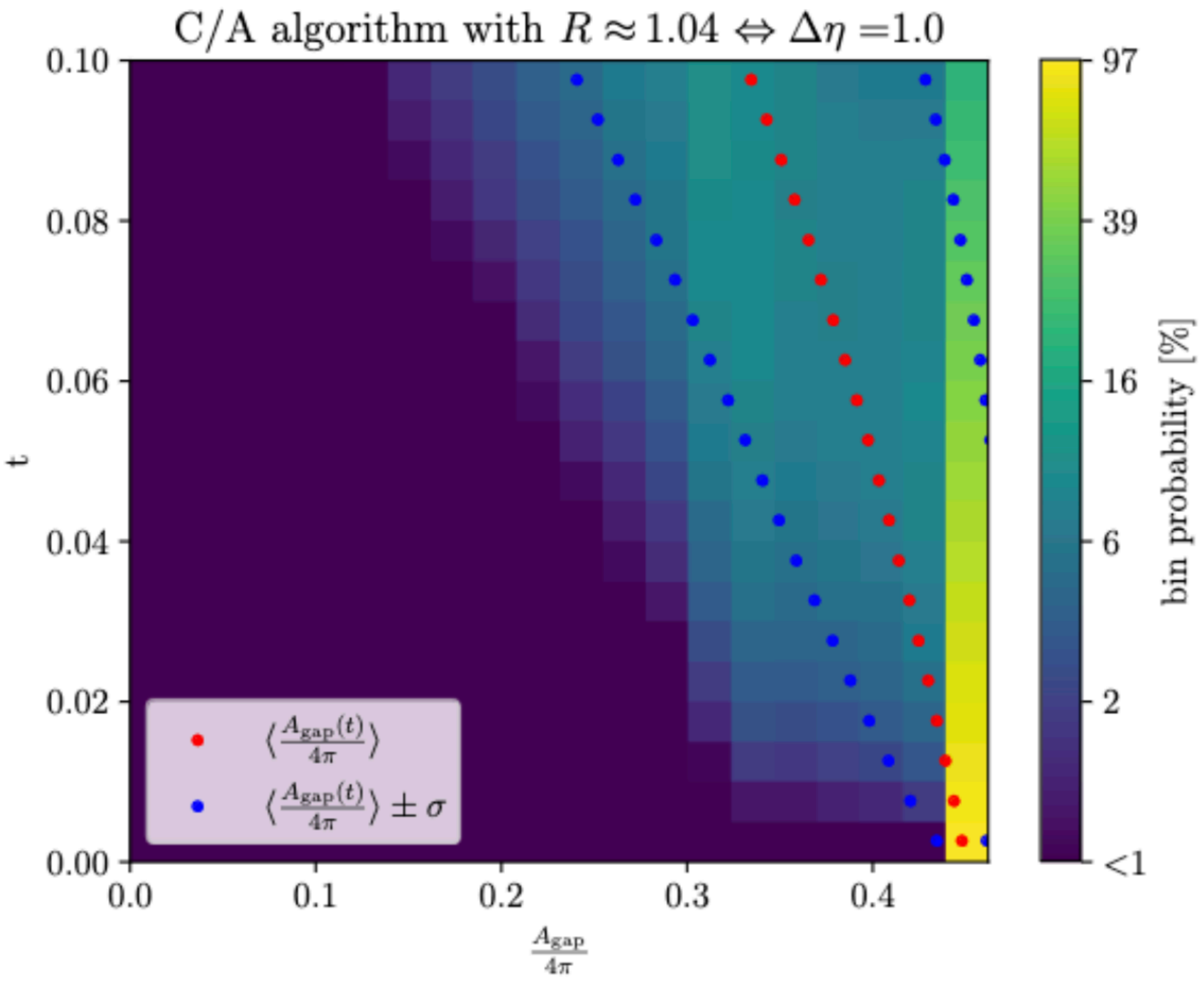
$$\Sigma(t) = \frac{\sigma(t)}{\sigma_B}$$

Effective Gap Area

- Note, if there is no gap then there is no veto and there are no large logs
- With the clustering, the gap becomes smaller with each emission. At some shower time it should vanish completely and the shower should evolve unitarily.
- Cross section becomes independent of $t(Q_0)$

Effective Gap Area





How good is the primary approximation?

Just ignoring the non-global effects due to “new” dipoles (pretending the gluons are photons) gets you within 10% of the correct result for k_t -clustering!

Note that 10% is also roughly the effect of subleading color or NLL corrections!

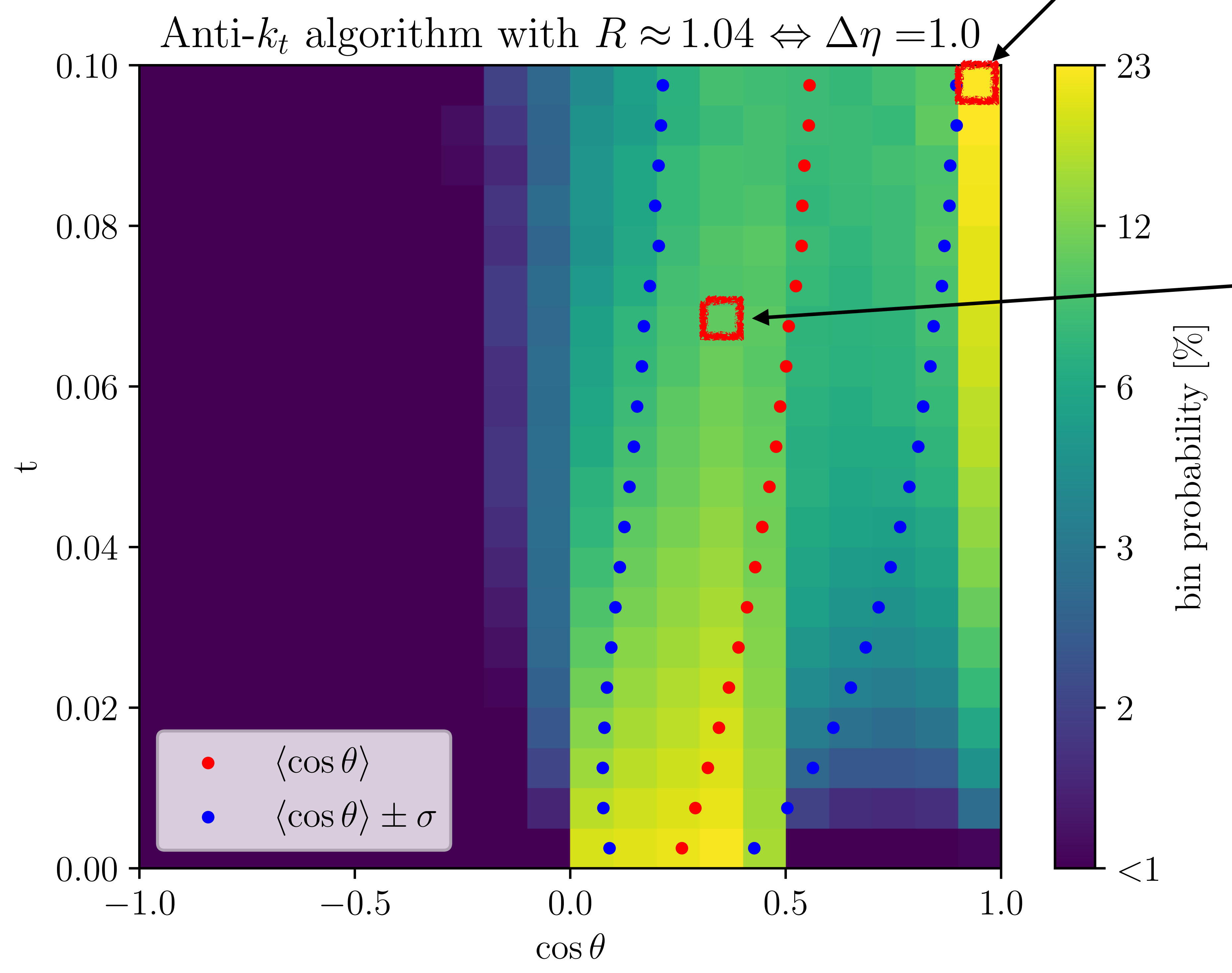
$\Delta\eta$	anti-k_t[%]	C/A[%]	k_t [%]
2.0	66.6 ± 1.5	32.0 ± 1.4	9.1 ± 1.0
1.0	64.7 ± 1.0	17.8 ± 0.8	-4.7 ± 0.6
0.2	42.8 ± 0.6	3.9 ± 0.5	-1.2 ± 0.5

The table lists the ratio $(\Sigma_{\text{primary}}(t) - \Sigma_{\text{LL}}(t))/\Sigma_{\text{LL}}(t)$ at $t = 0.07$

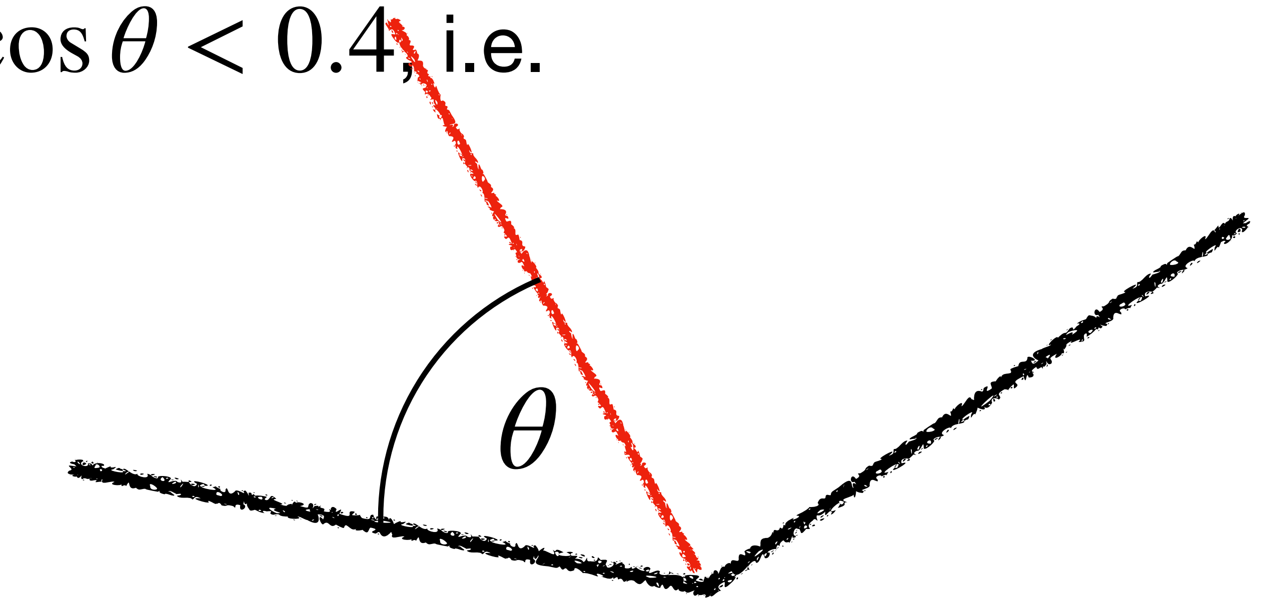
Reduction of NG Effects

The strong suppression of the gap fraction due to NGL is driven by collinear emissions into the gap:

Without clustering, collinear emissions soon dominate



At shower times of $t \sim 0.07$ roughly 10% of the gluons emitted into the gap are emitted with $0.3 < \cos\theta < 0.4$, i.e. $66^\circ < \theta < 73^\circ$



The plots show at which angle from the emitting dipole gluons are radiated into the gap.

Reduction of NG Effects

The strong suppression of the gap fraction due to NGL is driven by collinear emissions into the gap:

Without clustering, collinear emissions soon dominate

For k_t clustering, there is a strict bound on collinear emission

The plots show at which angle from the emitting dipole gluons are radiated into the gap.

Conclusion

- Presented a **first factorisation theorem** applicable to NG observables with clustering effects
  - Derived the 1-Loop anomalous dimension
  - Simplified sequential clustering algorithms at LL
- LL result was implemented in a parton shower. Our result agrees with results previously calculated with different methods. Using the shower we
  - analysed how the “effective gap” seen by emissions shrinks with larger t (smaller Q_0)
  - shed light on how clustering suppresses the importance of collinear emissions

Outlook

- calculate the two-loop anomalous dimension to go to NLL
- expand our analysis to more general, possibly double logarithmic, observables (like jet masses)
- Analyze the effect of subleading color
- look at hadron collider observables, including also SLL

Back-up slides

Global, Non-Global, Clustering, Super-Leading

What is what?

- All of these are logarithms of $\frac{Q}{Q_0} \gg 1$

Global	Non-Global	Super-leading	Clustering
Towers of $\alpha_S \log\left(\frac{Q}{Q_0}\right)$ expected from naive exponentiation of the fixed order result. We only need to consider the primary hard partons as emitters. (Concept does not really make sense beyond LL)	Even at LL, non-abelian strongly ordered emissions destroy the naive exponentiation starting from α_S^2 . With every additional emission one gets new dipoles that can again radiate and generate new "non-global" logs.	At hadron colliders, starting at α_S^3 and beyond LC, collinear singularities do no longer cancel exactly between real and virtual emissions due to Glauber exchanges. Starting from α_S^4 these effects are super-leading.	If the shape of the gap changes with every emission, then even abelian emissions do not exponentiate. Leading to additional non-global logs at α_S^2 .

- In general, one has all of those and they mix. Instead of listing which types of logs exists for a given observable, one should rather state which logs are absent.

RG Evolution and Resummation

- The resummed cross section becomes

$$\sigma(Q_0) = \sum_{l=M}^{\infty} \Big\langle \boldsymbol{\mathcal{H}}_l\left(\{\underline{n}\}, \{\underline{z}\}, Q, \mu_h\right) \sum_{m \geq l} \boldsymbol{U}_{lm}\left(\{\underline{n}\}, \{\underline{z}\}, \mu_s, \mu_h\right) \otimes_z \boldsymbol{\mathcal{S}}_m\left(\{\underline{n}\}, \{\underline{z}\}, Q_0, \mu_s\right) \Big\rangle ,$$

  with the evolution kernel $\boldsymbol{U}\left(\{\underline{n}\}, \{\underline{z}\}, \mu_s, \mu_h\right) = \mathbf{P} \exp\left[\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \boldsymbol{\Gamma}^H(\{\underline{n}\}, \{\underline{z}\}, \mu)\right]$

- We implement this equation in a Parton shower with the shower time

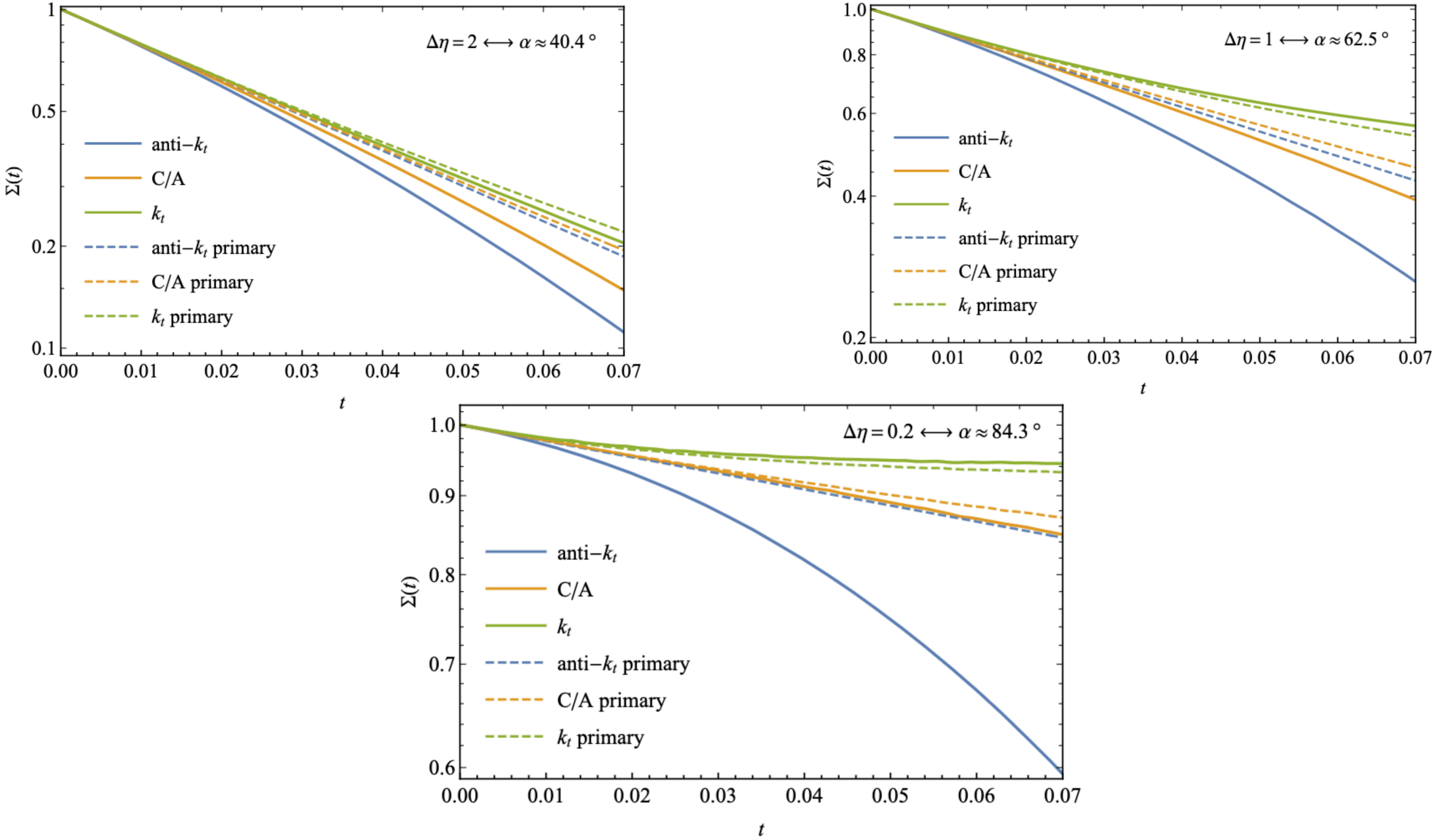
$$\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu}\, \boldsymbol{\Gamma}^H = \int_{\alpha(\mu_s)}^{\alpha(\mu_h)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha}{4\pi} \boldsymbol{\Gamma}^{(1)} = \frac{1}{2\beta_0} \ln \frac{\alpha(\mu_s)}{\alpha(\mu_h)} \boldsymbol{\Gamma}^{(1)} = t\, \boldsymbol{\Gamma}^{(1)}$$

- Define $\boldsymbol{\mathcal{H}}_m(t) \equiv \boldsymbol{\mathcal{H}}_M(\{\underline{n}\}, \{\underline{z}\}, Q, \mu_h)\, \boldsymbol{U}_{Mm}(\{\underline{n}\}, \{\underline{z}\}, \mu_h, \mu_s)$

- Iterative solution $\boldsymbol{\mathcal{H}}_{M+1}(t) = \int_0^t dt'\, \boldsymbol{\mathcal{H}}_M(t')\, \boldsymbol{R}_M\, e^{(t-t')\boldsymbol{V}_{M+1}}$

- And combine everything as $\sigma_{\rm LL}(t) = \sum_{m=2}^{\infty} \langle \boldsymbol{\mathcal{H}}_m(t) \otimes_z \mathbf{1} \rangle$

- At leading color, one can reformulate the solution as a parton shower (LL: Becher et. al.: 1803.07045,2006.00014, NLL: Becher, Schalch, Xu: 2307.02283)



Δη = 2 ⟷ α ≈ 40.4 °
Σ(t)
t
anti−kt
C/A
kt
anti−kt primary
C/A primary
kt primary
Δη = 1 ⟷ α ≈ 62.5 °
Δη = 0.2 ⟷ α ≈ 84.3 °

Strongly Ordered Clustering

- Note: If the jet clustering on $\{p_1, \ldots, p_m\}$ yields the jets $\{P_1, \ldots, P_{n_J}\}$, then the clustering on $\{p_1, \ldots, p_m, p_{m+1}\}$ either

  1. yields the same jets $\{P_1, \ldots, P_{n_J}\}$
  2. or $\{P_1, \ldots, P_{n_J}, p_{m+1}\}$

- In case 1., the “in” condition is satisfied
- In case 2., we only need to check the new jet
- How to simplify the clustering, taking into account the strung ordering?

Factorisation Theorem

Factorisation for fixed cones:

$$\sigma(Q, Q_0) = \sum_{m=M}^{\infty} \langle \boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, Q, \mu) \otimes \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

Note, that
$\Theta_{\text{in}}(\{\underline{p}\}) = \Theta_{\text{in}}(n_1)\Theta_{\text{in}}(n_2)\dots$
is very simple

$$\boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, Q) = \frac{1}{2Q^2}\left(\prod_{i=1}^{m}\int \frac{dE_i E_i^{d-3}}{\tilde{c}^{\epsilon}(2\pi)^2}\right)\widetilde{\boldsymbol{\mathcal{H}}}_m(\{\underline{p}\})\times (2\pi)^d \delta(Q - E_{\text{tot}})\delta^{d-1}(\vec{p}_{\text{tot}})\Theta_{\text{in}}(\{\underline{p}\})$$

$$\boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q_0) = \sum\!\!\!\!\!\!\int_X \langle 0| \boldsymbol{\mathcal{S}}_1^{\dagger}(n_1)\dots\boldsymbol{\mathcal{S}}_m^{\dagger}(n_m)|X\rangle\langle X|\boldsymbol{\mathcal{S}}_1(n_1)\dots\boldsymbol{\mathcal{S}}_m(n_m)|0\rangle\theta(Q - 2E_{\text{out}})$$