"Proof" that GEO600/LIGO operates at its ultimate quantum limit.

Refs

1) <u>Parameter estimation theory</u>

Kay S.M. – "Fundamentals of statistical signal processing"

2) <u>Quantum limits</u>

Demkowicz, Jarzyna, Kołodyński – "QLs in optical interferometry" Prog. Optics 60 2015 arXiv:1405.7703

3) <u>Fundamental bound for GEO600</u>

Demkowicz, Banaszek, Schnabel – Phys Rev A 88, 041802(R) 2013

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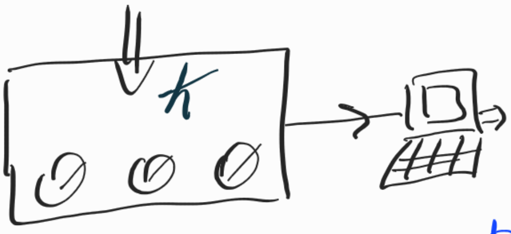
[1] Parameter estimation @ core of any sensor

<u>Sensing experiment</u>

φ (parameter)



$\to X$ $\begin{cases} \text{dig. } x \in \{i\}_{i=1}^{D} \\ \text{and. } \end{cases}$

$\{x \mid \in \mathbb{R}_{(a,b)}\}$

outcome (random variable)

$\Rightarrow$ repeat $\times\,\nu$ times

$\underline{X} = \{x_i\}_{i=1}^{\nu}$ measurement data

- data distributed according to $p(\underline{x} \mid \varphi)$

• AIM: "most accurate" estimate of φ

estimator: $\tilde{\varphi}(\underline{x})$ (random variable)

Frequentist approach (vs Bayesian)

a) $\nu \to \infty$ limit available

b) "sensing" around a known value $\varphi \approx \varphi_0$ (local estimation)

$\Rightarrow$ consequence: φ – deterministic (but unknown)

$$p(\underline{x}|\varphi) \equiv p_\varphi(x) \text{ — likelihood}$$

TASK:

$\Rightarrow$ minimise mean square error (MSE)

$$\Delta^2 \tilde{\varphi} = \int d\underline{x}\, p(\underline{x}|\varphi)\left(\tilde{\varphi}(\underline{x}) - \varphi\right)^2$$

(depends on φ)

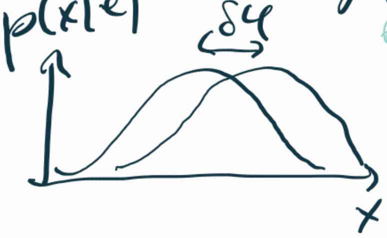
$\Rightarrow$ ultimate limit for (locally) unbiased estimators

unb: $\int d\underline{x}\, p(\underline{x}|\varphi)\, \tilde{\varphi}(\underline{x}) = \varphi$

l. unb: $\frac{\partial}{\partial\varphi} \int d\underline{x}\, p(\underline{x}|\varphi)\, \tilde{\varphi}(\underline{x}) = 1$ $\left(\text{around } \varphi \approx \varphi_0\right)$

ok, for regular $p(\underline{x}|\varphi)$:

e.g. $\frac{\partial}{\partial\varphi} \int d\underline{x}\, p(\underline{x}|\varphi) = 0 \Leftrightarrow \int d\underline{x} \frac{\partial}{\partial\varphi} p(\underline{x}|\varphi) = 0$



vs

{ not always: "German tank problem"

$\Updownarrow$

$\varphi \to$ max of distribution

Cramér-Rao-Bound (CRB)

$$\int d\underline{x}\, \frac{\partial p(\underline{x}|\varphi)}{\partial\varphi} \left(\tilde{\varphi}(\underline{x}) - \varphi\right) = \underset{\text{unb}}{1} - \underset{\text{reg}}{0} = 1$$

$$\int dx\, \underbrace{\sqrt{p(\underline{x}|\varphi)}\,\frac{\partial \ln p(\underline{x}|\varphi)}{\partial\varphi}}_{f(\underline{x})}\; \underbrace{\sqrt{p(\underline{x}|\varphi)}\,\left(\tilde\varphi(\underline{x})-\varphi\right)}_{g(\underline{x})} = 1$$

CS: $$\left(\int d\underline{x}\, f(\underline{x})\, g(\underline{x})\right)^2 \le \int d\underline{x}\, f(\underline{x})^2\, g(\underline{x})^2$$

$$1 \le \underbrace{\int d\underline{x}\, p(\underline{x}|\varphi)\left(\tilde\varphi(\underline{x})-\varphi\right)^2}_{\Delta^2\tilde\varphi \ \text{(MSE)}}\; \underbrace{\int d\underline{x}\, p(\underline{x}|\varphi)\left[\frac{\partial}{\partial\varphi}\ln p(\underline{x}|\varphi)\right]^2}_{F[p(\underline{x}|\varphi)]}$$

Fisher Information

$$F[p(\underline{x}|\varphi)] = \left\langle \left(\frac{\partial}{\partial\varphi}\ln p(\underline{x}|\varphi)\right)^2 \right\rangle = -\left\langle \frac{\partial^2}{\partial\varphi^2}\ln p(\underline{x}|\varphi)\right\rangle$$

so for iid: $p(\underline{x}|\varphi) = \prod_{i=1}^{\nu} p(x_i|\varphi)$

FI is additive (information) $F[p(\underline{x}|\varphi)] = \nu F[p(x|\varphi)]$

CRB

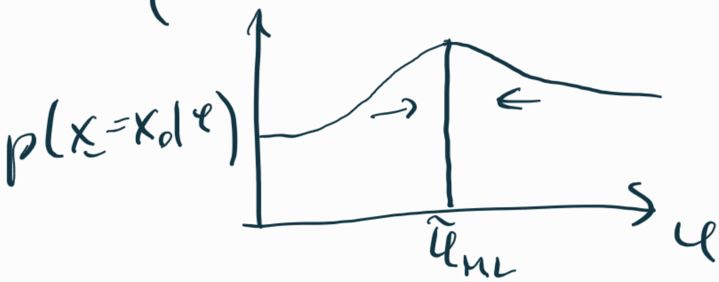
$$\boxed{\Delta^2\tilde\varphi \geq \frac{1}{\nu F[p(x|\varphi)]}}$$

{ locally unbiased estimators }

Efficient estimator that (almost) always works as $\nu \to \infty$:

Max-likelihood estimator (MLE)

$$\tilde{\varphi}_{ML}(\underline{x}_0) = \underset{\varphi}{\arg\max}\; p(\underline{X}=\underline{x}_0|\varphi) = \underset{\varphi}{\arg\max}\; \underbrace{\ln p(\underline{X}=\underline{x}_0|\varphi)}_{\text{log-likelihood}}$$



Facts:
- MLE is biased for finite ν, but becomes unbiased as $\nu \to \infty$.
- in fact as $\nu \to \infty$:

$$\tilde{\varphi}_{ML}(\underline{x}) \longrightarrow \mathcal{N}\left(\varphi, \frac{1}{\nu F[p(x|\varphi)]}\right)$$

Example: Gaussian distⁿ → estimation of mean $\varphi \equiv \mu$

$$X \sim \mathcal{N}(\mu, \sigma^2) \qquad p(X|\mu) \sim e^{-(X-\mu)^2/2\sigma^2}$$

MLE:

$$\tilde{\mu}_{ML}(\underline{x}) = \underset{\mu}{\arg\max} \prod_{i=1}^{\nu} e^{-(x_i-\mu)^2/2\sigma^2}$$

"log" (

$$= \arg\max \sum_{i=1}^{\nu} -\frac{(x_i-\mu)^2}{2\sigma^2}$$

a) $$\frac{\partial}{\partial\mu}\left(\sum_i -(x_i-\mu)^2\right) = 0 \Rightarrow \tilde{\mu}_{ML}(\underline{x}) = \frac{\sum_{i=1}^{\nu} x_i}{\nu}$$

MLE is the "average"

b) $$\langle \tilde{\mu}_{ML}(\underline{x}) \rangle = \left\langle \sum_i \frac{x_i}{\nu} \right\rangle = \mu \Rightarrow$$ MLE is always (any ν) unbiased

c) MSE of $\tilde{\mu}_{ML}$:

$$\Delta^2 \tilde{\mu}_{ML} = \langle (\tilde{\mu}_{ML} - \mu)^2 \rangle = \left\langle \left(\frac{\sum_i x_i}{\nu} - \mu \right)^2 \right\rangle =$$

$$= \left\langle \frac{\left(\sum_i (x_i - \mu)\right)^2}{\nu^2} \right\rangle = \frac{\nu \,\mathrm{Var}[x]}{\nu^2} = \frac{\sigma^2}{\nu}$$

CRB $\quad F[p(x|\mu)] = -\left\langle \frac{\partial^2}{\partial\mu^2} \log p(x|\mu) \right\rangle =$

$$= \left\langle \frac{\partial^2}{\partial\mu^2} \frac{(x-\mu)^2}{2\sigma^2} \right\rangle = \frac{1}{\sigma^2}$$

$$\Rightarrow \quad \Delta^2 \tilde{\mu} \geq \frac{1}{\nu F} = \frac{\sigma^2}{\nu}$$

(achieved by MLE for any ν)

N.B. $\frac{\sum_i x_i}{\nu} \xrightarrow[\nu\to\infty]{} \mathcal{N}\left(\mu, \frac{\sigma^2}{\nu}\right) \Rightarrow$ Central Limit Theorem (special case of an estimation problem)

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2 Quantum parameter estimation

Q. Mechanics:

$$p(x|\varphi) = \mathrm{Tr}\{\rho_\varphi M_x\}$$

Born's rule

system state

POVMs

(finite dimensions: $\rho_\varphi \in B(\mathcal{H}_d)$, $\{M_x\}_{x=1}^{D}$)

$M_x \geq 0, \; \sum_x M_x = \mathbb{1}$

Quantum Fisher Information (QFI)

$$F_Q[\rho_\varphi] := \max_{M_x} F[p(x|\varphi)]$$

optimise over all possible measurements

allowed by QM.

Carl Helstrom 1976 (also Braunstein-Caves PhysRevLett 1994)

$$F_Q[\rho_\varphi] = \mathrm{Tr}\{\rho_\varphi L^2\} \quad \left\{ \leftarrow \text{quant.} \quad F = \left\langle \left(\frac{\partial}{\partial\varphi} \ln p(x|\varphi) \right)^2 \right\rangle \right.$$

where $\frac{\partial \rho_\varphi}{\partial \varphi} = \frac{1}{2}(\rho_\varphi L + L\rho_\varphi)$ L - symmetric logarithmic derivative $\underline{\text{SLD}}$

Pure state

$$\rho_\varphi = |\psi_\varphi\rangle\langle\psi_\varphi| \quad \left\{ \begin{array}{l} \text{recall} \\ \text{local. } \varphi \approx \varphi_0 \end{array} \right. \Rightarrow \dot\rho \equiv \left.\frac{\partial\rho_\varphi}{\partial\varphi}\right|_{\varphi=\varphi_0}$$

$$|\dot\psi\rangle \equiv \left.\frac{\partial|\psi_\varphi\rangle}{\partial\varphi}\right|_{\varphi=\varphi_0}$$

$$\Rightarrow \dot\rho = |\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi|$$

by inspection: $\quad L = 2(|\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi|)$

but $\langle\psi|\psi\rangle = 1 \Rightarrow \langle\dot\psi|\psi\rangle + \langle\psi|\dot\psi\rangle = 0$

so $\frac{1}{2}(\rho L + L\rho) = |\psi\rangle\langle\psi|(|\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi|) +$
$+ (|\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi|)(|\psi\rangle\langle\psi| =$
$= |\psi\rangle\langle\dot\psi| + |\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|\underbrace{(\langle\psi|\dot\psi\rangle + \langle\dot\psi|\psi\rangle)}_{0}$
$= \dot\rho$ //

$$\Rightarrow F_Q[\psi_\varphi] = \mathrm{Tr}\{\rho L^2\} = \mathrm{Tr}\{\dot\rho L\} \quad \left\{ \begin{array}{l}\text{recall} \\ \dot\rho = \frac{1}{2}(\rho L + L\rho)\end{array}\right.$$

$$= \mathrm{Tr}\{(|\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi|)\, 2(|\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi|)\}$$

$$= 2(\langle\psi|\dot\psi\rangle^2 + \langle\dot\psi|\dot\psi\rangle + \langle\dot\psi|\dot\psi\rangle + \langle\dot\psi|\psi\rangle^2\}$$

$$= 4(\langle\dot\psi|\dot\psi\rangle - |\langle\psi|\dot\psi\rangle|^2)$$

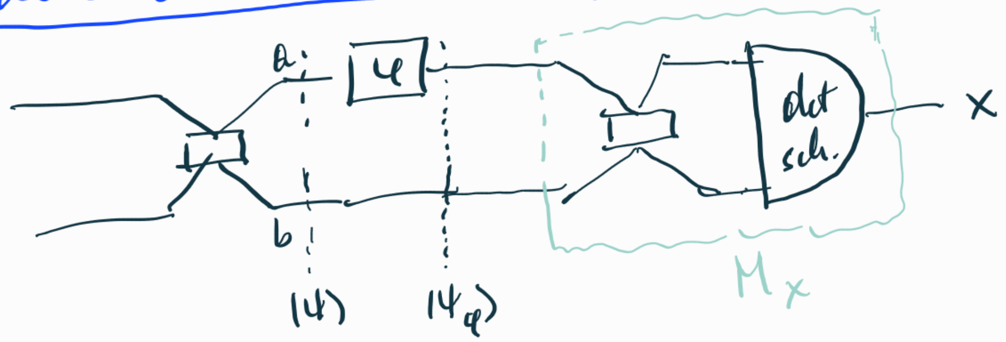
Unitary encoding

$$|\psi_\varphi\rangle = e^{-i\hat H\varphi}|\psi_0\rangle \quad \left\{ \begin{array}{l}\text{think of Schrödinger eq with time-indep. Hamiltonian} \\ |\psi_t\rangle = e^{-i\hat H t}|\psi_0\rangle\end{array}\right.$$

$$\Rightarrow \quad |\dot\psi\rangle = -i\hat H|\psi\rangle$$

$$\Rightarrow \quad F_Q[\psi_\varphi] = 4\langle\psi|\hat H^2|\psi\rangle - \langle\psi|\hat H|\psi\rangle^2$$
$$= 4\,\Delta^2\hat H\big|_{\psi_\varphi} \qquad (@\ \varphi=\varphi_0)$$

Mach-Zehnder Interferometer



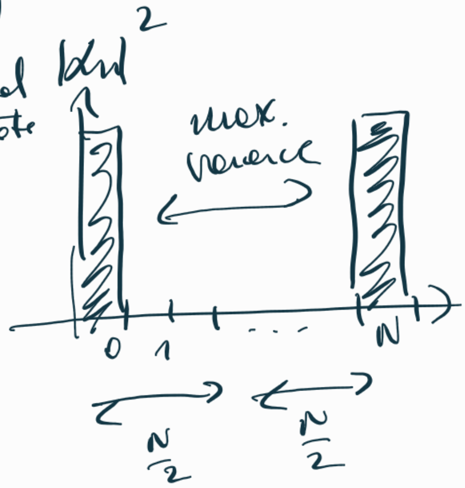
- state of N photons

$$|\psi\rangle = \sum_{n=0}^{N} \alpha_n\,|n\rangle_a\,|N-n\rangle_b$$

$$\rightarrow \quad |\psi_\varphi\rangle = e^{-i\varphi\hat H}|\psi\rangle = \sum_{n=0}^{N}\alpha_n\,e^{-i\varphi n}\,|n,N-n\rangle$$

$$\hat H = \tfrac{1}{2}(\hat n_a - \hat n_b) = \tfrac{1}{2}(\hat a^\dagger\hat a - \hat b^\dagger\hat b)$$

$$\Rightarrow \quad F_Q[\psi_\varphi] = 4\,\Delta^2\hat H\big|_{\psi_\varphi} \Rightarrow$$



$\Rightarrow$ "NOON state"

$$|\psi\rangle = \frac{1}{\sqrt2}\big(|N0\rangle + |0N\rangle\big)$$

$\Rightarrow$ maximally entangled in between the photons

$$|\psi\rangle = \frac{1}{\sqrt2}\big(|a\rangle_1|a\rangle_2\cdots|a\rangle_N + |b\rangle_1|b\rangle_2\cdots|b\rangle_N\big)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\ldots \right)$$

$$\Rightarrow F_Q[|\psi\rangle] = 4\langle(\underbrace{\hat{H}}_{\frac{N}{2}} - \underbrace{\langle\hat{H}\rangle}_{0})^2\rangle = N^2$$

$$\Rightarrow \Delta^2\tilde{\varphi} \geq \frac{1}{\nu F_Q} = \frac{1}{\nu}\cdot\frac{1}{N^2}$$

Heisenberg Limit

vs

photon in a classical state (e.g. coherent) $\rightarrow$ independent objects

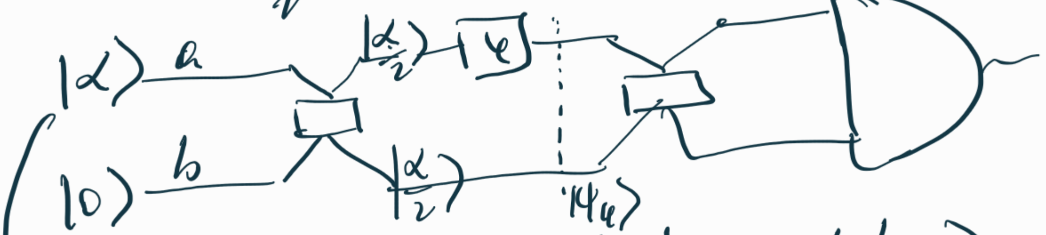
F_Q is also additive (on tensor products)

$$F_Q = N$$

$$\Rightarrow \Delta^2\tilde{\varphi} \geq \frac{1}{\nu F_Q} = \frac{1}{\nu}\cdot\frac{1}{N}$$

Standard Quantum Limit SQL

Easy exercise on quantum optics



coherent beam: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|n\rangle$$

$$\bar{N} = |\alpha|^2$$

Jordan-Schwinger map

$$\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})$$

$\Rightarrow \quad \Delta^2 H = \Delta^2 J_z = \frac{\ }{4}$

$$\left\{ \begin{aligned} J_x &= \ldots \\ J_y &= \frac{i}{2}(\hat b^\dagger \hat a - \hat a^\dagger \hat b) \\ J_z &= \frac{1}{2}(\hat a^\dagger \hat a - \hat b^\dagger \hat b) \end{aligned} \right.$$

$\Rightarrow \quad F_Q = |\alpha|^2 = \bar N$

$\Rightarrow \quad \Delta^2 \tilde\varphi \geq \frac{1}{\nu}\cdot\frac{1}{\bar N}$ SQL with mean number of photons

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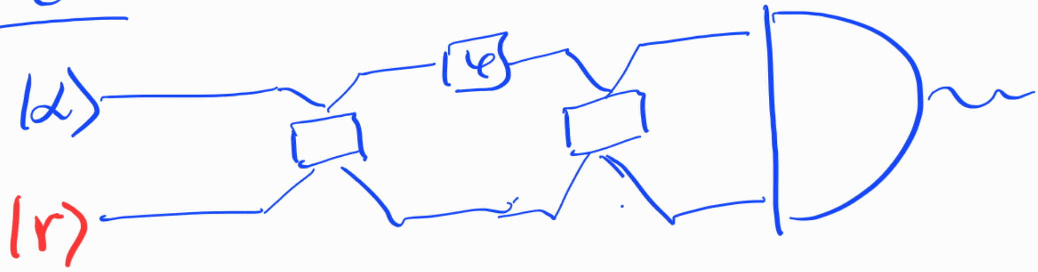
OK, but LIGO uses "only" squeezed light

LIGO



squeezed state $\Rightarrow$ Why not more exotic states with stronger interphoton entanglement?

$\bar N \to \bar N^2$ increase

$P = 260$ kW $\Rightarrow$ huge!

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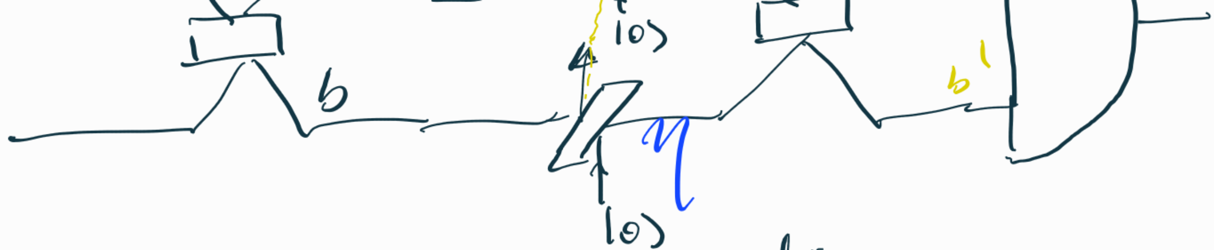
[3] Fundamental limit on GEO600 (LIGO)

Answer: Presence of imperfect detection efficiency deems it "useless"!

Lossy MZ interferometer:



Quantum Channel perspective

single photon $\leftrightarrow$ (dual rail) qubit

$$|\psi\rangle = \alpha|a\rangle_1 + \beta|b_1\rangle \equiv \alpha|0\rangle + \beta|1\rangle$$

LMZ $\Rightarrow$ qubit $\to$ qubit channel

$$\Lambda_\varphi : \rho_{in} \to \rho_{out}(\varphi)$$

Kraus op's

CPTP: completely positive trace-preserving map $\Longleftrightarrow$ $\rho_{out} = \sum_i K_i \rho_{in} K_i^\dagger$

any can be represented in the Kraus form

$$\rho_{in} = |\psi\rangle\langle\psi| \xrightarrow{\Lambda_\varphi} \begin{cases} U_\varphi \rho_{in} U_\varphi^\dagger, & \text{with prob } \eta^2 \\ |c'\rangle\langle c'|, & \text{with prob } 1-\eta^2 \end{cases}$$

$$\rho_{out}(\varphi) = \eta^2 U_\varphi |\psi\rangle\langle\psi| U_\varphi^\dagger + (1-\eta^2)|c'\rangle\langle c'|$$

Heisenberg limit is an "elusive" (measure-zero) phenomenon as $N \to \infty$

Convex set of all CPTP maps:

one can prove

$$F_Q[\Lambda_\varphi[|\psi\rangle]] \leq \frac{N}{\varepsilon_+ \varepsilon_-}$$

Example

Lossy M-Z: $F_Q[\rho_\varphi] \leq \frac{\eta}{1-\eta} N$

(actually need use Kraus rep^s)

$\Rightarrow$ no matter the input state

$$\Delta^2 \tilde{\varphi} \geq \frac{1-\eta}{\eta} t \cdot \frac{1}{N} \equiv \frac{1-\eta}{\eta} \cdot \frac{1}{\bar{N}}$$

(convexity of QFI)

mean number of photons

Example: GEO600 PRA 88, 041802 (2013)

strain in gravitational-wave detector

$$h = \frac{c}{L} \cdot \frac{1}{\sqrt{2}\, g(\Omega)} \cdot \varphi$$

L: length of arms

$\frac{1}{\tau} := (\text{time})^{-1}$

freq. of signal

$$g(\Omega) = \sqrt{\frac{T}{2 - T - 2\sqrt{1-T}\cos(2\Omega\tau)}}$$

Michelson interferometer

effective amplification factor ... as a Mach-Zehnder interf.

{T – power transmissivity of the signal-recycling mirror}

Ultimate limit with $\eta < 1$

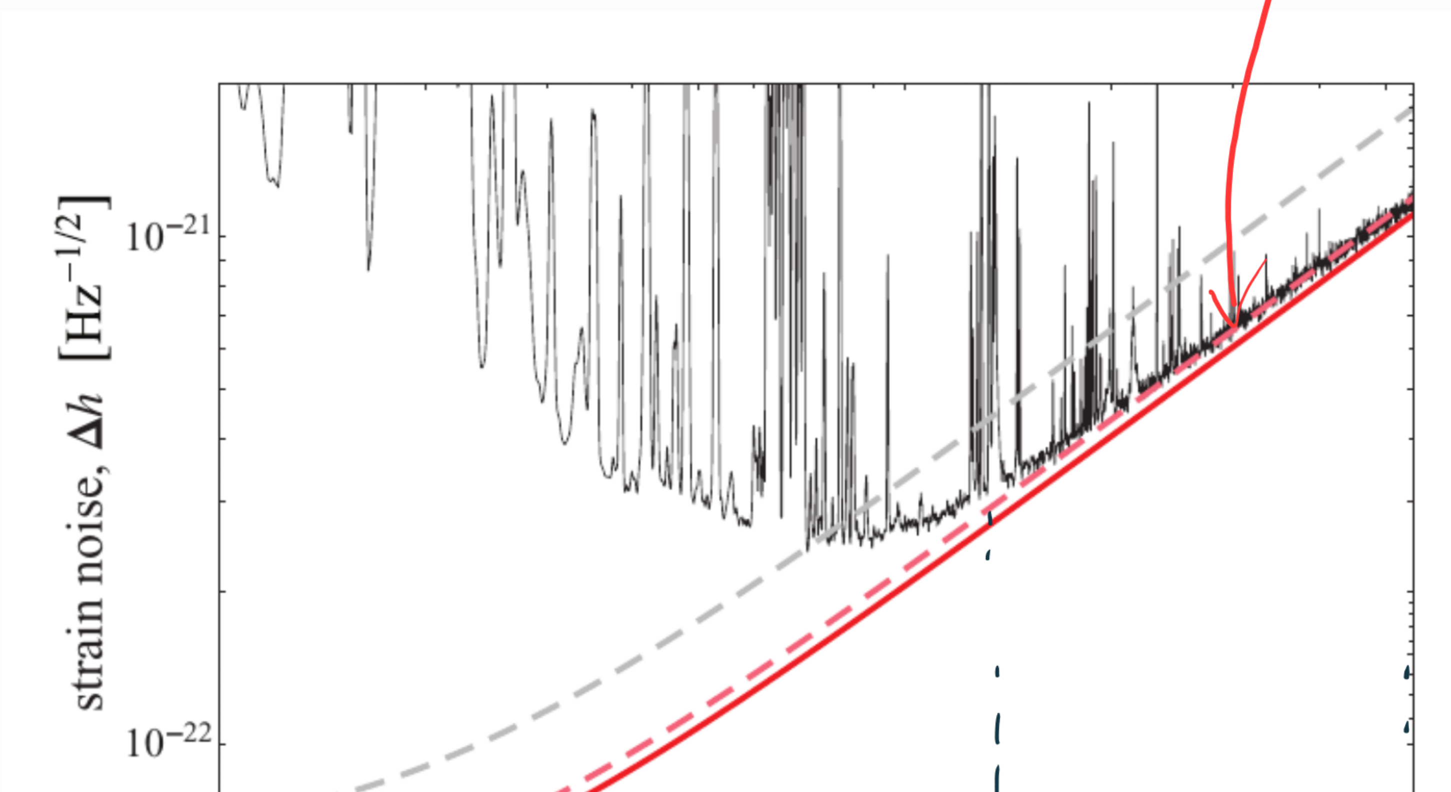
$$\Delta^2 \tilde{h} = \frac{c^2}{l^2} \frac{1}{2 g(\Omega)^2} \Delta^2 \tilde{\varphi}$$

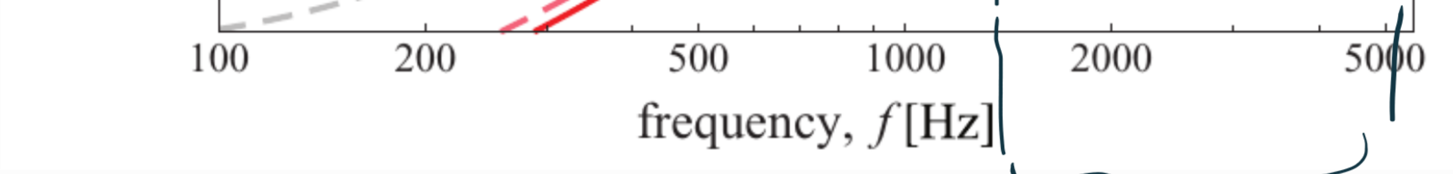
$$\geq \frac{c^2}{l^2} \frac{1}{2 g(\Omega)^2} \left(\frac{1-\eta}{\eta} \right) \cdot \frac{1}{\underbrace{(P/\hbar\omega_0)}_{\dot{N}}}$$

For GEO600:

$\frac{2\pi}{\omega_0} = \lambda_0 = 1064\,nm$, $l = 1200\,m$, $P = 3.7\,kW$

$\underbrace{\eta = 0.62}_{62\%}$, $T = 1.9\%$





optimal operation

"quantum noise" dominated region $\Rightarrow$ e.g. { Black-hole / neutron star merger signals }

<u>Conclusion</u>: at high signal-frequencies LIGO can only be improved by improving overall detection efficiency η
