Ch-21 Elements of Spacetime geometry

A point in spacetime has a location in space and a location in time, in other words, a point in spacetime represents an event, something that happens at a particular place at a particular time. Points P_A and P_B refer to events that we shall call E_A and E_B. The quantity ΔS_{AB}^2 cannot be the square of the distance between two events, because it can be negative.

$$\Delta S_{AB}^2 = -(\tau_A - \tau_B)^2 + (x_A^1 - x_B^1)^2 + \dots + (x_A^0 - x_B^0)^2$$

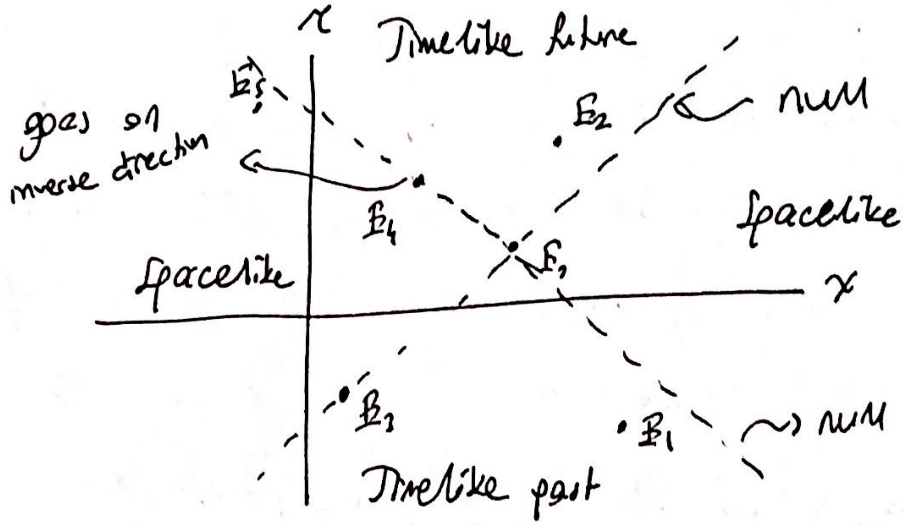
But if it's not the distance between events, then what does this interval signify?

It tells us about the causal structure of the spacetime - which event can be the cause of another event, and which cannot.

Normally, without taking special relativity into account, we would just use time to decide the question of causality, assuming that the time between two events is absolute and the same for all observers. If event E_A occurs at time t_A and event E_B occurs at time t_B, if $t_A < t_B$, then event E_A could be the cause of event E_B, but if $t_B < t_A$, then the causal relationship is reversed. But this view doesn't take into account the speed of light, which is finite and same for all observers. When we take into account the speed of light, we lose absolute time and learn that time is relative, that two inertial observers will not agree when two events occur at the same time.

So what happens to causality when the passage of time is relative? The answer is encoded into the spacetime geometry by the sign of the Minkowski interval, which is unchanged by a Lorentz transformation, and hence the same for all inertial observers.

There are three possibilities for the sign of Δs_{AB}^2, and these three possibilities divide up the spacetime around each point into the three Lorentz invariant regions described below:



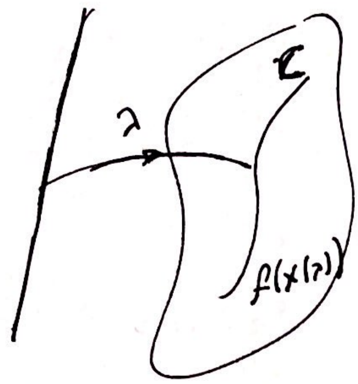
Every point in the Minkowski spacetime has a light cone in which causality encoded.

i.) $\Delta S_{AB}^2 > 0$: Events E_A and E_B are separated by a spacelike interval. There exists a Lorentz boost at some velocity β to a frame where events E_A and E_B happen at the same time, but there exists no Lorentz transformation to a frame where the two events happen at the same place.

ii.) $\Delta S_{AB}^2 = 0$: Events E_A and E_B are separated by a lightlike or null interval. This is the path a beam of light or a massless particle would take to get from event E_A to event E_B. There exists no Lorentz transformation at any velocity β to a frame where events E_A and E_B happen at the same time or the same place.

iii-) $\Delta S_{AB}^2 < 0$: Events E_A and E_B are separated by a timelike interval. There exists a Lorentz transformation at some velocity β to a frame where events E_A and E_B happen at the same place, but there exists no Lorentz transformation to a frame where the two events happen at the same time.

- Vectors on a Manifold :



$$\frac{df(x(\lambda))}{d\lambda} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} = X[f]$$

$$X = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} = a^i \partial_i$$

$\{\hat{e}_i\} \in T_p(M) \rightarrow$ basis vectors

$\{\bar{e}^i\} \in T_p^*(M) \rightarrow$ dual basis vectors

$$\langle \hat{e}_j , \bar{e}^i \rangle = \delta^i_j$$

$\langle \frac{\partial}{\partial x^j}, dx^i \rangle = \delta^i_j$, where dx^i : basis one-form

$$X \in T_p(M) \rightarrow X = a^i \frac{\partial}{\partial x^i}$$

$$\bar{\omega} \in T_p^*(M) \longrightarrow \bar{\omega} = \omega_i dx^i$$

$$\langle \bar{v}, \bar{\omega} \rangle = \langle v^i \partial_i , \omega_j dx^j \rangle = v^i \omega_j \langle \partial_i , dx^j \rangle = v^i \omega_i$$

Metric tensor: $g : T_p(M) \otimes T_p(M) \longmapsto \mathbb{R}$

$$g = g_{ij}\, dx^i \otimes dx^j$$

$g_{ij} = g(\partial_i, \partial_j)$ and $g(\vec{u}, \vec{v}) = g_{ij} u^i v^j$

- Vectors in spacetime:

$$\vec{V} = V^\mu \frac{\partial}{\partial x^\mu} = V^\tau \frac{\partial}{\partial \tau} + V^i \frac{\partial}{\partial x^i}$$

A Lorentz boost $L(\beta)$ with velocity component β^i in the i direction of frame S with coordinates $(x^0, x^1, \dots, x^D)$ to frame $\tilde{S}$ with coordinates $(\tilde{x}^0, \tilde{x}^1, \dots, \tilde{x}^D)$ has components:

$$L^{\tilde{0}}_{\ 0} = \gamma$$

$$L^{\tilde{0}}_{\ i} = -\gamma\beta^i \quad , \quad L^{\tilde{i}}_{\ 0} = -\gamma\beta^i$$

$$L^{\tilde{i}}_{\ j} = (\gamma - 1)\frac{\beta^i\beta^j}{\beta^2} + \delta^{ij}$$

with the inverse transformation obtained by sending $\beta^i \to -\beta^i$.

The new components are then

$$\tilde{V}^\mu = L^{\tilde{\mu}}_{\ \nu} V^\nu$$

~~The new components are then~~

The scalar product of two vectors $\vec{u}$ and $\vec{v}$ is

$$\vec{u}\cdot\vec{v} = \eta_{\mu\nu} v^\mu u^\nu = -v^0 u^0 + \sum_{i=1}^{D} v^i u^i$$

$$\vec{v}^2 = \vec{v}\cdot\vec{v} = \eta_{\mu\nu} v^\mu u^\nu = -(v^0)^2 + \sum_{i=1}^{D} (v^i)^2 = \begin{cases} <0, & \text{timelike} \\ =0, & \text{null} \\ >0, & \text{spacelike} \end{cases}$$

Since $\vec{v}^2$ is a scalar and hence the same for all observers.

~~Timelike vectors:~~

- Velocity and Momentum in spacetime:

u: timelike vector

$g(u,u) < 0 \rightarrow$ u is tangent to curves representing the world lines of objects traveling through time.

The curve $\rightarrow$ $C(\lambda) = \left(t(\lambda), x^1(\lambda), \dots, x^D(\lambda)\right)$ with $D = d-1$

λ: proper time

At each point on this curve the Lorentz invariant line element is

$$-d\lambda^2 = -d\tau^2 + \sum (dx^i)^2$$

Dividing both sides by $d\lambda^2$ gives

$$-1 = -\left(\frac{d\tau}{d\lambda}\right)^2 + \sum \left(\frac{dx^i}{d\lambda}\right)^2$$

We can also write this as

$$u^\mu = \frac{dx^\mu}{d\lambda} \rightarrow u^0 = \frac{d\tau}{d\lambda}, \quad u^i = \frac{dx^i}{d\lambda}$$

$$u = u^\mu \frac{\partial}{\partial x^\mu} \rightarrow \text{spacetime vector tangent to the curve } C(\lambda)$$

$$u^2 = u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$$

For $d=4$, u is called the four-velocity. For general d, we will call it the spacetime velocity.

$$\frac{dx^i}{d\lambda} = \frac{dx^i}{d\tau}\frac{d\tau}{d\lambda}, \quad d\tau = \gamma\, d\lambda, \quad \lambda\text{: proper time}$$

from time dilation

$$= \gamma \frac{dx^i}{d\tau}$$

The spacetime velocity vector is then revealed to be

$$u = u^\mu \partial_\mu = u^0 \partial_0 + u^i \partial_i \quad , \quad u^\mu = \frac{dx^\mu}{d\lambda}$$

$$= \frac{dx^0}{d\lambda} \partial_0 + \frac{dx^i}{d\lambda} \partial_i$$

$$= \frac{dx^0}{d\tau} \frac{d\tau}{d\lambda} \partial_0 + \frac{dx^i}{d\tau} \frac{d\tau}{d\lambda} \partial_i \quad , \quad x^0 = \tau \ , \ \tau = ct \quad \frac{dx^i}{d\tau} = \beta^i = \frac{1}{c} \frac{dx^i}{dt}$$

$$= \gamma \frac{\partial}{\partial \tau} + \gamma \beta^i \frac{\partial}{\partial x^i} \quad , \quad u^\mu = (\gamma, \gamma \beta^i)$$

If the curve $C(\lambda)$ is a path of a particle or object in spacetime, then the tangent vector must represent a spacetime generalization of velocity. Normally in Newtonian physics the momentum is $\vec{p} = m\vec{v}$. If we generalize this to flat spacetime in d dimensions, then we should write

$$p = mu$$

and using $u^2 = \eta_{\mu\nu} u^\mu u^\nu = -1 \quad , \quad u^\mu = (\gamma, \gamma\beta^i)$

$$p^2 = \eta_{\mu\nu} p^\mu p^\nu = -(p^0)^2 + |\vec{p}|^2 = -m^2$$

$$p^0 = \gamma m$$
$$\vec{p} = \gamma m \vec{\beta}$$

Expanding γ for small β gives, $\gamma = (1-\beta^2)^{-\frac{1}{2}}$

$$p^0 \sim m\left(1+\frac{1}{2}\beta^2+\dots\right) = m + \frac{1}{2}m\frac{v^2}{c^2}$$

$$= \frac{1}{c^2}\left(mc^2 + \frac{1}{2}mv^2\right)$$

$$\vec{p} \sim m\left(1+\frac{1}{2}\beta^2+\dots\right)\frac{\vec{v}}{c} = m\frac{\vec{v}}{c} + \dots$$

The time component p^0 of the spacetime momentum looks like the kinetic energy of the object whose world line is $C(\lambda)$, but there is the extra term mc^2 to account for. This term is called the rest energy of the object in question. The time component of the momentum is the relativistic energy of the object, with a contribution from the kinetic energy and a contribution from the mass of the object at rest. This is what lies ~~[illegible]~~ behind Einstein's famous equation

$$E = mc^2,$$

which is what we get in the limit $\beta \to 0$ if we make the assignment $p^0 = \frac{E}{c^2} = \gamma m$.

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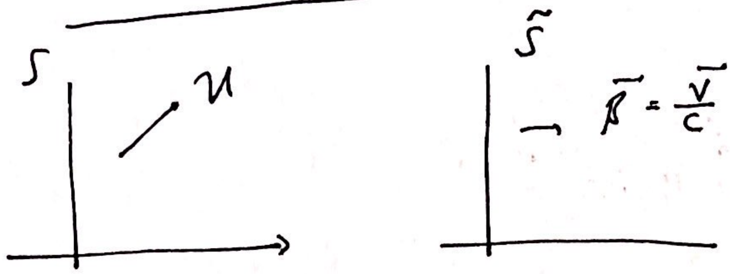
- Lorentz boost of velocity



u with Minkowski coordinate components $(\gamma_u, \gamma_u \vec{\beta_u})$

$$\gamma_u = \frac{1}{\sqrt{1-|\beta_u|^2}} \quad , \quad \vec{\beta_u} = \frac{\vec{u}}{c}$$

$$\tilde{u}^{\mu} = L^{\tilde{\mu}}{}_{\nu} u^{\nu}$$

$$\tilde{u}^{0} = L^{\tilde{0}}{}_{\nu} u^{\nu} = L^{\tilde{0}}{}_{0} u^{0} + L^{\tilde{0}}{}_{i} u^{i}$$

$$= \gamma u^0 - \gamma \vec{\beta} \cdot \vec{u} = \gamma \gamma_u - \gamma \gamma_u \vec{\beta} \cdot \vec{\beta_u}$$

$$= \gamma \gamma_u \left(1 - \vec{\beta} \cdot \vec{\beta_u}\right)$$

Since $\tilde{u}^0 = \tilde{\gamma}_u$, we see that the lorentz boost rule for γ_u is

$$\tilde{\gamma}_u = \tilde{u}^0 = \gamma \gamma_u \left(1 - \vec{\beta} \cdot \vec{\beta_u}\right) = \frac{\left(1 - \vec{\beta} \cdot \vec{\beta_u}\right)}{\sqrt{1-|\vec{\beta_u}|^2}\sqrt{1-|\vec{\beta}|^2}}$$

The space components transform in a more complicated manner, with

$$\tilde{u}^i = \tilde{L}^i_{\ \mu} u^\mu = \tilde{L}^i_{\ 0} u^0 + \tilde{L}^i_{\ j} u^j$$

$$= -\gamma\beta^i u^0 + \left((\gamma-1)\frac{\beta^i\beta^j}{\beta^2} + \delta^{ij} \right) u^j$$

$$= u^i - \gamma\beta^i u^0 + (\gamma-1)\frac{\vec{\beta}\cdot\vec{u}}{\beta^2}\beta^i$$

In this form it's hard to see that this is a Lorentz boost. For simplicity let's work in $d=3$ with $\vec{\beta} = \beta\hat{e}_x$, so that

$$\vec{\beta}\cdot\vec{u} = \beta u^x .$$

We then get

$$\cancel{\tilde{u}^x = u^x - \gamma\beta u^0 + (\gamma-1)\frac{\beta u^x}{\beta^2}\beta}$$

$$\tilde{u}^x = u^x - \gamma\beta u^0 + (\gamma-1)\frac{\vec{\beta}\cdot\vec{u}}{\beta^2}\beta$$

$$= u^x - \gamma\beta u^0 + (\gamma-1)\frac{\beta u^x}{\beta^2}\beta$$

$$= \cancel{u^x} - \gamma\beta u^0 + \gamma u^x - \cancel{u^x}$$

$$= \gamma u^x - \gamma\beta u^0 = \gamma(u^x - \beta u^0) \qquad , u = (\gamma_u, \gamma_u\vec{\beta}_u)$$

$$= \gamma\left(\gamma_u (\beta_u)^x - \beta\gamma_u\right)$$

$$= \gamma\gamma_u\left((\beta_u)^x - \beta\right) \quad , \quad (\beta_u)^x = \frac{u^x}{c}$$

$$\tilde{u}^y = u^y = \gamma_u (\beta_u)^y$$

which is the usual formulas for a Lorentz boost in one dimension (here in the x direction).

The components of the transformed velocity become

$$(\tilde{\beta}_u)^x = \frac{\tilde{u}^x}{\tilde{v}^0} = \frac{\gamma\gamma_u\left((\beta_u)^x - \beta\right)}{\gamma\gamma_u\left(1 - \vec{\beta}\cdot\vec{\beta}_u\right)} = \frac{(\beta_u)^x - \beta}{1 - \vec{\beta}\cdot\vec{\beta}_u}$$

$$(\beta_u)^y = \frac{\tilde{u}^y}{\tilde{v}^0} = \frac{\gamma_u (\beta_u)^y}{\gamma\gamma_u\left(1 - \vec{\beta}\cdot\vec{\beta}_u\right)} = \frac{(\beta_u)^y}{\gamma\left(1 - \vec{\beta}\cdot\vec{\beta}_u\right)} .$$

Notice that although the relative motion between S and $\tilde{S}$ is constrained to the x direction, the object's velocity in the y direction is changed by the transformation.