SPECIAL RELATIVITY

Ch-1 / ~~From Pythagoras to~~ spacetime geometry:

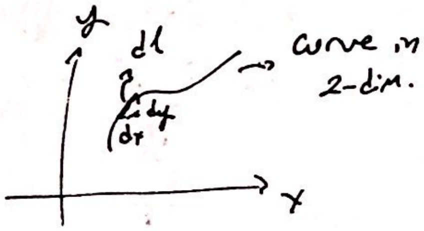
D - dimensional space;

$$\vec{r} = \sum_{i=1}^{D} x^i \hat{e}_i \quad , \quad \hat{e}_i \cdot \hat{e}_j = \delta_{ij} \rightarrow \text{Orthonormal basis}$$

Any vector $\vec{V}$,

$$\vec{V} = \sum_{i=1}^{D} V^i \hat{e}_i \quad , \quad V^i = \vec{V} \cdot \hat{e}_i$$

Infinitesimal line element; (metric)



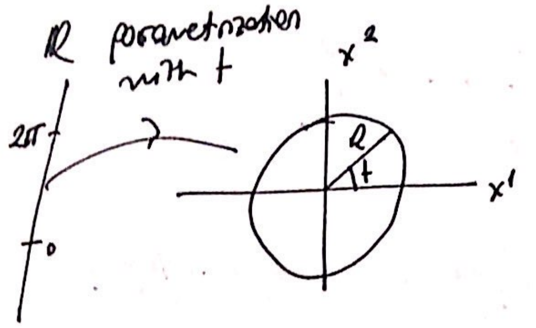
metric in D-dim. $\rightarrow dl^2 = \delta_{ij}\, dx^i dx^j$

A curve in D-dimensional Euclidean space

$$\Delta L = \int_{P_1}^{P_2} dl \quad , \quad P_1 = x(t_1) \quad , \quad P_2 = x(t_2)$$

$$dl = \sqrt{\delta_{ij}\, dx^i dx^j} = \sqrt{\delta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt \quad , \quad \frac{dx^i}{dt} = \dot{x}^i$$

$$\Delta L = \int_{t_1}^{t_2} \sqrt{\delta_{ij}\, \dot{x}^i \dot{x}^j}$$

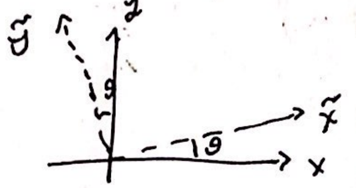


$$x^1 = R\cos t$$
$$x^2 = R\sin t$$

$$\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 = (-R\sin t)^2 + (R\cos t)^2$$
$$= R^2(\sin^2 t + \cos^2 t) = R^2$$

$$\Delta L = \int_{t_1}^{t_2} \sqrt{\delta_{ij}\,\dot{x}^i\dot{x}^j}\,dt = \int_0^{2\pi} R\,dt = 2\pi R \rightarrow$$ Circumference of a circle of radius R.

* Rotation preserve the Euclidean metric:



$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{aligned} \tilde{x} &= x\cos\theta + y\sin\theta \\ \tilde{y} &= -x\sin\theta + y\cos\theta \end{aligned}$$

$$R \in SO(2) \rightarrow R^T I R = I$$

$\det R = 1$ (special) $\quad R^T R = I$ (orthogonal)

* Infinitesimal Rotations:

$$\tilde{x} = x\cos\theta + y\sin\theta$$

$$\tilde{y} = -x\sin\theta + y\cos\theta$$

A very small rotation with $\theta \sim 0$ can be written

$$\cos\theta \sim 1 \quad , \quad \sin\theta \sim \theta$$

$$\tilde{x} \simeq x + \theta y + \mathcal{O}(\theta^2)$$
$$\tilde{y} \simeq y - \theta x + \mathcal{O}(\theta^2)$$
$$\Rightarrow \quad R(\theta) \approx \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

$$\Rightarrow \quad R(\theta) \simeq I + \theta \Gamma + \cdots \quad , \quad \Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightsquigarrow \text{generator of the rotation transformation}$$

Consider

$$e^{\alpha x} \simeq 1 + \alpha x + \cdots$$

A non-infinitesimal rotation $R(\theta)$ can be obtained from the exponential of the generator matrix Γ, $e^{\theta\Gamma} \simeq I + \theta\Gamma + \cdots$

⚠ An <u>antisymmetric generator</u> matrix will always lead to a rotation ~~matrix~~ when we take the exponential to get the full transformation.

Now let's call the new generator matrix which is symmetric ($L^T = L$) and the new transformation parameter ξ.

$$L(\xi) \simeq I + \xi l + \dots \quad , \quad l = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$L(\xi) = e^{\xi l} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix}$$

⚠ The action of $L(\xi)$ on a set of rectangular coordinates axes is not to rotate them but to skew them like scissors.

$$\theta : 0 \to 2\pi \qquad \text{BUT} \qquad \xi : -\infty \to \infty$$

$$L^{-1}(\xi) = L(-\xi) = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}$$ → Transformation in the opposite direction.

$\det L = 1$ same as $\det R = 1$.

But

$$L^T \eta L = \eta \quad , \quad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Recall: $R^T I R = I$

$$L \in SO(1,1)$$

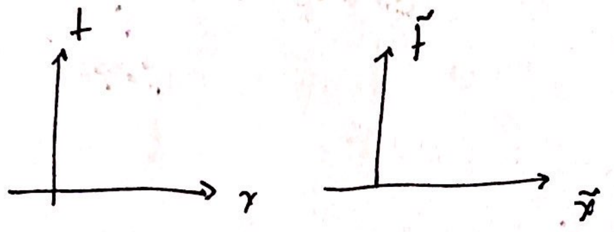
* Could a line element include time?

$$\vec{r} = \vec{r}(t) = x^i(t)\,\hat{e}_i$$

In 1-dim: free particle $\rightarrow$ $\dfrac{d^2 x(t)}{dt^2} = 0$

This equation is invariant under the transformation

$$\begin{aligned}\tilde{t} &= t \\ \tilde{x} &= x - vt\end{aligned} \quad \Rightarrow \quad \frac{d^2 \tilde{x}(t)}{d\tilde{t}^2} = 0$$



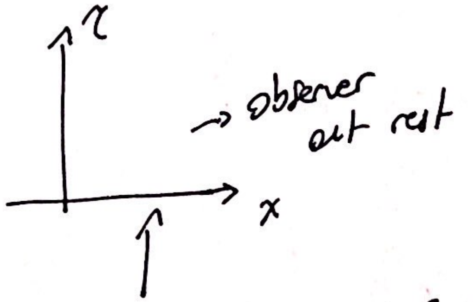
Let's say $\tau = ct$, $[\tau] = \dfrac{\text{length}}{\text{time}} \times \text{time} = [\text{length}]$

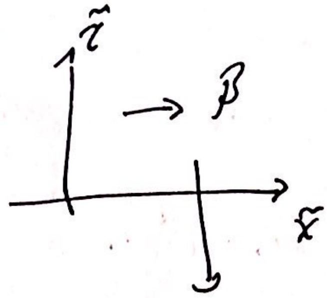
The line element

$$ds^2 = -d\tau^2 + dx^2$$

is invariant under the coordinate transformation given by $L(\xi)$.

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu \quad , \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$





, $\beta = \frac{v}{c}$

metric : $ds^2 = -d\tau^2 + dx^2$

$dx = 0$

$ds^2 = -d\tau^2 = -c^2 dt^2$

For the observer in this frame

$$\frac{d\tilde{x}}{d\tilde{\tau}} = -\beta$$

$$ds^2 = -d\tilde{\tau}^2 + d\tilde{x}^2 \underset{\uparrow \; d\tilde{x} = -\beta d\tilde{\tau}}{=} -d\tilde{\tau}^2 + \beta^2 d\tilde{\tau}^2$$

$$-d\tau^2 = -d\tilde{\tau}^2(1-\beta^2) \Rightarrow d\tilde{\tau}^2 = \frac{d\tau^2}{(1-\beta^2)}$$

$$\Rightarrow d\tilde{\tau} = \frac{d\tau}{\sqrt{1-\beta^2}} \Rightarrow d\tilde{\tau} = \gamma d\tau \, , \; \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Recall that $\tilde{\tau} = \tau \cosh\zeta - x \sinh\zeta$

$\tilde{x} = -\tau \sinh\zeta + x \cosh\zeta$

$d\tilde{\tau} = d\tau \cosh\zeta - dx \sinh\zeta$, $dx = 0 \Rightarrow d\tilde{\tau} = d\tau \cosh\zeta$

So $\cosh\zeta = \gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$d\tilde{x} = dx\cosh\eta - d\tau\sinh\eta \quad , \quad dx = 0$$

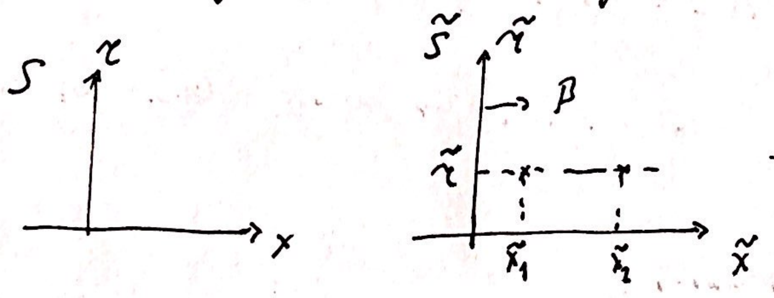
$$\Rightarrow \quad d\tilde{x} = -d\tau\sinh\eta \;\to\; d\tilde{x} = -\frac{d\tilde{\tau}}{\gamma}\sinh\eta$$

$$d\tilde{\tau} = \gamma d\tau$$

$$\Rightarrow \quad \underbrace{\gamma\frac{d\tilde{x}}{d\tilde{\tau}}}_{-\beta} = -\sinh\eta \quad \Rightarrow \quad \sinh\eta = \gamma\beta$$

$$\Rightarrow \quad L(\beta) = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \quad \Rightarrow \text{Lorentz Transformation.}$$

* Relativity of simultaneity:



$$\Delta\tilde{\tau} = 0$$

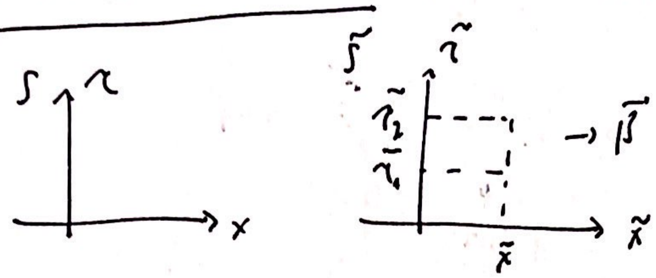
$$\to \Delta\tilde{x} = \tilde{x}_2 - \tilde{x}_1 \neq 0$$

$$\begin{pmatrix} \tau \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} \tilde{\tau} \\ \tilde{x} \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \tau &= \gamma\tilde{\tau} + \gamma\beta\tilde{x} \\ x &= \gamma\beta\tilde{\tau} + \gamma\tilde{x} \end{aligned}$$

$$\begin{aligned} \tau_1 &= \gamma\tilde{\tau}_1 + \gamma\beta\tilde{x}_1 \\ \tau_2 &= \gamma\tilde{\tau}_2 + \gamma\beta\tilde{x}_2 \end{aligned} \quad \Rightarrow \quad \tau_2 - \tau_1 = \gamma\big(\overbrace{\cancel{\tilde{\tau}_2 - \tilde{\tau}_1}}^{\Delta\tilde{\tau} = 0}\big) + \gamma\beta(\tilde{x}_2 - \tilde{x}_1)$$

$$\Rightarrow \quad \Delta\tau = \tau_2 - \tau_1 = \gamma\beta(\tilde{x}_2 - \tilde{x}_1) = \gamma\beta\Delta\tilde{x} \;\checkmark$$

- Time dilation:



$$\Delta\tilde{x} = 0$$

$$\Delta\tilde{\tau} = \tilde{\tau}_2 - \tilde{\tau}_1 \neq 0$$

$$\begin{pmatrix} \tau \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \tilde{\tau} \\ \tilde{x} \end{pmatrix}$$

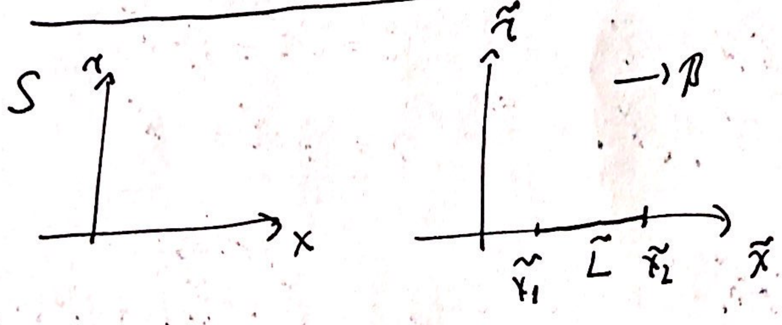
$$\tau = \gamma\tilde{\tau} + \gamma\beta\tilde{x} \rightarrow \begin{array}{l} \tau_2 = \gamma\tilde{\tau}_2 + \gamma\beta\tilde{x}_2 \\ \tau_1 = \gamma\tilde{\tau}_1 + \gamma\beta\tilde{x}_1 \end{array}$$

$$\Delta\tau = \tau_2 - \tau_1 = \gamma(\tilde{\tau}_2 - \tilde{\tau}_1) + \gamma\beta\cancel{(\tilde{x}_2 - \tilde{x}_1)} = \gamma\Delta\tilde{\tau}$$

$$\Delta\tau = \gamma\Delta\tilde{\tau}, \quad \gamma > 1$$

$\Delta\tau > \Delta\tilde{\tau}$ → moving clocks run slowly.

- Length contraction:



$$, \Delta\tilde{\tau} = 0$$

$$\begin{pmatrix} \tilde{\tau} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix}$$

$$\tilde{\tau} = \gamma\tau - \gamma\beta x \qquad\qquad \tilde{x}_2 = -\gamma\beta\tau_2 + \gamma x_2$$

$$\tilde{x} = -\gamma\beta\tau + \gamma x \quad \Rightarrow \quad \tilde{x}_1 = -\gamma\beta\tau_1 + \gamma x_1$$

$$\tilde{x}_2 - \tilde{x}_1 = -\gamma\beta(\tau_2 - \tau_1) + \gamma(x_2 - x_1) \quad \ldots\ldots \quad 1$$

$$\tilde{\tau}_2 = \gamma\tau_2 - \gamma\beta x_2$$

$$\tilde{\tau}_1 = \gamma\tau_1 - \gamma\beta x_1 \quad \Rightarrow \quad \tilde{\tau}_2 - \tilde{\tau}_1 = 0 = \gamma(\tau_2 - \tau_1) - \gamma\beta(x_2 - x_1)$$

$$\Rightarrow \gamma(\tau_2 - \tau_1) = \gamma\beta(x_2 - x_1) \quad \ldots\ldots \quad 2$$

Let's put 2 in 1 ;

$$\underbrace{\tilde{x}_2 - \tilde{x}_1}_{\tilde{L}} = -\beta\left(\gamma\beta\underbrace{(x_2 - x_1)}_{L}\right) + \gamma\underbrace{(x_2 - x_1)}_{L}$$

$$\tilde{L} = -\gamma\beta^2 L + \gamma L = \gamma L(1 - \beta^2) = L\frac{1}{\sqrt{1-\beta^2}}(1-\beta^2)$$

$$= L\sqrt{1-\beta^2} = \frac{L}{\gamma}$$

$$\Rightarrow \quad \tilde{L} = \frac{L}{\gamma} \rightarrow L = \gamma\tilde{L} \quad , \gamma > 1$$

$$\Rightarrow L > \tilde{L} \rightarrow \text{length contraction.}$$