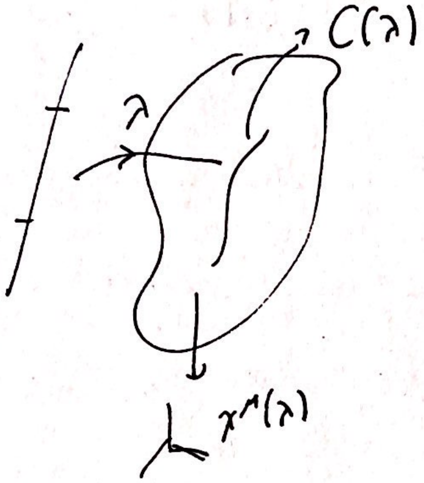
Ch-31 Mechanics in spacetime

* Equation of motion in spacetime:

$$\vec{F} = m\vec{a} = m\,\frac{d^2\vec{x}}{dt^2}$$

Relativistic acceleration:



$$u^\mu = \frac{dx^\mu}{d\lambda}$$

$$a^\mu = \frac{du^\mu}{d\lambda}$$

$$a^\mu = \frac{du^\mu}{d\lambda} = \frac{\partial u^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} = u^\alpha \partial_\alpha u^\mu \quad , \quad \partial^\alpha = \eta^{\alpha\beta}\partial_\beta$$

$a = u \cdot \partial u \rightarrow$ a in a basis independent manner.

The components a^μ of the acceleration 'a' transform as a spacetime vector.

$$\tilde{a}^\mu = L^{\tilde{\mu}}_{\ \nu}\, a^\nu .$$

Spacetime acceleration 'a' is always orthogonal to spacetime velocity u.

Proof:

$$u.u = -1 = \eta_{\mu\nu} u^{\mu} u^{\nu}$$

$$u.a = u.(u.\nabla)u = \eta_{\mu\nu} u^{\mu} (u^{\alpha}\partial_{\alpha} u^{\nu})$$

$$= u^{\alpha} \frac{1}{2} \partial_{\alpha} (\eta_{\mu\nu} u^{\mu} u^{\nu}) = \frac{1}{2} u^{\alpha} \partial_{\alpha} (-1) = 0 .$$

Since u is the tangent vector to the world line $C(\lambda)$, and $u.a = 0$, the spacetime acceleration vector a must be normal vector to the world line $C(\lambda)$.

* Relativistic Force:

World line of a free particle:

For there are no forces;

$$\frac{du}{d\lambda} = u.\partial u = 0 .$$

One obvious solution is $u = (1, 0, \dots, 0)$

To get the equation for the world line of this particle in terms of the Minkowski coordinates $x^{\mu}(\lambda)$, we need to solve the equation

$$u = \frac{dx}{d\lambda}.$$

The world line of our particle is

$$x^\mu(\lambda) = (\lambda, 0, \ldots, 0).$$

This particle is at rest relative to the coordinate frame we have chosen. It moves forward in time, but it stays at the same location in space.

We know that a <u>translation</u> or a <u>Lorentz transformation</u> of any solution will yield another solution.

Let's define a translation vector

$$\Delta x^\mu = (c^0, c^1, \ldots, c^D) \quad \text{where } c^\mu \text{: constant}$$

and the sum of $x^\mu(\lambda) + \Delta x^\mu$ is also a solution.

$$x^\mu(\lambda) = (\lambda + c^0, c^1, \ldots, c^D).$$

This is still a particle at rest, but it rests somewhere else now.

What about solutions for a particle not at rest?

* Lorentz transformation $L^{\tilde{\alpha}}_{\mu} x^{\mu}(\lambda)$ of any solution is also a solution.

* So we can pick any velocity $\vec{\beta}$ that we desire and make a Lorentz transformation of our solution for the particle at rest and get a particle traveling at velocity $\vec{\beta}$.

We do this by transforming to a frame $\tilde{S}$ traveling at velocity $-\vec{\beta}$ relative to the frame S in which our particle appears to be at rest.

$\tilde{S}$ traveling at speed $-\beta$ in the x^1 direction

particle is at rest in S.

$$\tilde{\tau}(\lambda) = \gamma\tau(\lambda) + \gamma\beta x^1(\lambda) = \gamma\lambda \quad , x^1 = 0$$

$$\tilde{x}^1(\lambda) = \gamma\beta\tau(\lambda) + \gamma x^1(\lambda) = \gamma\beta\lambda$$

$$\tilde{x}^i(\lambda) = x^i(\lambda) = 0 \quad , \quad i = 2, \ldots, D.$$

So in frame $\tilde{S}$

$$x^{\mu}(\lambda) = (\gamma\lambda, \gamma\beta\lambda, 0, \ldots, 0).$$

Δ We can apply any number of translation and Lorentz boosts at some velocity $\vec{\beta}$ to our original solution and get almost all of the possible solutions without having to solve differential equations.

* World line of a uniformly accelerated object:

In two spacetime dimensions, look at the set of Lorentz-invariant submanifolds of flat spacetime that satisfy

$$-\tau^2 + x^2 = \alpha^2 \quad \rightarrow \text{(hyperbolic eqn)}$$

where $\alpha \in \mathbb{R}$ is a constant.

The solution to this equation is

$$\tau(\lambda) = \alpha \sinh\left(\frac{\lambda}{\alpha}\right), \quad x(\lambda) = \alpha \cosh\left(\frac{\lambda}{\alpha}\right).$$

This solution is a timelike curve that passes through the event $(0, \alpha)$ at $\lambda = 0$. The tangent vector to this world line is given by

$$u(\lambda) = \left(\frac{d\tau}{d\lambda}, \frac{dx}{d\lambda}\right) = \left(\cosh(\lambda/\alpha), \sinh(\lambda/\alpha)\right).$$

The acceleration for this world line is given by

$$a(\lambda) = \left(\frac{d^2\tau}{d\lambda^2}, \frac{d^2x}{d\lambda^2}\right) = \left(\frac{1}{\alpha}\sinh(\lambda/\alpha), \frac{1}{\alpha}\cosh(\lambda/\alpha)\right).$$

$$a \cdot a = \eta_{\mu\nu} a^\mu a^\nu = -(a^0)^2 + (a^1)^2 = -\frac{1}{\alpha^2}\sinh^2(\lambda/\alpha) + \frac{1}{\alpha^2}\cosh^2(\lambda/\alpha)$$

$$= \frac{1}{\alpha^2}.$$

The magnitude of this acceleration vector is <u>constant</u>, so this world line represents a particle or object undergoing constant acceleration in spacetime.