LECTURE NOTES ON QUANTUM FIELD THEORY

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Steven Weinberg wrote 3 volumes on Quantum Field Theory. The first 2 volumes cover almost everything one needs to know about field theory. So after these two books, it should be illegal by international law for people like me to write a book on field theory. But the fact of the matter is that Weinberg probably wrote these books for a highly gifted physicist like himself. The reader should consider the current manuscript as a peraparation to read Weinberg's books.

These notes are a raw summary of my lecture notes. Its current form is not to be distributed as it is highly incomplete. References and some figures will be added later. I welcome corrections and suggestions.

Acknowledgments will be added later, but at this stage let me note that my previous students Ibrahim Gullu, Suat Dengiz and Emel Altas gave me a great deal of help in writing up my lecture notes. of course, all mistakes belong to me, but still I cannot claim originality of my mistakes as I have learned this topic from other books and sources. I was lucky to do my PhD in Minnesota between the years 1994 and 1999 where there were many experts in quantum field theory and particle physics. I took courses on this topic from Yutaka Hosotani (my advisor), from Mikhail Shifman (he did so many good things in non-perturbative QFT, he himself looks non-perturbative), Larry McLerran, Joseph Kapusta, and the late Mikhail Voloshin (whose particle theory course was some kind of robust poetry). Again, I will add a full acknowledgment later.

Textbooks:
A Modern Introduction to Quantum Field Theory M. Maggiore
An Introduction to quantum field theory M. Peskin and D.V. Schroeder
Quantum Field Theory in a Nutshell A. Zee

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CONTENTS

I. BROAD OVERVIEW OF QFT

A. Why do we need QFT?

This section is a brief discussion of quantum field theory and is not meant to be completely clear or explicit.

𝒰nlike the religious textbooks, most expositions on quantum field theory start with the apparently apologetic question: *why do we need QFT?* [1] The question is based on the premise that we already have the good old quantum mechanics (the relativistic and non-relativistic versions), and aren't these good enough ? The answer is they are not. First of all, it is clear that, just like one needs a quantum theory for the electron, one also needs a quantum theory for the fields, such as the electromagnetic field. In fact, historically, this is how quantum field theory started in 1926 with the paper by Born, Heisenberg and Jordan, who set out to write a quantum theory for the electromagnetic field. They made simplifying assumptions such as working in 1+1 dimensions and with a scalar field instead of the four-vector potential (ignoring the polarization of light). But they got the correct result that quantized electromagnetic fields yield photons (then known as light quanta).

Secondly, special relativity is a powerful constraint on all possible theories: its consequences are not just only perturbative improvements of results in the usual $\mathcal{O}(\frac{v^2}{c^2})$ form. We know, for example that a theory of gravity consistent with the (local) special relativity principles gave rise to general relativity based on the geometry of spacetime which is very different from the Newtonian gravity. Similarly, the constraints from special relativity on quantum mechanics yield, not just some perturbative corrections, but highly restrictive theories: these are the quantum field theories and there are only a couple of such theories in four dimensions. Let us expound upon this a little bit.

1. In the usual (say non-relativistic) quantum mechanics, given a potential $\hat{V}$ energy, the equation to solve is the Schrödinger equation

$$\left(\frac{\hat{p}^2}{2m} + \hat{V}\right)\psi = i\hbar\frac{\partial}{\partial t}\psi. \tag{1}$$

In all the processes or phenomena that can be dealt with this equation, the particle number or type does not change. But in real life this is not correct at all, consider even the simplest case of an atom absorbing a photon and getting excited: $A + \gamma \to A^*$. The initial photon-atom system *disappears* and a single exited atom *appears,* so this process cannot be defined by a wave function ψ (or the corresponding abstract state $|\psi\rangle$). Of course with some semi-classical approximation, one can approximately "understand" this absorption process within quantum mechanics, see Section 5.8 of Sakurai & Napolitano for such a computation. But a fully satisfactory understanding requires quantum field theory of both matter and radiation.

2. We need a theory based on quantum principles which is also consistent with special relativity. These are the bare minimums, there can be more constraints such as locality or cluster decomposition (the fact that spacelike separated experiments should give uncorrelated results.) For example, even a cursory look suggests that uncertainty relations coupled with the fact that energy can be converted to matter (mass) and matter can be converted to energy, forces one to abandon single particle theories as viable theories of nature. Quantum

[1] I suggest two nice and brief expositions for further reading: S. Weinberg, "What is quantum field theory, and what did we think it is?," [hep-th/9702027]; F. Wilczek, "Quantum field theory," Rev. Mod. Phys. **71**, S85 (1999) [hep-th/9803075].

Field Theory is the only possible "offspring" of the marriage between quantum mechanics and special relativity. QFTs are very restrictive theories based on certain symmetries and properties, such as *Lorentz invariance, quantum mechanics, renormalizability.* Unlike in the case of quantum mechanics, the interactions and the particles that can exist are dictated by QFT itself. This point is really significant: the potential in (1) can be anything in principle, but in QFT there are only a couple of possibilities.

Note that General Relativity does not conform to the principles of quantum mechanics and there is no QFT version of it. Therefore, we shall only deal with QFTs in (flat) Minkowski spacetime devoid of gravity. We are not completely hopeless about a quantum theory that has gravity in it. Such a theory probably cannot be simply obtained from the quantization of each interaction separately, but it is a unified theory of all possible interactions and fields all at once. String theory is a candidate for such a theory which is also a candidate for a theory of “Quantum Gravity", but it is a radically different theory (based on extended, non-point-like degrees of freedom) that exists only in more than 4 dimensions.

B. A Naive but useful picture

Recall the following naive picture:

1. In classical mechanics, the basic objects of interest (for example, the phase space degrees of freedom) are the position $\vec{x}$ and momenta $\vec{p}$, or their generalized versions at a given moment of universal time t.

2. In quantum mechanics, $\hbar$ appears, and one upgrades the basic “observables" to operators $\hat{\mathbf{x}}, \hat{\mathbf{p}}$ that may not commute, which satisfy relations like $[\hat{x}^i, \hat{p}^j] = i\hbar\delta^{ij}$ and the basic object of interest becomes an abstract state $|\psi\rangle$ of which the position-space representation is the wave-function $\psi(t, \vec{x})$ that obeys the Schrödinger equation.

3. In QFT, the speed of light also enters the picture and we have $(\hbar, c)$. Then $\vec{x}, t$, the position and time, are parameters, treated almost equally, unlike the case of QM, but the single particle function $\psi(t, \vec{x})$ becomes an operator $\hat{\psi}(t, \vec{x})$, a quantum field, and loses its direct interpretation as the probably amplitude. This procedure is sometimes called "second quantization", but it is a misnomer: the wavefunction is not really quantized. For example, the QFT for the Maxwell's electrodynamics would boil down to making the electric and magnetic fields as operators, and these are not wavefunctions in quantum mechanics. The notion of a wavefunction still exists in QFT but it becomes very complicated to work with it, as we shall explain below.

Note that in special relativity $(t, \vec{x})$ are on equal footing (up to the usual signature difference). So to respect this equality, when we pass on to quantum field theory, we really either have to make time an operator, just like $\vec{x}$ or downgrade the position operator $\hat{\mathbf{x}}$ to be a parameter just like time. Making time an operator, naively, comes with a heavy price : Hamiltonian is unbounded from below and one loses the notion of the ground state, which is of course not physical. But in special relativity, there are more than one time that we can choose and in principle, we could use two different times and upgrade the coordinate time to be an operator such as $\hat{x}^\mu(\tau)$ (where τ is, for example, the proper time) but we shall not do that. For us time and position, $(t, \vec{x})$, are parameters labeling the spacetime points (events) and they are not observables, or operators.

Another important issue is the idea of a wavefunction: now that we have apparently upgraded the usual single particle wavefunction (recall the above discussion though) to be an operator, a quantum field, have we lost the all important notion of a state or the wave function in quantum mechanics ? We have not: we can still have a wavefunction, let us call it a big Ψ which is a functional of the quantum field $\hat{\psi}$ as $\Psi[\hat{\psi}]$. Moreover, we can write a Hamiltonian $\hat{H}$, which is also a functional of the quantum field and its derivatives, and a quantum field theory version of the Schrödinger equation, which of course necessarily involves integrals over the position space as the value of the quantum field at each spatial point is a degree of freedom. For example, the Schrödinger field equation for a free massless scalar field theory in this form (in the Schrödinger *picture* where operators do not depend on time) would read (See the discussion in the book by Hatfield)

$$\hat{H}\Psi[\hat{\psi}] = i\hbar\frac{\partial}{\partial t}\Psi[\hat{\psi}], \tag{2}$$

where the Hamiltonian operator is given in terms of an integral summing all the degrees of freedom at each point in space:

$$\hat{H} = \frac{1}{2}\int d^3x \left(\hat{\pi}^2 + \hbar^2|\nabla\psi(\vec{x})|^2\right), \tag{3}$$

with the canonical momentum operator defined as

$$\hat{\pi} := \frac{\hbar}{i}\frac{\delta}{\delta\hat{\psi}(\vec{x})}, \tag{4}$$

which is analogous to the usual point particle canonical momentum operator: $\hat{p}_i := \frac{\hbar}{i}\frac{\partial}{\partial \hat{q}^i}$. While all this is fine, in quantum field theory this way of "doing things" is too cumbersome: instead we will not talk about wave functions (or functionals more properly), but we will introduce procedures of calculating any desired (and hopefully calculable) *transition amplitudes* using the quantum fields themselves. For bound state problems, of course quantum field theory works just as fine, but we shall not deal with the bound state problems: in fact, quantum field theoretical corrections (perturbations) to bound states, such as the Lamb-shift in the hydrogen atom, can be handled as scattering problems via the Feynman diagram techniques (which itself is a way of organizing or doing the perturbation theory). There are of course cases when naive scattering theory does not work, which is the case in the low energy limit of Quantum Chromodynamics (QCD), i.e. the theory of strong interactions; such cases require different non-perturbative approaches or lattice theory techniques.

Let us recap by stating what we need: we need to incorporate

$$\text{energy} \to \text{mass} \to \text{energy} \tag{5}$$

conversions and the probability and uncertainty principles of quantum mechanics. For this purpose, the first attempt that comes into mind is to write a relativistic version (or versions) of the Schrödinger equation, such as the Klein-Gordon (spin-0), Dirac (spin-1/2) and Rarita-Schwinger (spin-3/2) equations. But these equations will still be single particle equations which will not allow us deal with creation or annihilation of particles. Nevertheless, these relativistic equations will not be completely useless, we shall recycle them and interpret them as *operator equations* and not equations for wave functions. One more time: we do not quantize wave functions !

C. Indistinguishability and the point-like nature of fields, effective theories

Before we do anything we should make three pertinent remarks:

1. The fact that all fundamental particles are *indistinguishable* allows us to actually construct microscopic theories of nature. All the electrons are the same, (they can differ in location, energy and spin orientation, but for example, they do not age, they do not smell, they do not have shapes) and one only needs an electron quantum-field to describe all the electrons in the Universe ![2] Moreover, trying to describe an electron, one necessarily incorporates a positron. So a single quantum field suffices for electrons and positrons. Indistinguishability in 4 dimensions splits the quantum systems into two classes that have different statistics: fermions and bosons. Fermions have anti-symmetric, bosons have symmetric wave functions under the interchange of two particles. Fermions turn out to have half-integer spins while bosons have integer spins. This fact is the essence of the spin-statistics connection. For bosons and fermions different quantization rules will be needed. Quantum field theory partly explains the spin-statistics connection. I say "party" because it does this for the *fundamental* quantum fields and not for extended or composite objects. The way QFT will provide a proof of the spin-statistics connection will be through causality which we shall explain. [NOTE: give a footnote here reminding the Bose-statistics and its motivation and how it was found.]

2. Secondly, thinking about special relativity, we should realize that we can only study point-like objects. How could an extended rigid object be consistent with special relativity and quantum mechanics? We really don't have consistent quantum mechanics of extended objects save the fundamental strings (and some other solitonic multidimensional extended objects in string theory) which live in $26, 10$ or 11 dimensions depending on the type of string theory one considers. In addition to $\hbar, c$, string theory introduces another parameter, the string tension. Having said this, one might wonder about the proton, neutron or atoms which are extended objects. Well they are composite objects not fundamental particles. And it is correct that we still do not know exactly (not on a lattice simulation) how a *proton* really emerges from fundamental quarks and gluons.

3. We might accept the view that all theories, including the quantum field theories, are *effective theories* in the sense that they are valid up to some given energy scale. Beyond that energy scale, the theory can change in field and symmetry content or can be replaced altogether dramatically. For example, The Standard Model of particle physics can be replaced by a Grand Unified theory (GUT) combining strong, weak and electromagnetic interactions, say around the 10^{16}GeV scale, and perhaps with string theory at the 10^{18}GeV scale. Effective theories are not required to be renormalizable: one can include in them all interactions and terms that are consistent with some assumed (or observed) symmetry. This effective field theory approach is a more modest approach that somewhat lessens the great pleasure one gets assuming that we hit the ultimate building blocks, laws and symmetries of Nature. It is a humbling feeling but a really required one: reality (whatever it means) is energy-dependent. We should not ask what en electron is, but ask what an electron does at a given energy E. And if that energy is too large to ignore the gravity effects, we do not know what it does. So it is clear that we can only trust our theory about the electrons up to some energy scale, call it Λ and we have the dimensionless parameter $\frac{E}{\Lambda}$ which hides our ignorance of high energy physics, when we are working at low energies. Of course this approach does not deter us the least in building theories: they work remarkably well and reproduce all the possible measurements such as the muon's or the electron's dipole moment.

[2] In some sense Newton's second law (lex Secunda) is makes sense just because there is a single point-like object only characterized by mass and no other property such as a detailed shape.

Properties of QFTs

1. Particle number and species are not conserved. So these theories are intrinsically many (∞) particle theories. This is quite easy to see in a pedagogical example: take a particle and try to localize it in a cube of side L, then its uncertainty in momentum is at least $\Delta p = \frac{\hbar}{2L}$ and this gives an uncertainty in the energy which is at least $\Delta E = \frac{\hbar c}{2L}$. If L is small then this uncertainty in the energy can be of the order of $2mc^2$ which is the minimum amount needed to create a particle/anti-particle pair. So, we can no longer speak of a single particle that we wanted to localize. Recalling that $\lambda = \frac{\hbar}{mc}$ is the reduced Compton size of a massive particle with mass m, one realizes that this is the scale at which the particle that we started with should not be seen as a single particle: it really is a low-energy summary to our "vulgar eyes" of a very complicated soup of fields.[3] In some sense, *localization* comes with this price: we need infinite degrees of freedom.

   Example: The famous β-decay

   $$\text{neutron} \rightarrow \text{proton} + \text{electron} + \text{anti-neutrino}. \tag{6}$$

   Here we should note that for example the electron does not pre-exist, it is created during the process from the available energy. Everything in the above process is a field (be it composite or fundamental), and the particles are, at best, "bundles of energy and momentum of these fields" as Weinberg puts it. But we are so accustomed to the particles that we will not do away with them altogether.

2. A *renormalizable* QFT is supposed to work at arbitrarily large (even at infinite) energies. But even at low energies QFT can be used to understand condensed matter systems and phenomena such as superconductivity.

3. QFT is a universal language. Its results worked remarkably in mathematics, Donaldson, Seiberg-Witten theories are just two examples. The basic idea is to probe the geometry and topology of a manifold by writing a quantum field theory in that manifold. Of course this requires a lot extra structure other than the manifold. For example, it needs a fibre bundle etc. but, the crux of the idea is that consistency of quantum fields on a manifold can be used to detect the topology of a manifold. [NOTE: remind the Atiyah-Singer Index]

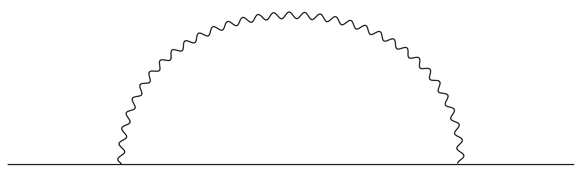
4. Note also that QFT is dimension-dependent. In higher and lower dimensions, interactions among particles change or cease to exist. In this course, we will mostly work in the $3+1$ and $2+1$ dimensions. Of course, sometimes we will cheat: to make sense of some of the expressions in the calculations, especially the loop calculations that involve integrals over arbitrary momenta or spacetime position, we will act as if the theory is defined on n dimensions where n can be any real number, not necessarily an integer and of course in the end we shall take the correct $n \rightarrow 4$ limit. This procedure is called dimensional regularization.

We need to explain the idea of renormalizability and renormalization a little bit here. It is at the heart of quantum field theory, only after, I think, Dyson's work in 1949, we understood this property of QFTs in a rigorous way. The basic idea is that in QFT, the parameters, coupling constants defining the theory should either be dimensionless or should have positive mass dimension (in natural units of $\hbar = 1 = c$).

[3] Recall that at a much larger scale, the de Broglie scale, the wave nature of the particle enters into the picture which is at $\lambda_{\text{de Broglie}} = \frac{\hbar}{mv}$.

Renormalization:

We argued that principles of QM should be incorporated in QFT. One of these is, certainly, the uncertainty principle which can be taken as $\triangle E \triangle t \geq \hbar$. Crudely speaking, energy of a particle is measured precisely only for $\triangle t \to \infty$.[4] That leads to the idea of virtual particles: These particles do not obey the relation $E = \sqrt{p^2c^2 + m^2c^4}$, i.e. they are not on-shell.



Because of these interactions the *calculated* mass of an electron becomes infinite. Of course this makes no sense, the way out is to *regularize* the theory, that is to find a way to make the results of the relevant integrals finite so that one can add and subtract these numbers. And after regularization (which can be done in various ways, such as by putting cut-offs in the integrals, dimensionally continuing the integrals to n dimensions etc.), one can "renormalize" the coupling constants and the masses in the theory so that the original "bare" values of these constants have large parts that take away the divergences coming from the perturbative calculations such as the one shown in the above Feynman diagram. For example, for the mass of the electron at one loop in Quantum Electrodynamics (QED), we have the following expression (which we shall calculate later)

$$m_{\text{exp}} = m_{\text{bare}} \left(1 + \frac{3\alpha}{4\pi} \log \frac{\Lambda^2}{m_{\text{bare}}^2} \right). \tag{7}$$

Because of the log-term, which we like a lot in QFT, and the smallness of the number in front of the log term (which is around 0.0017) we can take the cut-off Λ^2 to be quite a large number, and keep m_{exp} and m_{bare} pretty close to each other. Of course if we let $\Lambda^2 \to \infty$, then so must we let the bare mass to infinity to get a finite number. Then we say that the mass of the electron is renormalized. The great thing about the renormalizable QFTs is that there are only a couple of such parameters to be renormalized usually at the first few orders in perturbation theory (that is first few loop levels in the Feynman diagrams.)

The idea of renormalization makes QFT powerful: we will see that basic merger of QM+SR allow the existence of $m^2 > 0, m^2 = 0$ and spin $= 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...$ On the other hand, for the particles in $D = 3 + 1$, renormalization works only for spin-0, $\frac{1}{2}$, 1 and $\frac{3}{2}$ and no more ! As experimentally found fundamental examples of these particles, we have the following: spin-0 (Higgs particle), spin$-\frac{1}{2}$ (electron, muon, tau,3 type of neutrinos, 6 type of quarks) , spin-1(photon, W-boson, Z-boson, gluons). There are some other particles that often appear in the (theory) literature which lack explicit experimental conformation (the dilaton, inflaton, axion, Rarita-Schwinger field *etc.*)

[4] We should add two remarks: 1) the "true" uncertainty relation $\triangle p_x \triangle x \geq \frac{\hbar}{2}$, can be rigorously derived and interpreted as measurements of position and momentum at the *same time*; 2) the energy-time uncertainty relation is somewhat loose (as time is not an observable) and refers to the fact that a system loses resemblance to itself after a time $\triangle t \geq \frac{\hbar}{\triangle E}$ if its energy is uncertain by $\triangle E$.

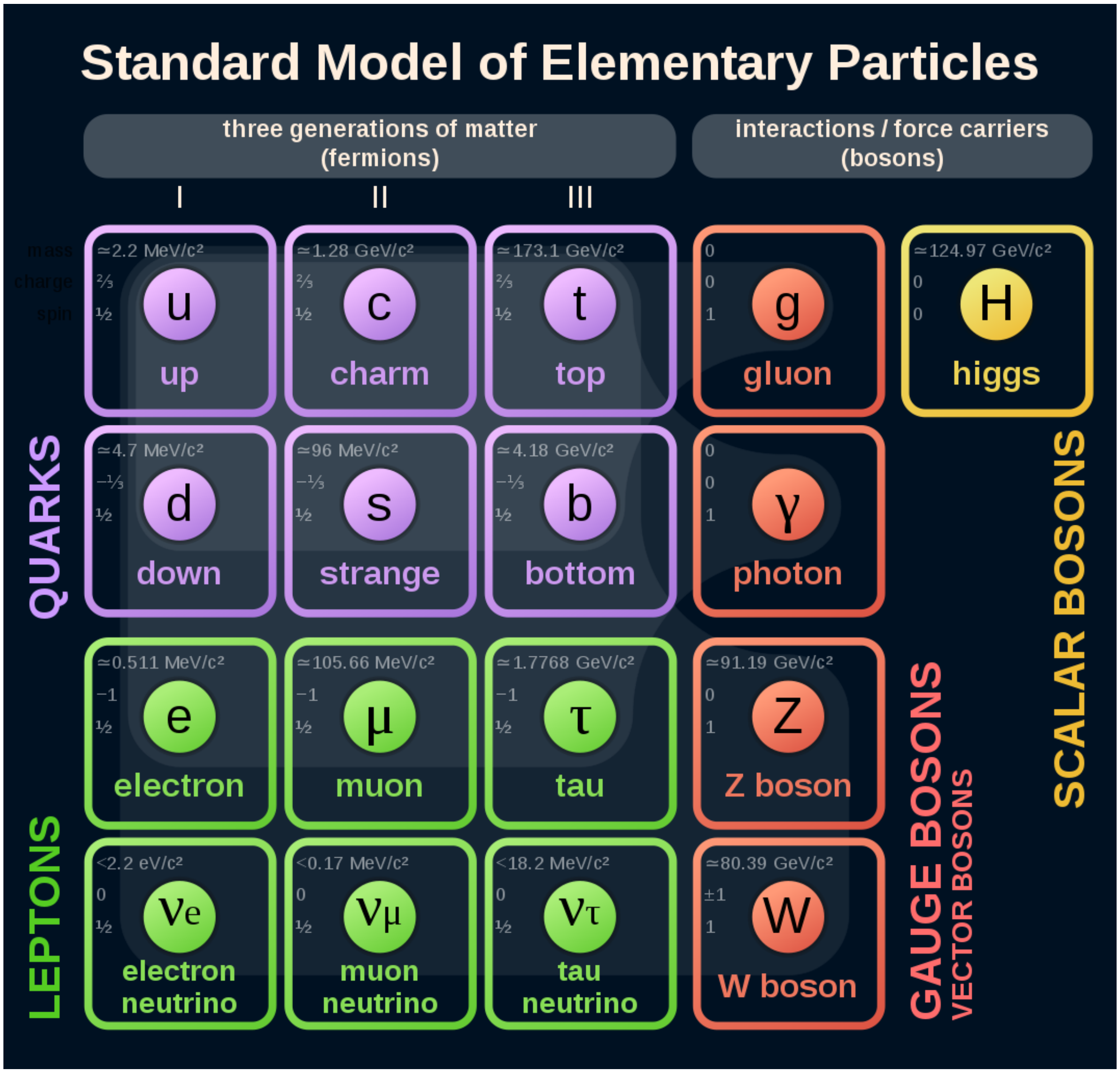


Figure 1. Fundamental particles in the standard model of particle physics. Figure credit: Wikipedia Commons

D. A brief excursion to QED, Weak Interactions and QCD

Quantum Electrodynamics (QED) (1950s)

Many people have contributed to quantum electrodynamics (no idea is born in a vacuum, and any part of QFT is no exception, please forget the idealized lone geniuses that create a theory from nothing), but the physicists whose final touches made the theory as it is today are Richard Feynman (1918-1988), Sin-ItiroTomonaga (1906-1979), Julian Schwinger (1918-1994) and Freeman Dyson (1923-2020) [5] For a fascinating account of the topic, one should read the book "*QED and the men who made it*" by Silvan Schweber. [I usually suggest books to my students. Very few of them read these books. Why do I still keep suggesting ? Because very few of them read.)

Without worrying about the direction of time, QED really starts with the basic *interaction*

[5] Dyson passed away in 2020 at the age of 97. I met him once, he gave a talk on how to stop large meteors that can hit the earth and destroy the civilization.

vertex between a charged particle (say the electron) and the propagation of photons and charged particles.

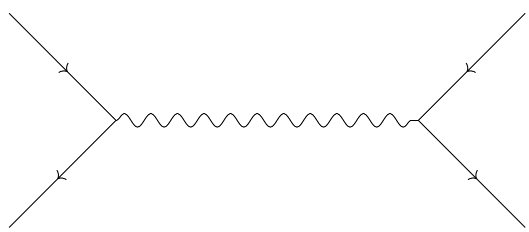
This is really where we start: interaction is the paradigm. Interaction naively means that the "current" attributed to the electron (say $j^\mu(x)$) is multiplied with the vector potential $A^\mu(x)$ of the photon at the spacetime point as $A_\mu(x)j^\mu(x)$.[6] But as we shall not know where in spacetime the interaction takes place exactly, to take into account all possibilities we have an integral over this as

$$S_{int} = \int d^4x A_\mu(x) j^\mu(x). \tag{8}$$

This type of vertex (interaction) is the only allowed one by the symmetries (gauge and Lorentz symmetries) and the renormalization property of the theory. It will be clear soon that every other process regarding the electric charge and the photon interactions can be produced from the above basic vertex. [7] Naively this comes from the following transition matrix element

$$\mathcal{M}_{fi} = \langle final | e^{-\frac{i}{\hbar}\int d^4x \hat{A}_\mu(x)\hat{j}^\mu(x)} | initial \rangle. \tag{9}$$

Expanding the exponential in Taylor series (and there are some technical details about operator ordering which we do not bother right now), one can obtain the possible perturbative terms which are nicely organized as Feynman diagrams. For example the lowest non-trivial diagram would be the following one which describes electron-electron interactions at the lowest order .



This diagram comes from the second order expansion of (9) in the Taylor series which reads

$$\left(-\frac{i}{\hbar}\right)^2 \int d^4x \hat{A}_\mu(x)\hat{j}^\mu(x) \int d^4x' \hat{A}_\mu(x')\hat{j}^\mu(x'). \tag{10}$$

Later on we shall see that this expression describes (in a somewhat unified way) some other possible interactions that involve electrons, positrons and photons, but one of the terms corresponds to the electron-electron interaction (the last picture) via the exchange of a photon. Such a diagram is called a *tree-level* diagram (as it does not include any loops). Since the "electronic-current" j^μ has the electric charge e in it, this interaction (10) is the first order term that arises as an expansion in the fine structure constant

$$\alpha \equiv \frac{e^2}{4\pi\hbar c} \simeq \frac{1}{137}. \tag{11}$$

[6] Soon we shall see that the electron current is given in terms of the electronic field as $j^\mu(x) = e\bar{\psi}\gamma^\mu\psi(x)$.

[7] Note that for any charged fundamental field, that is a non-composite field, such as the μ, τ, quark and W boson fields, interaction with a photon is given by the same vertex with the corresponding current. For composite objects, such as the proton one this type of vertex is just and approximation.

As the sign convention for the charge should not matter, we expect the symmetry $e \to -e$, so even powers of the electric charge should appear in the calculations.

Actually, we will see that the fine structure constant is not really a constant given by Sommerfeld[8] value the but *runs with energy* and its lowest possible value is

$$\alpha(0) = \frac{1}{137.03599911(46)}. \tag{12}$$

It is independently measured with quantum Hall effect as the quantized resistance in quantum Hall effect reads $R = \frac{h}{e^2\nu}$ with $\nu = 1, 2, 3, ...$ for integer quantum Hall effect and $\nu = 1/3, 2/5, 3/7, ...$ for fractional quantum Hall effect.

Since QED has α as a small parameter (at small energies) that allows us to compute processes in perturbation theory, we can express any computation as

$$C(E) = \Sigma_{n=0}^{\infty} a_n \alpha(E) \qquad\qquad \text{with finite } a_n \tag{13}$$

Generically QFT best works if we can apply perturbation theory in some small parameter. For that purpose one needs a dimensionless small number, which is not always available in QFT. For example in strong interactions at low energies there is no such small parameter and one must resort to non-perturbative techniques, such as expanding around the inverse number of colors 1/3 or, say generically $1/N_c$. In gravity we also do not have a dimensionless small coupling constant and a naive approach to perturbative QFT of General Relativity does not work. But there are proposals to do a perturbation theory in the inverse number of dimensions, that is 1/4 or $1/D$ where D is the number of spacetime dimensions.

At this point, one might ask what the equations of QFT are? In principle, as in other theories, one might look for a single (or a set of) partial differential equations that describe the theory. We will get back to this but let us say that QFT is not best described by a (set) of differential equation. Rather, we know a way to calculate how different processes contribute to a certain scattering process and this will be nicely done with the path integral techniques first suggested by Dirac, then developed by Feynman.

E. Example of a QED Calculation: the gyromagnetic ratio of the electron

The "old" definition of the gyromagnetic ratio, denoted as γ is

$$\gamma \equiv \frac{\text{magnetic dipole moment}}{\text{angular momentum}} = \frac{\mu}{L}. \tag{14}$$

For a current carrying loop (or a circulating charge) one has

$$\mu \equiv IA = \frac{e}{t}\pi r^2 = \frac{eL}{2m} \Rightarrow \vec{\mu} = \frac{e}{2m}\vec{L}. \tag{15}$$

So classically the gyromagnetic ratio of a bound electron is $\gamma = \frac{e}{2m_e}$. Now let us define the dimensionless gyromagnetic ratio of the electron g as $\gamma = \frac{e}{2m_e} g_e$. So we have

$$\begin{aligned} g_e &= 1 \qquad\qquad \text{(Classical)} \\ g_e &= 2 \qquad\qquad \text{(Dirac equation which assumes minimal interaction)} \end{aligned} \tag{16}$$

[8] I have to say this: Nobel Prize could not win Arnold Sommerfeld (1868-1951). There is hardly any corner of physics this man did not contribute to in a significant way.

Note that one often finds the statement that the Dirac equation predicts g_e to be 2, while this is true and historically Dirac was motivated by this, there is an important caveat. Dirac equation together with the minimal coupling assumption yield this result, otherwise one can almost find any number one desires. Minimal coupling turns out to be the only option in quantum field theory as dictated by renormalization, but at the level of a single particle equation, such as the Dirac equation, one does not have this requirement. One is expected to reproduce the Lorentz force in the classical limit, but this does not uniquely constrain the theory. Let us expound in this a little bit: minimal coupling of an a charged particle with the photon field requires the substitution

$$p_\mu \to p_\mu - \frac{e}{c} A_\mu, \tag{17}$$

and when this is carried out in the free Dirac equation, one arrives at the coupled electron photon fields (actually, generically one has also positrons around, but we are simplifying the discussion). In the non-relativistic limit, when electron's rest mass energy is much larger than its kinetic energy, the equation one obtains from the Dirac equation is that of the Schrödinger-Pauli equation with $g = 2$. Pauli assumed this value to explain the spectrum of the hydrogen atom under a magnetic field, but Dirac was motivated to explain this value from theory. And in some sense he did explain it. But of course, had Dirac chosen a non-minimal coupling of the fields, allowed by symmetries, say of the following form

$$p_\mu \to p_\mu - \frac{e}{c} A_\mu + \kappa F_{\mu\nu} p^\nu, \tag{18}$$

with some κ, he would have predicted a different value for g. As we shall see the last term in this equation is not allowed by the renormalization criterion. In any case, the ultimate arbiter in this business is the experiment. It is true that some kind of ambiguous notion of beauty is often a useful guide to construct theories, as much amplified by Dirac, but Nature apparently has a different sense of beauty. For example we have the following result regarding the gyromagnetic ratio of the elctron:

$$\begin{aligned}
\text{Experimental :} \qquad & \frac{g_e - 2}{2} = 0.001159652187(4) \\
\text{QED :} \qquad & \frac{g_e - 2}{2} = \frac{\alpha}{2\pi} - (0.328478965...)(\frac{\alpha}{\pi})^2 \\
& \qquad + (1.17611...)(\frac{\alpha}{\pi})^3 - (1.434...)(\frac{\alpha}{\pi})^4 \\
& \qquad = 0.001159652140(5)(4)(27)
\end{aligned} \tag{19}$$

QED calculation comes from 891 Feynman diagrams. Theory and experiment agree in 10 digits. The difference might be due to the hadronic contributions which we had to calculate.

Historical remark In the famous Shelter Island conference in NY in June 1-4 1947 of which the topic was " On the foundations of Quantum Mechanics", Isidor I. Rabi presented the results of his experiments on the hyperfine structure of the hydrogen and deuterium and noted that the experiments deviate from Dirac's theory as follows

$$\mu = 1.0013 \mu_{Dirac} \tag{20}$$

From the theory side, Breit suggested that the difference comes from the order α radiative corrections. Scwhinger immediately calculated this to be

$$\mu = \frac{e\hbar}{2mc}\left(1 + \frac{\alpha}{2\pi}\right) = 1.001162 \mu_{Dirac} \tag{21}$$

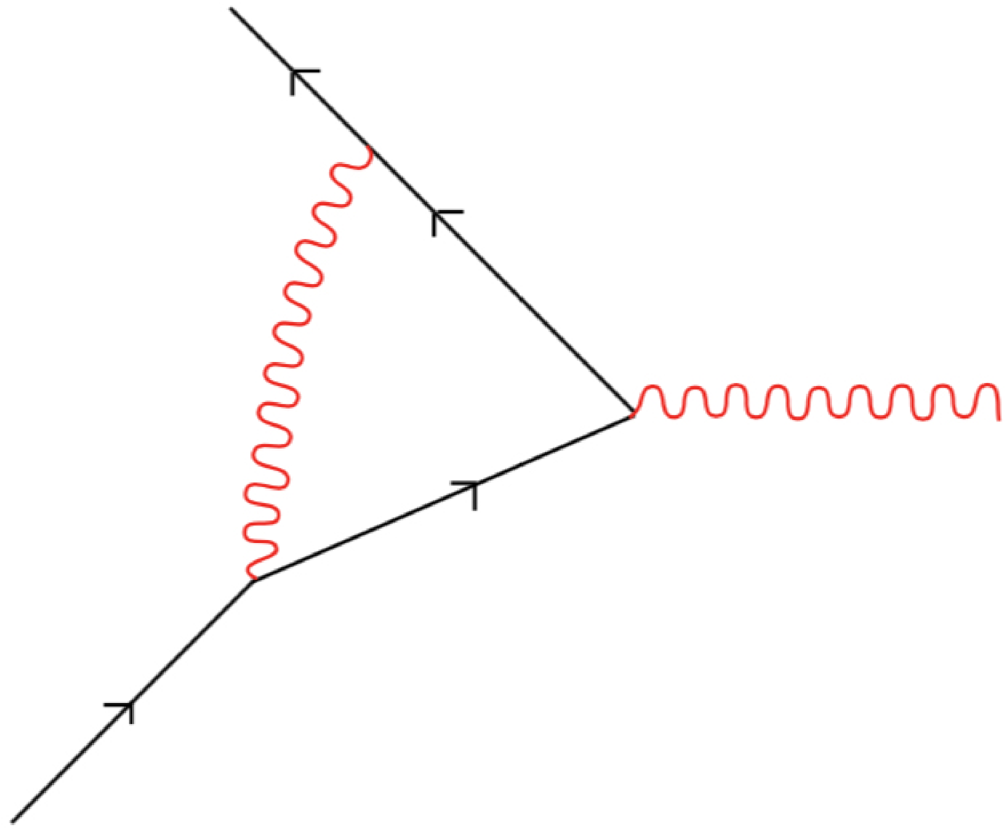


Figure 2. An example of a one-loop contribution to the gyromagnetic ratio g_e of electron. Schwinger showed in 1948 that , after renormalization, this diagram gives $\frac{\alpha}{2\pi}$ which was consistent with the experiments of the day done by Rabi.

F. Lamb Shift issue

Darwin (1928) and Gordon (1928) calculated the exact energy spectrum in the hydrogen atom within the context of Dirac's theory and got

$$E_{nj} = mc^2 \left(1 + \frac{\alpha^2}{\left(n - j - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - \alpha^2} \right)^2} \right)^{-\frac{1}{2}} \tag{22}$$

In the spectroscopic notation $n^{2s+1)}L_j$, this equation gives exact degeneracy of the $2s_{1/2}$ and $2p_{1/2}$ states. But in late 1930s experiments began to indicate a splitting of these two states : $\Delta\nu_{2s_{1/2}-2p_{1/2}} \approx 1000$ MHz.

Major question: What is wrong with Dirac's equation? Why does it give exact degeneracy of these two states ?

Any computation in perturbation theory gave ∞ which of course does not make sense.

Willis Lamb in the Shelter Island conference presented the results of a remarkable experiment and argued that there is a definite splitting of the $2s_{1/2}$ and $2p_{1/2}$ states.

The experiment (as described by Weinberg) goes as follows. Hydrogen atoms come out of an oven there are many atoms in the excited $2s_{1/2}$ and $2p_{1/2}$ states. $2p_{1/2}$ are unstable and decay

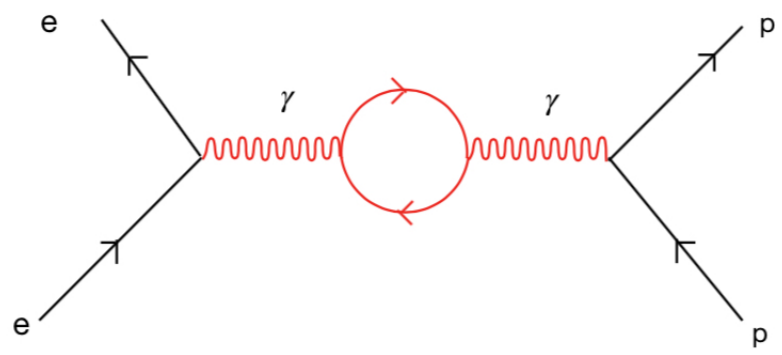


Figure 3. A radiative (loop) contributions to the Lamb shift: vacuum polarization.

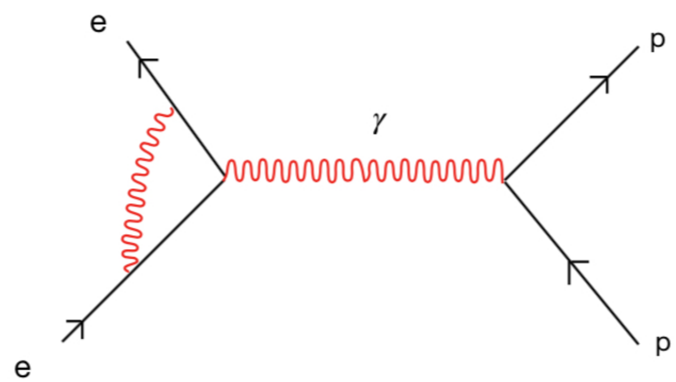


Figure 4. A radiative (loop) contributions to the Lamb shift: anomalous magnetic moment.

rapidly to the ground state $1s$ by one photon emission (that is Lyman α series). Bu the $2s_{1/2}$ state is metastable. These meta stable states pass through magnetic filed $\vec{B}$ whose magnitude can be changed. This magnetic field gives rise to a Zeeman splitting of the $2s_{1/2}$ and $2p_{1/2}$ states. of course this Zeeman splitting is contributes to any existing splitting between these two states. Then the beam of these states pass through a microwave of $\nu = 10$GeV. At a certain magnetic field, the microwave field produces resonant transition from the $2s$ state to the $2p$ state which then makes a transition to the $1s$ state. So one can measure the Zeeman plus any intrinsic splitting of these two states and it turns out the splitting corresponds to a 1000 MHz frequency.

So this experiment immediately attracted attention. The word was now " Just because something is infinite, it does not mean it is zero !" . Weinberg recalls this from his post-doc days in Copenhagen.

What does that mean ? I noted above that the computations yielded infinite results, if you ignored these, you got the correct result ! That is zero. Butthe Lamb shift experiment proved that the radiative corrections do not yield zero, but a small number. So one must be able to compute this from theory.

After the Shelter Island conference, theoreticians started to find a way to explain the observed Lamb shift (Lamb got the Nobel prize in 1955.) Hans Bethe (literally on the train back home)

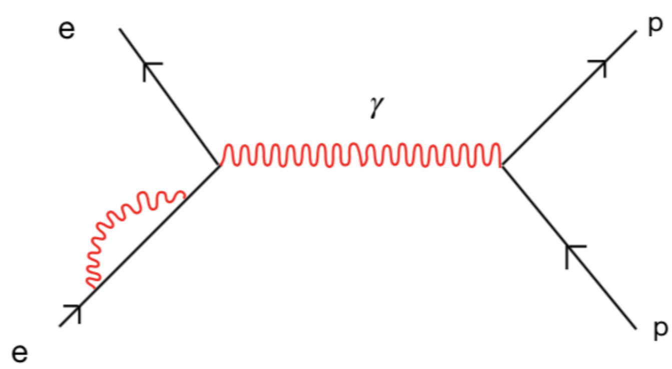


Figure 5. A radiative (loop) contributions to the Lamb shift: electron mass renormalization.

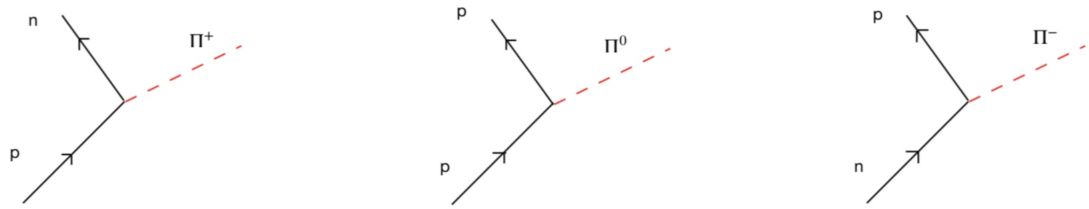


Figure 6. Some strong interaction diagrams in the early version of the theory, that is pre-QCD.

II. QUANTUM CHROMODYNAMICS BASED ON YANG-MILLS THEORY (1954)

Initial idea was to consider the neutron and proton as two different states (isospin states) of a single particle called the *nucleon*. This was suggested by Heisenberg in 1932. Just switch of the electromagnetic interactions, the strong interactions do not change if neutrons and protons are interchanged.

$$|nucleon\rangle = \begin{pmatrix} p \\ n \end{pmatrix},$$

The mass of the proton and neutron are close to each other $M_p \sim M_n \sim 1GeV/c^2$. In fact the difference is about twice the mass of the electron: $M_e \sim 0.5MeV/c^2$; $M_n - M_p \sim 1MeV/c^2$.

Yang and Mills (in 1954) [9] [10] "gauged" the global isospin symmetry suggested by Heisenberg

$$S \begin{pmatrix} p \\ n \end{pmatrix},$$

where S is a 2×2 complex matrix to a local one. That gave rise to 3 pions (π^+, π^-, π^0) which were already known then. Gauge group is $SU(2)$; the group of special (that means determinant $+1$), unitary (that means $S^\dagger = S$) matrices.

---

[9] Chen Ning Yang and Robert Mills, "Conservation of Isotopic Spin and Isotopic Gauge Invariance, Physical Review 96, no. 191 (1954). Mills remarked, "I am only a name attached to a brilliant idea of Yang's . See Sheldon Glashow article in the "Inference" website.

[10] Note also that Abdus Salam's graduate student Ronald Shaw also found the Yang-Mills theory in the same year as Yang and Mills, and it also seems perhaps a little earlier, but neither Salam nor Shaw realized the importance of the work and hence he did not publish it. A short anecdote: Poor Shaw came to METU to teach physics and he was bitten by a stray dog in the campus, which is a canonical treatment of the students and faculty by some dogs around here. See Micheal Atiyah's paper about the late Shaw.

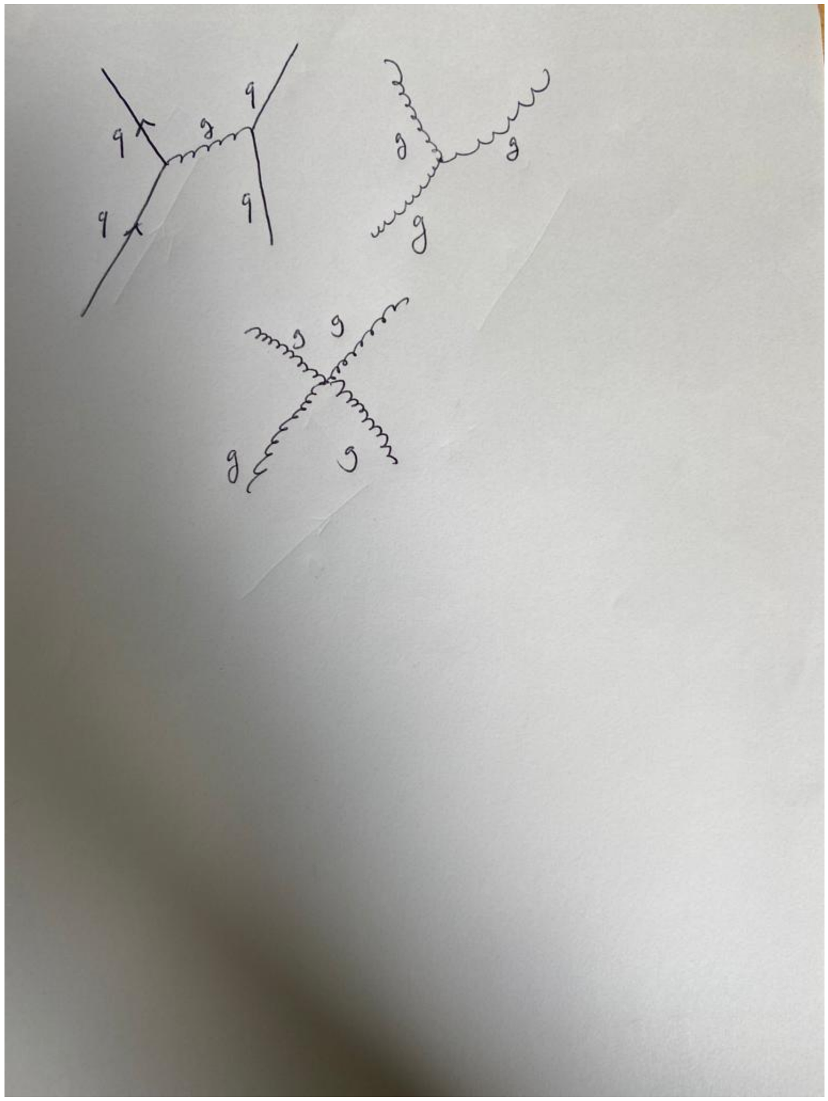


Figure 7. Some strong interaction diagrams in QCD.

Pions have masses around $\frac{1}{10}$ of the nucleon. Cecil F. Powel (1903-1969) disovered the pion (or Yukawa's meson) in 1947 and got the Nobel prize in 1950. Perihan Tolun was his student, she went to Bristol in 1955 and came to METU in 1966.

Pauli did not like YM theory: the theory strictly has massless gauge bosons and in those days, no particle besides the photon was expected to be massless. There were also problems with $\pi - \pi$ scattering amplitudes.

Later Yang-Mills theory became fundamental to QCD and the Electroweak theory.

$$\begin{gathered} 6 \text{ quarks, } 8 \text{ gluons}: \text{ gauge group is } SU(3) \\ \text{This is the color symmetry!} \end{gathered} \tag{23}$$

Quarks are in the fundamental representations, they come in 3 colors. Gluons are in the adjoint representations and carry 2 colors.

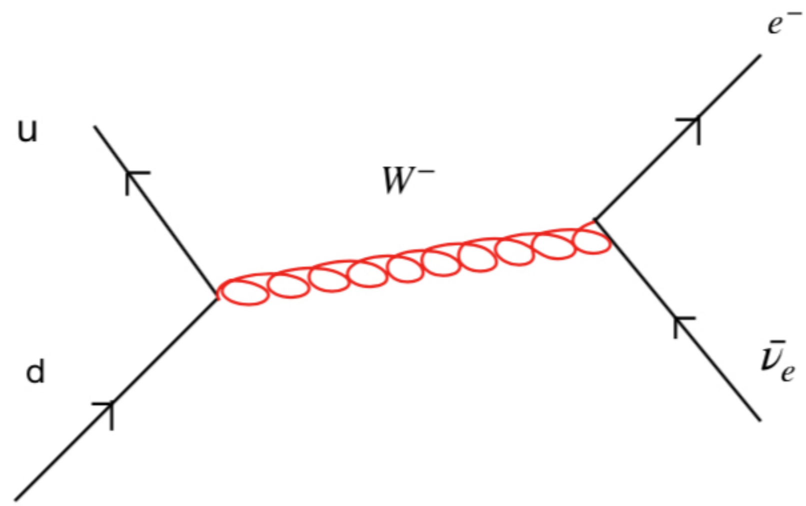


Figure 8. A charged weak interaction diagrams in Electroweak theory.

III. WEAK INTERACTIONS

Recall the β-decay

$$n \to p^+ + e^- + \bar{\nu}_e \tag{24}$$

Early Fermi theory (1933-1934) (*Attempt at a theory of β-rays*, Z. Physik 88 (1934) 161), utilizing Pauli's neutrino hypothesis, of four-fermion interaction gave a very nice approximate model for the β-decay, but [11] did not work in high energies. After a long history, starting from Yukawa in 1935 (*On the Interaction of Elementary Particles*, Proc. Phys. Math. Soc. of Japan, 17 (1935) 48.), Oskar Klein (*Mesons and Nucleons*, Nature 4101 (1948) 897), Pontecorvo (*Nuclear Capture of Mesons and the Meson Decay*, Phys. Rev. 72 (1947) 246) and following with the works of Schwinger and his PhD student Glashow, it was understood that weak and electromagnetic interactions should be treated together, that is they must be unified. The story of weak interactions would take us too far from our course now, so I shall skip it for now and come back to it later. But the main issue here is to understand how weak interactions can be made short-ranged, that is how the intermediate vector bosons that carry the interaction can be made massive. Once this was understood in 1964 with the discovery of the Anderson-Higgs-Kibble-Guralnik-Brout-Englert mechanism, or shortly, the "Higgs Mechanism", it took about 15 more years to write down a correct model for electro-weak theory.

In the fundamental theory, β-decay boils down to a flavor-changing interaction: that is the dype of the quark changes as follows:

$$d(-\frac{1}{3}e) \to u(\frac{2}{3}e) + e^- + \bar{\nu}_e \tag{25}$$

Here the fundamental vertex is depicted in figure 7. The mass of the charged intermediate vector boson is $M_W \sim 80GeV/c^2$. This is a charged vertex.

[11] Fermi's paper was rejected from the journal Nature on the basis that it was too speculative. Fermi published in Nuevo Cimento and Zeitschrift fur Physik

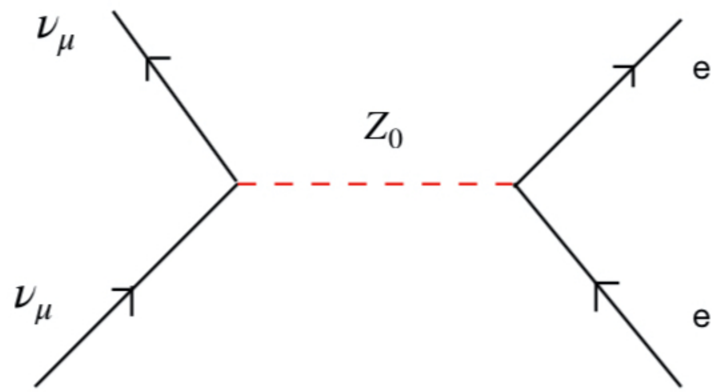


Figure 9. A neutral weak interaction process in Electroweak theory. This scattering was detected in 1973 by the Gargamelle detector at CERN.

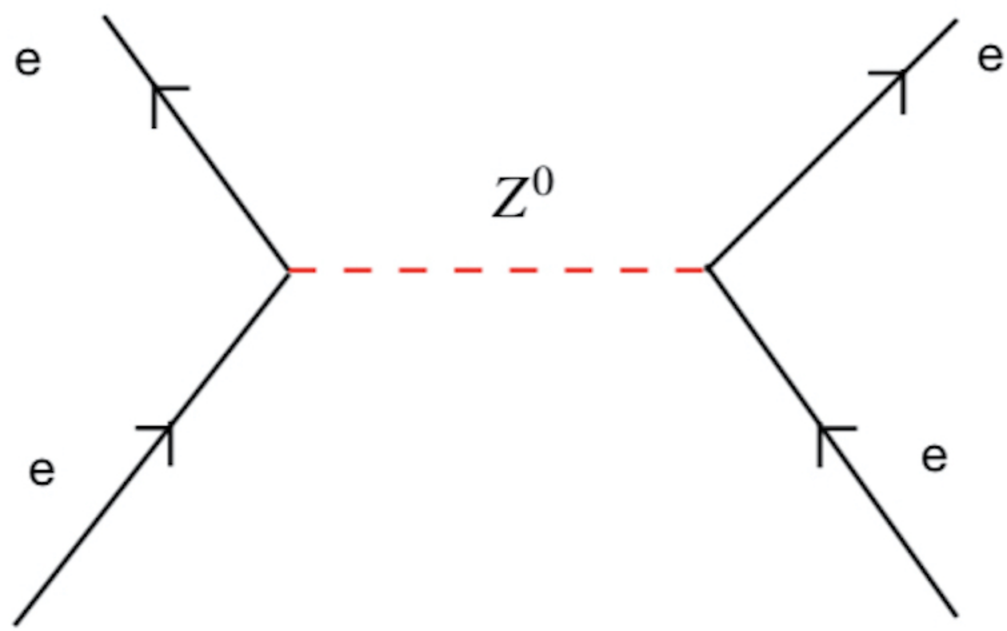


Figure 10. Electron electron scattering with a via a Z boson.

Neutral vertices are shown in figure 7. The mass of the Z-boson, which is very much like a massive photon is $M_Z \sim 94GeV/c^2$. Theory of Weak Interaction is based on an $SU(2) \times U(1)$ gauge theory: Yang-Mills Higgs theory. Note that $U(1)$ factor here is not the gauge symmetry of electromagnetism. What happens in this theory is that the symmetry of the theory, that is, $SU(2) \times U(1)$, is not realized by the vacuum of the theory: the symmetry is spontaneously broken as $SU(2) \times U(1) \to U(1)_{EM}$. To be able to understand this symmetry breaking, we should write down the Lagrangian and compute the lowest order excitations in the theory. We shall do that later.

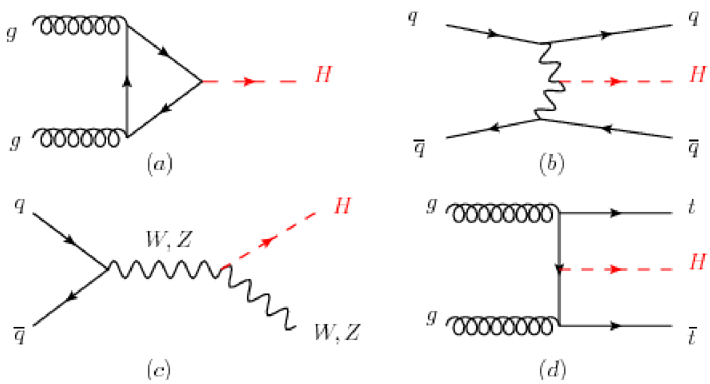


Figure 11. Some Higgs production diagrams. Credit CERN server

1. Higgs Mechanism

Pauli's objection to YM gauge bosons with a mass was well-founded. It was difficult to give mass to the gauge bosons. One needed another particle whose field fills all the universe. Only in 2012 was this particle found at $M_{Higgs} \sim 125GeV/c^2$.

Why was it so hard? Because in QM you only have the probability of creation of a particle. For Higgs this probability is rather low.

So the theory we are looking for should be relativistic, it should be able to describe particle creation and disappearance etc.

Standard Model of particle physics is based on a QFT with the following symmetry group

$$\underbrace{SU(3)_{color}}_{\text{Strong Interaction}} \times \underbrace{SU(2)_{weak} \times U(1)}_{\text{Electroweak}} \tag{26}$$

Here, the Strong Interaction part is confined, we do not exactly understand the details. On the other side, the symmetry is broken in the Electroweak sector.

<u>Running of the Coupling Constants and GUT</u>:

SUSY is need for exact unification.

$$p^+ \to e^+ + \pi^0 \quad \text{or} \quad p^+ \to \bar{\nu}_\mu + \pi^+ \tag{27}$$

Baryon number and Lepton number may change. How about gravity?

$F = -\frac{GM_1M_2}{r^2}$ is replaced by 1 graviton exchange but the theory does not make sense at higher energies. For example when loops come into picture, one gets divergences.

In QFT interaction is the paradigm but in gravity geometry dictates. Kaluza-Klein and string theories in higher dimensions try to unify forces in terms of geometry.

<u>QUESTIONS AND DIMENSIONS</u>

1. What sets the size of room temperature?

2. What sets the size of the universe?

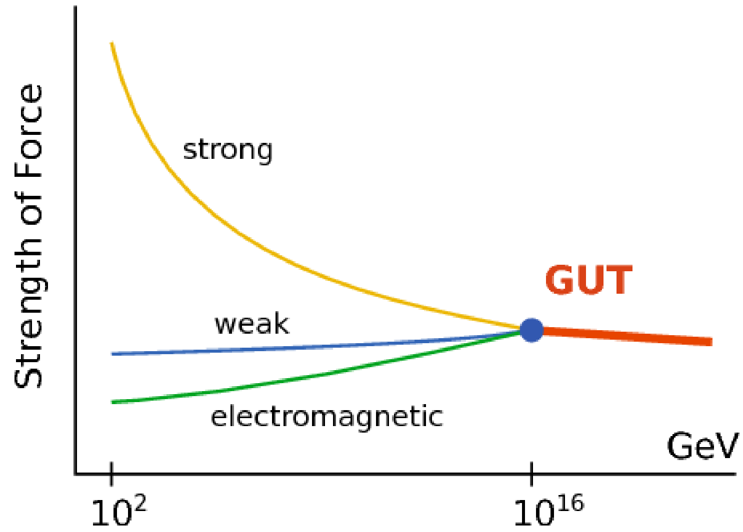


Figure 12. Running of coupling constants. Credit CERN server

3. What sets the size of an atom?

4. What sets the size of a proton?

$$\frac{\text{Size of the proton}}{\text{Size of the atom}} \sim 10^{-5} \tag{28}$$

$\alpha(0) \simeq \frac{1}{137}$. QED does not explain this number. But once this value is given, it can predict how it will be at any other energy (E).

Choose $\hbar = c = 1$: Natural units. Recall that $c \equiv 299792458 m/s$ is defined not measured(?). Now the Planck units:

$$\hbar = 1, \quad c = 1, \quad G_N = 1 \tag{29}$$

So in natural units $E = mass$

$$E_e = m_e = 0.5 MeV \tag{30}$$

Note

$$\hbar c = 1 = 6.58211899(16) \times 10^{-16}\ eV.s \times 300 \times 10^6 m/s \simeq 200\ MeV.fm \tag{31}$$

Then,

$$1fm \cong \frac{1}{200\ MeV} \tag{32}$$

Here, it is also known that $1fm = 10^{-13}\ cm$. (Actually the better value is $1fm \cong \frac{1}{197.3\ MeV}$). Now LHC, we have

$$7\ TeV = 7 \times 10^6\ MeV = 3.5 \times 10^4 \times (200\ MeV) \tag{33}$$

So

$$X_{\text{LHC}} = \frac{10^{-13}\ cm}{3.5 \times 10^4} \sim 3 \times 10^{-18}\ cm. \tag{34}$$

Let us discuss some typical sizes in QED and QCD

Compton radius of the electrons: In the rest frame of the electron we have its mass as the only sole. So

$$r_c = \frac{1}{m_e} \simeq \frac{200\ MeV.fm}{0.5\ MeV} = 400fm = 4 \times 10^{-11}\ cm \tag{35}$$

classically r_c made its first appearance in the X-ray scattering of electrons. $[\lambda_{X-ras} = (0.1 - 100)A^{\circ}]$

$$\lambda^{'} - \lambda = r_c(1 - cos\theta) \tag{36}$$

Note that this relation is obtained from pure kinematics, so the strength of the interaction does not enter the picture.

Size of the Atom: Of course the size of the atom should be inversely related to the strenght of the interaction. It also should not depend much on the mass of the proton. So

$$r_B = \frac{1}{\alpha m_e} = 137 \times r_c \cong 0.5A^{\circ}. \tag{37}$$

We can also find the energy. Using the virial theorem

$$E = -\frac{1}{2}V \text{and} \quad V = -\frac{\alpha}{r_B} \tag{38}$$

then

$$E = \frac{\alpha}{2r_B} = \frac{\alpha^2 m_e}{2} = \frac{1}{(137)^2} \times \frac{0.5}{2} MeV = 13.6\ eV. \tag{39}$$

Electron-Photon scattering

Consider the electron to be Initially at rest ($E_i = w$)

$$e^- + \gamma \to e^- + \gamma \tag{40}$$

The energy of the final photon is fixed by θ, w and m_e. So here the relevant mass parameters are w and m_e. We would like to find the total cross section

1. Consider $w \ll m_e$ case first.

$$\sigma \sim \text{length}^2 \tag{41}$$

Note also that the amplitude is $O(e^2)$ and the cross-section is $O(e^4)$. So

$$\sigma \sim \frac{\alpha^2}{m_e^2} \tag{42}$$

Define $r_0 = \frac{\alpha}{m_e} \simeq 2.8 \times 10^{-13}$ cm which is the Classical electron radius. So $\sigma \sim r_0^2$. Or exact computation gives

$$\sigma_T = \frac{8}{3}\pi r_0^2, \tag{43}$$

which is *Thomson cross-section*. When classical EM field is used, the size of an electron is seen as r_0. Note here we talk about scattering amplitudes, cross-sections etc. instead of forces and so on.

2. Consider now the opposite $w \gg m_e$. We cannot let $m_e \to 0$ in this case. There are divergences. In the high energy limit, the result turns out to be

$$\sigma_T \sim \frac{2\pi\alpha^2}{s} \ln(\frac{s}{m_e^2}) \tag{44}$$

where $s = p^2 = (k + p_e)^2$. So in QFT, divergences spoil naive dimensional analysis.

   (a) $r_B = \frac{1}{\alpha m_e} = 0.510^{-8} cm = 0.5 A^\circ$

   (b) $r_c = \frac{1}{m_e} = 410^{-11} cm = 0.004 A^\circ$

   (c) $r_{classical} = \frac{\alpha}{m_e} = 2.810^{-13} cm$

IV. KLEIN-GORDON AND DIRAC FIELDS

Goal: We need a Lorentz covariant/invariant formulation of QM. Recall that in QM we have the following axioms:[12]

1. We have a wave function $\psi(q_i, s_i; t)$ which is a $\mathbb{C}$ function of all *classical* DOF (positions) and time and any additional DOF such as spin s_i.

   Note: ψ has no direct physical interpretation, but $\infty > |\psi|^2 \geq 0$ is interpreted as the probability density of the system having q_i, s_i at time t.[13]

2. Every physical observable is represented by a linear Hermitian operator, such as

$$p_i \to \frac{\hbar}{i}\frac{\partial}{\partial q^i}, \quad H \to i\hbar\frac{\partial}{\partial t} \tag{45}$$

3. A physical system is an eigenstate of the operator $\hat{\Omega}$ if

$$\hat{\Omega}\phi_n = w_n \phi_n; \quad w_n \in \mathbb{R} \text{ and } \hat{\Omega} \text{ is Hermitian,} \tag{46}$$

   where w_n is the eigenvalue.

4. A wave function can be expanded in a complete set of eigenfunctions $\{\phi_n\}$ of a complete set of mutually commuting observables {operators $\hat{\Omega}_k$}

$$\psi = \sum_n a_n \phi_n \tag{47}$$

   where $|a_n|^2$ denotes the probability of finding the system at the state ϕ_n when $\hat{\Omega}_k$ are measured.

   Orthogonality/orthonormality means

$$\sum_s \int d^k q \; \phi_n^*(q_1, ..., q_k, s; t)\phi_m(q_1, ..., q_k, s; t) = \delta_{mn}. \tag{48}$$

---

[12] See Bjorken and Drell "Relativistic Quantum Mechanics"

[13] This part is a little advanced, but let me still note it. In general the wave function is not a function from the four dimensional manifold to the complex numbers. It atually is a *section* of a complex line bundle with the spacetime as the base space. This bundle theory describtion is required when we have a non-trivial manifold on which the quantum system is described.

5. The results of a measurement of a physical observable is any of its eigenvalues.

$$\psi = \sum_n a_n \phi_n, \quad \hat{\Omega}\phi_n = w_n \phi_n \tag{49}$$

Measurement of $\hat{\Omega}$ will yield w_n with probability $|a_n|^2$

$$\hat{\Omega}\psi = \sum_n a_n \hat{\Omega}\phi_n = \sum_n a_n w_n \phi_n. \tag{50}$$

The average of many measurements of the observable $\hat{\Omega}$ on identically prepared systems is

$$\langle\hat{\Omega}\rangle_\psi = \sum_s \int \psi^* \hat{\Omega}\psi d^n q = \sum_n |a_n|^2 w_n. \tag{51}$$

Note: Note that if we do not have identically prepared systems (the so called a pure ensemble), then we need to introduce the notion of a density matrix to define the expectation values. This works as follows, for a generic mixed ensemble we define

$$\hat{\rho} = \sum_m P_m |\psi_m\rangle\langle\psi_m|, \tag{52}$$

where P_m is the probability of finding the system in the corresponding state. Then the expectation value of an operator reads $\langle\hat{\Omega}\rangle = \mathrm{Tr}(\hat{\rho}\hat{\Omega})$.

6. The time-development of a physical system is expressed by the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi \tag{53}$$

In 1926, this equation appeared in a paper of Schrodinger. $\hat{H}$ is a Hermitian operator with no explicit time dependence for closed system $\partial_t \hat{H} = 0$. Here for non-relativistic systems one can take the Hamiltonian operator from classical physics as

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \hat{V}(\vec{q}) \tag{54}$$

Separation of variables yield

$$\hat{H}\psi = E\psi \Rightarrow \psi_n(\vec{q}, t) = e^{-iE_n t/\hbar}\psi_n(\vec{q}) \tag{55}$$

so that

$$|\psi_n(\vec{q}, t)|^2 = |\psi_n(\vec{q})|^2 \tag{56}$$

Therefore energy eigenstates are called *stationary states*. There is *no* non-trivial time-dependence. Moreover, the conservation of probability is

$$i\hbar\frac{d}{dt}\int dV \psi^* \psi = \int dV (\hat{H}\psi)^* \psi - \psi^* \hat{H}\psi = 0 \tag{57}$$

One obtains time-dependence only when one considers the superposition of energy eigenstates.

$$\psi(\vec{q}, t) = \sum_n e^{-iE_n t/\hbar}\psi_n(\vec{q}) + \int dE c(E, q) e^{-iEt/\hbar} \tag{58}$$

A little Digression: One thing we usually sweep under the rug is the following: The Hilbert space, that is the space of states, is not really just the separable inner product space $\mathcal{H}$ with a countable (albeit infinite) basis. If this were the case, we could not account for the unbounded observables with continuous spectra, such as, say the position operator. The proper mathematical setting of the space that takes into account these operators is called the *Rigged Hilbert space* (that is the equipped Hilbert space) introduced by Gelfand in 1960s. So in this space, we do have a mathematically rigorous understanding of Dirac-delta function type normalizations such as $\langle x|x'\rangle = \delta(x - x')$. Knowing that there is a mathematically well-defined setting for the unbounded operators in quantum mechanics is reassuring.

A. RELATIVITY AND RELATIVISTIC NOTATION

Consider two infinitesimally close events in four dimensional spacetime, which we take to be flat, covered with global time and Cartesian coordinates. The coordinates of these two events are given as (x, y, z, t) and $(x + dx, y + dy, z + dz, t + dt)$ then the "line element" (or the interval) is given as

$$ds^2 = c^2 dt^2 - d\vec{x}^2. \tag{59}$$

This is an "invariant" in the sense that all *inertial observers* record the same number (ds^2), which can be of any sign or zero. Note that we made a choice of the metric. This is the Minkowski spacetime $\mathbb{R}^{3,1}$ or $\mathbb{R}^{1,3}$ and we have chosen the mostly-negative sign convention.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{where} \quad x^\mu = (ct, \vec{x}) \quad \text{and} \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \tag{60}$$

The other, mostly plus metric convention, is just as good. As of today, there is no way to physically distinguish between these two choices. The minus signs can become a headache: one must be very careful. For example, we will always keep the spatial indices up as in

$$\vec{p} \cdot \vec{p} = \sum_i p^i p^i \neq \sum_i p_i p^i. \tag{61}$$

The metric gives us the causal relations between all of the events in the spacetime. Here, $ds^2 > 0$ is time-like (trajectory of massive particles is a time-like trajectory); $ds^2 = 0$ is null (light-like); $ds^2 < 0$ is space-like. In a more geometric way, for time-like trajectories, the tangent vector at each point on the trajectory has a magnitude less than the speed of light, for null curves, the speed is always the speed of light. Note that if two points are space-like separated then they are not in causal contact, space-like curves have tangent vectors with magnitude always larger than the speed of light. It is a minor point, but we still must note that the characteristic of curves do not change over the space-time : we do not allow a time-like curve to become null at a later time.

The Minkowski spacetime, $\mathbb{R}^{3,1}$ has 10 *continuous* symmetries:[14]

$$\left.\begin{array}{l} 3 - \text{rotations} \\ 3 - \text{boosts} \\ 4 - \text{translations} \end{array}\right\} \text{Gives us all inertial observers.} \tag{62}$$

You can also see these symmetries as the symmetries that leave the speed of light intact, but these are not the only symmetries that leave the speed of light intact: conformal symmetries, which include the above symmetries, in general will do that. But the Nature is not conformally invariant, even though conformal symmetry (allowing scalings of and special conformal transformations) appear in some physical systems in phase transition, and conformal field theory, especially the two dimensional one, for which the conformal group is infinite dimensional, is of extreme importance. We shall come back to this later.

Note that we raise and lower with the inverse metric and the metric :

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -\vec{x}), \quad x^0 \equiv ct = x_0. \tag{63}$$

(In a curved spacetime, we should not lower the indices of the coordinates, as they are not the components of a covariant vector, but rather functions. However in flat spacetime, we are luck,

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[14] $\mathbb{R}^{D-1,1}$ has $\frac{D(D+1)}{2}$ symmetries, D translations, $D-1$ boosts and $\frac{(D-1)(D-2)}{2}$ rotations.

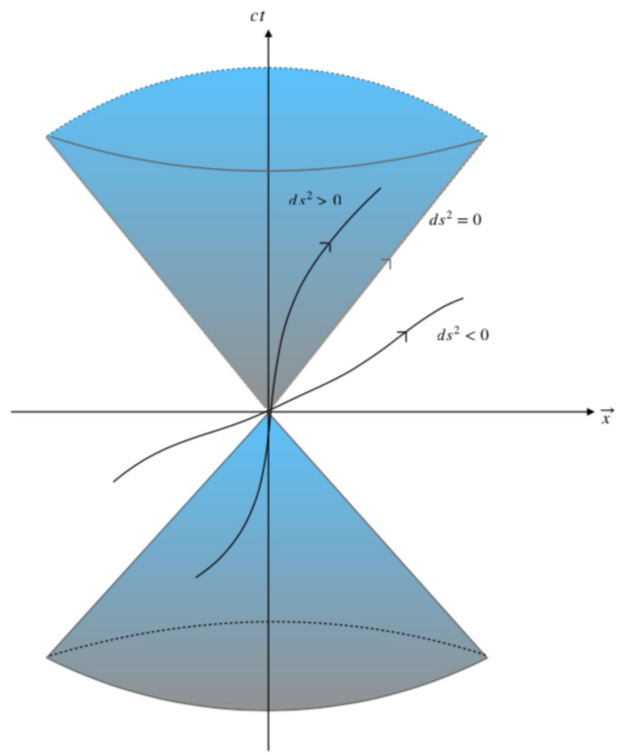

Figure 13. Light-cone centered at at the origin of Minkowski spacetime. Note that unlike the case in non-relativistic physics, "now" is a single spacetime point in Minkowski spacetime; and no two events have the same now. The 4 dimensional spacetime is a collection of events; and like Weyl said, to avoid having a "yawning void", we assume something happens at each point so that we have a continium, instead of e discrete set of events. The metric tensor's main job is to provide causality to the spacetime.

which since the spaetime itself has a vector space structure, hence the coordinates can be treated as vectors.) We will need the first order and second order differential operators, so we have

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \Big(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\Big) = \Big(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\Big) = \Big(\frac{1}{c}\partial_t, \vec{\nabla}\Big). \tag{64}$$

So we have the following identification $(\vec{\nabla})^i \equiv \partial_i$. Note that if you are in doubt check $\partial_\nu x^\mu = \delta^\mu_\nu$. We also have the derivative operator with the up index:

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \eta^{\mu\nu}\partial_\nu = \Big(\frac{1}{c}\partial_t, -\vec{\nabla}\Big). \tag{65}$$

Then the d′Alembertian is defined as

$$\partial^2 \equiv \Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2, \tag{66}$$

which is frame-independent. (Note some books use $\Box^2$, but that is a little bit silly, sorry Jackson.). The energy-momentum (or 4-momentum) of a particle is

$$\begin{aligned} p^\mu &= \Big(\frac{E}{c}, \vec{p}\Big); \qquad p_\mu = \Big(\frac{E}{c}, -\vec{p}\Big) \\ p^2 &= p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \end{aligned} \tag{67}$$

For a single particle, p^μ is sufficient to define the energy properties, but for a system of particles, or for fields, we shall need a rank-2 tensor $T^{\mu\nu}$. One can easily understand this as follows: for many particles moving with different momenta with respect to a center, one should be able to encode the information of these relative momenta, which will give pressure. Hence, the notion of non-relativistic pressure must also appear in special relativity. Similarly, the notion of viscosity (both shear and bulk) will also appear in the energy-momentum tensor. This remark could appear advanced now, but I wrote it to invite you to think about relativistic generalizations of the quantities that are measured in the lab.

1. *KLEIN-GORDON EQUATION*

The equation has "many fathers" (a remark made by Pauli) , 8 of which are Schrödinger in 1926, Oskar Klein 1926, Fock, de Broglie, Walter Gordon, van den Dungen, de Donder, Kudar, 1926, but it is known as the Klein-Gordon equation, the name that I will also use.[15]

In QM, we upgrade the classical observables to Hermitian operators, so

$$p^\mu = i\hbar\partial^\mu; \qquad p_\mu = \eta_{\mu\nu}p^\mu = i\hbar\partial_\mu. \tag{68}$$

For a free relativistic particle

$$E \to H = \sqrt{\vec{p}^2c^2 + m^2c^4} \tag{69}$$

We upgrade the last equation as an operator acting on the wave function to get

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \sqrt{-\hbar^2c^2\vec{\nabla}^2 + m^2c^4}\,\psi(x) \tag{70}$$

where, x denotes space and time together. Later on we shall allow $\psi(x)$ to be real, but for now to emulate the Schrödinger wave function let us assume it to be complex: $\psi(x) \in \mathbb{C}$. Here, square-root of the derivative operator is pathological as it stands, because expanding in power series, one would get infinitely many derivative terms. This would make the theory highly non-local. One could of course go to the momentum space-version. But, instead, let us try to take the square of it

$$\begin{aligned} H^2\psi(x) &= -\hbar^2\frac{\partial^2}{\partial t^2}\psi(x) \\ \left(-\hbar^2\nabla^2 + m^2c^2\right)\psi(x) &= -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\psi(x) \\ \left(\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - \hbar^2\nabla^2 + m^2c^2\right)\psi(x) &= 0 \end{aligned} \tag{71}$$

which, in the covariant notation, is

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(x) = 0 \tag{72}$$

Observe that inverse of the reduced Compton wave-length of the particle appears in the equation. Dirac says in his lectures that Schrödinger first arrived at this relativistic equation before his non-relativistic one, but then dismissed it on the basis that the equation does not give the Bohr

[15] For the brief history of the equation, see H. Kragh, Equation with the many fathers. The Klein-Gordon equation in 1926, American Journal of Physics, **52**, (1984).

energies of the hydrogen atom. But Schrödinger realized that the non-relativistic limit of this equation could make sense. Hence he took the non-relativistic limit.

How does one reproduce Schrödinger equation from the KG equation? Clearly, $c \to \infty$ limit must be taken. But this naively gives a wrong result as the mass term in the equation blows up. So, let us separate the mass part first and define

$$\psi(x,t) \equiv e^{-imc^2 t/\hbar}\tilde{\psi}(x,t) \tag{73}$$

In some sense, we have separated the high-frequency mode and low frequency mode. Then in the $c \to \infty$ limit, we get the free non-relativistic Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\tilde{\psi}(x,t) = i\hbar\frac{\partial}{\partial t}\tilde{\psi}(x,t). \tag{74}$$

How can we get an interacting theory so that we can apply it to the hydrogen atom? We need to recall the interactions of charged particles with electric and magnetic fields. Let Φ be the electro-static potential and $\vec{A}$ be the magnetic potential. Both depend on $(\vec{x},t)$. The Lorentz force is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{75}$$

which is not relativistically covariant. Here we have the usual Maxwell's equations and the background fields are given as

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \qquad \text{and} \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{76}$$

But it can be easily converted into relativistically covariant equation, once we add also the power formula to get

$$\frac{d}{d\tau}p^\mu = qF^\mu{}_\nu u^\nu \quad \text{where} \quad F^\mu{}_\nu = \partial^\mu A_\nu - \partial_\nu A^\mu \tag{77}$$

where τ is the proper time (proper time makes sense as we assume that the charged particle is massive; in fact we do not have in Nature charged particles without a mass) and u^ν is the 4-velocity: $\gamma(c,\vec{v})$. To introduce the interaction of charged particles with background electric and magnetic fields, we are simply guided by the Lorentz force which for non-relativistic motion, requires the following substitution in the free particle Hamiltonian

$$H = \frac{p^2}{2m} \Rightarrow H = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} + e\,\Phi. \tag{78}$$

This is minimal coupling that gives rise to the Lorentz force. In QM and CM, there is no real argument, except simplicity that the minimal coupling scheme works, but in QFT there is in some sense a proof of this: renormalizability of the theory demands this, as we shall see. Guided by classical EM, we do the minimal coupling

$$E \to E - e\Phi, \qquad \vec{p} \to \vec{p} - \frac{e}{c}\vec{A} \tag{79}$$

So the coupled Klein-Gordon equation becomes

$$\Big[(\hat{H} - e\,\Phi)^2 - (\vec{p} - \frac{e}{c}\vec{A})^2c^2 - m^2c^4\Big]\phi(x) = 0 \tag{80}$$

Set again $\hat{H} \to i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \to \frac{\hbar}{i}\vec{\nabla}$. For the hydrogen-like atoms one can take the nucleus at rest and set $\vec{A} = 0$ and $\Phi = -\frac{Ze}{r}$. This is what Schrödinger did, but at the end KG does not give the correct spectrum. Note that this is a rather long problem, I will add the solution later. For stationary states, one assumes $\phi(x,t) = \exp(-iEt/\hbar)\phi(x)$ and solves the second order time-independent differential equation.

2. *Problem of Negative Probabilities in Klein-Gordon Theory*[16]

Let us consider the free KG equation

$$\Big(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\Big)\psi(x) = 0, \qquad (p^2 - m^2)\phi(p) = 0, \tag{81}$$

and assuming $\psi(x)$ to be complex, carry out the following analysis:

$$\begin{aligned} \psi^*\Big(\partial^2 + \frac{m^2c^2}{\hbar^2}\Big)\psi(x) = 0 \\ \psi\Big(\partial^2 + \frac{m^2c^2}{\hbar^2}\Big)\psi^*(x) = 0 \end{aligned} \tag{82}$$

Subtracting them yields

$$\psi^*\partial^2\psi - \psi\partial^2\psi^* = 0 \Rightarrow \partial_\mu\left[\psi^*\partial^\mu\psi - \psi\partial^\mu\psi^*\right] = 0 \tag{83}$$

which is a statement of the the local conservation of a four-current as $\partial_\mu J^\mu = 0$, where we define the current to be

$$J^\mu := \frac{i\hbar}{2m}\Big[\psi^*\partial^\mu\psi - \psi\partial^\mu\psi^*\Big] \tag{84}$$

The coefficient $\frac{i\hbar}{2m}$ was chosen to get the Schrödinger currents as $c \to \infty$. Now $J^\mu \equiv (c\rho, \vec{J})$ then

$$\begin{aligned} \partial_\mu J^\mu = \partial_0 J^0 + \partial_i J^i = 0 \\ \frac{1}{c}\frac{\partial(c\rho)}{\partial t} + \partial_i J^i = 0 \\ \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \end{aligned} \tag{85}$$

Then the total charge is

$$Q \equiv \int_\Sigma d^3x\, J^0(t,\vec{x}) \tag{86}$$

so that

$$\begin{aligned} \frac{dQ}{dt} &= \int_\Sigma d^3x\,\frac{\partial}{\partial t}J^0(t,\vec{x}) \\ &= -c\int_\Sigma d^3x\,\vec{\nabla}\cdot\vec{J} \\ &= -c\int_{S^2} d\vec{S}\cdot\vec{J} \\ &= 0 \end{aligned} \tag{87}$$

We used the Stokes theorem and assumed that $J \to 0$ faster than $\frac{1}{r^2}$. So the following density, when integrated yields a conserved quantity

$$\rho_{KG} = \frac{J^0}{c} = \frac{i\hbar}{2mc}(\psi^*\dot{\psi} - \psi\dot{\psi}^*) \tag{88}$$

[16] In field quantum field theory, this problem will disappear since the KG field will not be the wave function of the system.

But it is clear that this could be negative or positive. Hence we cannot interpret ρ_{KG} as a probability density. Recall that in Schrödinger theory we have

$$\rho_S = \psi^* \psi \geq 0, \tag{89}$$

which is amenable to probability density interpretation. Note that when $c \to \infty$, we can get ρ_S from ρ_{KG} using our earlier presentation.

The ramifications of (88) not being positive definite was huge, both from the historical point of view and from the interpretation of the Klein-Gordon field. We shall study these. As a single particle wave equation, KG does not seem to be compatible with QM as we just saw. In 1934 Pauli and Weisskopf re-interpreted the KG equation as a *field theory* equation. ρ_{KG} is interpreted as the total number of particles minus anti-particles.

$$N = \int d^3x \, \rho_{KG}(x,t) = \# \text{ of particles} - \# \text{ of anti-particles} \tag{90}$$

thus $\frac{dN}{dt} = 0$. This does not mean that the Quantum field theory does not have a wave-function in the quantized version of the Klein-Gordon field. As we argued before, there is a wave-function Ψ which is a function of the $\psi(x)$ field as $\Psi[\psi]$.

3. Plane-wave Solutions of the KG equation

To appreciate the problems of a single particle interpretation of the relativistic wave equations, let us study the KG equation in a little more detail. Let us consider the plane-wave ansatz:

$$\psi(x) = Ae^{-ip\cdot x/\hbar} = Ae^{-i(Et-\vec{p}\cdot\vec{x})/\hbar} \tag{91}$$

and plug it into the free KG theory. Then we have

$$\partial^2\psi = -\frac{p_\mu p^\mu}{\hbar^2}\psi \tag{92}$$

So that the dispersion relation is $-p^2 + m^2c^2 = 0$ with two possible energy solutions $E_\pm = \pm\sqrt{p^2c^2 + m^2c^4}$. In free theory, one might simply ignore the negative energy solutions as they seem to get faster and faster to lower the energy. To keep them to rest, one must apply forces. In an interacting theory, we cannot simply take the positive energy particles, we must take them all as the Hilbert space otherwise would be incomplete. Note that this aspect of the Klein-Gordon equation motivated Dirac to search for a better equation for the electron. It will turnout that this motivation is untenable but it did lead to a remarkable discovery.

Consider the interaction of the KG-field field with the electromagnetic field A^μ is given as

$$\Big[(p^\mu - \frac{e}{c}A^\mu)(p_\mu - \frac{e}{c}A_\mu) - m^2c^2\Big]\psi(x) = 0 \tag{93}$$

where $p_\mu = i\hbar\partial_\mu$. Again let us do our usual trick to get a conserved current directly from the field equations[17]

$$\begin{aligned}\psi^*\Big[(p^\mu - \frac{e}{c}A^\mu)(p_\mu - \frac{e}{c}A_\mu) - m^2c^2\Big]\psi(x) = 0\\ \psi\Big[(p^\mu + \frac{e}{c}A^\mu)(p_\mu + \frac{e}{c}A_\mu) - m^2c^2\Big]\psi^*(x) = 0\end{aligned} \tag{94}$$

Subtract these two equations

$$\psi^* p_\mu p^\mu \psi - \psi p_\mu p^\mu \psi^* - \frac{e}{c}\psi^* A^\mu p_\mu \psi - \frac{e}{c}\psi A^\mu p_\mu \psi^* - \frac{e}{c}\psi^* p^\mu(A_\mu\psi) - \frac{e}{c}\psi p^\mu(A_\mu\psi^*) = 0 \tag{95}$$

So

$$J^\mu \equiv -\frac{ie\hbar}{2m}(\psi^*\partial^\mu\psi - \psi\partial^\mu\psi^*) + \frac{e^2}{mc}A^\mu\psi^*\psi \tag{96}$$

By defining the $D^\mu := i\hbar\partial^\mu - \frac{e}{c}A^\mu$, one gets

$$J^\mu \equiv -\frac{e}{2m}(\psi^* D^\mu\psi - \psi D^\mu\psi^*) \tag{97}$$

now $\partial_\mu J^\mu$. So from $J^\mu = (c\rho, \vec{J})$, we have the current density

$$\rho(\vec{x}, t) = \frac{ie\hbar}{2mc^2}(\psi\partial_t\psi^* - \psi^*\partial_t\psi) + \frac{e^2}{mc^2}A^0\psi^*\psi \tag{98}$$

[17] This should already tell you that the corresponding current will be conserved only upon use of field equations, i.e. it is not conserved by fiat.

Consider now the Coulomb potential energy

$$eA_0 = -\frac{Ze^2}{r}, \qquad \vec{A} = 0 \tag{99}$$

A stationary state will be of the form

$$\psi(\vec{x}, t) = \psi(\vec{r})e^{-iEt/\hbar}; \qquad \partial_t \psi = -\frac{iE}{\hbar}\psi \tag{100}$$

Then (98) reduces to

$$\rho(\vec{x}, t) = \frac{e}{mc^2}(E + eA^0)\psi^*\psi \tag{101}$$

Suppose our charge is e, then, for $E + eA^0 > 0$, ρ and e have the same sign. This is possible for $E > 0$ since $eA^0 < 0$.

But for $E + eA^0 < 0$, ρ and e differ in sign. This happens for strong fields. So in strong fields charge is created.

4. A solved exercise

Question: For at least free KG particles, can we not simply consider the positive energy solutions?

Let us restrict our solutions to the positive energy ones by taking a generic superposition of such solutions as

$$\phi(x) = N \int d^4p e^{-ip\cdot x}\delta^{(4)}(p^2 - m^2)\theta(p_0)\varphi(p), \tag{102}$$

where N is a normalization factor which I shall not worry about.

Check that KG equation is satisfied. (I have set $c = \hbar = 1$)

$$(\partial^2 + m^2)\phi(x) = 0 \tag{103}$$

must be satisfied!

$$\partial^2\phi = -N \int d^4p\ p^2 e^{-ip\cdot x}\delta^{(4)}(p^2 - m^2)\theta(p_0)\varphi(p) \tag{104}$$

Note that this is Lorentz invariant. So clearly KG is satisfied. Now let us try to carry out the p^0 integral. To do so we need to work on the Dirac-delta function

$$\delta(p^2 - m^2) = \delta(p_0^2 - \vec{p}^2 - m^2) \tag{105}$$

Recall that[18]

$$\delta[f(x)] = \sum_a \frac{\delta(x - x_a)}{|f'(x_a)|} \tag{106}$$

So

$$\delta(p^2 - m^2) = \frac{\delta(p^0 - \sqrt{\vec{p}^2 + m^2}\ )}{2\sqrt{\vec{p}^2 + m^2}} + \frac{\delta(p^0 + \sqrt{\vec{p}^2 + m^2}\ )}{2\sqrt{\vec{p}^2 + m^2}} \tag{107}$$

[18] You can easily prove this, just start from $\int df(x)\delta(f(x))$ and assume $f(x)$ has n zeros. Then make a change of integration parameter around each zero.

Then

$$\phi(x) = N \int d^3\vec{p}\; e^{i\vec{p}\cdot\vec{x}} \frac{\varphi(\vec{p},\vec{p}_0)}{2\sqrt{\vec{p}^2+m^2}} \int_{-\infty}^{\infty} dp_0 e^{-ip_0 t}\theta(p_0)\Big[\delta(p^0 - \sqrt{\vec{p}^2+m^2}\ ) + \delta(p^0 + \sqrt{\vec{p}^2+m^2}\ )\Big] \tag{108}$$

By defining $w_{\vec{p}} \equiv +\sqrt{\vec{p}^2+m^2}$, so we get

$$\phi(x) = N \int \frac{d^3\vec{p}}{2w_{\vec{p}}}\, e^{-iw_p t + i\vec{p}\cdot\vec{x}} \varphi(\vec{p}, w_{\vec{p}}) \tag{109}$$

Remark:by the looks of it, $\frac{d^3\vec{p}}{2w_{\vec{p}}}$ does not seem to be Lorentz invariant, but it actually is Lorentz invariant. You can check it by doing a Lorentz transformation.

b) Among positive solutions, let us define an inner product

$$\langle \phi, \psi \rangle_t \equiv i \int_t d^3x \Big[\phi^*(x)\partial_0\psi(x) - \psi(x)\partial_0\phi^*(x)\Big] \tag{110}$$

Then one can show that

$$\langle \phi, \psi \rangle_t = \frac{|N|^2}{4} \int_t \frac{d^3p}{w_p} |\varphi(p)|^2 \geq 0 \tag{111}$$

c) With the above inner product, let us show that the naive position operator $\vec{x} = i\vec{\nabla}_p$, is not Hermitian.

$$\begin{aligned} \langle \phi, \vec{x}\psi \rangle &= i \int_t \frac{d^3p}{w_p}\ \psi^*(\vec{p})\vec{\nabla}_p \phi(\vec{p}) \\ &= \int_t \frac{d^3p}{w_p}\ \Big[\Big(- i\vec{\nabla}_p + i\frac{\vec{p}}{\vec{p}^2+\vec{m}^2}\Big)\psi^*\Big]\phi(\vec{p}) \\ &\neq \langle \vec{x}\phi, \psi \rangle \end{aligned} \tag{112}$$

But the following "Newton-Wigner operator" is Hermitian

$$\vec{x} = i\vec{\nabla}_p - \frac{i\vec{p}}{2(\vec{p}^2+\vec{m}^2)}, \tag{113}$$

as you can easily check. This exercise shows us that when we throw away the negative energy states, it is not clear what we mean by $\vec{x}$ that appears in the wave function.

B. DIRAC EQUATION

Dirac opened a new window in Modern Physics and Mathematics with the paper "The quantum theory of the electron" Proc. Roy. Soc. $A117$ 610 1928. Let me directly quote from the abstract of that paper so that you understand his motivations.

> The new quantum mechanics, when applied to the problem of the structure of the atom with point-charge electrons, does not give results in agreement with experiment. The discrepancies consist of "duplexity" phenomena, the observed number of stationary states for an electron in an atom being twice the number given by the theory. To meet the difficulty, Goudsmit and Uhlenbeck have introduced the idea of an electron with a spin angular momentum of half a quantum and a magnetic moment of one Bohr magneton. This model for the electron has been fitted into the new mechanics by Pauli, and Darwin, working with an equivalent theory, has shown that it gives results in agreement with experiment for hydrogen-like spectra to the first order of accuracy. The question remains as to why Nature should have chosen this particular model for the electron instead of being satisfied with the point-charge. One would like to find some incompleteness in the previous methods of applying quantum mechanics to the point-charge electron such that, when removed, the whole of the duplexity phenomena follow without arbitrary assumptions. In the present paper it is shown that this is the case, the incompleteness of the previous theories lying in their disagreement with relativity, or, alternatively, with the general transformation theory of quantum mechanics. It appears that the simplest Hamiltonian for a point-charge electron satisfying the requirements of both relativity and the general transformation theory leads to an explanation of all duplexity phenomena without further assumption. All the same there is a great deal of truth in the spinning electron model, at least as a first approximation. The most important failure of the model seems to be that the magnitude of the resultant orbital angular momentum of an electron moving in an orbit in a central field of force is not a constant, as the model leads one to expect.

Dirac wanted to obtain a Lorentz invariant generalization such that it gives *positive* probability density and free of negative energies. But more than that he wanted to get the spin of the electron that was introduced by Pauli into the Schrödinger equation by hand.

Before everything else, Dirac realized the following identity (like many of us)

$$\left(p_1^2 + p_2^2 + p_3^2\right)^2 I = \sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3 \tag{114}$$

where σ_i are Pauli matrices and I is the 2×2 unit matrix. Dirac says

> That was a pretty mathematical result. I was quite excited over it. It seemed that it must be of some importance.

of course the next thing is to find the relativistic version of this expression, so one really needs to find what the right-hand side is for

$$\left(p_0^2 - p_1^2 - p_2^2 - p_3^2\right)^2 I = ? \tag{115}$$

Let us quote Dirac on this

> It took me quite a while before I suddenly realized that there was no need to stick to the quantities σ_i with just 2 rows and columns. Why not go to 4 rows and columns?

That insight led to the solution of the problem as follows: take the Schrödinger equation as

$$i\hbar\,\partial_t\psi = \hat{H}\psi \tag{116}$$

and instead of taking the square of the Hamiltonian as in the case of the Klein-Gordon example, keep the time derivative to be the first derivative, then make an ansatz such that Hamiltonian is also linear in the space derivatives as demanded by special relativity:

$$\hat{H} = c\,(\vec{\alpha}\cdot\vec{p} + \beta mc) \tag{117}$$

$\vec{\alpha}$ and β are four different matrices to be determined. The number of matrices is equivalent to the number of spacetime dimensions, but we do not assume that they are 4×4 matrices in four dimensions.[19]

So we need some how $I\sqrt{p^2c^2 + m^2c^4} \sim c\,(\vec{\alpha}\cdot\vec{p} + \beta mc)$. Then as a consistency condition, we must have

$$\begin{aligned} \left(p^2c^2 + m^2c^4\right) I &= c^2\,(\vec{\alpha}\cdot\vec{p} + \beta mc)^2 \\ \left(p^2 + m^2c^2\right) I &= (\vec{\alpha}\cdot\vec{p})^2 + \vec{\alpha}\cdot\vec{p}\beta mc + \beta mc\,\vec{\alpha}\cdot\vec{p} + \beta^2m^2c^2 \end{aligned} \tag{118}$$

from which one deduces $\beta^2 = I$ and

$$(\vec{\alpha}\cdot\vec{p})^2 = \alpha^i p^i \alpha^j p^j = \frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)p^ip^j. \tag{119}$$

Then one obtains

$$\frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i) = \delta^{ij}, \qquad \{\alpha^i, \beta\} = 0. \tag{120}$$

So altogether we have the following anti-commuting objects

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}; \qquad \{\beta, \beta\} = 2; \qquad \{\alpha^i, \beta\} = 0 \tag{121}$$

We call this algebra as the "Dirac algebra". Observe some properties of α^i and β:

1. α^i and β are Hermitian by construction since $\hat{H}$ is Hermitian.
2. $(\alpha^i)^2 = \beta^2 = I$, that means their eigenvalues are ± 1:

$$\alpha^i v^i = \lambda^i v^i \Rightarrow (\alpha^i)^2(v^i)^2 = (\lambda^i)^2(v^i)^2 \qquad \text{So} \qquad (\lambda^i)^2 = 1 \tag{122}$$

3. They are traceless since, $\alpha^i = -\beta\alpha^i\beta \Rightarrow \mathrm{Tr}\,\alpha^i = 0$. Similarly, $\mathrm{Tr}\beta = -\mathrm{Tr}(\alpha^i\beta\alpha^i) = 0$

They are not necessarily diagonal, but a Hermitian Matrix can always be diagonalized.
Proof: Let $A^\dagger = A$ and $\lambda_1, ..., \lambda_n$ are all real:

$$A\vec{U}_i = \lambda_i\vec{U}_i \tag{123}$$

[19] We would like to eventually solve the problem for generic D dimensions and we shall derive first the relations these matrices must satisfy.

where $\vec{U}_i$ are orthonormal eigenvectors. Then

$$A = U\Lambda U^* \qquad \text{where} \qquad U = [\vec{U}_1, ..., \vec{U}_n] \quad \text{and} \quad \Lambda = \text{diag}(\lambda_1, ..., \lambda_n) \tag{124}$$

Here U is unitary.

Since Tr $\equiv$ Sum of eigenvalues, then the number of positive and negative eigenvalues are equal to each other (since eigenvalues are ± 1.)

$$\text{So we have EVEN dimensional matrices} \tag{125}$$

The smallest even number $N = 2$ does not work, since we know that there are only 3 mutually anti-commuting Pauli matrices, we cannot satisfy the full Dirac algebra with 2×2 matrices. The next candidate is $N = 4$. So we have

$$\begin{aligned} i\hbar\,\partial_t\psi = H\psi = c\,(\vec{\alpha}\cdot\vec{p} + \beta mc)\psi \\ \Big(\frac{i\hbar}{c}\partial_t - \frac{\hbar}{i}\vec{\alpha}\cdot\vec{\nabla} - \beta mc\Big)\psi(x) = 0 \\ \Big(i\hbar\frac{\partial}{\partial x^0} - \frac{\hbar}{i}\alpha^k\frac{\partial}{\partial x_k} - \beta mc\Big)\psi(x) = 0 \end{aligned} \tag{126}$$

Multiply with β from the left

$$\Big(\beta i\hbar\frac{\partial}{\partial x^0} - \frac{\hbar}{i}\beta\alpha^k\frac{\partial}{\partial x_k} - mc\Big)\psi(x) = 0 \tag{127}$$

Define

$$\gamma^0 = \beta \quad \text{and} \quad \gamma^k = \beta\alpha_k \tag{128}$$

then

$$\begin{aligned} \Big(i\hbar\gamma^0\frac{\partial}{\partial x^0} - \frac{\hbar}{i}\gamma^k\frac{\partial}{\partial x_k} - mc\Big)\psi(x) = 0 \\ (i\hbar\gamma^0\partial_0 + i\hbar\gamma^k\partial_k - mc)\psi(x) = 0 \end{aligned} \tag{129}$$

This can be compactly written as

$$\boxed{(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0.} \tag{130}$$

This is the free Dirac equation. Furthermore in terms of Feynman-slash notation, Dirac equation turns into

$$(\gamma^\mu p_\mu - mc)\psi(x) = 0 \quad \Rightarrow \quad (\not{p} - mc)\psi(x) = 0 \tag{131}$$

So clearly, in four spacetime dimensions, $\psi(x)$ is a four column-vector object with complex components which we need to understand better [20]

$$\{\gamma^0, \gamma^0\} = 2 \quad \{\gamma^0, \gamma^k\} = 0 \tag{132}$$

[20] When this first appeared it puzzled people. For example von Neumann wrote (1928) "That a quantity with 4 components is not a 4-vector, has never happened in relativity theory". Ehrenfest, who was good at giving names, named this object " a spinor".

also

$$
\begin{aligned}
\{\gamma^k, \gamma^l\} &= \{\beta\alpha_k, \beta\alpha_l\} \\
&= \beta\alpha_k\beta\alpha_l + \beta\alpha_l\beta\alpha_k \\
&= -\alpha_k\alpha_l - \alpha_l\alpha_k \\
&= -\{\alpha_k, \alpha_l\} \\
&= -2\delta_{kl}
\end{aligned}
\tag{133}
$$

In fact, we can raise the right-hand side as

$$\{\gamma^k, \gamma^l\} = -2\delta^{kl} \tag{134}$$

So altogether we have

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.} \tag{135}$$

Recall that $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. This is called *Clifford algebra* in mathematics found by William Kingdon Clifford (1845 – 1879). He was also the first to suggest that gravity should have a geometric theory. He translated Riemann's paper to English and added a one page comment where he even talks about gravitational waves. 11 days after he died Einstein was born. He has some writings on Ethics. I quite like one of his sayings " it is wrong always, everywhere, and for anyone, to believe anything upon insufficient evidence."

We can consider the Dirac or Clifford algebra generic d-dimensions with $2^{[d/2]}$ dimensional matrices where $[d/2]$ is the integer.

A little digression

Looking at (135), one might wonder if one can extend the algebra to a curved spacetime by suggesting that one has x dependent γ matrices that satisfy the following algebra

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x), \tag{136}$$

then you are not alone. That occurred to many people and indeed it is a correct question. To couple spinors to gravity fields, it turns out that one must introduce something like the square-root of the metric tensor, basically a local orthonormal frame that satisfies the following

$$g_{\mu\nu}(x) = \eta_{ab}e^a_\mu(x)e^b_\nu(x), \tag{137}$$

where η_{ab} is the flat metric. Comparing the last equation with (136), one can see that one must have $\gamma_\mu(x) = \gamma_a e^a_\mu(x)$. I used the covariant metric components, but you get the idea. In this formalism, one can couple fermions to gravity, otherwise, using just the metric only, we cannot do that. We will come back to how this is done in the "Fermions in curved backgrounds " chapter.

1. *Examples of Dirac Matrices*

Every set of matrices that satisfy the Clifford algebra will do our job. But from the physical point of view, one set is more suitable than the others. For example if one is interested in non-relativistic limit, one chooses a set that Dirac found in his fist paper on the subject. For ultra-relativistic cases, some other set works better. Each set is called a representation. Let us note some of the well-known ones. We are still working in 4 dimensions, and the γ-matrices below are 4×4 matrices.

Let us write the 2×2 Pauli matrices explicitly as they are used to build γ matrices.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Spinor representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^k = -\gamma_k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix},$$

2. Standard representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k = -\gamma_k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

3. Majorana Representation[21]

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$$

   This is a very interesting representation and appears a lot as it gives a real spinor field instead of a complex one.

4. Observe that we can relate various representations by the following transformation: $\gamma'^\mu = S\gamma^\mu S^{-1} with \quad \det S \neq 0$, γ'^μ also satisfy the anti-commutation relations as can be easily seen:

$$\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu = S\gamma^\mu S^{-1}S\gamma^\nu S^{-1} + S\gamma^\nu S^{-1}S\gamma^\mu S^{-1} = 2\eta^{\mu\nu} \tag{138}$$

5. As we noted before Dirac wave function has 4-components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \psi_i \in \mathbb{C}.$$

   Note that if we were working in 3-dimensions, we would have a two-component wave function.[22]

6. $(\gamma^0)^\dagger = \gamma^0$, since $\gamma^0 = \beta$ and β was Hermitian. Also, $(\gamma^k)^\dagger = -\gamma^k$, since $\gamma^k = \beta\alpha_k$; β, α_k are Hermitian but they anti-commute.

[21] Ettore Majorana was born in 1906 and disappeared into thin air in 1938. It is not clear what exactly happened to him. He took a boat and never landed.

[22] Note that Dirac, in 1928, considers the duplexity of the spectrum as a main problem. He says that he has an explanation which is not *ad hoc*. The Dirac theory for the electron has many successes: It is Lorentz covariant (as we shall see); it contains the electron spin with gyromagnetic ratio $g_e = 2$ together with the assumption of minimal coupling; it predicts the anti-particle positron which was experimentally observed; it gives the correct fine structure of the hydrogen atom energy levels.

7. For a massless particle $m = 0$ and $E = |\vec{p}|c$. So the β term in the Dirac Hamiltonian disappears and we are left with a simpler equation:

$$i\hbar\,\partial_t\psi = c\,\vec{\alpha}\cdot\vec{p}\,\psi \tag{139}$$

Since

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \qquad \alpha_i = \begin{cases} +\sigma_i \\ -\sigma_i \end{cases}$$

one gets

$$i\hbar\,\frac{\partial}{\partial x^0}\psi = \pm i\hbar\,\vec{\sigma}\cdot\vec{\nabla}\psi \tag{140}$$

This is the Weyl equation (written by Weyl in 1929 one year after Dirac's paper) used for massless neutrinos and excitations for graphene. In graphene $c = 10^6\,m/s$! (2010 Nobel Prize). Note the following weird fact: one atom-thick graphene absorbs a significant amount of incident white light (around $2.3\% = \pi\alpha$, see Geim et. al relevant publication in Nature. fine structure appears here.)

8. Dirac Conjugate $\bar{\psi}$, set $\hbar = c = 1$: Note that

$$\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \tag{141}$$

Taking the Hermitian conjugation (namely, transposition and complex conjugation) of the Dirac equation[23]

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \tag{142}$$

one has

$$\psi^\dagger\Big(-i(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu - m\Big) = 0 \tag{143}$$

which can be recast as

$$\psi^\dagger\gamma^0\gamma^0\Big(i(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu + m\Big) = 0 \tag{144}$$

Let us now multiply by γ^0 from the right

$$i\psi^\dagger\gamma^0\gamma^0(\gamma^\mu)^\dagger\gamma^0\overleftarrow{\partial}_\mu + m\,\psi^\dagger\gamma^0 = 0 \tag{145}$$

Before proceeding let us first prove the following relation $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$:

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \begin{cases} \gamma^0 & \text{if} \quad \mu = 0 \\ \gamma^k & \text{if} \quad \mu = k \quad \text{since} \quad (\gamma^k)^\dagger = -\gamma^k \end{cases}$$

So we have

$$i\psi^\dagger\gamma^0\gamma^\mu\overleftarrow{\partial}_\mu + m\,\psi^\dagger\gamma^0 = 0 \tag{146}$$

---

[23] There is a subtle point, we do the Hermitian conjugation in the finite matrix space, not the x space, so we do not do anything to the partial derivative operator ∂_μ which is anti-Hermitan in the field space.

By defining $\bar{\psi} \equiv \psi^\dagger \gamma^0$ which is called "the Dirac conjugate", one finally gets

$$\bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0 \tag{147}$$

Note that that there is actually the identity matrix hitting m which we conventionally do not write

$$I_{4\times 4} = \begin{pmatrix} \vec{I} & 0 \\ 0 & \vec{I} \end{pmatrix}$$

To proceed further and construct the conserved current, let us take the Dirac equation and its conjugate

$$\begin{aligned} \bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0 \ \ /\psi \\ \bar{\psi}/ \ \ (i\gamma^\mu \partial_\mu - m)\psi = 0 \end{aligned} \tag{148}$$

and subtract[24]

$$\partial_\mu(\underbrace{\bar{\psi}\gamma^\mu\psi}_{J^\mu}) = 0 \quad \Rightarrow \quad J^\mu = \bar{\psi}\gamma^\mu\psi \tag{149}$$

Note that this is conserved on-shell, meaning the field equation has to be used.

$$\int d^3x\, \partial_\mu J^\mu = \partial_0 \int d^3x\, J^0 + \underbrace{\int d^3x\, \partial_i J^i}_{\text{convert it to a surface integral}} = 0 \tag{150}$$

Then the number

$$N = \int d^3x\, J^0(\vec{x}, t), \tag{151}$$

is a constant in time, i.e. a conserved quantity when field equations are used. Furthermore, as Dirac demanded, it is positive definite

$$J^0 = \psi^\dagger\gamma^0\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0. \tag{152}$$

So Dirac cured the problem of negative probability density problem inflicted the Kaluza-Klein theory , hence he interpreted the Dirac spinor ψ as a single particle wave function that is exactly analogous to the Schrödinger wave function. But later it will turn out that this interpretation is untenable, it must be interpreted as a quantum field, and one really has the following conserved quantity, not the probability of a single particle not disappearing during collisions:

$$\int d^3x\, eJ^0(t, \vec{x}) = e\,(\#\ \text{ electrons} - \#\ \text{ positrons}) \tag{153}$$

Note that in components we should write the Dirac equation as

$$\sum_b \Big(i(\gamma^\mu)_{ab}\partial_\mu - m\delta_{ab}\Big)\psi_b(x) = 0; \qquad a, b = 1, ..., 4 \tag{154}$$

[24] Note in standard representation $\bar{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$

C. Some "Problems" with the Dirac Equation

1. *Zitterbewegung (trembling motion or quivering motion)*

Schrödinger noted in 1930 that the position of the electron fluctuates with a frequency of $\frac{2mc^2}{\hbar}$ which is 1.6×10^{21} Hz in Dirac's theory. Note that γ-ray radiation corresponds to $> 10^{19}$ Hz. So a free electron should emit a huge amount of highly energetic γ rays immediately. How is that possible ? The electron would radiate all of its energy away in a short amount of time. As we shall see, it arises from the interference of positive and negative energy eigenstates:

In the Heisenberg picture, by assuming that the operator $\hat{Q}$ does not have explicit time dependence, it satisfies the usual Heisenberg equation (which was first written by Dirac actually):

$$\frac{d\hat{Q}(t)}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{Q}]. \tag{155}$$

Consider the position operator $\hat{x}^k(t)$ of the free electron and calculate the coordinate velocity operator

$$\begin{aligned} \hat{v}^k := \frac{d\hat{x}^k(t)}{dt} &= \frac{i}{\hbar}[\hat{H}, \hat{x}^k(t)] \\ &= \frac{i}{\hbar} c\alpha^i \underbrace{[p^i, \hat{x}^k(t)]}_{-i\hbar\delta^{ik}} \\ &= c\,\alpha^k. \end{aligned} \tag{156}$$

Clearly this is rather bizarre, eigenvalues of the velocity operator are only $\pm c$. The massive electron, which should never move with the speed of light in classical relativity theory, seems to move always with the speed of light. Breit (1928) noted that $\frac{d\hat{x}^k(t)}{dt}$ is a "weird object". Dirac must have been very worried when he realized this.[25] Note also that the velocity components do not commute with each other and with the Hamiltonian:

$$[\hat{v}^x, \hat{v}^y] \neq 0 \quad \text{and} \quad [\hat{H}, \hat{v}^i] \neq 0. \tag{157}$$

This says if you are given a free electron with a known energy, you cannot measure its speed, and in unlike the non-relativistic quantum mechanics, velocity components are not compatible observables, and so only one component of the velocity can be known at a given time. Note that, to add salt to the injuery, the momenum operators p_i commute with each other and hence can be measured at the same time. Now let us look at the time evolution of the velocity operators:

$$\frac{d\hat{v}^k}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{v}^k] = \frac{i}{\hbar}[\hat{H}, c\,\hat{\alpha}^k]. \tag{158}$$

It is easy to show that the right-hand side becomes

$$\frac{d\hat{v}^k}{dt} = \frac{2i}{\hbar}(c^2\hat{p}^k - \hat{v}^k\hat{H}), \tag{159}$$

where $\hat{p}^k$ and $\hat{H}$ are time-independent, so we can easily integrate this equation

$$\frac{d}{dt}\left[\hat{v}^k e^{2i\hat{H}t/\hbar}\right] = \frac{2ic\hat{p}^k}{\hbar}\, e^{2i\hat{H}t/\hbar}, \tag{160}$$

[25] In 1928, Heisenberg said "The saddest chapter in modern physics is and remains the Dirac theory".

which becomes

$$\hat{v}^k(t) = \Big(\hat{v}^k(0) - \frac{c^2\hat{p}^k}{\hat{H}}\Big)e^{-2i\hat{H}t/\hbar} + \frac{c^2\hat{p}^k}{\hat{H}}. \tag{161}$$

Note that this is an operator equation, as such , one might wonder what $\hat{p}^k\hat{H}$ means and whether it makes sense or not. Since $E = 0$ is not in the spectrum of the Hamiltonian for $m \neq 0$, inverse of the Hamiltonian makes sense; and since the $\hat{p}^k$ commutes with the Hamiltonian, the expression $\hat{p}^k\hat{H}$ makes sense as one can write it as $\hat{p}^k\frac{1}{\hat{H}}$ or the other way around.

So, again, "velocity operator" is not a constant of motion, even though the momentum p_i are constants of motion for this free particle case. Then from $\frac{d\hat{x}^k}{dt} = \hat{v}^k(t)$, one obtains the operator equation for the position operator

$$\hat{x}^k(t) = \hat{x}^k(0) + c^2\hat{p}^k\hat{H}^{-1}t + \frac{i\hbar}{2}\Big(\hat{v}^k(0) - c^2\hat{p}^k\hat{H}^{-1}\Big)\hat{H}^{-1}e^{-2i\hat{H}t/\hbar}. \tag{162}$$

The first two terms are understandable since their expectation values give the trajectory of the wave packet according to the classical physics. (See Sakurai)

$$x_k^{\text{classical}}(t) = x_k^{\text{classical}}(0) + \frac{p_k c^2}{E}t \tag{163}$$

But the 3^{rd} term in (162) gives rapid oscillations which are called Zitterbewegung. Dirac seems to believe for their existence.

Sakurai in section (3.7) shows that the expectation value of the position and velocity operators, in the Schrödinger picture yield violent oscillations that come from the interference of positive and negative energy modes of generic wave packet solutions. Violent fluctuations of the free electron are of the order of $\hbar/(mc) = 3.9 \times 10^{-11}$ cm. The reason why Dirac takes these fluctuations seriously could be that we can compute the effects of Zitterbewegung for the hydrogen atom. Since the electron fluctuates rapidly, the Columb potential of the proton on the electron also fluctuates as

$$V(\vec{x} + \delta\vec{x}) = V(\vec{x}) + (\delta x)^i\nabla^i V + \frac{1}{2}(\delta x)^i(\delta x)^j\frac{\partial^2 V}{\partial x^i \partial x^j} + \mathcal{O}((\delta x)^3). \tag{164}$$

The third term yields a Darwin-type Dirac delta function potential over a time inverval which contributes to the energy levels of the s-states (here we take $(\delta x)^i = \hbar/(mc)$).

2. Coupling of Dirac's Field to Electromagnetic Fields

The free Dirac equation is

$$(\gamma^\mu p_\mu - mc)\psi(x) = 0. \tag{165}$$

The vector potential is $A^\mu = (\Phi, \vec{A})$, then the Dirac equation *minimally* coupled to the electromagnetic field is defined as

$$\Big(\gamma^\mu(p_\mu - \frac{e}{c}A_\mu) - mc\Big)\psi(x) = 0, \qquad e < 0 \tag{166}$$

Observe that this equation is of the form

$$\Big(\gamma^\mu p_\mu - mc\Big)\psi(x) = \frac{e}{c}\gamma^\mu A_\mu\psi(x), \tag{167}$$

Consider the simplest case with $\vec{A} = 0, \Phi = 0$, and for $m \neq 0$, we cen go the rest frame of the particle such that $\vec{p} = 0$, then the equation reduces to

$$(i\hbar\gamma^0\partial_t - mc^2)\psi(x) = 0, \tag{168}$$

which in the standard representation turns into

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = mc^2\begin{pmatrix}1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}.$$

Therefore, we have four independent solutions

$$\psi_{1,2} = Ne^{-imc^2t/\hbar} \qquad \text{and} \qquad \psi_{3,4} = Ne^{imc^2t/\hbar} \tag{169}$$

So generically the rest-frame solutions are

$$\psi^1 = e^{-imc^2t/\hbar}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \psi^2 = e^{-imc^2t/\hbar}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\psi^3 = e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \psi^4 = e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

ψ^1 and ψ^2 are called positive energy solutions since, conventionally we define positive energy plane waves as

$$\psi \sim e^{-ip_\mu \cdot x^\mu/\hbar} = e^{-iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar} \tag{170}$$

ψ^3 and ψ^4 are called negative energy solutions. This nomenculature is provisional, we shall reinterpret the latter solutions as anti-electrons (positrons). Initially, people called the negative energy solutions as "donkey electrons" because they do the opposite of what you tell them to do: you eed to apply force to them to keep them at rest.

3. Non-relativistic limit

Let us use the standard representation of the γ matrices and consider the non-relativistic limit of the coupled theory: For this purpose let us split the Dirac spinor as

$$\psi^1 = \begin{pmatrix}\phi\\ \chi\end{pmatrix}; \qquad \phi \text{ and } \chi \text{ are two-component spinors} \tag{171}$$

and

$$\alpha^i = \begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix}, \qquad \beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Now the non-interacting Dirac equation is

$$i\hbar\frac{\partial}{\partial t}\psi = (c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi \tag{172}$$

and the interacting Dirac equation is

$$(i\hbar\frac{\partial}{\partial t} - e\Phi)\psi = \Big(c\,\vec{\alpha}\cdot(\vec{p} - \frac{e}{c}\vec{A}) + \beta mc^2\Big)\psi. \tag{173}$$

By defining the kinetic momentum as $\vec{\pi} := \vec{p} - \frac{e}{c}\vec{A}$, one gets

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}0 & c\,\vec{\pi}\cdot\vec{\sigma}\\ c\,\vec{\pi}\cdot\vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} + \begin{pmatrix}mc^2 + e\Phi & 0\\ 0 & -mc^2 + e\Phi\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix}, \tag{174}$$

which becomes two coupled equations in the two component spinors:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = c\,\vec{\sigma}\cdot\vec{\pi}\begin{pmatrix}\chi\\ \phi\end{pmatrix} + mc^2\begin{pmatrix}\phi\\ -\chi\end{pmatrix} + e\Phi\begin{pmatrix}\phi\\ \chi\end{pmatrix}. \tag{175}$$

Let us write try to seperate the high-frequency part before we take the $c \to \infty$ limit

$$\begin{pmatrix}\phi\\ \chi\end{pmatrix} \qquad \equiv \qquad e^{-imc^2t/\hbar} \qquad \underbrace{\begin{pmatrix}\tilde{\phi}\\ \tilde{\chi}\end{pmatrix}}_{\text{slowly varying functions of time}} \tag{176}$$

Then we have

$$mc^2e^{-\frac{imc^2}{\hbar}t}\begin{pmatrix}\tilde{\phi}\\ \tilde{\chi}\end{pmatrix} + e^{-\frac{imc^2}{\hbar}t}i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\tilde{\phi}\\ \tilde{\chi}\end{pmatrix} = c\,\vec{\sigma}\cdot\vec{\pi}\,e^{-\frac{imc^2}{\hbar}t}\begin{pmatrix}\tilde{\chi}\\ \tilde{\phi}\end{pmatrix} + mc^2\,e^{-\frac{imc^2}{\hbar}t}\begin{pmatrix}\tilde{\phi}\\ -\tilde{\chi}\end{pmatrix} + e\Phi\,e^{-\frac{imc^2}{\hbar}t}\begin{pmatrix}\tilde{\phi}\\ \tilde{\chi}\end{pmatrix}$$

So we get

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\tilde{\phi}\\ \tilde{\chi}\end{pmatrix} = c\,\vec{\sigma}\cdot\vec{\pi}\begin{pmatrix}\tilde{\chi}\\ \tilde{\phi}\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \tilde{\chi}\end{pmatrix} + e\Phi\begin{pmatrix}\tilde{\phi}\\ \tilde{\chi}\end{pmatrix} \tag{177}$$

Let us consider "small kinetic energies" $\partial_t\tilde{\phi} = \partial_t\tilde{\chi} = 0$ and "small interaction" $e\Phi$. Then the second equation gives

$$\tilde{\chi} = \frac{\vec{\sigma}\cdot\vec{\pi}}{2mc}\tilde{\phi} \tag{178}$$

So in the standard representation, $\tilde{\chi}$ is small compared to $\tilde{\phi}$ in the $c \to \infty$ limit. Then the first equation gives the expected Schrödinger-Pauli equation which was our goal, as it was Dirac's immediate goal in 1928 :[26]

$$i\hbar\frac{\partial}{\partial t}\tilde{\phi} = \Big(\frac{(\vec{\sigma}\cdot\vec{\pi})^2}{2m} + e\Phi\Big)\tilde{\phi} \tag{179}$$

Exercise: To reduce the last equation into the usual form, let us first prove the following:

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \vec{\sigma}^2 - \frac{e\hbar}{c}\vec{\sigma}\cdot\vec{B} \tag{180}$$

[26] Note that Dirac said he was too scared to try to solve his equation for the hydrogen atom as he was worried he would get the wrong spectrum.

Proof:

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \sigma^i\sigma^j\pi^i\pi^j \tag{181}$$

since $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, we have

$$\begin{aligned}(\vec{\sigma}\cdot\vec{\pi})^2 &= \vec{\pi}^2 + i\epsilon^{ijk}\sigma^k\pi^i\pi^j \\ &= \vec{\pi}^2 + \frac{i}{2}\epsilon^{ijk}\sigma^k[\pi^i,\pi^j].\end{aligned} \tag{182}$$

Note that

$$[\pi^i,\pi^j] = [(p^i - \frac{e}{c}A^i),(p^j - \frac{e}{c}A^j)] = -\frac{e}{c}[p^i,A^j] - \frac{e}{c}[A^i,p^j]. \tag{183}$$

Since $[x^i,p^j] = i\hbar\,\delta^{ij}$ and $[A^i,p^j] = i\hbar\partial^j A^i$, we have

$$[\pi^i,\pi^j] = \frac{i\hbar e}{c}F^{ij}. \tag{184}$$

Then we get

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \vec{\pi}^2 - \frac{e\hbar}{2c}\,\epsilon^{ijk}\sigma^k F^{ij} \tag{185}$$

Define

$$B^k = \frac{1}{2}\epsilon^{ijk}F^{ij} \tag{186}$$

Hence, we get

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \vec{\sigma}^2 - \frac{e\hbar}{c}\vec{\sigma}\cdot\vec{B}. \tag{187}$$

Now, let us substitute this into and $\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$ into (179)

$$i\hbar\frac{\partial}{\partial t}\varphi = \Big(\frac{(\vec{p}-\frac{e}{c}\vec{A})^2}{2m} - \frac{e\hbar}{2mc}\,\vec{\sigma}\cdot\vec{B} + e\Phi\Big)\varphi, \tag{188}$$

where wrote $\varphi = \tilde{\phi}$. This is Schrödinger-Pauli two component spin theory which excited Dirac a great deal as he argued that Nature has chosen the non-point like electron due to special relativity. This misconception unfortunately affected people like Bohr and Pauli as they kept thinking that the spin propert of the elctron arises as quantum property when the elctron is bound to the atom. They thought that a fre electron does not have spin. Observe that the term in the middle of the right-hand side corresponds to the magnetic field and magnetic dipole interaction

$$U = -\vec{\mu}\cdot\vec{B} \quad \Rightarrow \quad \mu = \frac{e\hbar}{2m}\vec{\sigma} = \frac{e}{m}\vec{S}. \tag{189}$$

Compare it with the classical result

$$\vec{\mu} = \frac{e}{2m}\vec{L} \tag{190}$$

So

$$\vec{\mu} = g\frac{e}{2m}\vec{S} \tag{191}$$

So, the Dirac equation gives $g = 2$ for the gyromagnetic ratio! But, as we noted, it really comes from the assumption of minimal coupling, and in Nature, as we noted before, it is slightly above 2 and so dirac's equation cannot be the ultimate theory of the electron.

Example: Weak Uniform Magnetic Field

Consider a weak field uniform magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and choose the transverse gauge and no static field:

$$\vec{\nabla} \cdot \vec{A} = 0, \qquad \vec{A} = \frac{1}{2}\vec{B} \times \vec{r}, \qquad \Phi = 0. \tag{192}$$

Then we have

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\varphi &= \Big[\frac{\vec{p}^2}{2m} - \frac{e}{2mc}\vec{p}\cdot\vec{A} - \frac{e}{2mc}\vec{A}\cdot\vec{p} - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + \frac{e^2}{2mc^2}\vec{A}^2\Big]\varphi \\ &= \Big[\frac{\vec{p}^2}{2m} - \frac{e}{2mc}\vec{p}\cdot(\frac{1}{2}\vec{B}\times\vec{r}) - \frac{e}{2mc}(\frac{1}{2}\vec{B}\times\vec{r})\cdot\vec{p} - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + \frac{e^2}{2mc^2}\vec{A}^2\Big]\varphi \end{aligned} \tag{193}$$

Note:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{B}\cdot(\vec{A}\times\vec{C}) \quad \text{So} \quad \vec{p}\cdot(\vec{B}\times\vec{r}) = -\vec{B}\cdot(\vec{p}\times\vec{r}) = -\vec{B}\cdot\vec{L} \tag{194}$$

Hence we get

$$i\hbar\frac{\partial}{\partial t}\varphi = \Big(\frac{\vec{p}^2}{2m} - \frac{e}{2mc}(\vec{L}+2\vec{S})\cdot\vec{B} + \frac{e^2}{2mc^2}\,\vec{A}^2\Big)\varphi \tag{195}$$

Here the orbital and spin angular momenta are defined as

$$\vec{L} = \vec{r}\times\vec{p} \qquad \text{and} \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma} \tag{196}$$

Note that the $\vec{L}\cdot\vec{B}$ term gives rise to paramagnetism. The $\vec{A}^2$ term gives rise to diamagnetism.[27] Typically

$$\frac{\text{dia}}{\text{para}} \sim 10^{-10}B \quad (in \quad Gauss) \tag{197}$$

This is actually not hard to see: Take $\vec{B} = B\hat{z}$. Then

$$\frac{\Big\langle \frac{e^2}{2mc^2}\,\vec{A}^2 \Big\rangle}{\Big\langle \frac{e}{2mc}(\vec{L}+2\vec{S})\cdot\vec{B} \Big\rangle} = \frac{\frac{e^2B_0^2}{8mc^2}\,\langle x^2+y^2\rangle}{\frac{eB_0}{2mc}\,\langle L_z+2S_z\rangle} \tag{198}$$

Here $\langle x^2+y^2\rangle \sim a^2$ where a is the Bohr radius; $\langle L_z+2S_z\rangle \sim \hbar$. Then

$$\frac{\text{dia}}{\text{para}} = \frac{e}{4c}\frac{a^2B}{\hbar} = \frac{e^2}{4\hbar c}\frac{B}{e/a^2} \simeq 1.1\times 10^{-10}\ B\ (\text{in } gauss) \tag{199}$$

Experimentally 10^5 *Gauss* fields are achievable. Then if $\langle L_z+2S_z\rangle \neq 0$ then diamagnetism is negligible. But for metal electrons

$$\chi_{\text{Landau}} = -\frac{1}{3}\chi_{\text{Pauli}} \quad \text{where} \quad \chi : \text{susceptibility} \tag{200}$$

[27] Ferromagnetism is a collective effect of many spins and does not appear here.

Also, on the surface of neutron stars $B \sim 10^{12}$ *Gauss*. Let us now compare the paramagnetic term with the Coulomb energy

$$\frac{\frac{e}{2mc}\langle L_z + 2S_z\rangle B}{e^2/a} = 2 \times 10^{-10}\ B\ (\text{in } gauss) \tag{201}$$

Note also that $-\frac{e}{2mc}\vec{L}\cdot\vec{B}$ term leads to the normal Zeeman effect whereas $-\frac{e}{mc}\vec{S}\cdot\vec{B}$ leads to anomalous Zeeman effect.

Every material shows paramagnetism and diamagnetism. These are very small effects, even smaller than gravity effects. But one can actually levitate objects (tiny frogs for example) using the effects of diamagnetism. Superconductors also show diamagnetism. Magnetism is a huge and beautiful subject, we cannot do justice to it here.

The coupled Dirac equation was

$$\Big(i\gamma^\mu(\partial_\mu + ieA_\mu) - m\Big)\psi = 0 \tag{202}$$

where we set $\hbar = c = 1$. Define $D_\mu := \partial_\mu + ieA_\mu$ which is called the (gauge) covariant derivative. Let us see why it has this cool name. Observe the following as we make a sapcetime-dependent phase transformation of the Weyl-type $\psi \to e^{i\Lambda(x)}\psi$ Then the usual partial derivative does not transform like the field itself:

$$\partial_\mu\psi^{'} = e^{i\Lambda}\partial_\mu\psi + i\,\psi\, e^{i\Lambda}\partial_\mu\Lambda. \tag{203}$$

The covariant derivative transforms as

$$\begin{aligned}(D_\mu\psi)^{'} &= (\partial_\mu + ieA^{'}_\mu)e^{i\Lambda}\psi \\ &= e^{i\Lambda}\partial_\mu\psi + i\,\psi\, e^{i\Lambda}\partial_\mu\Lambda + ieA^{'}_\mu e^{i\Lambda}\psi\end{aligned} \tag{204}$$

So if we allow the gauge field to transform as $A^{'}_\mu = A_\mu - \frac{1}{e}\partial_\mu\Lambda$, which we can do, since the difference of the new and old gauge fields does not appear in the field strength, then the covaraint derivative transforms just like the field itself:

$$(D_\mu\psi)^{'} = e^{i\Lambda}D_\mu\psi. \tag{205}$$

Let us check the commutator of two covariant derivatives in generically different directions

$$[D_\mu, D_\nu]\psi = [\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu]\psi = ie(\partial_\mu A_\nu - ie\partial_\nu A_\mu)\psi. \tag{206}$$

Hence we get the commutator of the fields as

$$[D_\mu, D_\nu] = ieF_{\mu\nu}. \tag{207}$$

Note that if you are familiar with Riemannian geometry, a similar formula is valid there for the covariant derivative ∇_μ which reads as

$$[\nabla_\mu, \nabla_\nu]V^\sigma = R_{\mu\nu}{}^{\sigma}{}_{\rho}V^\rho, \tag{208}$$

where $R_{\mu\nu}{}^{\sigma}{}_{\rho}$ is the Riemann curvature tensor. So in some sense $F_{\mu\nu}$ is an analog of the Riemann tensor. It can be interpreted as a "curvature" of some geometry. But it is now not obvious what is the corresponding geometry. It turns out, after many years of missing the similarities between gauge theory and geometry, people eventually realized that the proper geometric setting for gauge theories is the fiber bundle setting, which we shall describe later. $F_{\mu\nu}$ becomes curvature of the connection (gauge field) in the principal $U(1)$ bundle over the spacetime manifold.

1. Digression 1: (from Sakurai's advanced QM)

   In non-relativistic QM, to account for the electron spin and magnetic field interaction, W. Pauli in an adhoc way introduced the interaction Hamiltonian

$$H^{(\text{spin})} = -\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \tag{209}$$

which clearly does not arise from minimal interaction

$$\vec{p} \to \vec{p} - \frac{e}{c}\vec{A} \tag{210}$$

But Feynman noted the following

$$H = \frac{\vec{p}^2}{2m} = \frac{(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{2m} \tag{211}$$

Now do minimal coupling

$$H = \frac{\vec{\sigma}}{2m}\cdot(\vec{p} - \frac{e}{c}\vec{A})\,\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}) \tag{212}$$

Since

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}) \tag{213}$$

which holds even when $\vec{A}$ and $\vec{B}$ are operators. Then

$$H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \tag{214}$$

2. <u>Digression 2</u>: The Dirac equation is

$$\Big(\gamma^\mu(p_\mu - \frac{e}{c}A_\mu) - mc\Big)\psi = 0 \tag{215}$$

Now

$$\frac{dx^k}{dt} = c\alpha^k \qquad \text{and define} \qquad \vec{\pi} = \vec{p} - \frac{e}{c}\vec{A} \tag{216}$$

From

$$\frac{d\vec{\pi}}{dt} = \frac{i}{\hbar}[H, \vec{\pi}], \tag{217}$$

One obtains

$$\frac{d\vec{\pi}}{dt} = e(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}) \tag{218}$$

where we assumed $\partial_t\vec{\pi} = 0$ and defined

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi \qquad \text{and} \qquad \vec{B} = \vec{\nabla}\times\vec{A} \tag{219}$$

3. <u>Digression 3</u>: Foldy-Wouthysen (1949)

Large and small components mix because of the $\vec{\alpha}$ matrices. Is it possible to transform the Hamiltonian into a form which does not mix these parts?

Consider the free particle case

$$H = \vec{\alpha}\cdot\vec{p} + \beta m \tag{220}$$

Define $U = e^{iS}$ with

$$S = -\frac{i}{2}\beta\frac{\vec{\alpha}\cdot\vec{p}}{p}\tan^{-1}(\frac{p}{m}) \qquad \text{where} \qquad p = |\vec{p}| \tag{221}$$

and

$$e^{\pm iS} = \frac{E + m \pm \beta(\vec{\alpha}\cdot\vec{p})}{\sqrt{2E(E+m)}} \tag{222}$$

Now

$$\begin{aligned} H\psi &= E\psi \\ e^{iS}He^{-iS}e^{iS}\psi &= Ee^{iS}\psi \end{aligned} \tag{223}$$

then

$$H^{'} = e^{iS}He^{-iS} = \beta E_p \qquad \text{where} \qquad E_p^2 = p^2 + m^2 \tag{224}$$

so this is a method of actually finding the non-relativistic limit. e^{-iS} would give the ultra-relativistic limit. This is not possible to do when the interactions are turned in

KLEIN PARADOX: (1929)

One can solve the 1+1 dimensional Dirac equation for a step potential. This is not a difficult exercise. One finds that $|J_R| > |J_I| >$: the reflected current is bigger than the incident current. You get more than what you send. The only explanation seems to be that positron-electron pairs are created due to the strong potential. The electron is repulsed by the potential while the positron is attracted to it.

Actually Klein paradox is more fascinating if one considers the atomic nucleus as it was understood in those days: the neutron was not known, so there were electrons and protons inside the nucleus. With a high nuclear potential, the Dirac equation yields creation of more electrons and this was a puzzle.

Klein-Nishina Formula: (1929) (I will get back to this part)

$$\frac{d\sigma_c}{d\Omega} = \frac{r_e^2}{2}\frac{1}{(1+\mathcal{E}(1-\cos\theta))^2}\left[1+\cos\theta+\frac{\mathcal{E}^2(1-\cos\theta)^2}{1+\mathcal{E}(1-\cos\theta)}\right]$$

V. LORENTZ AND POINCARE TRANSFORMATIONS

Recall that the Minkowski space line-element is $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. The Poincare transformations that keep this line element intact and the metric components intact are given as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{225}$$

where $\Lambda^\mu{}_\nu$ is a constant 4×4 real matrix and a^μ is a constant vector. Here is an important point x'^μ and x^μ refer to the same spacetime event say p which is a point in the spacetime. For two nearby points we have

$$dx'^\mu = \Lambda^\mu{}_\nu dx^\nu. \tag{226}$$

The invariance of ds^2 leads to

$$\eta_{\mu\nu}dx'^\mu dx'^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta dx^\alpha dx^\beta = \eta_{\alpha\beta}dx^\alpha dx^\beta. \tag{227}$$

so we have a constraint on the possible $\Lambda^\mu{}_\nu$ matrices :

$$\Lambda^\mu{}_\alpha\eta_{\mu\nu}\Lambda^\nu{}_\beta = \eta_{\alpha\beta}, \tag{228}$$

or in compact matrix form

$$\Lambda^T\eta\,\Lambda = \eta, \tag{229}$$

where $\Lambda = (\Lambda^\sigma{}_\nu)$ and $\Lambda^\mu{}_\alpha = (\Lambda^T)_\alpha{}^\mu$. Then we can take the determinant to get

$$\det(\Lambda^T\eta\,\Lambda) = \det\eta, \tag{230}$$

since

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

we have

$$\det(\Lambda^T)\,\det\Lambda = 1 \tag{231}$$

and since $\det(\Lambda^T) = \det\Lambda$, so we arrive at

$$(\det\Lambda)^2 = 1 \quad\Rightarrow\quad \det\Lambda = \begin{cases} +1 \\ -1 \end{cases}$$

Consider the η_{00} component to understand the types of matrices we shall get:

$$\Lambda^\mu{}_0\eta_{\mu\nu}\Lambda^\nu{}_0 = \eta_{00} = 1 \quad\Rightarrow\quad (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1, \tag{232}$$

so we have

$$\Lambda^0{}_0 = \pm\sqrt{1 + (\Lambda^i{}_0)^2}. \tag{233}$$

Hence we have a bifurcation: either ${\Lambda^0}_0 \geq 1$ or ${\Lambda^0}_0 \leq -1$. The ${\Lambda^0}_0 > 1$ and $\det\Lambda = +1$ sector is called the *Proper Orthochronous Lorentz transformations.*

Λ matrices form in general a sub-group $GL(4, \mathbb{R})$ matrices. Let us observe that

$$\begin{aligned} \Lambda^T \Lambda &= B \\ (\Lambda^T \Lambda)_{\alpha\beta} &= B_{\alpha\beta} \\ (\Lambda^T)_\alpha{}^\nu \Lambda_{\nu\beta} &= B_{\alpha\beta} \\ \Lambda^\nu{}_\alpha \Lambda_{\nu\beta} &= B_{\alpha\beta} \\ \Lambda^\nu{}_\alpha \eta_{\mu\nu} \Lambda^\mu{}_\beta &= B_{\alpha\beta} = \eta_{\alpha\beta} \end{aligned} \tag{234}$$

So we have proven that $(\Lambda^T \Lambda)_{\alpha\beta} = \eta_{\alpha\beta}$. Therefore,

$$(\Lambda^T \Lambda)^\alpha{}_\beta = \delta^\alpha{}_\beta \quad \Rightarrow \quad \Lambda^T \Lambda = \mathbb{I} \quad \Rightarrow \quad \Lambda^T = \Lambda^{-1} \tag{235}$$

Thus Λ is an orthogonal matrix so $\Lambda \in O(1,3)$. $O(n)$ are the group of orthogonal $n \times n$ matrices. Consider the spatial case: $x'^i = L^i{}_j x^j$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} & & \\ & L & \\ & & \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad L \in O(3)$$

As we noted, the Lorentz (and its extended version Poincare) transformations form a group. Let us recall the basic properties of a group.

A group G is a set with, say, an operation (or multiplication) $\cdot$ and $g_1, g_2 \in G$, then

1. $g_1 \cdot g_2 \in G$, this is the closure property.
2. There is an identity element e such that
$$g \cdot e = e \cdot g = g.$$
3. Inverse element : $\forall g, \quad \exists g^{-1}$ such that $g \cdot g^{-1} = e = g^{-1} \cdot g$
4. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ associativity

In the case of the Lorentz group: each Lorentz transformation is an element of the group. So each symmetry operation is a group element. How many possible Lorentz transformations are there ? The answer is uncountably many! So we have a continuous differentiable set with a group property. It is called a *Lie group* which is a differentiable manifold with a group structure comptatible with differentiability.

$$L = O(1,3) = \left\{ \Lambda \in GL(4, \mathbb{R}) \mid \Lambda^T \eta \, \Lambda = \eta \right\} \tag{236}$$

with the Lie Algebra $\mathfrak{o}(1,3)$

$$\mathfrak{o}(1,3) = \left\{ a \in M_{4\times4}(\mathbb{R}) \mid a^T = -\eta \, a \, \eta \right\}, \tag{237}$$

where $M_{4\times4}(\mathbb{R})$ is the set of 4×4 matrices which are not necessarily invertable. So these matrices do not form a group but they form and algebra, that is a vector space with over the field of real numbers endowed with a bilinear product between its elements.

Proof: Let $\Lambda \in O(1,3)$, it can be written as

$$\Lambda(t) = \exp(ta) \qquad t \in \mathbb{R} \tag{238}$$

and by definition it should satisfy

$$\Lambda^T(t)\eta\Lambda(t) = \eta. \tag{239}$$

Taking the derivative of both sides near $t = 0$, we have

$$\frac{d}{dt}[\Lambda^T(t)\eta\Lambda(t)]_{t=0} = 0 \tag{240}$$

hence the claim follows:

$$a^T\eta + \eta\, a = 0. \tag{241}$$

1. *Classification of various subsets of the Lorentz Group*

We saw that the Lorentz group naturally falls into four components. Let us note these components:

1. Proper Lorentz transformations

$$L_+ = SO(1,3) = \{\Lambda \in O(1,3) \mid \det\Lambda = +1\} \tag{242}$$

L_+ is a subgroup of L as can be easily checked.

2. Improper Lorentz transformations

$$L_- = \{\Lambda \in O(1,3) \mid \det\Lambda = -1\} \tag{243}$$

L_- is not a subgroup of L since the identity element is not in L_-.

3. Orthochronous Lorentz transformations (ortho= correct, straight)

$$L^\uparrow = \{\Lambda \in O(1,3) \mid {\Lambda^0}_0 \geq 1\} \tag{244}$$

$L^\uparrow$ is a subgroup of L.

4. Non-orthochronous Lorentz transformations.

$$L^\downarrow = \{\Lambda \in O(1,3) \mid {\Lambda^0}_0 \leq -1\} \tag{245}$$

$L^\downarrow$ is not a subgroup of L.

Restricted Lorentz group is denoted as $L_+^\uparrow = L^\uparrow \cap L_+$

$$L_+^\uparrow = \{\Lambda \in O(1,3) \mid \det\Lambda = +1 \text{ and } {\Lambda^0}_0 \geq 1\} \tag{246}$$

So the restricted Lorentz group does not contain the space and time reflections. So we have excluded the following two matrices:

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Here T and P are time-reversal and parity, respectively such that

$$L_-^\downarrow = TL_+^\uparrow \qquad L_-^\uparrow = PL_+^\uparrow \qquad L_+^\downarrow = TPL_+^\uparrow \tag{247}$$

A. Dirac Spinor under Lorentz Transformations

When Dirac found his equation, several questions emerged: what kind of a mathematical object is the Dirac spinor ? (A little earlier Ehrenfest gave the name spinor to the Pauli's two component wave function.) For example von Neumann noted that for the first time in relativity theory, there arose a four component object which is not a vector. So let us answer the following question:

How does the Dirac spinor transform under Lorentz transformations ?

Under Lorentz transformations $x^{'} = \Lambda x$, let us assume that the Dirac spinor transforms according to

$$\psi(x) \to \psi^{'}(x^{'}) = S(\Lambda)\psi(x) \tag{248}$$

where $S(\Lambda)$ is a 4×4 matrix to be determined.

Note that

$$\psi^{'}(\Lambda x) = S(\Lambda)\psi(x) \tag{249}$$

Or we could write the last equation as

$$\psi^{'}(x^{'}) = S(\Lambda)\psi(\Lambda^{-1}x^{'}) \tag{250}$$

In order to be physically acceptable, $S^{-1}(\Lambda)$ should also exist

$$\begin{aligned} \psi(x) &= S^{-1}(\Lambda)\psi^{'}(\Lambda x) \\ \psi(\Lambda^{-1}x^{'}) &= S^{-1}(\Lambda)\psi^{'}(\Lambda x) \end{aligned} \tag{251}$$

Since we also have

$$\psi(x) = S(\Lambda^{-1})\psi^{'}(\Lambda x) \tag{252}$$

we arrive at

$$S^{-1}(\Lambda) = S(\Lambda^{-1}) \tag{253}$$

Therefore, under Lorentz transformations, free Dirac equation transforms as

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \quad \Rightarrow \quad (i\gamma^{'\mu}\partial^{'}_{\mu} - m)\psi^{'}(x^{'}) = 0 \tag{254}$$

Before going further, let us first prove:

$$\partial^{'}_{\mu} = \Lambda_{\mu}{}^{\nu}\,\partial_{\nu} \tag{255}$$

where $\Lambda_{\mu}{}^{\nu}$ is the inverse of $\Lambda^{\mu}{}_{\nu}$. Because we would like to keep

$$\partial_{\mu}x^{\mu} = \delta^{\mu}{}_{\nu} \tag{256}$$

so

$$\begin{aligned} \partial^{'}_{\nu}x^{'\mu} &= \delta^{\mu}{}_{\nu} \\ \partial^{'}_{\nu}(\Lambda^{\mu}{}_{\alpha}x^{\alpha}) &= \Lambda_{\nu}{}^{\beta}\Lambda^{\mu}{}_{\alpha}\underbrace{\partial_{\beta}x^{\alpha}}_{\delta_{\beta}{}^{\alpha}} = \delta^{\mu}\nu \end{aligned} \tag{257}$$

hence we get

$$\Lambda_{\nu}{}^{\alpha}\Lambda^{\mu}{}_{\alpha} = \delta^{\mu}{}_{\nu} \tag{258}$$

Now the transformation of the Dirac equation under Lorentz transformations is

$$(i\gamma'^{\mu}\Lambda_{\mu}{}^{\nu}\partial_{\nu} - m)S(\Lambda)\psi(x) = 0 \tag{259}$$

or it can be written as

$$S(\Lambda)\Big[i\Lambda_{\mu}{}^{\nu}S^{-1}(\Lambda)\gamma'^{\mu}S(\Lambda)\partial_{\nu} - m\Big]\psi(x) = 0 \tag{260}$$

So we want

$$\Lambda_{\mu}{}^{\nu}S^{-1}(\Lambda)\gamma'^{\mu}S(\Lambda) = \gamma^{\nu} \tag{261}$$

to have covariance. Or

$$\begin{aligned} \Lambda^{\rho}{}_{\nu}\gamma^{\nu} &= \Lambda^{\rho}{}_{\nu}\Lambda_{\mu}{}^{\nu}S^{-1}(\Lambda)\gamma'^{\mu}S(\Lambda) \\ &= (\Lambda^{T}\Lambda)^{\rho}{}_{\mu}S^{-1}(\Lambda)\gamma'^{\mu}S(\Lambda) \end{aligned} \tag{262}$$

so that

$$\Lambda^{\rho}{}_{\nu}\gamma^{\nu} = S^{-1}(\Lambda)\gamma'^{\rho}S(\Lambda) \tag{263}$$

There is an important point not mentioned in the books

$$\gamma'^{\mu} = U^{\dagger}\gamma^{\mu}U \qquad : \quad \begin{array}{l}\text{all such matrices are equivalent} \\ \text{up to unitary transformations } U^{\dagger} = U^{-1}\end{array} \tag{264}$$

Every observer can use the same γ-matrices.

$$\Lambda^{\rho}{}_{\nu}\gamma^{\nu} = S^{-1}(\Lambda)\gamma^{\rho}S(\Lambda) \tag{265}$$

Now consider an infinitesimal Lorentz transformation such that

$$\Lambda^{\mu}{}_{\nu} \backsimeq \delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu} \quad \text{such that} \quad |\epsilon^{\mu}{}_{\nu}| \ll 1 \tag{266}$$

Recalling that

$$\begin{aligned} \Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} &= \eta_{\alpha\beta} \\ (\delta^{\mu}{}_{\alpha} + \epsilon^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + \epsilon^{\nu}{}_{\beta}) &= \eta_{\alpha\beta} \end{aligned} \tag{267}$$

At the first order, we have

$$\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0 \tag{268}$$

So ϵ is an anti-symmetric matrix with 6 independent parameters in $3+1$ dimensions

1. 3 Lorentz boosts
2. 3 rotations

In D-dimensions ϵ has $\frac{D(D-1)}{2}$ parameters. For $D=2$ it has just 1 parameter, that is the Lorentz boost and of course there is no rotation. For $D=3$, it has 3 parameters, 1 rotation and 2 Lorentz boosts.

So we have

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}{}_{\nu} x^{\nu} + O(\epsilon^2) \tag{269}$$

Suppose the matrix that transforms the Dirac spinor for infinitesimal transformations is written as

$$S(\Lambda) \equiv \mathbb{I} + \frac{i}{2}\epsilon_{\mu\nu} M^{\mu\nu} + O(\epsilon^2) \tag{270}$$

where $\mathbb{I}$ is the 4×4 identity matrix and $M^{\mu\nu}$ is 4×4 matrix. (So there are six 4×4 M-matrices in $D = 4$. We sould ge the algebra of these matrices and find all of them.) Therefore, using a short-hand notation when necessary, from (265)we have

$$\begin{aligned} S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda) &= (\mathbb{I} - \frac{i}{2}\epsilon M)\gamma^{\mu}(\mathbb{I} + \frac{i}{2}\epsilon M) \\ &= \gamma^{\mu} - \frac{i}{2}\epsilon M \gamma^{\mu} + \frac{i}{2}\epsilon \gamma^{\mu} M + O(\epsilon^2) \\ &= \gamma^{\mu} - \frac{i}{2}\epsilon_{\rho\sigma}(M^{\rho\sigma} - \gamma^{\mu} M^{\rho\sigma}) \\ &= (\delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu})\gamma^{\nu} = \gamma^{\mu} + \epsilon^{\mu}{}_{\nu}\gamma^{\nu} \end{aligned} \tag{271}$$

so that we get

$$-\frac{i}{2}\epsilon_{\rho\sigma}[M^{\rho\sigma}, \gamma^{\mu}] = \epsilon^{\mu}{}_{\nu}\gamma^{\nu} \tag{272}$$

To get rid of the arbitrary ϵ's, let us do the following

$$-\frac{i}{2}\epsilon_{\rho\sigma}[M^{\rho\sigma}, \gamma^{\mu}] = \eta^{\mu\rho}\epsilon_{\rho\sigma}\gamma^{\sigma} = \frac{1}{2}\epsilon_{\rho\sigma}(\eta^{\mu\rho}\gamma^{\sigma} - \eta^{\mu\sigma}\gamma^{\rho}) \tag{273}$$

So then we have

$$\boxed{[M^{\rho\sigma}, \gamma^{\mu}] = i(\eta^{\mu\rho}\gamma^{\sigma} - \eta^{\mu\sigma}\gamma^{\rho})} \tag{274}$$

This looks like a nice equation, but still how are we going to find $M^{\rho\sigma}$? It is clear that we $M^{\rho\sigma}$?, which is anti-symmetric, can be written as an anti-symmetric product of two γ matrices. So, we can make the obvious ansatz $M^{\mu\nu} = a[\gamma^{\mu}, \gamma^{\nu}]$ and determine the coefficient a. Which eventually gives us

$$M^{\mu\nu} = -\frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}] \tag{275}$$

which satisfies the algebra. It is customary to define the following six matrices[28]

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}] \tag{276}$$

then

$$M^{\mu\nu} = -\frac{1}{2}\sigma^{\mu\nu} \tag{277}$$

[28] At this stage, please do check your understanding, it should be clear that for each $\mu \neq \nu$, $M^{\mu\nu}$ is a matrix, of which the components are $(M^{\mu\nu})_{\alpha\beta}$.

So with the $\sigma^{\mu\nu}$ matrices, the infinitesimal Lorentz transformation that acts on the Dirac 4-spinor reads as

$$S(\epsilon) = \mathbb{I} - \frac{i}{4}\epsilon_{\mu\nu}\sigma^{\mu\nu} \tag{278}$$

For a large Lorentz transformations we can multiply many of these small transformations as

$$x^{'} = \Lambda x \quad ; \quad x^{''} = \Lambda\Lambda x \quad ; \quad x^{'''} = \Lambda^3 x \quad ; \cdots ; \quad x^N = \Lambda^N x \tag{279}$$

and

$$\psi^{(N)}(x^{'}) = S(\Lambda)^N \psi(x) \tag{280}$$

Define $\epsilon_{\mu\nu} = \frac{\omega_{\mu\nu}}{N}$ where $\omega_{\mu\nu}$ is large. Then consecutive Lorentz transformations yield

$$\lim_{N\to\infty} \left(\mathbb{I} - \frac{i}{4}\frac{\omega_{\mu\nu}}{N}\sigma^{\mu\nu}\right)^N \equiv e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \tag{281}$$

So

$$S(w) = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \tag{282}$$

$S(w)$ is a complicated 4×4 matrix, but we can compute it once the γ matrices are chosen.

We still need to show will how $\omega_{\mu\nu}$ are related to the *rotations* and *boosts*.

So what have we accomplished up to this point?

Dirac spinor transforms as

$$\psi^{'}(x^{'}) = e^{-\frac{i}{4}w_{\mu\nu}\sigma^{\mu\nu}}\psi(x) \tag{283}$$

for Lorentz transformations $x^{'} = \Lambda(w)x$.

1. *"Cousins" of γ-matrices.*

4×4 matrices as a vector space needs 16 linearly independent matrices to form a basis. Whata re these ? In the language of Zee, let us talk about the "cousins" of γ-matrices.

$$\underbrace{\mathbb{I}}_{1}, \qquad \underbrace{\gamma^\mu}_{4}, \qquad \underbrace{\gamma^\mu\gamma^\nu}_{6} \quad \mu \neq \nu, \qquad \underbrace{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3}_{1}, \qquad \underbrace{\gamma^\mu\gamma^5}_{4} \tag{284}$$

So

$$\boxed{\{\mathbb{I}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}} \tag{285}$$

is the linearly independent set.

Now let us work out the components of the $\sigma^{\mu\nu}$ matrices in the standard representation.

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \quad \Rightarrow \quad \sigma^{0i} = \frac{i}{2}[\gamma^0, \gamma^i] = i\gamma^0\gamma^i \tag{286}$$

In the standard representation

$$\sigma^{0i} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Also

$$\sigma^{ij} = i\gamma^i\gamma^j = i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = i\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix}$$

Since $i \neq j \quad \sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k$ and $\{\sigma^i, \sigma^j\} = 2i\epsilon^{ijk}\sigma^k$. So

$$\sigma^{0i} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad , \sigma^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{287}$$

2. DIRAC BILINEARS

Check how the Dirac conjugate transforms

$$\begin{aligned}\bar{\psi}'(x') = (\psi^\dagger\gamma^0)' &= \psi'^\dagger(x')\gamma^0 \\ &= \Big(S(\Lambda)\psi(x)\Big)^\dagger\gamma^0 \\ &= \psi^\dagger(x)S^\dagger(\Lambda)\gamma^0\end{aligned} \tag{288}$$

So we have

$$\bar{\psi}'(x') = \bar{\psi}(x)\gamma^0 S^\dagger(\Lambda)\gamma^0 \tag{289}$$

Now

$$S^\dagger(\Lambda) = \mathbb{I} - \frac{i}{2}\epsilon_{\mu\nu}(M^{\mu\nu})^\dagger = \mathbb{I} + \frac{i}{4}\epsilon_{\mu\nu}(\sigma^{\mu\nu})^\dagger \tag{290}$$

and

$$S^{-1}(\Lambda) = \mathbb{I} + \frac{i}{4}\epsilon_{\mu\nu}\sigma^{\mu\nu} \tag{291}$$

Observe that

$$\begin{aligned}&(\sigma^{kj})^\dagger = (i\gamma^k\gamma^j)^\dagger = \sigma^{kj} \\ &(\sigma^{0k})^\dagger = -\sigma^{0k}\end{aligned} \tag{292}$$

So $\sigma^{\mu\nu}$ is not Hermitian!!! That means

$$S^\dagger(\Lambda) \neq S^{-1}(\Lambda) \tag{293}$$

So $S(\Lambda)$ is not unitary!!! How about

$$\begin{aligned}\gamma^0 S^\dagger(\Lambda)\gamma^0 &= \mathbb{I} + \frac{i}{4}\epsilon_{\mu\nu}\underbrace{\gamma^0(\sigma^{\mu\nu})^\dagger\gamma^0}_{=\sigma^{\mu\nu}} \\ &= S^{-1}(\Lambda)\end{aligned} \tag{294}$$

Then we have

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}(\Lambda) \tag{295}$$

so one arrives at

$$\begin{aligned}\bar{\psi}\psi \quad \to \quad \bar{\psi}'(x') &= \bar{\psi}(x)S^{-1}S\psi(x) \\ &= \bar{\psi}\psi(x)\end{aligned} \tag{296}$$

which is *Lorentz-invariant*. There are 16 bilinears that we can construct $\bar{\psi}\Gamma\psi$ where Γ is one of these matrices.

Let us discuss the space reflections and time reflections which we have dismissed before.

1. Space Reflections $\mathcal{S}$ and Parity transformations $\mathcal{P}$

$$\vec{x}^{'} = -\vec{x}^{'}, \quad t^{'} = t \quad \Rightarrow \quad x^{'\mu} = (t^{'}, \vec{x}^{'}) = (t, -\vec{x}^{'}) \tag{297}$$

A true vector changes sign under parity just like $\vec{x}$, but a pseudo vector such as $\vec{L} = \vec{r} \times \vec{p}$ does not change sign. For example, under parity $\vec{E}$ changes sign but $\vec{B}$ does not change sign.

$$m\frac{d^2\vec{p}}{dt^2} = e(\vec{E} + \vec{v} \times \vec{B}), \qquad \vec{E} \cdot \vec{B} : \text{ pseudoscalar} \tag{298}$$

Note that for parity we have

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \Rightarrow \eta^{\mu\nu}$$

Let $\mathcal{S} = \mathcal{P}$ for reflections

$$\psi^{'}(x^{'}) = \mathcal{P}\psi(x) \tag{299}$$

and we assume that the Dirac equation is invariant. Then

$$\mathcal{P}^{-1}\gamma^{\nu}\mathcal{P} = \Lambda^{\nu}{}_{\mu}\gamma^{\mu} = \sum_{\mu} \eta^{\mu\nu}\gamma^{\mu} \tag{300}$$

Consider $\nu = 0$

$$\mathcal{P}^{-1}\gamma^{0}\mathcal{P} = \gamma^{0} \tag{301}$$

it can be easily seen that

$$\mathcal{P} = e^{i\delta}\gamma^{0} \tag{302}$$

where δ is some arbitrary phase. Let us check:

$$\mathcal{P}^{-1}\gamma^{i}\mathcal{P} = e^{-i\delta}\gamma^{0}\gamma^{i}\gamma^{0}e^{i\delta} = -\gamma^{i} \tag{303}$$

Note that $\mathcal{P}^{-1} = \mathcal{P}^{+}$ so $\mathcal{P}$ is unitary for parity transformation. Then we have

$$\psi^{'}(x^{'}) = \psi^{'}(-\vec{x}, t) = e^{i\delta}\gamma^{0}\psi(\vec{x}, t) \tag{304}$$

Take $\delta = 0$ and consider the standard representation

$$\psi^{'}(x^{'}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_{up} \\ \psi_{down} \end{pmatrix} = \begin{pmatrix} \psi_{up} \\ -\psi_{down} \end{pmatrix}$$

<u>Important</u>: The upper component has *even* and the lower component has *odd* parity. PARTICLES HAVE INTRINSIC PARITIES!!!

<u>Note-1</u>: Note also $\Lambda^{0}{}_{0} = +1$ and $\det\Lambda = -1$.

<u>Note-2</u>: $\bar{\psi}^{'}(x^{'}) = \psi^{'+}\gamma^{0} = \psi^{+}\gamma^{0}e^{-i\delta}\gamma^{0} = e^{-\delta}\bar{\psi}(x)\gamma^{0}$

<u>Note-3</u>: $e^{+}e^{-}$ positronium has $(-1)^{l+1}$ parity; photon's parity is -1.

2. Time reflections $x^{'} = (t^{'}, \vec{x}) = (-t, \vec{x})$. Note that $\Lambda^0{}_0 = -1$ and $\Lambda^i{}_i = +1$. So

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = -\eta^{\mu\nu}$$

Note again Dirac equation is invariant under these transformations. OK . Set $\mathcal{S} = \mathcal{T}$, so

$$\psi^{'}(x^{'}) = \psi^{'}(t^{'}, \vec{x}^{'} = \psi^{'}(-t, \vec{x}) = \mathcal{T}\psi(x) \tag{305}$$

$$\mathcal{T}^{-1}\gamma^\nu\mathcal{T} = -\sum_\mu \eta^{\mu\nu}\gamma^\nu \tag{306}$$

check $\nu = 0$

$$\mathcal{T}^{-1}\gamma^0\mathcal{T} = -\gamma^0 \quad ; \quad \mathcal{T}^{-1}\gamma^i\mathcal{T} = \gamma^i \tag{307}$$

Recall $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\{\gamma^5, \gamma^\mu\} = 0$ as well as $(\gamma^5)^2 = 1$. Try $\mathcal{T} = c\gamma^5\gamma^0$. Eventually we get

$$\mathcal{T} = e^{i\eta}\gamma^5\gamma^0. \tag{308}$$

Question: Show that if $\psi(x)$ satisfies the Dirac equation, then $\gamma^5\psi(x)$ satisfies the same equation with $m \to -m$.

Proof: The Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{309}$$

then

$$(i\gamma^\mu\partial_\mu + m)\gamma^5\psi = \gamma^5(-i\gamma^\mu\partial_\mu + m)\psi(x) = 0 \tag{310}$$

So for massless case both $\psi(x)$ and $\gamma^5\psi(x)$ are solutions.

B. Lorentz and Poincare Symmetries (Chp-2 of Maggiore)

A Lie group is a group whose elements g depend in a *continuous* and *differentiable* way on a set of real parameters.

$$\theta^a, \quad a = 1, 2, 3, \cdots N \tag{311}$$

So a Lie group is a group at the same time it is a differentiable manifold. Without loss of generality, the coordinates θ^a can be chosen in such a way that

$$\underbrace{g(0)}_{\text{origin of the manifold}} = \underbrace{e}_{\text{identity element of the group}} \tag{312}$$

Example: Circle (S^1) and $g(\theta) = e^{i\theta}$

A linear representation R of a group is an *operation* that assigns to generic, abstract element g, the linear operator $\mathcal{D}_R(g)$ such that

$$g \quad \to \quad \mathcal{D}_R(g) \tag{313}$$

with the properties

1. $\mathcal{D}_R(e) = \mathbb{I}$ where $\mathbb{I}$ is the identity operator.

2. $\mathcal{D}_R(g_1)\mathcal{D}_R(g_2) = \mathcal{D}_R(g_1 g_2)$

So the mapping preserves the group property. The operators act on a vector space which is called the basis of the representation R. Typically we have matrix representations, i.e., $\mathcal{D}_R(g)$ are matrices and the base space is finite dimensional.

$$\mathcal{D}_R(g) : V \quad \rightarrow \quad V, \qquad \dim(V) = \text{dimension of the representation.} \tag{314}$$

Then g, the element of the group is represented by an $n \times n$ matrix $\mathcal{D}_R(g)^i{}_j \;\; i, j = 1, \cdots, n$.

Example:

FIG 22!!!!

Easier Example: Here is an abstract group $\{e, b\}$. So it has two elements. The *multiplication* table is given as

$$e \cdot b = b \quad ; \quad b \cdot b = e. \tag{315}$$

Consider a representation $[1, -1]$ under the usual multiplication

$$1(-1) = -1, \qquad (-1)(-1) = 1 \tag{316}$$

which acts on

$$\left.\begin{aligned} 1 \cdot x &= x \\ -1 \cdot x &= -x \end{aligned}\right\} \text{mirror reflection.} \tag{317}$$

This is a 1-dimensional representation.

Consider a two-dimensional representation

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

under matrix multiplication

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This acts on

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

A representation R is called *reducible* if it has an invariant subspace i.e.,

$$\mathcal{D}_R(g)\begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{pmatrix} = \begin{pmatrix} v_2 \\ v_1 \\ -- \\ v_3 \\ v_4 \\ . \\ . \\ . \\ v_n \end{pmatrix}, \qquad \forall\ \mathcal{D}_R(g)$$

Complete reducible representation: If $\mathcal{D}_R(g)$ can be written in block diagonal form (in a suitable basis) then we have a completely reducible representation

$$\mathcal{D}_R(g) = \begin{pmatrix} \square & & & & & \\ & \square & & & & \\ & & \square & & & \\ & & & . & & \\ & & & & . & \\ & & & & & . \end{pmatrix}$$

Two representations R and $R^{'}$ are called equivalent if there is a matrix S, independent of g, such that $\forall\ g$ we have

$$\mathcal{D}_R(g) = S^{-1}\mathcal{D}_{R'}(g)S, \qquad \forall\ g \tag{318}$$

equivalent representations correspond to a change of basis.

Important: When we change the representation, in general, the explicit form an the dimension of $\mathcal{D}_R(g)$ will change but the Lie algebra will not change.

By the assumption of smoothness

$$\mathcal{D}_R(\theta) \simeq \mathbb{I} + i\theta_a T_R^a \qquad \text{near the identity} \tag{319}$$

Here "i" is just a choice, if T turns out to be Hermitian, then $\mathcal{D}_R$ will be unitary. It of course may not be !!!

$$T_R^a \equiv -i\frac{\partial \mathcal{D}_R}{\partial \theta^a}\ |_{\theta=0} \tag{320}$$

where T_R^a are generators of the group in the representation R.

Lie group theory books show that all the group manifold connected to the identity can be obtained as

$$\mathcal{D}_R(g(\theta)) = e^{i\theta_a T_R^a} \tag{321}$$

Let us specifically talk about matrix representation. Given two matrices

$$\mathcal{D}_R(g_1) = e^{i\alpha_a T_R^a}, \qquad \mathcal{D}_R(g_2) = e^{i\beta_a T_R^a} \tag{322}$$

We should have

$$\mathcal{D}_R(g_1)\mathcal{D}_R(g_2) = \mathcal{D}_R(g_1 g_2) \tag{323}$$

which is

$$e^{i\alpha\cdot T_R} e^{i\beta\cdot T_R} = e^{i\delta(\alpha,\beta)\cdot T_R} \tag{324}$$

Since in general $e^A e^B \neq e^{A+B}$ so $\delta_a \neq \alpha_a + \beta_a$. Taking the Logarithm and expanding with

$$\log(1+x) \simeq x - \frac{x^2}{2} \tag{325}$$

In fact first expand the exponentials before taking log !!!

$$\begin{aligned} \log\Big[\mathbb{I} + i\vec{\alpha}\cdot\vec{T}_R - \frac{1}{2}(\vec{\alpha}\cdot\vec{T}_R)^2\Big]\Big[\mathbb{I} + i\vec{\beta}\cdot\vec{T}_R - \frac{1}{2}(\vec{\beta}\cdot\vec{T}_R)^2\Big] &= \log\Big[\mathbb{I} + i\vec{\delta}\cdot\vec{T}_R - \frac{1}{2}(\vec{\delta}\cdot\vec{T}_R)^2\Big] \\ \log\Big[\mathbb{I} + i(\vec{\alpha}+\vec{\beta})\cdot\vec{T}_R - \frac{1}{2}(\vec{\alpha}\cdot\vec{T}_R)^2 - \vec{\alpha}\cdot\vec{T}_R\ \vec{\beta}\cdot\vec{T}_R - \frac{1}{2}(\vec{\beta}\cdot\vec{T}_R)^2\Big] &= \log\Big[\mathbb{I} + i\vec{\delta}\cdot\vec{T}_R - \frac{1}{2}(\vec{\delta}\cdot\vec{T}_R)^2\Big] \end{aligned} \tag{326}$$

which becomes

$$i(\vec{\alpha}+\vec{\beta})\cdot\vec{T}_R - \frac{1}{2}(\vec{\alpha}\cdot\vec{T}_R)^2 - \vec{\alpha}\cdot\vec{T}_R\ \vec{\beta}\cdot\vec{T}_R - \frac{1}{2}(\vec{\beta}\cdot\vec{T}_R)^2 + \frac{1}{2}((\vec{\alpha}+\vec{\beta})\cdot\vec{T}_R)^2 = i\vec{\delta}\cdot\vec{T}_R - \frac{1}{2}(\vec{\delta}\cdot\vec{T}_R)^2 \tag{327}$$

So we have

$$\frac{1}{2}(\vec{\beta}\cdot\vec{T}_R)(\vec{\alpha}\cdot\vec{T}_R) - \frac{1}{2}(\vec{\alpha}\cdot\vec{T}_R)(\vec{\beta}\cdot\vec{T}_R) = i(\vec{\delta}-\vec{\alpha}-\vec{\beta})\cdot\vec{T}_R \tag{328}$$

which is

$$\beta^a\alpha^b[T_R^a, T_R^b] = 2i(\delta^c - \alpha^c - \beta^c)T_R^c \tag{329}$$

$\alpha,\ \beta$ are arbitrary so

$$\delta^c - \alpha^c - \beta^c = f^c{}_{ab}\beta^c\alpha^b \tag{330}$$

so we have

$$[T_R^a, T_R^b] = i f_c{}^{ab} T_R^c \tag{331}$$

which is the Lie algebra of the group. Here $f_c{}^{ab}$ is structure constants. These are independent even though T depend on the representation. To find the representations of the Lie algebra, one finds all the matrices T_R satisfying the above commutation relation. No further requirement comes from the higher order terms.

$T_R^a = 0$ satisfies the algebra. This is called the trivial representation. Note also that from the representation of the algebra, we can go to the representation of the group by the expansion. Any continuous group is in this form.

Example: Defining representation

$$SO(3) \quad : \quad 3\times 3 \ \text{ real orthogonal matrices with } \det g = +1 \tag{332}$$

In another words $g \in SO(3)$; $\det g = +1$

$$g^T g = 1 = g g^T \quad ; \quad g^T = g^{-1} \tag{333}$$

So what is the Lie algebra?

Solution: From smoothness property, we can write

$$g = \mathbb{I} + A + O(A^2) + \cdots \tag{334}$$

here A is 3×3 matrix. We also have

$$(\mathbb{I} + A^T)(\mathbb{I} + A) = \mathbb{I} \quad \Rightarrow \quad A^T = -A \tag{335}$$

Then A has 3 independent real parameters, so does g. Near the identity, we have

$$g(\epsilon_1, \epsilon_2, \epsilon_3) = \begin{pmatrix} 1 & \epsilon_1 & \epsilon_2 \\ -\epsilon_1 & 1 & \epsilon_3 \\ -\epsilon_2 & -\epsilon_3 & 1 \end{pmatrix}$$

Then

$$T^1 = \frac{1}{i}\frac{\partial g}{\partial \epsilon_1} |_{\epsilon_i=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^2 = \frac{1}{i}\frac{\partial g}{\partial \epsilon_2} |_{\epsilon_i=0} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$T^3 = \frac{1}{i}\frac{\partial g}{\partial \epsilon_3} |_{\epsilon_i=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

check that

$$[T^i, T^j] = i\epsilon^{ijk}\, T^k \tag{336}$$

is satisfied. This is the angular momentum algebra in Quantum Mechanics

$$[J^i, J^k] = i\epsilon^{ijk}\, J^k \tag{337}$$

Note that ϵ^{ijk} are the structure constants. Also recall that

$$[L^i, L^j] = i\hbar\epsilon^{ijk} L^k \tag{338}$$

and

$$\vec{L} = \vec{r} \times \vec{p} \quad ; \quad L^i = \epsilon^{ijk} x^k p^k = -i\hbar\epsilon^{ijk} x^j \frac{\partial}{\partial x_k} \tag{339}$$

So the representation need not be a matrix of finite dimensions. We will discuss this later.

1. Some Remarks About Group Theory

A group is called Abelian if all its elements commute between themselves, otherwise the group is non-Abelian.

Casimir Operators play important role in the study of representations. These operators are constructed from T^a and commute with all T^a's. In each irreducible representation, the Casimir operators are proportional to identity and the proportionality constant labels the representation. For example, in the angular momentum algebra

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \tag{340}$$

the quadratic Casimir operator is

$$J^2 = j(j+1)\mathrm{I} \qquad \text{with } j = 0, 1/2, 1, \cdots \tag{341}$$

Compact vs non-Compact Groups: A Lie group, considered as a manifold is a *compact* Lie group if it is a compact manifold.

Compact manifold: Every open cover has a finite subcover. Spatial rotations is a compact Lie group. Lorentz group is non-compact.

Theorem: Non-compact groups have no *unitary representations* of finite dimension (except for the trivial representation.)

So for a physical representation, we need an infinite dimensional representation. Infinite dimensional representations are introduced by defining the Hilbert space of one-particle states.

2. (From 2.2 of Maggiore) Lorentz Group (again)

$$x^\mu \to x^{'\mu} = \Lambda^\mu{}_\nu x^\nu \quad \text{ leaves } \eta_{\mu\nu} x^\mu x^\nu \text{ invariant.} \tag{342}$$

$$O(1,3) \quad \text{or} \quad O(3,1) \quad \text{and} \quad \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu, \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \tag{343}$$

Example: Boost $t^{'} = \gamma(t + vx)$ and $x^{'} = \gamma(x + vt)$ with $\gamma = (1 - v^2)^{-1/2}$ and $-1 < v < 1$. Or in Rindler coordinates

$$v = \tanh\eta \qquad \Rightarrow \qquad \begin{aligned} t^{'} &= \cosh\eta t + \sinh\eta x \\ x^{'} &= \sinh\eta t + \cosh\eta x \end{aligned} \tag{344}$$

where η: rapidity. Since $0 \leq |v|$ which is a non-compact interval.

C. The Lorentz Algebra

Consider an abstract element of the Lorentz group

$$\Lambda = e^{-\frac{i}{2} w_{\mu\nu} J^{\mu\nu}} \tag{345}$$

Consider a set of objects ϕ^i with $i = 1, 2, \cdots, n$ transforming in the representation R of the Lorentz group

$$\phi^i \to \left[e^{-\frac{i}{2}w_{\mu\nu}J_R^{\mu\nu}}\right]^i{}_j\,\phi^j \tag{346}$$

Since $w_{\mu\nu}$ is $n \times n$ matrices so $J_R^{\mu\nu}$ is $n \times n$ matrices. So under an infinitesimal transformation we have

$$\delta\phi^i = -\frac{i}{2}w_{\mu\nu}(J_R^{\mu\nu})^i{}_j\phi^j \tag{347}$$

So in the weird object $(J^{\mu\nu})^i{}_j$. Note that i, j denote the matrix element and μ, ν denote the matrix. All the physical quantities can be classified according to their transformation under the Lorentz group: Scalar: invariant under the Lorentz transformation

$$\phi' = \phi = e^{-\frac{i}{2}w_{\mu\nu}J^{\mu\nu}}\phi \tag{348}$$

$J = 0$ so we have scalars transforming under the *trivial* representation of the Lorentz group. Contravariant 4-vector $V^\mu \to \Lambda^\mu{}_\nu V^\nu$

$$V'^\mu = (e^{-\frac{i}{2}w_{\alpha\beta}J^{\alpha\beta}})^\mu{}_\nu\, V^\nu \tag{349}$$

so

$$\delta V^\mu = -\frac{i}{2}\epsilon_{\alpha\beta}(J_R^{\alpha\beta})^\mu{}_\nu\, V^\nu \tag{350}$$

At the same time, we know that

$$\delta V^\mu = \epsilon^\mu{}_\nu V^\nu \tag{351}$$

then

$$\epsilon^\mu{}_\nu = -\frac{i}{2}\epsilon_{\alpha\beta}(J_R^{\alpha\beta})^\mu{}_\nu \tag{352}$$

which can also be written as

$$\frac{1}{2}\epsilon_{\alpha\beta}(\eta^{\mu\alpha}\delta^\beta{}_\nu - \eta^{\mu\beta}\delta^\alpha{}_\nu) = -\frac{i}{2}\epsilon_{\alpha\beta}(J_R^{\alpha\beta})^\mu{}_\nu \tag{353}$$

So we have

$$(J_R^{\alpha\beta})^\mu{}_\nu = i(\eta^{\mu\alpha}\delta^\beta{}_\nu - \eta^{\mu\beta}\delta^\alpha{}_\nu) \tag{354}$$

(Is there a minus sign???) This representation is *irreducible* since a generic Lorentz transformation mixes all 4 components of a 4-vector.

Using this explicit representation we can find the Lie algebra of the Lorentz group which does not depend on the representation:

$$\begin{aligned}
([J^{\mu\nu}, J^{\rho\sigma}])^\alpha{}_\beta &= (J^{\mu\nu})^\alpha{}_\lambda(J^{\rho\sigma})^\lambda{}_\beta - (J^{\rho\sigma})^\alpha{}_\lambda(J^{\mu\nu})^\lambda{}_\beta \\
&= -(\eta^{\mu\alpha}\delta^\nu{}_\lambda - \eta^{\nu\alpha}\delta^\mu{}_\lambda)(\eta^{\rho\lambda}\delta^\sigma{}_\beta - \eta^{\sigma\lambda}\delta^\rho{}_\beta) - \mu\nu \leftrightarrow \rho\sigma \\
&= -\eta^{\mu\alpha}\eta^{\rho\lambda}\delta^\nu{}_\lambda\delta^\sigma{}_\beta + \eta^{\mu\alpha}\eta^{\sigma\lambda}\delta^\nu{}_\lambda\delta^\rho{}_\beta + \cdots \text{the rest} \\
&= i\Big[\eta^{\rho\nu}(J^{\mu\sigma})^\alpha{}_\beta - \eta^{\rho\mu}(J^{\nu\sigma})^\alpha{}_\beta - \eta^{\sigma\nu}(J^{\mu\rho})^\alpha{}_\beta + \eta^{\sigma\mu}(J^{\nu\rho})^\alpha{}_\beta\Big]
\end{aligned} \tag{355}$$

So we have

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\Big[\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\Big] \tag{356}$$

Lie algebra of the group $SO(1,3)$, it is denoted as $SO(1,3)$. It is convenient to use two spatial vectors J^i and K^i defined as

$$J^i \equiv \frac{1}{2}\epsilon^{ijk}J^{jk} \quad , \quad K^i \equiv J^{i0} \tag{357}$$

This will help us to find the representations of $SO(1,3)$ with these definitions $SO(1,3)$ algebra becomes

$$[J^i, J^j] = i\epsilon^{ijk}J^k \quad ; \quad [J^i, K^j] = i\epsilon^{ijk}K^k \quad ; \quad [K^i, K^j] = -i\epsilon^{ijk}K^k \tag{358}$$

Recall that the first commutation relation is the Lie algebra of $SO(3)$ or $SU(2)$. Also introduce

$$\theta^i \equiv \frac{1}{2}\epsilon^{ijk}w^{jk} \qquad (w_{12} = w^{12} = \theta^3 \text{ etc.}) \quad \text{and} \quad \eta^i \equiv w^{i0} = -w_{i0} \tag{359}$$

Then

$$\begin{aligned} \frac{1}{2}w_{\mu\nu}J^{\mu\nu} &= \sum_{i=1}^{3} w_{i0}J^{i0} + w_{12}J^{12} + w_{13}J^{13} + w_{23}J^{23} \\ &= -\vec{\eta}\cdot\vec{K} + \vec{\theta}\cdot\vec{J} \end{aligned} \tag{360}$$

So the Lorentz transformation reads

$$\Lambda = \exp\{-i\vec{\theta}\cdot\vec{J} + i\vec{\eta}\cdot\vec{K}\} \tag{361}$$

What are our sign convent s? Choose the infinitesimal transformations say a rotation around the z-axis with small angle θ

$$\begin{aligned} \delta x^\mu &= -i\theta(J^{12})^\mu{}_\nu x^\nu \quad \text{since} \quad \theta^3 = \frac{1}{2}\epsilon^{3jk}w^{jk} = w^{12} \\ &= \theta(\eta^{1\mu}\delta^2{}_\nu - \eta^{2\mu}\delta^1{}_\nu)x^\nu \end{aligned} \tag{362}$$

Now

$$\delta t = 0 \quad ; \quad \delta x = -\theta y \quad ; \quad \delta y = +\theta x \tag{363}$$

FIG 23!!!!

$$\begin{aligned} x^{'} &= r\cos(\theta+\varphi), \qquad x = r\cos\varphi \\ y^{'} &= r\sin(\theta+\varphi), \qquad y = rsin\varphi \end{aligned} \tag{364}$$

then

$$\delta x = -\theta y \quad , \quad \delta y = \theta x \tag{365}$$

Consider the boost along x

$$\delta x^\mu = i\eta(J^{10})^\mu{}_\nu\, x^\nu, \qquad \delta t = \eta x, \qquad \delta x = +\eta t \tag{366}$$

Recall the $\vec{J}, \vec{K}$ vectors. So here is the BIG question: How are we going to find the representations of $SO(1,3)$ algebra, now that we know the algebra?

It is more convenient to split $J^{\mu\nu}$ into two spatial vectors

$$J^i \equiv \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad K^i \equiv J^{i0} \tag{367}$$

We can easily obtain the following relations

$$[J^i, J^j] = i\epsilon^{ijk}J^k \quad ; \quad [J^i, K^j] = i\epsilon^{ijk}K^k \quad ; \quad [K^i, K^j] = -i\epsilon^{ijk}J^k \tag{368}$$

Now let us do the following simplification

$$\vec{J}^{\pm} = \frac{1}{2}(\vec{J} \pm i\vec{K}) \tag{369}$$

Then

$$\begin{aligned}([J^{+i}, J^{+j}]) &= \frac{1}{4}[J^i + iK^i, J^j + iK^j] \\ &= \frac{1}{4}i\epsilon^{ijk}J^k - \frac{1}{2}\epsilon^{ijk}K^k + \frac{1}{4}\epsilon^{ijk}J^k \\ &= \frac{1}{2}i\epsilon^{ijk}(J^k + iK^k)\end{aligned} \tag{370}$$

so that

$$[J^{+i}, J^{+j}] = i\epsilon^{ijk}J^{+k} \tag{371}$$

we also have

$$[J^{-i}, J^{-j}] = i\epsilon^{ijk}J^{-k} \quad ; \quad [J^{+i}, J^{-j}] = 0 \tag{372}$$

So we have split the $SO(1,3)$ algebra to two $SU(2)$ algebras. Recall the $SU(2)$algebra

$$[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = i\epsilon^{ijk}\frac{\sigma^k}{2} \quad \Rightarrow \quad [S^i, S^j] = i\epsilon^{ijk}S^k \tag{373}$$

SU(2) matrices:

$$U^+ = U^{-1} \quad \det U = \mathbb{I} \tag{374}$$

So $U^+U = \mathbb{I}$. Let

$$U = \mathbb{I} + A + O(A^2) \tag{375}$$

Then

$$(\mathbb{I} + A^+)(\mathbb{I} + A) = \mathbb{I} \quad \Rightarrow \quad A^+ = -A \tag{376}$$

so

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{8 real entries.}$$

and

$$A^{+} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = - \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So

$$\begin{aligned} a^* &= -a \quad \Rightarrow \quad \text{real part of a is zero.} \\ d^* &= -d \quad \Rightarrow \quad \text{real part of d is zero} \end{aligned} \tag{377}$$

Therefore there are 4 real parameters.

$SO(3)$ is locally $SU(2)$, or their Lie algebras are equivalent. But to take spinors into account we must consider the rotation group to be $SU(2)$ So locally

$$SO(1,3) \sim SU(2) \times SU(2) \tag{378}$$

<u>representations of $SU(2)$</u>: Quantum Mechanics of Angular Momentum

$$[J^i, J^j] = i\epsilon^{ijk} J^k \tag{379}$$

$\vec{J}^2$ is our quadratic Casimir operator

$$[\vec{J}^2, \vec{J}] = 0 \tag{380}$$

Define

$$J_+ \equiv J_x + iJ_y \quad ; \quad J_- = J_x - iJ_y \tag{381}$$

Then

$$[J_z, J_\pm] = \pm J_\pm \quad ; \quad [J^2, J_\pm] = 0 \tag{382}$$

Also we have

$$J^2 = J_+J_- + J_z^2 - J_z = J_-J_+ + J_z^2 + J_z \tag{383}$$

Introduce

$$J^2 \mid \lambda, m\rangle = \lambda \mid \lambda, m\rangle, \qquad J_z \mid \lambda, m\rangle = m \mid \lambda, m\rangle \tag{384}$$

Note also that

$$J^2(J_\pm \mid \lambda, m\rangle) = \lambda(J_\pm \mid \lambda, m\rangle) \qquad \text{Since } J^2 \text{ commutes with } J_\pm \tag{385}$$

How about

$$\begin{aligned} J_z J_\pm \mid \lambda, m\rangle &= (J_\pm J_z J_\pm) \mid \lambda, m\rangle \\ &= (m \pm 1) J_\pm \mid \lambda, m\rangle \end{aligned} \tag{386}$$

More clearly

$$J_z J_\pm \mid \lambda, m\rangle = (m \pm 1) J_\pm \mid \lambda, m\rangle \tag{387}$$

So clearly

$$J_{\pm} \mid \lambda, m\rangle \;\sim \mid \lambda, m\pm\rangle \tag{388}$$

Since $\langle J^2\rangle \geq \langle J_z^2\rangle$ since these are Hermitian operators we have $\lambda \geq m^2$; $-\sqrt{\lambda} \leq m \leq \sqrt{\lambda}$ so m is bounded from above and below.

So by repeated application of J_- on $\mid \lambda, m\rangle$, we can reduce it as much as we like but then we could violate the lower bound. So at same m_1 we must have

$$J_- \mid \lambda, m_1\rangle = 0 \tag{389}$$

and similarly we must have

$$J_+ \mid \lambda, m_2\rangle = 0 \tag{390}$$

OK, now start from state $\mid \lambda, m_1\rangle$ and apply J_+ many times to reach $\mid \lambda, m_2\rangle$ so

$$m_2 - m_1 = k \qquad k \geq 0 \tag{391}$$

here by k we mean k times J_+ was used.OK so

$$J^2 \mid \lambda, m_1\rangle = \lambda \mid \lambda, m_1\rangle \tag{392}$$

$$(J_+J_- + J_z^2 - J_z) \mid \lambda, m_1\rangle = \lambda \mid \lambda, m_1\rangle \tag{393}$$

$$m_1^2 - m_1 = \lambda \tag{394}$$

Similarly we have

$$\lambda = m_2^2 + m_2 \tag{395}$$

$$m_2^2 + m_2 = m_1^2 - m_1 \tag{396}$$

$$(m_2 - m_1)(m_2 + m_1) = -(m_2 + m_1) \tag{397}$$

$$(m_2 + m_1) \quad \underbrace{(m_2 - m_1 + 1)}_{k+1 \text{ since } k\geq 0,\; k+1>0} \quad = 0 \tag{398}$$

So we arrive at

$$m_2 = -m_1 \tag{399}$$

Hence we get

$$m_2 - m_1 = k \Rightarrow m_2 = \frac{k}{2} \equiv j = 0, 1/2, 1, \cdots \quad \text{(Since k is an integer)} \tag{400}$$

Then

$$\lambda = m_2^2 + m_2 = j(j+1) \tag{401}$$

and the number of possible m values are

$$m = -j, -j+1, -j+2, \cdots 0, 1, 2, \cdots, +j \qquad j \geq 0 \tag{402}$$

So there are $2j+1$ m-values.

Since $J_- = J_+^+$

$$J_+ \mid \lambda, m\rangle = \eta \mid \lambda, m+1\rangle \quad \Rightarrow \quad \langle \lambda, m \mid J_- = \eta^* \langle \lambda, m+1 \mid \tag{403}$$

Then

$$\begin{gathered} \langle \lambda, m \mid J^2 \mid \lambda, m\rangle = \langle \lambda, m \mid J_- J_+ + J_z^2 + J_z \mid \lambda, m\rangle \\ j(j+1) = \eta^* \eta + m(m+1) \end{gathered} \tag{404}$$

which is

$$|\eta|^2 = j(j+1) - m(m+1) \tag{405}$$

Take η to be real and positive, then

$$J_+ \mid \lambda, m\rangle = \sqrt{(j+m+1)(j-m)} \mid \lambda, m+1\rangle \tag{406}$$

$$J_- \mid \lambda, m\rangle = \sqrt{(j-m+1)(j+m)} \mid \lambda, m-1\rangle \tag{407}$$

Summary: Representation of $SU(2)$ will be $(2j+1) \times (2j+1)$-dimensional matrices $[J^i, J^j] = i\epsilon^{ijk} J^k \quad j = 0, 1/2, 1, 3/2, \cdots$

$j = 0 \;\; J^i = 0$ this is the trivial representation.

$j = 1/2 \Rightarrow 2 \times 2$ matrices: Pauli matrices.

$j = 1 \Rightarrow 3 \times 3$ matrices which are found before.

$j = 3/2 \Rightarrow 4 \times 4$ matrices: Find them.

Consider the $j = 1/2$ representation, which is called the spinorial representation.

$$J^i = \frac{\sigma^i}{2} \tag{408}$$

Spinorial is the fundamental representation of $SU(2)$ since all the other representation can be obtained from this one by tensor products. [So physically with spin-1/2 particles we can construct composite system with all possible integer or half integer spins.]

So that means we have to study a little bit how angular momenta can be added.

1. *Clebsch-Gordon Coefficients*

Generically say we have $[\vec{J}_1, \vec{J}_2] = 0$ and $\mid j_1, m_1\rangle$ and $\mid j_2, m_2\rangle$ be the eigenstates of $\vec{J}_1^2$ and $\vec{J}_{1z}$ and $\vec{J}_2^2$ and $\vec{J}_{2z}$, respectively.

<u>OPTION A</u>: Choose the base eigenkets as the eigenkets of $\vec{J}_1^2, \vec{J}_2^2, \vec{J}_{1z}, \vec{J}_{2z}$ as

$$| \, j_1 j_2; m_1 m_2\rangle \tag{409}$$

These four operators obviously commute with each other and so we can do that.

$$\begin{aligned} \vec{J}_1^2 \, | \, j_1 j_2; m_1 m_2\rangle &= j_1(j_1+1) \, | \, j_1 j_2; m_1 m_2\rangle \\ \vec{J}_2^2 \, | \, j_1 j_2; m_1 m_2\rangle &= j_2(j_2+1) \, | \, j_1 j_2; m_1 m_2\rangle \\ \vec{J}_{1z} \, | \, j_1 j_2; m_1 m_2\rangle &= m_1 \, | \, j_1 j_2; m_1 m_2\rangle \\ \vec{J}_{2z} \, | \, j_1 j_2; m_1 m_2\rangle &= m_2 \, | \, j_1 j_2; m_1 m_2\rangle \end{aligned} \tag{410}$$

<u>OPTION B</u>: Choose the brackets to be the simultaneous eigenkets of $\vec{J}^2, \vec{J}_1^2, \vec{J}_2^2, \vec{J}_z$. Again this is a commuting set of observables. The corresponding base kets

$$| \, j_1 j_2; jm\rangle \tag{411}$$

The notation is obvious. Sometimes we just denote this as $| \, jm\rangle$ since j_1 and j_2 are understood. Note

$$[\vec{J}_1, \vec{J}_2] \neq 0. \tag{412}$$

Consider a unitary transformation between these two bases

$$| \, j_1 j_2; jm\rangle = \sum_{m_1} \sum_{m_2} | \, j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2 \, | \, j_1 j_2; jm\rangle \tag{413}$$

So we have used the relation of identity

$$\sum_{m_1} \sum_{m_2} | \, j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2 \, |= 1 \tag{414}$$

for given j_1 and j_2. The elements of this transformation matrix

$$\langle j_1 j_2; m_1 m_2 \, | \, j_1 j_2; jm\rangle \tag{415}$$

are called Clebsch-Gordon Coefficients. Note that

$$(J_z - J_{1z} - J_{2z}) \, | \, j_1 j_2; jm\rangle = 0 \tag{416}$$

multiplying this with $\langle j_1 j_2; m_1 m_2 \, |$ gives

$$m = m_1 + m_2 \tag{417}$$

FIG 24!!!!

So the projection of the momenta in the z-direction is simply added!!! So $\langle j_1 j_2; m_1 m_2 \, | \, j_1 j_2; jm\rangle$ vanishes for $m \neq m_1 + m_2$. Secondly, the coefficients vanish unless

$$| \, j_1 - j_2 \, | \leq j \leq j_1 + j_2 \tag{418}$$

which is clear from the vector picture. This can be proven by several ways but let us note that the dimensionality of the space spanned by $\{| \, j_1 j_2; m_1 m_2\rangle\}$ is

$$N = (2j_1 + 1)(2j_2 + 1) \tag{419}$$

On the other hand for the (j, m) counting we have $2j+1$ for each m_j and

$$N = \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1) \tag{420}$$

Clebsch-Gordon coefficients form unitary matrix. (By convention, they can be taken to be real.) So $\langle j_1 j_2; jm \mid j_1 j_2; m_1 m_2\rangle$ is the same as $\langle j_1 j_2; m_1 m_2 \mid j_1 j_2; jm\rangle$.

A *real*, unitary matrix is orthogonal:

$$\sum_j \sum_m \langle j_1 j_2; m_1 m_2 \mid j_1 j_2; jm\rangle \langle j_1 j_2; m_1' m_2' \mid j_1 j_2; jm\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{421}$$

Example: Addition of spin-$\frac{1}{2}$ angular momenta:

$$\vec{S} = \vec{S}_1 + \vec{S}_2 \tag{422}$$

There are 4-states

$$\begin{aligned} \mid ++\rangle &\equiv \mid +\rangle \mid +\rangle \\ \mid --\rangle &\equiv \mid -\rangle \mid -\rangle \\ \mid +-\rangle &= \mid +\rangle \mid -\rangle \\ \mid -+\rangle &= \mid -\rangle \mid +\rangle \end{aligned} \tag{423}$$

An important question is that: Can we really distinguish the last two states for identical particles? The answer is NO!! Now

$$S_z = S_{1z} + S_{2z} \tag{424}$$

such that

$$S_z \mid ++\rangle = \hbar \mid ++\rangle \quad , \quad S_z \mid --\rangle = -\hbar \mid --\rangle \quad , \quad S_z \mid +-\rangle \;= 0 \quad , \quad S_z \mid -+\rangle = 0 \tag{425}$$

So as expected from our general formalism, total spin is

$$\mid S_1 - S_2 \mid \le S \le S_1 + S_2 \quad \Rightarrow \quad 0 \le S \le 1, \quad \text{so} \quad S = 0 \text{ or } 1! \tag{426}$$

Then

$$m = -1, 0, 1 \tag{427}$$

Therefore $\mid sm\rangle$ basis reads

$$\begin{aligned} \mid 1,1\rangle &= \mid ++\rangle \\ \mid 1,0\rangle &= \frac{1}{\sqrt{2}}\Big(\mid +-\rangle + \mid -+\rangle\Big) \\ \mid 1,-1\rangle &= \mid --\rangle \\ \mid 0,0\rangle &= \frac{1}{\sqrt{2}}\Big(\mid +-\rangle - \mid -+\rangle\Big) \end{aligned} \tag{428}$$

Recall that $m = m_1 + m_2$. Check

$$S_z \mid 1,0\rangle = (S_{1z} + S_{2z}) \frac{1}{\sqrt{2}}\Big(\mid +-\rangle + \mid -+\rangle\Big) \tag{429}$$

Check also

$$S^2 \mid 1,0\rangle = \sqrt{2}\,\hbar^2 \mid 1,0\rangle \tag{430}$$

Summary:

1. The finite dimensional representation of the Lorentz algebra can be labeled by two half-integers (j_-, j_+).

2. The dimension of the representation (j_-, j_+) is $(2j_-+1)(2j_++1)$. This is in general complex dimension.

3. The generator of rotations $\vec{J}$ is related to $\vec{J}^+$ and $\vec{J}^-$ by

$$\vec{J} = \vec{J}^+ + \vec{J}^- \tag{431}$$

So *addition of angular momenta* yield for the representation (j_-, j_+) we have all the possible spin j in the integer steps

$$\mid j_- - j_+ \mid \leq j \leq j_+ + j_- \tag{432}$$

The representations are in general complex.

Simplest Cases:

(a) $(0,0)$ has dimension one. $\vec{J}^{\pm} = 0, \vec{J} = 0, \vec{K} = 0$. It is the scalar representation.

(b) $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ both have dimension 2 (complex 2×2 matrices). They are spinorial representations. A spinor in $(\frac{1}{2}, 0)$ is denoted as $(\psi_L)_\alpha$ with $\alpha = 1, 2$. A spinor in $(0, \frac{1}{2})$ is denoted as $(\psi_R)_\alpha$ with $\alpha = 1, 2$.

Here $(\psi_L)_\alpha$ is Left-handed Weyl spinor and $(\psi_R)_\alpha$ is Right-handed Weyl spinor. (In the Literature $(\psi_L)_\alpha$ is denoted by $\bar{\chi}_{\dot{\alpha}}$ with dotted and undotted notations.)

So how do $\vec{J}$ and $\vec{K}$ act on Weyl spinors?

Example: $(\frac{1}{2}, 0)$ on this representation $\vec{J}^-$ is represented by a 2×2 matrix, while $\vec{J}^+ = 0$. So the solution of $[J^{-i}, J^{-j}] = i\epsilon^{ijk} J^{-k}$ can be $\vec{J}^- = \frac{\vec{\sigma}}{2}$. Then

$$\vec{J} = \vec{J}^+ + \vec{J}^- = \frac{\vec{\sigma}}{2}, \qquad \vec{K} = -(\vec{J}^+ - \vec{J}^-) = i\frac{\vec{\sigma}}{2} \tag{433}$$

where $\vec{K}$ is non-Hermitian leading to non-unitary representation. (Recall that non-compact groups have no unitary finite dimensional representations!)

So then

$$\psi_L \ \rightarrow \ \Lambda_L \psi_L \tag{434}$$

under Lorentz transformations. More precisely,

$$\psi_L^{'} = e^{(-i\vec{\theta} - \vec{\eta}) \cdot \frac{\vec{\sigma}}{2}} \psi_L. \tag{435}$$

For the $(0, \frac{1}{2})$ representation, we have

$$\vec{J} = \frac{\vec{\sigma}}{2}, \qquad \vec{K} = -i\frac{\vec{\sigma}}{2} \tag{436}$$

such that

$$\psi_R \;\rightarrow\; \Lambda_R\psi_R = e^{(-i\vec{\theta}+\vec{\eta})\cdot\frac{\vec{\sigma}}{2}}\psi_R. \tag{437}$$

Here $\Lambda_{L,R}$ are complex. So $\psi_{R,L}$ are complex.

<u>Check first that</u>: $\sigma^2\sigma^i\sigma^2 = -\sigma^{i*}$

Recall that

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then let us show that

$$\sigma^2\Lambda_L^*\sigma^2 = \Lambda_R \tag{438}$$

which is

$$\sigma^2\, e^{(i\vec{\theta}-\vec{\eta})\cdot\frac{\vec{\sigma}^*}{2}}\,\sigma^2 = \Lambda_R \tag{439}$$

Then

$$\sigma^2\psi_L^* \;\rightarrow\; \sigma^2(\Lambda_L\psi_L)^* = \sigma^2\Lambda_L^*\sigma^2\sigma^2\psi_L^* = \Lambda_R\sigma^2\psi_L^* \tag{440}$$

So if $\psi_L \in (\frac{1}{2},0)$, then clearly $\sigma^2\psi_L^*$ is a right-handed spinor and so $\sigma^2\psi_L^* \in (0,\frac{1}{2})$. Needless to say that neutrinos s in the Standard Model is a left-handed Weyl-spinor.

<u>CHARGE CONJUGATION</u>:

$$\psi_L^c \equiv i\sigma^2\psi_L^* \tag{441}$$

So the charge conjugation transforms a left-handed spinor into a right-handed one. On a right-handed spinor, we have

$$\psi_R^c \equiv -i\sigma^2\psi_R^* \tag{442}$$

where ψ_R^c is a left-handed spinor. Observe that the "i" in the definition gave us the following expected relation

$$(\psi_L^c)^c \equiv (i\sigma^2\psi_L^*)^c = -i\sigma^2(i\sigma^2\psi_L^*)^* = \psi_L \tag{443}$$

where we have used $\sigma^{2*} = -\sigma^2$. We will discuss the physical meaning later.

<u>NEXT</u>: $(\frac{1}{2},\frac{1}{2})$ has complex dimensions 4.

$$\mid \frac{1}{2}-\frac{1}{2} \mid\leq j \leq \frac{1}{2}+\frac{1}{2} \quad\Rightarrow\quad 0\leq j\leq 1 \tag{444}$$

So we have

$$j = 0 \;\text{ or }\; j = 1 \tag{445}$$

A generic element in this representation can be written as

$$\Big((\psi_L)_\alpha, (\xi_R)_\beta\Big) \qquad \alpha, \beta = 1, 2 \tag{446}$$

where $(\psi_L)_\alpha$ and $(\xi_R)_\beta$ are independent Weyl spinors.

Explicitly, what is the relation between these 4 complex quantities and four components of a 4-vector?

OK, given a right-handed spinor ξ_R, we can form a left-handed spinor ξ_L by

$$\xi_L \equiv -i\sigma^2 \xi_R^* \tag{447}$$

And similarly from ψ_L, we can form

$$\psi_R \equiv i\sigma^2 \psi_L^* \tag{448}$$

Then define

$$\sigma^\mu \equiv (\mathbb{I}, \sigma^i) \qquad \text{so that} \qquad \bar{\sigma}^\mu = (\mathbb{I}, -\sigma^i) \tag{449}$$

Then we can show that

$$\xi_R^+ \sigma^\mu \psi_R \quad \text{and} \quad \xi_L^+ \bar{\sigma}^\mu \psi_L \tag{450}$$

are contravariant 4-vectors.

<u>Lets Check</u>: Under Lorentz transformations

$$V^\mu \equiv \xi_R^{'+} \sigma^\mu \psi_R^{'} = \xi_R^+ \Lambda_R^+ \sigma^\mu \Lambda_R \psi_R \tag{451}$$

Now

$$\Lambda_R = e^{(-i\vec{\theta}+\vec{\eta})\cdot\frac{\vec{\sigma}}{2}} \quad \Rightarrow \quad \Lambda_R^+ = e^{(i\vec{\theta}+\vec{\eta})\cdot\frac{\vec{\sigma}}{2}} \tag{452}$$

So

$$e^{(i\vec{\theta}+\vec{\eta})\cdot\frac{\vec{\sigma}}{2}}\, \sigma^\mu\, e^{(-i\vec{\theta}+\vec{\eta})\cdot\frac{\vec{\sigma}}{2}} = \tag{453}$$

$(1,0)$ and $(0,1)$ representations. Both have complex dimension 3. These correspond to self-dual and anti-self dual tensors $A^{\mu\nu}$. Here solve problem 2.5.

VI. FIELD REPRESENTATIONS

We would like to construct a Lorentz invariant field theory

$$x^\mu \to x^{'\mu} = \Lambda^\mu{}_\nu\, x^\nu, \quad \phi(x) \to \phi^{'}(x^{'}) \tag{454}$$

A. Scalar Fields

$$\phi^{'}(x^{'}) = \phi(x) \qquad \text{definition} \tag{455}$$

Note that $x^{'}$ and x denote the *same* point in different reference frames. So *numerical value of a scalar field at a point P is Lorentz invariant.* Functional form of $\phi(x)$ changes to keep $\phi(x)$ invariant.

Consider an infinitesimal Lorentz transformation

$$x^{'\rho} = x^\rho + \delta x^\rho \quad \text{where} \quad \delta x^\rho \equiv \epsilon^\rho{}_\sigma x^\sigma \tag{456}$$

We also know that

$$\delta x^\rho = -\frac{i}{2}\epsilon_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma \tag{457}$$

and

$$(J^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma) \tag{458}$$

By definition under this transformation

$$\delta\phi \equiv \phi^{'}(x^{'}) - \phi(x) = 0 \tag{459}$$

So the scalar representation corresponds to the trivial representation $J^{\mu\nu} = 0$ of the group. Instead consider the variation at a fixed coordinate

$$\delta_L\phi \equiv \phi^{'}(x) - \phi(x) \tag{460}$$

where L refers to the Lie derivative. So now we are comparing different DOF. And there are infinitely many. So we need an ∞ dimensional representation of the Lorentz group. We have to actually say why this might be relevant at all.

$$S \simeq \int d^4x\, \partial_\mu\phi(x)\partial^\mu\phi(x) \tag{461}$$

and

$$S^{'} \simeq \int d^4x'\, \partial'_\mu\phi'(x')\partial'^\mu\phi'(x') \simeq \int d^4x\, \partial_\mu\phi'(x)\partial^\mu\phi'(x) \tag{462}$$

So we really need to know this quantity $\phi'(x)$. So how do we find the generators?

$$\begin{aligned}
\delta_0\phi(x) &= \phi^{'}(x^{'} - \delta x) - \phi(x) \\
&= \phi^{'}(x^{'}) - \delta x^\rho \partial_\rho \phi^{'}(x^{'}) - \phi(x) \\
&= -\delta x^\rho\partial_\rho\phi(x) = \frac{i}{2}\epsilon_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma\partial_\rho\phi(x) \\
&= -\frac{1}{2}\epsilon_{\mu\nu}(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma)x^\sigma\partial_\rho\phi(x) \\
&= -\frac{1}{2}\epsilon_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\phi(x) \\
&\equiv -\frac{i}{2}\epsilon_{\mu\nu}L^{\mu\nu}\phi(x)
\end{aligned} \tag{463}$$

So

$$L^{\mu\nu} \equiv -i(x^\nu \partial^\mu - x^\mu \partial^\nu) = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{464}$$

Therefore under the Lorentz transformations, the scalar field has a non-zero Lie derivative. Now, define $p^\mu = i\partial^\mu$, then

$$L^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu \tag{465}$$

which is the generalization of $\vec{L} = \vec{r} \times \vec{p}$ relation. Now

$$L^{ij} = x^i p^j - x^j p^i \quad \Rightarrow \quad L^i \equiv \frac{1}{2}\epsilon^{ijk} L^{jk} = \epsilon^{ijk} x^j p^k \tag{466}$$

How about L^{0i}?

$$L^{0i} = tp^i - p^0 x^i = tm\gamma v^i - m\gamma x^i = -m\gamma(x^i - tv^i) \tag{467}$$

where $r^i = (x^i - tv^i)$ is the position of the particle. So $\gamma m \vec{r}$ is the relativistic mass of the particle. If we had many particles it would be

$$\frac{\sum \gamma_i m_i r_i}{\sum m_i} \sim \text{Center of Mass.} \tag{468}$$

Of course every object have special relativistic partner. How about Torque ($\tau^{\mu\nu}$)?

$$\tau^{\mu\nu} = x^\mu f^\nu - x^\nu f^\mu \tag{469}$$

So

$$\tau^{0i} = tf^i - x^i f^0 = tf^i - x^i \vec{F} \cdot \vec{v} \tag{470}$$

Here f^i is impulse. So this is the relativistic version of impulse.

B. Weyl Fields

Under the Lorentz transformations $x' = \Lambda x$, left-handed Weyl field transforms according to

$$\psi_L(x) \;\rightarrow\; \psi_L'(x') \equiv \Lambda_L \psi_L(x) \tag{471}$$

where $\Lambda_L = e^{(-i\vec{\theta} - \vec{\eta})\cdot\frac{\vec{\sigma}}{2}}$. In the classical theory $\psi_L(x)$ is an ordinary commuting c-number.

$$\psi_R(x) \;\rightarrow\; \psi_R'(x') \equiv \Lambda_R \psi_R(x) \tag{472}$$

where $\Lambda_R = e^{(-i\vec{\theta} + \vec{\eta})\cdot\frac{\vec{\sigma}}{2}}$. So we have

$$\begin{aligned} \delta_0 \psi_L &\equiv \psi_L'(x) - \psi_L(x) \\ &= \psi_L'(x' - \delta x) - \psi_L(x) \\ &= \psi_L'(x') - \delta x^\rho \partial_\rho \psi_L'(x) - \psi_L(x) \\ &= (\Lambda_L - 1)\psi_L(x) - \delta x^\rho \partial_\rho \psi_L(x) \end{aligned} \tag{473}$$

Note that the last term is exactly as in the scalar field case. So the 2×2 matrices

$$\Lambda_L = e^{-\frac{i}{2}\epsilon_{\mu\nu} S^{\mu\nu}} \simeq \mathbb{I} - \frac{i}{2}\epsilon_{\mu\nu} S^{\mu\nu} \tag{474}$$

Note also that $S^{i0} = i\frac{\sigma^i}{2} \;\Rightarrow\; S^i \equiv \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{\sigma^i}{2}$. (For right-handed one $S^{i0} = -i\frac{\sigma^i}{2}$). Hence, we obtain

$$\delta_0\psi_L = -\frac{i}{2}\epsilon_{\mu\nu}J^{\mu\nu}\psi_L(x) \tag{475}$$

So we have

$$J^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu} \tag{476}$$

Thus angular momentum splits into orbital and spin parts.

C. Dirac Fields

Consider a parity transformation $(t,\vec{x}) \to (t,-\vec{x})$. Every true vector should transform like $\vec{x}$ under parity. $\vec{K}$ being a true vector transforms as $\vec{K} \to -\vec{K}$ under parity. (By the way, how do we know $\vec{K}$ is a true vector ?) We require the expectation value to transform like a vector

$$\langle\alpha \mid V_i \mid \alpha\rangle \to \sum R_{ij}\langle\alpha \mid V_j \mid \alpha\rangle \tag{477}$$

where $U^+V_iU = \sum_j R_{ij}V_j$, whose infinitesimal version is the commutation relation. So under parity transformation $J^i_+ \leftrightarrow J^i_-$. Recall that

$$\vec{J}^+ = \frac{\vec{J}+i\vec{K}}{2}, \qquad \vec{J}^- = \frac{\vec{J}-i\vec{K}}{2} \tag{478}$$

So $(j_-, j_+) \leftrightarrow (j_+, j_-)$. Therefore ψ_L and ψ_R are not parity eigenstates.

If $j_- = j_+$ then the representation respects the parity symmetry otherwise parity is not a good, conserved quantum number. In nature, parity is violated by weak interactions. Left and right handed spinors enter the theory in a different way. But the affects of the weak interaction appear at $O(100\ GeV)$ which is much higher than the strong interaction and the Electromagnetic interaction scale. Strong and EM interactions respect and conserve parity. So $(0,0)$ scalar representation respects parity (where does the pseudoscalar fit into this picture? Well recall that we are looking at the representations of the connected part of the Lorentz group)

Parity violation in weak interaction 1956 (Lee and Yang gave the theory) Wa (1957) did the experiment.

Radioactive Cobalt 60 undergoes a β-decay. She found that most of the emitted electrons came out in the direction of the nuclear spin

FIG 25!!!!

Dirac field written (in the chiral basis) is

$$\psi = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} : \quad 4 \text{ complex components}$$

such that

$$\psi \to \Lambda_D\psi \quad \text{where} \quad \Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$$

Under parity $x^\mu \to x^{'\mu} = (t, -\vec{x})$

$$\begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \to \begin{pmatrix} \psi_R(x^{'}) \\ \psi_L(x^{'}) \end{pmatrix}$$

So

$$\psi \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(x^{'})$$

Recall that we defined the operation of charge conjugation on Weyl spinors

$$\psi_L^c = i\sigma^2 \psi_L^*, \quad \psi_R^c = -i\sigma^2 \psi_R^* \tag{479}$$

So a left-handed spinor is transformed into a right-handed one and vice versa. Given a Dirac spinor ψ, we can define a new Dirac spinor

$$\psi^c = \begin{pmatrix} -i\sigma^2 \psi_R^* \\ i\sigma^2 \psi_L^* \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \psi^*$$

Note that x^μ are unchanged under charge conjugation. Note also that

$$(\psi^c)^c = \psi \tag{480}$$

D. Majorana Spinors

(E. Majorana 1906-1938) A Majorana spinor is a Dirac spinor in which ψ_L and ψ_R are not independent. But

$$\psi_R = i\sigma^2 \psi_L^* \tag{481}$$

Now

$$\psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix}$$

Note that even though it has 4 components, it has the same number of DOF as a Weyl spinor.

Unlike the case of a complex scalar field or the complex vector field the condition

$$\psi_D^* = \psi_D \tag{482}$$

is *not* Lorentz invariant since ψ_D is not real! But

$$\psi_M^c = \begin{pmatrix} -i\sigma^2 i\sigma^2 \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix} = \psi_M$$

So

$$\psi_M^c = \psi_M \tag{483}$$

is the "reality" condition. Majorana fields are in some sense real Dirac fields.

E. Vector Fields

$$x^{\mu} \to x^{'\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \quad V^{\mu}(x) \to V^{'\mu}(x^{'}) = \Lambda^{\mu}{}_{\nu} V^{\nu}(x) \tag{484}$$

Since a vector field belongs to the $(\frac{1}{2}, \frac{1}{2})$ representation, it has $j_{-} = j_{+}$ and so it is a basis for parity.

VII. THE POINCARE GROUP

Sometimes called the Inhomogeneous Lorentz group $ISO(1,3)$. Up to now we have ignored translations. Let us include them in our picture. Recall that the interval $(x-y)^2$ is invariant under $x^{\mu} \to x^{'\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$. (Λ, a) are constants.

Say we make two consecutive transformations

$$x^{'\mu} = \Lambda_1 x + a_1 \quad ; \quad x^{''} = \Lambda_2 x^{'} + a_2 \tag{485}$$

Then

$$x^{''} = \Lambda_2 \Lambda_1 x + \Lambda_2 a_1 + a_2 \tag{486}$$

So let (Λ, a) be an element of $ISO(3,1)$.

Closure:

$$(\Lambda_2, a_2) \cdot (\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) \in ISO(3,1) \tag{487}$$

Note that this is a non-Abelian group. Here the identity is $(\mathbb{I}_{4\times 4}, 0)$.

Inverse

$$(\Lambda_2, a_2) \cdot (\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) = (\mathbb{I}_{4\times 4}, 0) \tag{488}$$

So

$$\Lambda_2 = \Lambda_1^{-1} \quad \text{and} \quad a_2 = -\Lambda_1^{-1} a_1 \tag{489}$$

So inverse of (Λ_1, a_1) is $(\Lambda_1^{-1}, \Lambda_1^{-1} a_1)$.

Associativity:

$$(\Lambda_2, a_2) \cdot (\Lambda_1, a_1) \cdot (\Lambda, a) : \qquad \text{easy to see} \tag{490}$$

$ISO(3,1) = \mathcal{P}$ is a semi-direct product of the Lorentz group and the translations. Just like the Lorentz group it also splits into 4 pieces

$$\mathcal{P}_{+}^{\uparrow}, \quad \mathcal{P}_{+}^{\downarrow}, \quad \mathcal{P}_{-}^{\uparrow}, \quad \mathcal{P}_{-}^{\downarrow} \tag{491}$$

Consider $\mathcal{P}_{+}^{\uparrow}$ what is the Lie algebra?

Consider a faithful representation $(\Lambda, a) \to g(\Lambda, a)$ acting on a vector space $\mathcal{V}$ such that

$$g(\Lambda_2, a_2) \cdot g(\Lambda_1, a_1) = g(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) \quad \text{and} \quad g^{-1}(\Lambda, a) = g(\Lambda^{-1}, -\Lambda^{-1} a) \tag{492}$$

Then near the identity, we can write

$$g(\Lambda, a) \equiv \mathbb{I} - \frac{i}{2}\epsilon_{\mu\nu}J^{\mu\nu} + ia_\mu\mathcal{P}^\mu + O(\epsilon^2, a^2) \tag{493}$$

Here $J^{\mu\nu}$ and $\mathcal{P}^\mu$ are generators of the Lorentz transformations and translations, respectively. Then consider

$$g^{-1}(\Lambda,0)\cdot g(\Lambda_1,a_1)\cdot g(\Lambda,0) = g^{-1}(\Lambda,0)\cdot g(\Lambda_1\Lambda,a_1) = g(\Lambda^{-1},0)\cdot g(\Lambda_1\Lambda,a_1) = g(\Lambda^{-1}\Lambda_1\Lambda,a_1) \tag{494}$$

Consider this for infinitesimal Λ_1 and a_1. Then

$$g^{-1}(\Lambda,0)\Big[\mathbb{I} - \frac{i}{2}\epsilon_1 J + ia_1\mathcal{P}\Big]g(\Lambda,0) = \mathbb{I} - \frac{i}{2}\Lambda^{-1}\epsilon_1\Lambda J + i\Lambda^{-1}a_1\mathcal{P} \tag{495}$$

So

$$\mathbb{I} - \frac{i}{2}\epsilon^1_{\mu\nu}g^{-1}(\Lambda,0)J^{\mu\nu}g(\Lambda,0) + ia'_\mu g^{-1}(\Lambda,0)\mathcal{P}^\mu g(\Lambda,0) = \mathbb{I} - \frac{i}{2}(\Lambda^{-1})^\sigma{}_\mu\epsilon^{\mu\nu}\Lambda_\nu{}^\rho J_{\sigma\rho} + i(\Lambda^{-1})^\sigma{}_\nu a^\nu\mathcal{P}_\sigma \tag{496}$$

So we have

$$\epsilon_{\mu\nu}g^{-1}(\Lambda,0)J^{\mu\nu}g(\Lambda,0) = \Lambda_\mu{}^\sigma\Lambda_\nu{}^\rho\epsilon^{\mu\nu}J_{\sigma\rho} \quad \text{and} \quad a_\mu g^{-1}(\Lambda,0)\mathcal{P}^\mu g(\Lambda,0) = \Lambda_\nu{}^\sigma a^\nu\mathcal{P}_\sigma \tag{497}$$

or dropping ϵ and a we have

1. $g^{-1}(\Lambda,0)J^{\mu\nu}g(\Lambda,0) = \Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho J^{\sigma\rho}$
2. $g^{-1}(\Lambda,0)\mathcal{P}^\mu g(\Lambda,0) = \Lambda^\mu{}_\sigma\mathcal{P}^\sigma$

Now consider Λ such that $\Lambda = \mathbb{I} + \epsilon$. The 1^{st} equation above yields

$$(\mathbb{I} + \frac{i}{2}\epsilon J)J^{\mu\nu}(\mathbb{I} - \frac{i}{2}\epsilon J) = (\delta^\mu{}_\sigma + \epsilon^\mu{}_\sigma)(\delta^\nu{}_\rho + \epsilon^\nu{}_\rho)J^{\sigma\rho} \tag{498}$$

which becomes

$$J^{\mu\nu} - \frac{i}{2}J^{\mu\nu}\epsilon J + \frac{i}{2}\epsilon J J^{\mu\nu} = J^{\mu\nu} + \epsilon^\nu{}_\rho J^{\mu\rho} + \epsilon^\mu{}_\sigma J^{\sigma\nu} \tag{499}$$

which then yields

$$[J^{\sigma\rho}, J^{\mu\nu}] = -i(g^{\mu\sigma}J^{\rho\nu} - g^{\mu\rho}J^{\sigma\nu} - g^{\nu\rho}J^{\mu\sigma} + g^{\nu\sigma}J^{\mu\rho}) \tag{500}$$

Now let us check the 2^{nd} equation

$$(\mathbb{I} + \frac{i}{2}\epsilon J)\mathcal{P}^\sigma(\mathbb{I} - \frac{i}{2}\epsilon J) = (\delta^\mu{}_\sigma + \epsilon^\sigma{}_\mu)\mathcal{P}^\mu \tag{501}$$

which gives

$$[J^{\alpha\beta}, \mathcal{P}^\sigma] = i(g^{\beta\sigma}\mathcal{P}^\alpha - g^{\alpha\sigma}\mathcal{P}^\beta) \tag{502}$$

It is also easy to show that $[\mathcal{P}^\mu, \mathcal{P}^\nu] = 0$ which is Abelian subalgebra. To find it let us do the following

$$g^{-1}(0,a)\cdot g(0,a_1)\cdot g(0,a) = g(0,a_1) \tag{503}$$

For small a_1, we have

$$g(0,-a)\cdot(\mathbb{I}+ia_1\mathcal{P})\cdot g(0,a)=\mathbb{I}+ia_1\mathcal{P} \tag{504}$$

For small a we have

$$(\mathbb{I}-ia\mathcal{P})\cdot(\mathbb{I}+ia_1\mathcal{P})\cdot(\mathbb{I}+ia\mathcal{P})=\mathbb{I}+ia_1\mathcal{P} \tag{505}$$

so we get

$$[\mathcal{P}^\mu,\mathcal{P}^\nu]=0 \tag{506}$$

So we have in our hands the full Poincare algebra. What are the representations of this algebra?

Important: What are the Casimir (quadratic) invariants of the Poincare group?

1. $\mathcal{P}^2=\mathcal{P}_\mu\mathcal{P}^\mu$ prove $[J_{\mu\nu},\mathcal{P}^2]=0$ and $[\mathcal{P}_\mu,\mathcal{P}^2]=0$.

2. The second Casimir invariant is defined from the Pauli-Lubanski polarization pseudovector

$$W_\mu\equiv\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{P}^\nu J^{\rho\sigma} \tag{507}$$

   Then $W_\mu W^\mu$ is the second Casimir Operator.

   (a) Show that $[\mathcal{P}_\mu,W_\nu]=0$

   (b) $[J_{\mu\nu},W_\rho]=-i(g_{\mu\rho}W_\nu-g_{\nu\rho}W_\mu)$. So W_μ is a vector under $L_+^\uparrow$.

   (c) $[W^2,J_{\mu\nu}]=0$ and $[W^2,\mathcal{P}_\mu]=0$

   So altogether, defining

$$\mathcal{P}^0\equiv H,\quad J^i\equiv\frac{1}{2}\epsilon^{ijk}J^{jk},\quad K^i=J^{i0} \tag{508}$$

   we have the Poincare algebra in $3+1$ dimensions as

$$\begin{array}{l}[J^i,J^j]=i\epsilon^{ijk}J^k\quad(\sim\vec{J}\times\vec{J}=i\vec{J})\quad;\quad[J^i,K^j]=i\epsilon^{ijk}K^k\quad;\quad[J^i,\mathcal{P}^j]=i\epsilon^{ijk}\mathcal{P}^k\\ [K^i,K^j]=-i\epsilon^{ijk}J^k\quad;\quad[\mathcal{P}^i,\mathcal{P}^j]=0\quad;\quad[K^i,\mathcal{P}^j]=iH\delta^{ij}\quad;\quad[J^i,H]=0\\ [\mathcal{P}^i,H]=0\quad;\quad[K^i,H]=i\mathcal{P}^i\end{array} \tag{509}$$

   Here J^i : generators of spatial rotations; K^i : Boosts; $\mathcal{P}^i$: translations in space; H : translations in time. J^i and $\mathcal{P}^i$ are conserved but K^i is not! Note also that

$$[K^i,K^j]=-i\epsilon^{ijk}J^k \tag{510}$$

   corresponds to Thomas Precession (1925 Nature). Explain this effect.

A. Representation on Fields

We have seen that fields provide an ∞-dimensional representation of the Lorentz group and the generators that act on the fields are of the form

$$J^{\mu\nu} = S^{\mu\nu} \bigoplus L^{\mu\nu} \tag{511}$$

where $S^{\mu\nu}$ depends on the spin of the field. Here

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) = x^\mu p^\nu - x^\nu p^\mu \quad \text{where} \quad p^\mu = i\partial^\mu \tag{512}$$

Now to obtain the representations of the full Poincare group, we must find the representations of p^μ.

We require that all fields, independent of their transformation properties under the Lorentz group, transform as *scalars* under spacetime translation. So as

$$x^\mu \to x^{'\mu} = x^\mu + a^\mu \quad \Rightarrow \quad \phi^{'}(x^{'}) = \phi(x) \tag{513}$$

where ϕ denotes a generic field, it could be a spinor, vector etc.

Consider an infinitesimal translation $x^{'\mu} = x^\mu + \epsilon^\mu$ where ϵ^μ is constant. So

$$\begin{aligned} \delta_0 \phi(x) &\equiv \phi^{'}(x) - \phi(x) \\ &= \phi^{'}(x^{'} - \epsilon) - \phi(x) \\ &= -\epsilon^\mu \partial_\mu \phi(x) \\ &= i\epsilon_\mu p^\mu \phi(x) \end{aligned} \tag{514}$$

A generic element of the translation group is $exp(-ia_\mu p^\mu)$.

B. Representation on One Particle States

Soon we will construct Poincare invariant Lagrangians. At the quantum level, the concept of particles emerge. So we would like to know how the Poincare group act on particle states.

Consider the Hilbert space of a free particle

$$\mid \vec{p}, s\rangle \tag{515}$$

where $\vec{p}$ is the momentum and s labels all the other quantum numbers such as the spin etc. of the particle. Also $\vec{p}$ is continuous and unbounded. So this Hilbert space is infinite dimensional.

A theorem by Wigner says that: On this Hilbert space any symmetry transformation can be represented by a unitary or anti-unitary operator.

So Poincare transformations can be represented by unitary matrices/operators and the generators $J^i, K^i, \mathcal{P}^i, H$ by Hermitian operators. How are the representations labeled? Of course with the Casimirs that we discussed

$$W_\mu W^\mu \quad \text{and} \quad \mathcal{P}_\mu \mathcal{P}^\mu \tag{516}$$

1.

$$\mathcal{P}_\mu \mathcal{P}^\mu = m^2 \quad \Rightarrow \quad m^2 = 0 \ ; \ m^2 > 0 \ ; \ m^2 < 0 \tag{517}$$

Note that for the $m^2 > 0$ case, one has either $m < 0$ (nobody has seen them) or $m > 0$. Note also that the $m^2 < 0$ case denote tachyons (no body has seen these either.).

2. <u>Massive Particles</u>: $m > 0$ and $m^2 > 0$ cases. We can go to the rest frame $p^\mu = (m, 0, 0, 0)$. Then

$$W^\mu = \frac{1}{2}\epsilon^\mu{}_{\alpha\beta\sigma}\mathcal{P}^\alpha J^{\beta\sigma} \tag{518}$$

$$W^0 = \frac{1}{2}\epsilon^0{}_{ijk}\mathcal{P}^i J^{ik} = 0 \tag{519}$$

$$W^i = \frac{1}{2}\epsilon^i{}_{\alpha\beta\sigma}\mathcal{P}^\alpha J^{\beta\sigma} = \frac{m}{2}\epsilon^i{}_{0jk}J^{jk} = -\frac{m}{2}\epsilon^{ijk}J^{jk} \tag{520}$$

So

$$W^0 = 0 \quad \text{and} \quad W^i = -mJ^i \tag{521}$$

Since $W_\mu W^\mu$ is Lorentz invariant, we have

$$W_\mu^2 = W_0^2 - \vec{W}^2 = -m^2\vec{J}^2 \tag{522}$$

therefore

$$W^2 = -m^2\vec{J}^2 \tag{523}$$

So on a one-particle state with mass m and $spin - j$ we have

$$-W^2 = m^2 j(j+1) \quad \text{where} \quad m \neq 0 \tag{524}$$

So we can label the particles with mass and spin of the mass is non-zero.Here the mass $m^2 > 0$ can acquire any value. But spin could be $0, 1/2, 1, 3/2, \cdots$. So a massive particle with spin j has $2j+1$ DOF. Note that this seems like the total spin but we have gone to the rest frame of the particle and so there should not be any orbital angular momentum.

Consider the case of the electron $j = 1/2$, it has 2 DOF so $j_z = 1/2$ or $j_z = -1/2$

FIG 26!!!!

$$\cos\theta = \frac{j_z}{|\ \vec{J}\ |} = \frac{1/2}{\sqrt{1/2(1+1/2)}} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \arccos(\frac{1}{\sqrt{3}}) \simeq 54^\circ \tag{525}$$

In general for a $spin - j$ particle

$$\theta = \arccos\Big(\frac{m}{\sqrt{j(j+1)}}\Big) \quad \text{where} \quad m = -j, -j+1, \cdots, 0, \cdots, +j. \tag{526}$$

For a

$$\begin{aligned} &\text{a massive photon}: \quad j = 1 \quad \text{so} \quad j_z = -1, 0, +1 \\ &\text{a massive graviton}: \quad j = 2 \quad \text{so} \quad j_z = -1, 0, +1, 2 \end{aligned} \tag{527}$$

$p^2 = 0$, the rest frame does not exist!!! So choose $p^\mu = (w, 0, 0, w)$. So the particle is going say in the $\hat{z}$-direction. The *little group* is the set of Poincare transformations that leaves p^μ invariant. One can use the representations of the little group to *induce* representations of the full group. These are called *induced representations*.

Recall that in the case of massive particles we have chosen $p^\mu = (w, 0, 0, 0)$. The little group is $SO(3)$. To include spinors we upgrade $SO(3)$ to $SU(2)$ and the representations of $SU(2)$ are used to induce the representations of the full Poincare group. When $p^\mu = (w, 0, 0, w)$, we can see that rotations in the $x-y$ plane leave the p^μ invariant. That is the $SO(2)$ group generated by J^3.

Technical Details: Actually the little group is a little bit larger. Consider the infinitesimal Lorentz transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu \quad ; \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \tag{528}$$

So

$$\Lambda^\mu{}_\nu p^\nu = p^\mu \quad \Rightarrow \quad \epsilon^{\mu\nu} p_\nu = 0 \tag{529}$$

Since $p_\nu = w(1, 0, 0, -1)$, we can write this as a matrix equation as

$$\begin{pmatrix} 0 & \epsilon^{01} & \epsilon^{02} & \epsilon^{03} \\ -\epsilon^{01} & 0 & \epsilon^{12} & \epsilon^{13} \\ -\epsilon^{02} & -\epsilon^{12} & 0 & \epsilon^{23} \\ -\epsilon^{03} & -\epsilon^{13} & -\epsilon^{23} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} -\epsilon^{03} \\ -\epsilon^{01} - \epsilon^{13} \\ -\epsilon^{02} - \epsilon^{23} \\ -\epsilon^{03} \end{pmatrix} = 0$$

So we obtain

$$\epsilon^{03} = 0, \qquad \epsilon^{01} + \epsilon^{13} = 0, \qquad \epsilon^{02} + \epsilon^{23} = 0 \tag{530}$$

Define

$$\epsilon^{01} = \alpha, \qquad \epsilon^{12} = \theta, \qquad \epsilon^{02} = \beta \tag{531}$$

Using these, we can see that the most general Lorentz transformations that leaves p^μ invariant is

$$\Lambda = e^{-i(\alpha A + \beta B + \theta C)} \tag{532}$$

where

$$A^\mu{}_\nu = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad B^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad C^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (J^3)^\mu{}_\nu$$

It is also easy to see that

$$A^\mu{}_\nu = (K^1 + J^2)^\mu{}_\nu, \qquad B^\mu{}_\nu = (K^2 - J^1)^\mu{}_\nu \tag{533}$$

Actually it is not hard to show that for massless particles, we have

$$-W_\mu W^\mu = w^2 \Big[(K^2 - J^1)^2 + (K^1 - J^2)^2 \Big], \quad m = 0 \tag{534}$$

So

$$-W_\mu W^\mu = w^2(A^2 + B^2) \tag{535}$$

We can easily show that

$$[J^3, A] = +iB \quad ; \quad [J^3, B] = -iA \quad ; \quad [A, B] = 0 \tag{536}$$

Formally this is the same algebra generated by the operators $p^x, p^y, L^z = xp^y - yp^x$ which describe the translations and rotations of a Euclidean plane where A and B play the role of translations.

This algebra is $ISO(2)$. $a^\mu{}_\nu$ and $B^\mu{}_\nu$ are not Hermitian (as expected they are 4×4 matrices and a finite dimensional representations of the non- compact Lorentz group algebra). But of course the above algebra can have ∞ dimensional operator representations.

Since $[A, B] = 0$, they can be diagonalized simultaneously. Then

$$A \mid \vec{p}; a, b\rangle = a \mid \vec{p}; a, b\rangle \quad \text{and} \quad B \mid \vec{p}; a, b\rangle = b \mid \vec{p}; a, b\rangle \tag{537}$$

Assume $a \neq 0$ and $b \neq 0$ then we can find a continuous set of eigenvalues. For example check the following state

$$\mid \vec{p}; a, b, \theta\rangle \equiv e^{-i\theta J^3} \mid \vec{p}; a, b\rangle \quad \text{with} \quad \theta \text{ is an arbitrary angle} \tag{538}$$

then

$$\begin{aligned} A \mid \vec{p}; a, b, \theta\rangle &\equiv A\, e^{-i\theta J^3} \mid \vec{p}; a, b\rangle \\ &= e^{-i\theta J^3} e^{i\theta J^3} A\, e^{-i\theta J^3} \mid \vec{p}; a, b\rangle \\ &= e^{-i\theta J^3}(A\cos\theta - B\sin\theta) \mid \vec{p}; a, b\rangle \\ &= e^{-i\theta J^3}(a\cos\theta - b\sin\theta) \mid \vec{p}; a, b\rangle \\ &= (a\cos\theta - b\sin\theta) e^{-i\theta J^3} \mid \vec{p}; a, b\rangle \end{aligned} \tag{539}$$

So we have

$$A \mid \vec{p}; a, b, \theta\rangle = (a\cos\theta - b\sin\theta) \mid \vec{p}; a, b, \theta\rangle \tag{540}$$

$$B \mid \vec{p}; a, b, \theta\rangle = (a\sin\theta + b\cos\theta) \mid \vec{p}; a, b, \theta\rangle \tag{541}$$

This means that unless $a = b = 0$, we find representations corresponding to massless particles with a continuous interval degree of freedom θ. But we do not know such physical representations. So we get $a = b = 0 \Rightarrow -W_\mu W^\mu = 0$. On the states with $A = B = 0$ the little group is $SO(2)$ or $U(1)$.

As for any Abelian groups, the irreducible representations of $SO(2)$ are 1-dimensional. Generators of $SO(2)$ is J^3, so 1-dimensional representations are labeled by the eigenvalue h of J^3.

h : represents the angular momentum in the direction of the propagationof the particle. It is called the *helicity*. It can be shown that h is quantized

$$h = 0, \pm\frac{1}{2}, \pm 1, \cdots \tag{542}$$

(Proof of quantization of h relies on the topology of the Lorentz groups. See Weinberg)

<u>So</u>: Massless particles have only *one* degree of freedom and are characterized by the value of h, their helicity!

On a state of helicity h, $U(1)$ rotation of little group is represented by

$$U(\theta) = e^{-i\theta h} \tag{543}$$

So from the point of view representation theory $+h$ and $-h$ are logically two different particles!

Since $h = \hat{p} \cdot \vec{J}$ then under parity $h \to -h$. So if you have a parity-invariant theory you should assemble together $+h$ and $-h$ helicities. This is the case for the photon $h = \pm$ and the graviton $h = \pm 2$. For neutrinos $h = -1/2$ is neutrinos and $h = +1/2$ is anti-neutrinos .

<u>Example of helicities</u>

1. A classical EM wave propagating in the $\hat{n} = (0, 0, 1)$ direction is defined by the linear polarizations

$$\vec{e}^1 = (1, 0, 0) \quad , \quad \vec{e}^2 = (0, 1, 0) \tag{544}$$

   Define circular polarization vectors $\vec{e}^{\pm} = \vec{e}^1 \pm i\vec{e}^2$. OK, BUT how do $\vec{e}^{\pm}$ transform under a rotation in the $x - y$ plane?

   FIG 27!!!!

$$\vec{e}^{1'} = \cos\theta\,\vec{e}^1 + \sin\theta\,\vec{e}^2 \quad \text{and} \quad \vec{e}^{2'} = -\sin\theta\,\vec{e}^1 + \cos\theta\,\vec{e}^2 \tag{545}$$

   So $\vec{e}^{\pm'} = e^{\mp i\theta}\vec{e}^{\pm}$. Compare at $\vec{e}^{\pm'} = U(\theta)\vec{e}^{\pm} = e^{-ih\theta}\vec{e}^{\pm}$ thus $h = \pm 1$.

2. Classical gravitational wave propagating in the $\hat{n}$- direction is described by h^{ij} (a symmetric, traceless tensor)

$$\eta^{ij} h_{ij} = 0 \quad , \quad \hat{n}^i h_{ij} = 0 \tag{546}$$

   with $h_{+,x}$ plus and cross polarizations

$$h^{ij} = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   Compute the transformation properties of h_+ and h_x under a rotation in the $x - y$ plane. Find the transformation properties of $h_x \pm ih_+$ and conclude that GR has *spin* $- 2$ particle.

$$h^{ij'} = R^i{}_k R^j{}_l h^{kl} \quad \Rightarrow \quad h' = R^T h R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h_+ & h_x \\ h_x & -h_+ \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   So we get

$$h_+' = h_+ \cos 2\theta + h_x \sin 2\theta \quad \text{and} \quad h_x' = h_x \cos 2\theta - h_+ \sin 2\theta \tag{547}$$

   Thus one gets

$$h_x' \pm ih_+' = e^{\pm 2i\theta} h_x \pm ih_+ \quad \Rightarrow \quad h = \pm 2 \tag{548}$$

3. How about a massless spinor?

<u>DIGRESSION</u>: <u>Tensor Representations</u>
By definition a tensor of rank 2 transform under Lorentz transformations as

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} \tag{549}$$

It has $4 \times 4 = 16$ components in $D = 4$ dimensions. This is a 16 dimensional representations of the Lorentz group. But this representation is reducible. Say

$$T^{\mu\nu} = +T^{\nu\mu} \quad \text{or} \quad T^{\mu\nu} = -T^{\nu\mu} \tag{550}$$

Then

$$T'^{\mu\nu} = \pm T^{\mu\nu} \tag{551}$$

So 16 dimensional representation reduces to 10 dimensional symmetric + 6 dimensional anti-symmetric ones. Define

$$S^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) \quad ; \quad A^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) \tag{552}$$

Note also $S = \eta_{\mu\nu}T^{\mu\nu}$ is invariant. So 10 dimensional symmetric representation decouples into 9 dimensional traceless part $(S^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}S)$ and a trace part

$$\begin{aligned} T^{\mu\nu} \in \Big(\mathbf{0}\bigoplus\mathbf{1}\Big)\bigotimes\Big(\mathbf{0}\bigoplus\mathbf{1}\Big) &= \Big(\mathbf{0}\bigotimes\mathbf{0}\Big)\bigoplus\Big(\mathbf{0}\bigotimes\mathbf{1}\Big)\bigoplus\Big(\mathbf{1}\bigotimes\mathbf{0}\Big)\bigoplus\Big(\mathbf{1}\bigotimes\mathbf{1}\Big) \\ &= \mathbf{0}\bigoplus\mathbf{1}\bigoplus\mathbf{1}\bigoplus\Big(\mathbf{0}\bigoplus\mathbf{1}\bigoplus\mathbf{2}\Big) \end{aligned} \tag{553}$$

We used

$$\underbrace{\mathbf{1}}_{j_1}\bigotimes\underbrace{\mathbf{1}}_{j_2}\underbrace{=}_{\Rightarrow}\underbrace{\mathbf{0}\bigoplus\mathbf{1}\bigoplus\mathbf{2}}_{|j_1-j_2|\le j\le j_1+j_2} \tag{554}$$

A long summary of what we have done for the Poincare group (Notes follow D.E. Soper). Rotations $p'^{i} = R^{ij}p^{j}$ where $R^T = R^{-1}$ which is $O(3)$. In QM, a rotation (active rotation) map each possible state $\mid \psi\rangle$ into a new state

$$\mid \psi'\rangle = U(R) \mid \psi\rangle \tag{555}$$

In order to preserve linear superposition principles of QM, $U(R)$ is a linear operator in Hilbert space. In order to preserve probabilistic interpretation we must also have

$$\langle\phi' \mid \psi'\rangle = \langle\phi \mid \psi\rangle \quad \Rightarrow \quad U(R)^{+} = U(R)^{-1} \tag{556}$$

So $U(R)$ is a linear operator. In some cases anti-unitary operators are also needed

$$\langle\phi' \mid \psi'\rangle = \langle\phi \mid \psi\rangle^{*} \tag{557}$$

$U(R_2)U(R_1) = U(R_2R_1)$ so operators U provide a representation of $O(3)$. Actually it suffice to have only "projective" representations

$$U(R_2)U(R_1) = e^{i\phi(R_2,R_1)}U(R_2R_1) \tag{558}$$

<u>infinitesimal Rotations and Finite Rotations</u>:

In order to study the unitary representations of the rotation group, let us consider

$$R = \mathbb{I} + \delta R \quad ; \quad R^T = \mathbb{I} + \delta R^T = R^{-1} \tag{559}$$

So

$$\delta R^T = -\delta R \tag{560}$$

Since this is anti-symmetric and real we can construct it from a combination of 3 matrices as

$$\delta R_{ij} = \delta\theta_k \, r^k_{ij} = -\delta\theta_k \, \epsilon_{ijk} \tag{561}$$

So

$$\delta R_{ij} \equiv -i\delta\theta_k \, J^k_{ij} \quad \text{where} \quad J^k_{ij} = -\epsilon_{ijk} \tag{562}$$

Therefore $R = \mathbb{I} + \delta R$ as given corresponds to an infinitesimal rotation through angle $\mid \delta\vec{\theta} \mid$ about the axis $\frac{\delta\vec{\theta}}{|\delta\vec{\theta}|}$.

<u>Exercise</u>: Show explicitly that if $\delta\theta_k = \delta\phi\delta_{k3}$ then

$$R = \mathbb{I} - i\theta_k J^k_{ij} \tag{563}$$

represents an infinitesimal rotation through angle $\delta\phi$ about the 3-axis.

How about a finite rotation through angle $\mid \vec{\theta} \mid$ about an axis $\frac{\vec{\theta}}{|\vec{\theta}|}$?

$$R(\lambda\vec{\theta} + \delta\lambda\,\vec{\theta}) = R(\delta\lambda\,\vec{\theta})R(\lambda\vec{\theta}) = [\mathbb{I} - i\delta\lambda\,\theta_k J_k + \cdots]R(\lambda\vec{\theta}) \tag{564}$$

Then

$$\frac{\partial R(\lambda\vec{\theta})}{\partial\lambda} = -i\theta_k J_k \, R(\lambda\vec{\theta}) \quad \Rightarrow \quad R(\vec{\theta}) = e^{-i\theta_k J_k} \tag{565}$$

VIII. CLASSICAL FIELD THEORY

1. Lagrangian and action formulation: $\{(\vec{x}_a(t)\}$ $a = 1, 2, \cdots N$ so we have N particles. Then

$$\mathcal{L} = \sum_a \frac{1}{2} m_a \dot{\vec{x}}_a^2 - V(\vec{x}) \tag{566}$$

Mikhail Vasilevich Ostragradoki 1801 – 1862 (Ukrainian).

It is actually a lot better to use generalized coordinates. So consider a classical system with N DOF described by a set of generalized coordinates $q_i(t)$ $i = 1, 2, \cdots, N$, then

$$\mathcal{L}(q, \dot{q}) = \sum_i \frac{m_i}{2} \dot{q}_i^2 - V(q) \quad \Rightarrow \quad S = \int_{t_1}^{t_2} dt \, \mathcal{L} \tag{567}$$

FIG 28!!!!

Now

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \tag{568}$$

So

$$\begin{aligned}\delta S = \int_{t_1}^{t_2} dt\, \delta\mathcal{L} &= \int_{t_1}^{t_2} dt\, \Big[\mathcal{L}(q_i + \delta q_i, \dot{q}_i + \delta\dot{q}_i) - \mathcal{L}(q_i, \dot{q}_i)\Big] \\ &= \int_{t_1}^{t_2} dt\, \Big[\frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i\Big] \\ &= \int_{t_1}^{t_2} dt\, \delta q_i \Big[\frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big] + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i\Big|_{t_1}^{t_2}\end{aligned} \tag{569}$$

So we get

$$\frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = 0, \qquad \text{Euler Lagrange equations } \; i = 1, 2, \cdots N \tag{570}$$

2. <u>Conservative Laws</u>:

   (a) <u>Homogeneity of time</u>: $t \to t + T_0$ is a symmetry $\frac{\mathcal{L}}{\partial t} = 0$. Then

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= \sum_i \Big(\frac{\partial\mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\ddot{q}_i\Big) \\ &= \sum_i \Big(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\;\dot{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\ddot{q}_i\Big) \\ &= \sum_i \frac{d}{dt}\Big(\dot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big)\end{aligned} \tag{571}$$

   So

$$\frac{d}{dt}\Big(\sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i - \mathcal{L}\Big) = 0 \tag{572}$$

   Since

$$\mathcal{H} \equiv \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i - \mathcal{L} \tag{573}$$

   so $\frac{dH}{dt} = 0$ thus *energy is conserved*. Define $p_i \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$.

   (b) <u>The Homogeneity of space</u>: $q_i \to q_i + a_i$ where a_i is constant. Then

$$\mathcal{L}(q_i + a_i, \dot{q}_i) = \mathcal{L}(q_i, \dot{q}_i) \tag{574}$$

   Then

$$\delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial q_i}a_i = \sum_i \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\;a_i \tag{575}$$

   Then define the total momentum as

$$p = \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \sum_i p_i \tag{576}$$

   a constant.

(c) Isotropy of Space: Let us go back to vector notation

$$\vec{x}_a \to \vec{x}_a + \delta\vec{\phi} \wedge \vec{x}_a \tag{577}$$

where $\delta\vec{\phi}$ is time-independent rotations. Now $\delta\vec{x}' = R\delta\vec{x}_a$, for a given a, we have

$$x'^{i} = R^i{}_j\, x^j = (\mathbb{I} + w)^i{}_j\, x^j \quad \Rightarrow \quad x'^{i} = x^i + w^i{}_j\, x^j \tag{578}$$

We know that R is orthogonal so $w^{ij} = -w^{ji}$. So let $w^{ij} = \epsilon^{ijk}\varphi_k$, then we have

$$x'^{i} = x^i + \epsilon^{ijk}\varphi_k x_j \quad \Rightarrow \quad \vec{x}' = \vec{x} + \vec{\varphi} \wedge \vec{x} \tag{579}$$

Assume the invariant of the Lagrangian under this transformations

$$\mathcal{L}[\vec{x}_a + \delta\vec{\phi} \wedge \vec{x}_a,\ \dot{\vec{x}}_a + \delta\vec{\phi} \wedge \dot{\vec{x}}_a] = \mathcal{L}[\vec{x}_a,\ \dot{\vec{x}}_a] \tag{580}$$

so

$$\begin{aligned} \frac{\delta\mathcal{L}}{\partial\phi^j} &= \sum_a \Big[\frac{\partial\mathcal{L}}{\partial x_a^i}\frac{\partial x_a^i}{\partial\phi^j} + \frac{\partial\mathcal{L}}{\partial\dot{x}_a^i}\frac{\partial\dot{x}_a^i}{\partial\phi^j}\Big] \\ &= \sum_a \Big[\frac{\partial\mathcal{L}}{\partial x_a^i}\epsilon^{ijk}x_a^k + \frac{\partial\mathcal{L}}{\partial\dot{x}_a^i}\epsilon^{ijk}\dot{x}_a^i\Big] \end{aligned} \tag{581}$$

Use Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\partial\phi^j} = \sum_a \frac{d}{dt}\Big[\epsilon^{ijk}x_a^k p_a^i\Big] = \frac{d\vec{J}}{dt} = 0 \tag{582}$$

where $\vec{J} = \sum_a \vec{x}_a \wedge \vec{p}_a$ is the total angular momentum. So

$$\frac{d\vec{J}}{dt} = 0 \quad \text{if} \quad \frac{\delta\mathcal{L}}{\partial\phi^j} = 0 \tag{583}$$

For D-dimensions, we will have

$$\mathcal{L}[x^i + w^i{}_j x^j,\ \dot{x}^i + w^i{}_j\,\dot{x}^j] = \mathcal{L}[x^i,\ \dot{x}^i] \tag{584}$$

So

$$\frac{\partial\mathcal{L}}{\partial w^l{}_m} = \frac{\partial\mathcal{L}}{\partial x'^i}\frac{\partial x'^i}{\partial w^l{}_m} + \frac{\partial\mathcal{L}}{\partial\dot{x}'^i}\frac{\partial\dot{x}'^i}{\partial w^l{}_m} \tag{585}$$

Note that

$$\frac{\partial x'^i}{\partial w^l{}_m} = \frac{\partial w^i{}_j}{\partial w^l{}_m}x^j = (\delta^i{}_l\delta^m{}_j - \delta^{im}\delta_{jl})x^j \tag{586}$$

So again using Euler-Lagrange equation

$$\frac{d}{dt}(p^i\delta^i{}_l x^m - p^i x^l \delta^{im}) = 0 \quad \Rightarrow \quad \frac{dJ^{ml}}{dt} = 0 \tag{587}$$

where $J^{ml} = p^l x^m - p^m x^l$.

1. Poisson Bracket

(Simeon Denis Poisson 1781-1840 student of Lagrange and Laplace)
Recall that

$$\mathcal{H} = \sum_a p_a \dot{q}_a - \mathcal{L}(p, \dot{q}, t) \tag{588}$$

But we can go to the new coordinates such that

$$\mathcal{H} = \mathcal{H}()p, q \tag{589}$$

Take the total differential of both equations

$$d\mathcal{H} = \sum_a (dp_a\, \dot{q}_a + p_a d\dot{q}_a) - \sum_a \Big(\frac{\partial \mathcal{L}}{\partial q_a} dq_a + \frac{\partial \mathcal{L}}{\partial \dot{q}_a} d\dot{q}_a\Big) \tag{590}$$

Since $p_a = \frac{\partial \mathcal{L}}{\partial \dot{q}_a}$, it turns out

$$d\mathcal{H} = \sum_a \Big(dp_a\, \dot{q}_a - \frac{\partial \mathcal{L}}{\partial q_a} dq_a\Big) \tag{591}$$

This is from 1^{st} equation. Do the same thing with 2^{nd} equation

$$d\mathcal{H} = \sum_a \Big(\frac{\partial \mathcal{H}}{\partial q_a} dq_a + \frac{\partial \mathcal{H}}{\partial p_a} dp_a\Big) \tag{592}$$

Therefore we have

$$\frac{\partial \mathcal{H}}{\partial q_a} = -\frac{\partial \mathcal{L}}{\partial q_a} \quad , \quad \dot{q}_a = \frac{\partial \mathcal{H}}{\partial p_a} \tag{593}$$

From Euler-Lagrange equations we also have

$$\dot{p}_a = \frac{\partial \mathcal{L}}{\partial q_a} \tag{594}$$

So then the Hamilton's equations are

$$\dot{p}_a = -\frac{\partial \mathcal{H}}{\partial q_a} \quad , \quad \dot{q}_a = \frac{\partial \mathcal{H}}{\partial p_a} \tag{595}$$

Definition of a Poisson Bracket

$$\Big\{A(q,p), B(q,p)\Big\}_{Poisson} = \sum_a \Big(\frac{\partial A}{\partial q_a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial q_a}\Big) \tag{596}$$

Examples:

$$\{q_a, \mathcal{H}\}_P = \sum_b \Big(\frac{\partial q_a}{\partial q_b}\frac{\partial \mathcal{H}}{\partial p_b} - \frac{\partial q_a}{\partial p_b}\frac{\partial \mathcal{H}}{\partial q_b}\Big) = \frac{\partial \mathcal{H}}{\partial p_a} \tag{597}$$

Similarly, one obtains $\{p_a, \mathcal{H}\}_P = -\frac{\partial \mathcal{H}}{\partial q_a}$. In general $Q = Q(q,p)$

$$\frac{dQ}{dt} = \sum_a \Big(\frac{\partial Q}{\partial q_a}\dot{q}_a + \frac{\partial Q}{\partial p_a}\dot{p}_a\Big) = \sum_a \Big(\frac{\partial Q}{\partial q_a}\frac{\partial \mathcal{H}}{\partial p_a} - \frac{\partial Q}{\partial p_a}\frac{\partial \mathcal{H}}{\partial q_a}\Big) \tag{598}$$

so we have

$$\frac{dQ}{dt} = \{Q, \mathcal{H}\}_P \tag{599}$$

Digression: From Bertshinger

1. Dynamical symmetries (time-independence of the Lagrangian etc.)
2. Non-dynamical symmetries: give rise to the mathematical identities.

Example of a non-dynamical symmetry: The action of a point particle

$$\begin{aligned} S[x^\mu(\tau)] &= \int_{\tau_1}^{\tau_2} \mathcal{L}_1 \Big[x^\mu(\tau), \dot{x}^\mu(\tau) \Big] \, d\tau \\ &= \int_{\tau_1}^{\tau_2} \Big[g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \Big]^{1/2} \, d\tau \end{aligned} \tag{600}$$

The action is invariant under the reparametrizations

$$\tau \to \tau^{'}(\tau) \quad \Rightarrow \quad d\tau = \frac{d\tau}{d\tau^{'}} d\tau^{'} \quad \Rightarrow \quad \frac{d}{d\tau} = \frac{d}{d\tau^{'}} \frac{d\tau^{'}}{d\tau} \tag{601}$$

So we get

$$\frac{dx^\mu}{d\tau^{'}} = \frac{dx^\mu}{d\tau} \frac{d\tau}{d\tau^{'}} \tag{602}$$

Observe that

$$d\tau^{'} \Big(\frac{dx^\mu}{d\tau^{'}} \frac{dx_\mu}{d\tau^{'}} \Big)^{1/2} = d\tau \Big(\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \Big)^{1/2} \tag{603}$$

So we have this invariance for arbitrary $\tau \to \tau^{'}(\tau)$ transformation so if $x^\mu(\tau)$ gives $\delta S = 0$. So does $y^\mu(\tau) = x^\mu(\tau^{'}(\tau))$.

Theorem: If the action $S[q(t)]$ is invariant under the infinitesimal transformations

$$t \to t + \epsilon(t) \quad \text{with} \quad \epsilon(t_1) = \epsilon(t_2) = 0 \tag{604}$$

at the end points, then the Hamiltonian *vanishes* identically.

Proof: Given a parametrized trajectory $q(t)$, we define a new parameterized trajectory

$$\bar{q}(t) \equiv q(t + \epsilon(t)) \tag{605}$$

So

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) \, dt \tag{606}$$

For small ϵ, we have

$$\bar{q}(t) = q(t + \epsilon) = q(t) + \dot{q}\epsilon \quad \Rightarrow \quad \frac{d\bar{q}}{dt} = \dot{q} + \frac{d(\epsilon \, \dot{q})}{dt} \tag{607}$$

Up to first order in ϵ, we have

$$S[q(t+\epsilon)] - S[q(t)] = \int_{t_1}^{t_2} \Big[\frac{\partial \mathcal{L}}{\partial t} \epsilon + \frac{\partial \mathcal{L}}{\partial q} \, \epsilon \, \dot{q}^i + \frac{\partial \mathcal{L}}{\partial \dot{q}} \, \frac{d(\dot{q} \, \epsilon)}{dt} \Big] \, dt \tag{608}$$

Recheck

$$\mathcal{L}\Big(q(t+\epsilon), \, \dot{q}(t+\epsilon), \, t+\epsilon \Big) = \mathcal{L}\Big(q(t) + \underbrace{\dot{q}\epsilon}_{\delta q}, \, \dot{q} + \underbrace{\frac{d(\epsilon \, \dot{q})}{dt}}_{\delta \dot{q}}, \, t + \epsilon \Big) \tag{609}$$

OK so this works. Then

$$S[q(t+\epsilon)] - S[q(t)] = \underbrace{[\mathcal{L}\,\epsilon]_{t_1}^{t_2}}_{=0} + \int_{t_1}^{t_2} \underbrace{\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}\Big)}_{\mathcal{H}} \frac{d\epsilon}{dt} dt \tag{610}$$

And so for arbitrary ϵ, $\mathcal{H}$. Note that I have used

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial q}\,\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\,\ddot{q} \tag{611}$$

Let us check for

$$\mathcal{L} = \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{612}$$

then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\dot{x}^\mu}{\mathcal{L}} \qquad \Rightarrow \qquad \mathcal{H} = \frac{\dot{x}^\mu}{\mathcal{L}}\dot{x}^\mu - \mathcal{L} = 0 \tag{613}$$

identically.

What does the vanishing of the Hamiltonian mean? This does not mean for example that the geodesic motion does not have a *Hamiltonian* formulation. It means that Lagrangian has non-dynamical degrees of freedom. These must be eliminated before we can construct a Hamiltonian. How can we do this?

1. We could set $\tau = t$ so the parameter is set to a coordinate.

2. Change Lagrangian so that the parametrization is lost

   $$\mathcal{L}_2 = \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) \tag{614}$$

   Then

   $$\frac{\partial \mathcal{L}_2}{\partial \tau} = 0 \tag{615}$$

   leads to a dynamical symmetry. Also

   $$\mathcal{H}_2 \equiv \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{616}$$

   is constant along *trajectory* which satisfies the field equations.

IX. ACTION IN FIELD THEORIES

$$S = \int_{t_1}^{t_2} dt\, \mathcal{L}\Big(\vec{x}_a(t), \dot{\vec{x}}_a(t)\Big) \qquad \text{for particles.} \tag{617}$$

In QM any kind of interaction makes sense. We can write down a potential and calculate whatever we want. It is possible that some interactions do not yield real or realistic energy spectra, but generically this is not the case. In QFT, the situation is quite different, renormalizability is an important criterion (Dyson).

But even before that if $\mathcal{L} = K - V$, in some field theories such a separation may not be easy. Also, we will see that Kinetic term for fermions and bosons are different!

In field theory

$$\left.\begin{aligned} t &\to t \\ a &\to x \end{aligned}\right\} \quad x^\mu = (t, \vec{x}) \text{ so } x^\mu \text{ will label the field.} \tag{618}$$

$$\begin{aligned} \vec{x}_a(t) &\longrightarrow \phi(x), \psi(x), A_\mu(x), \Psi^\mu(x), h_{\mu\nu}(x) \\ \sum_a &\longrightarrow \int d^4x \end{aligned} \tag{619}$$

So

$$S = \int d^4x \; \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{620}$$

where ϕ_a is some generic field. Since d^4x is a Lorentz scalar, $\mathcal{L}$ is a Lorentz scalar

$$d^4x' = J\, d^4x \tag{621}$$

Since

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \Rightarrow \quad J = \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| = \det\Lambda = 1 \tag{622}$$

Here $J = \det\Lambda = 1$ for proper orthochronous Lorentz transformations.

Now to be able to apply calculus of variations, let us divide the spacetime as

FIG 29 !!!!

Consider Σ_1 and Σ_2 as space-like hyper-surfaces.

$$S = \int_\Omega d^4x \; \mathcal{L}(\phi_a, \partial_\mu \phi_a) \quad \text{such that} \quad \partial\Omega = \Sigma_1 \cup \Sigma_2 \tag{623}$$

Vary the field in such a way that

$$\delta\phi_a \Big|_{\Sigma_1} = \delta\phi_a \Big|_{\Sigma_2} = 0 \tag{624}$$

then

$$\begin{aligned} \delta S &= \int_\Omega d^4x \; \delta\mathcal{L} = \int_\Omega d^4x \; \Big[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \Big] \\ &= \int_\Omega d^4x \; \delta\phi_a \; \Big[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Big) \Big] + \int_\Omega d^4x \; \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \Big) \end{aligned} \tag{625}$$

Recall the Stokes theorem

$$\int_\Omega d^n x \; \partial_\mu V^\mu = \int_{\partial\Omega} d\Sigma_\mu \; V^\mu \tag{626}$$

So the we have

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Big) = 0 \qquad \text{Euler-Lagrange equation.} \tag{627}$$

Define the conjugate momentum

$$\pi_a(x) \equiv \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_a(x)} \tag{628}$$

and the *Hamiltonian density* as

$$\mathcal{H}(x) \equiv \sum_a \pi_a(x)\, \partial_0 \phi_a(x) - \mathcal{L} \tag{629}$$

Then the total Hamiltonian reads

$$H = \int d^3x\ \mathcal{H}(x) \tag{630}$$

EXAMPLES

1. Real Scalar Field $(\lambda\phi^4$ theory)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi - V(\phi) \quad \text{where} \quad V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^2 \tag{631}$$

whose field equation is

$$\partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} = 0 \quad \Rightarrow \quad \partial_0^2 \phi - \vec{\nabla}^2 \phi + \frac{\partial V}{\partial \phi} = 0 \tag{632}$$

Explicitly the coupled KG theory

$$\mathcal{L} = \frac{1}{2}(\partial_0 \phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^2 \tag{633}$$

We took a $\phi \to -\phi$ symmetric theory ($\mathbb{Z}_2$ symmetry). We could also add a $J\,\phi$ source term.

2. Maxwell Field ($A_\mu(x)$)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \quad \text{where} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{634}$$

Note that we can also add $\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$. Then from the Euler-Lagrange equation

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} \Big) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{635}$$

one gets the in-homogeneous Maxwell equations

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{636}$$

But where are the homogeneous ones. From the definition of $F_{\mu\nu}$, we have the Jacobi identity

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \quad \Rightarrow \quad \epsilon^{\lambda\mu\nu\rho} \partial_\mu F_{\nu\rho} = 0 \quad \Rightarrow \quad \partial_\mu \tilde{F}^{\mu\lambda} = 0 \tag{637}$$

We could also write the Maxwell action in the first order form and assume A_μ and $F^{\mu\nu}$ as independent fields.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \tag{638}$$

Then Bianchi identity also follows. Heaviside Lorentz units $\frac{e^2}{4\pi\hbar c} = \alpha = \frac{1}{137}$. Recall Maxwell's equation

$$\begin{aligned} &a)\ \vec{\nabla}\cdot\vec{B} = 0 \quad , \quad && b)\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ &c)\ \vec{\nabla}\cdot\vec{E} = \rho \quad , \quad && d)\ \vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J} \end{aligned} \tag{639}$$

Note that

$$\begin{aligned} \vec{B} &= \vec{\nabla}\times\vec{A} \quad \text{solves } (a) \\ \vec{E} &= -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi \quad \text{solves } (b) \end{aligned} \tag{640}$$

Let us recall that

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad , \quad \partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \quad , \quad A^\mu = (\phi, \vec{A}) \tag{641}$$

Then

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{\partial A^i}{\partial t} + \partial_i\phi = \frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\phi = -\vec{E} \tag{642}$$

So $E^i = F^{i0}$. Meanwhile

$$F^{ij} = \partial^i A^j - \partial^j A^i \equiv -\epsilon^{ijk}B_k = \epsilon^{ijk}B_k \tag{643}$$

Note also that

$$B^k = \frac{1}{2}\epsilon^{ijk}F_{ij} = \frac{1}{2}\epsilon^{ijk}F^{ij} \tag{644}$$

So we get

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

A little bit about gauge invariance

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu\chi \quad \Rightarrow \quad F^{\mu\nu} \to F'^{\mu\nu} \quad \text{is intact.} \tag{645}$$

Now the field equation is

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \Rightarrow \quad \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = J^\nu \tag{646}$$

Choose the Lorentz gauge

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad \partial_0 A^0 + \partial_i A^i = 0 \quad \Rightarrow \quad \frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0 \tag{647}$$

Then we have

$$\Box A^{\nu} = J^{\nu} \tag{648}$$

In vacuum $\Box A^{\nu} = 0$, it looks like a vector version of massless KG equation. Note that the Lorentz gauge does not uniquely determine the gauge field. In this gauge we can still perform a gauge transformation

$$\partial_{\mu} A^{'\mu} = \partial_{\mu} A^{\mu} = 0 \quad \text{if} \quad A^{'\mu} = A^{\mu} + \partial^{\mu}\chi_2 \tag{649}$$

Then

$$\partial^2 \chi_2 = 0 \quad \Rightarrow \quad \Box\chi = 0 \quad \text{are harmonic functions.} \tag{650}$$

3. Proca Equation

$$(\Box + m^2) A^{\mu} = J^{\mu} \tag{651}$$

Take $A^0 = \phi(\vec{x})$ where $\vec{x}$ is time-independent and let $J^0 = q\delta^3(\vec{x})$, then

$$(-\vec{\nabla}^2 + m^2)\phi(\vec{x}) = q\delta^3(\vec{x}) \tag{652}$$

so we get

$$\phi(\vec{x}) = \frac{q}{4\pi \mid \vec{x} \mid} e^{-m|\vec{x}|} \qquad \text{as m} \to 0 \qquad \phi = \frac{q}{4\pi \mid \vec{x} \mid} \tag{653}$$

The dual tensor is

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

A Digression about the Magnetic Monopole:

FIG 30 !!!!

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

Then, one gets

$$(A_r)^N = 0, \qquad (A_\theta)^N = 0, \qquad (A_\varphi)^N = \frac{g}{r\sin\theta}(1-\cos\theta) \tag{654}$$

so

$$\vec{B} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & 0 & r\sin\theta A_\varphi \end{vmatrix} \quad , \quad 0 \le \theta \le \pi$$

Hence we get

$$\vec{B} = \frac{g}{r^2}\hat{r} \tag{655}$$

In the southern hemisphere

$$(A_r)^S = 0, \qquad (A_\theta)^S = 0, \qquad (A_\varphi)^S = -\frac{g}{r\sin\theta}(1+\cos\theta) \tag{656}$$

This gives the same $\vec{B}$ as above.

$$(A_\varphi)^N - (A_\varphi)^S = \frac{2g}{r\sin\theta} \tag{657}$$

In the overlapping region we have $\partial_\mu\Big(2g\varphi\Big)$. Recall

$$\vec{\nabla}\psi = \hat{r}\,\frac{\partial\psi}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi} \tag{658}$$

So now let us calculate the S.E. for an electron in the presence of a magnetic monopole

$$\text{In } R_a: \qquad \Big[\frac{1}{2m}(\vec{p} - e\vec{A}_a)^2 + V\Big]\psi_a = E\psi_a \tag{659}$$

$$\text{In } R_b: \qquad \Big[\frac{1}{2m}(\vec{p} - e\vec{A}_b)^2 + V\Big]\psi_b = E\psi_b \tag{660}$$

Since $\vec{A}_a$ and $\vec{A}_b$ differ by a gauge transformation, it is easy to see that

$$\psi_a = e^{\frac{i2eg\varphi}{\hbar}}\psi_b \tag{661}$$

as $\varphi \to \varphi + 2\pi$. Then

$$\frac{e2g2\pi}{\hbar} = n2\pi \quad \Rightarrow \quad 2eg = n\hbar \tag{662}$$

which is *Dirac's quantization condition.*

Discuss:

(a) Dyson

(b) 't Hooft-Polykavov monopole and monopole structure

(c) Duality

(d) Bound states in the Dirac monopole-electron problem.

Conserved angular momentum

$$\vec{L} = \vec{r} \times (\vec{p} - e\vec{A}) - eg\,\hat{r}, \qquad [L_x, L_y] = i\hbar L_z, \qquad [r^2, \vec{L}] = 0 \tag{663}$$

Monopole harmonics $Y_{q,l,m}$ which is related to Jacobi polynomials so

$$L^2\, Y_{q,l,m} = l(l+1)\hbar^2 Y_{q,l,m}, \qquad L_z Y_{q,l,m} = m\hbar\, Y_{q,l,m}, \tag{664}$$

so

$$l = 0,\, \frac{1}{2},\, 1, \cdots \quad ; \quad m \in (-l,\, l) \tag{665}$$

then allowed values

$$l = \mid q \mid,\, \mid q \mid +1,\, \mid q \mid +2, \cdots \tag{666}$$

4. Dirac Fields

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{667}$$

where ψ is a complex function. One can vary with respect to $(\psi_R, \psi_I), (\bar{\psi}, \psi)$ or (ψ^*, ψ). Take ψ and $\bar{\psi}$ as independent variables

$$\delta\bar{\psi}: \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu \bar{\psi}} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad \Rightarrow \quad (i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{668}$$

$$\delta\psi: \qquad \partial_\mu(\bar{\psi} i\gamma^\mu) + m\bar{\psi} = 0 \quad \Rightarrow \quad \bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0 \tag{669}$$

Do not forget that

$$\mathcal{L} = \sum_{\alpha,\beta} (\bar{\psi})_\alpha \Big[i(\gamma^\mu)_{\alpha\beta} - m\delta_{\alpha\beta} \Big] \psi_\beta \tag{670}$$

5. Schrödinger Field $\psi(x)$

$$\mathcal{L} = i\psi^+ \frac{\partial \psi}{\partial t} - \frac{1}{2m}(\vec{\nabla}\psi^+) \cdot \vec{\nabla}\psi - V(x)\psi^+\psi \tag{671}$$

then

$$\partial_0 \frac{\partial \mathcal{L}}{\partial\partial_0 \psi^+} + \nabla_k \Big(\frac{\partial \mathcal{L}}{\partial\partial_k \psi^+} \Big) - \frac{\partial \mathcal{L}}{\partial \psi^+} = 0 \tag{672}$$

$$-\frac{1}{2m}\nabla^2\psi - i\frac{\partial \psi}{\partial t} + V(x)\psi = 0 \tag{673}$$

6. Electrodynamics (electrons+photons)

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\Big(\gamma^\mu(i\partial_\mu - eA_\mu) - m\Big)\psi \\ &= -\frac{1}{4}F^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu\psi\end{aligned} \tag{674}$$

So then we have

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\mu\psi \quad \Rightarrow \quad \Big(\gamma^\mu(i\partial_\mu - eA_\mu) - m\Big)\psi = 0 \tag{675}$$

Gauge Invariance

Consider the free Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{676}$$

Observe that $\psi^{'}(x) = e^{i\alpha}\psi(x)$ gives $\mathcal{L}$ intact. Now upgrade this to a local symmetry

$$\psi^{'}(x) = e^{i\alpha(x)}\psi(x) \tag{677}$$

note we do not change x. Then

$$i\partial_\mu\psi^{'}(x) = -\partial_\mu\alpha\ \psi(x) + ie^{i\alpha}\partial_\mu\psi(x) \tag{678}$$

Define

$$\mathcal{D}_\mu \equiv \partial_\mu + ieA_\mu \quad ; \quad i\mathcal{D}_\mu = i\partial_\mu - eA_\mu \tag{679}$$

Then

$$(\mathcal{D}\psi)^{'} \equiv e^{i\alpha(x)}\mathcal{D}_\mu\psi \tag{680}$$

$$\partial_\mu\psi^{'} + ieA^{'}_\mu\psi^{'} = e^{i\alpha}(\partial_\mu\psi + ieA_\mu\psi) \quad \Rightarrow \quad A^{'}_\mu(x) = A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{681}$$

So then

$$\mathcal{L} = \bar{\psi}\Big(\gamma^\mu(i\partial_\mu - eA_\mu) - m\Big)\psi = \bar{\psi}i\gamma^\mu\mathcal{D}_\mu\psi \tag{682}$$

has local $U(1)$ gauge invariance. We need to add dynamics to the photon

$$\mathcal{L}_{Kinetic} = -\frac{1}{4}F^2_{\mu\nu} \tag{683}$$

Note that

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = eiF_{\mu\nu} \tag{684}$$

7. Complex Scalar Field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - V(\phi) \quad \text{where} \quad \phi = \frac{1}{2}(\phi_1 + i\phi_2) \tag{685}$$

In $D = 3+1$

$$V(\phi) = \frac{m^2}{2}\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2 \tag{686}$$

Since we have $\phi(x) \to \phi^{'}(x) = e^{i\alpha}\phi(x)$, a global $U(1)$ symmetry, we can gauge it

$$\partial_\mu \longrightarrow \mathcal{D}_\mu = \partial_\mu + ieA_\mu \tag{687}$$

such that

$$\mathcal{L} = (\mathcal{D}_\mu\phi)^*\mathcal{D}^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}^2 \tag{688}$$

Field equations

$$\delta\phi^* : \qquad \partial_\mu\mathcal{D}^\mu\phi + ieA_\mu\mathcal{D}^\mu\phi + \frac{\partial V}{\partial\phi} = 0 \quad \rightarrow \quad \mathcal{D}_\mu\mathcal{D}^\mu\phi = -\frac{\partial V}{\partial \phi} \tag{689}$$

To get the conserved current, look at the $-A_\mu J^\mu$ term in the Lagrangian

$$J^\mu = ie\Big(\phi^*\mathcal{D}^\mu\phi - \phi(\mathcal{D}^\mu\phi)^*\Big) \quad \text{such that} \quad J^{\mu *} = J^\mu \tag{690}$$

Check that

$$\partial_\mu J^\mu = 0 \qquad \text{on shell.} \tag{691}$$

Digression "Gauge Fields" R. Mills (1927-1999) Am. J. Phys. 57 (6) 1989

(a) Introduction: "The gauge principle". Every continuous symmetry should be a local one. [This may not work, see for example the EM type duality.]

This principle limits the interactions. All the force fields in nature seem to be of the gauge type! (This is a unification idea). But a real unification would be to have a single gauge field.

(b) The Beginnings of the gauge idea: Key ideas come from Noether, Weyl and London.

Emmy Noether (1882-1935) Mathematician known for her work on commutative rings and algebraic number theory. But math people do not know much about her work in physics. Noether proved her theorem in 1918 (Gottingen):

"For every continuous symmetry there is a conservation law and for every conservative law there is a symmetry".

Assumption is that there is a Lagrangian and the equations of motion are derivable from a Lagrangian. Dissipative forces are not included.

Weyl and London : Hermann Weyl (1885-1955). Friend of Noether in Gottingen. Deeply influenced by Einstein and the motion of general covariance. $x^\mu = f(x^\mu)$ Weyl wanted to unify/or apply a similar notion to Electromagnetism.

Weyl wanted to exploit *scale* invariance, the notion that all lengths can be scaled and yet physics remain intact. Weyl wanted to upgrade this *global* invariance to a *local* one and get electric charge out of it (1918). Einstein objected to Weyl's idea.

In 1927, after Schrodinger write his paper, Fritz London pointed out that the symmetry associated with conservation of electric charge is not *scale* invariance but a *phase* invariance, that is

$$\psi \to e^{i\alpha(x)}\psi \tag{692}$$

where ψ is wave function.

Important: Yes but all this is quantum mechanical, is there a way to get charge conservation in classical electrodynamics?

Answer: Yes, that is gauge invariance $A_\mu J^\mu$. Weyl and the words "gauge invariance"

Yang-Mills For almost 25 years, *local-gauge* invariance was seen to be a specific characteristic of Electromagnetic theory. It also meant *zero mass* for the photon. C.N. Yang (1922, alive as 2012) Nobel prize in 1957 visited BNL (in 1953-1954) where Mills was a post-doc. Yang was interested in the *isospin* conservation. BNL had the cosmotron (the biggest particle accelerator of protons at $2-3$ *Gev*). Mills was still writing his PhD thesis. Yang and Mills shared the same office.

Salam thought of all neutrinos are left-handed. Pauli did not like it.

In 1954 Ronald Shaw a student of Abdus Salam actually found the same set of equations in his unpublished thesis: "The problem of particle types and other contributions to the theory of elementary particles".

(c) The Gauge Philosophy: Local Symmetry

FIG 31 !!!!

(d) Conserved Quantities, Symmetries and Gauge Fields

FIG 32 !!!!

So for every true conservation law, there is a complete theory of gauge fields for which the given conserved quantity is the source. [Here continuity of the symmetry is important.]

A) Noether's theorem (in the QM context), to get the classical counterpart replace

$$[,] \to \{,\}_{Poisson} \tag{693}$$

Let $\hat{A}$ be a Hermitian linear operator. It has double role:

(a) $\hat{A}$ represents a dynamical variable.

(b) $\hat{A}$ is a generator of a class of transformations

Then the mean value of physical observable.

$$\langle \hat{A} \rangle = \int \psi^* \hat{A} \psi \tag{694}$$

We can also use $\hat{A}$ to generate a unitary transformation acting on state vector as

$$\psi \to \psi' = e^{-i\lambda \hat{A}} \psi \qquad \lambda \in \mathbb{R} \tag{695}$$

Example: Say $\hat{A} = p_z$ then ψ' differ from ψ by a displacement in the $\hat{z}$-direction

$$\psi' = \Big(1 - i\lambda \frac{\hbar}{i} \frac{\partial}{\partial z}\Big) \psi(z) = \psi(z - \lambda) \tag{696}$$

Consider the Hamiltonian

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad \Rightarrow \quad \psi(t) = e^{-i\hat{H}t} \psi(0) \tag{697}$$

$\hat{A}$ is conserved if $[\hat{A}, \hat{H}] = 0$.

B) Local Symmetry and the Gauge Fields: The Electromagnetic case. Consider $\hat{Q}$ as the electric charge operator

$$\psi' = e^{-i\lambda \hat{\alpha}} \psi = e^{-ine\theta} \psi \quad \text{where} \quad \theta \ \text{ is constant} \tag{698}$$

here Schrodinger or Dirac wave function.

$$\psi' \simeq (1 - ine\theta) \psi \tag{699}$$

Now make then $\theta \to (x)$

$$\psi'(x) = \Big(1 - ine\theta(x)\Big) \psi \tag{700}$$

We know that a gauge field will be introduced

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + ienA_\mu \tag{701}$$

$$\delta\psi = -ien\theta(x)\psi(x) \qquad\qquad i\partial_\mu \to i\mathcal{D}_\mu = i\partial_\mu - enA_\mu \tag{702}$$

1. Non-Abelian Case

Electric charge $\hat{Q}$ is replaced by a family of operators $\hat{T}_a$ for isospin $\hat{T}_1, \hat{T}_2, \hat{T}_3$

$$[T_i, T_j] = iC_{ijk} T_k \tag{703}$$

where C_{ijk} is the structure constant. So for $SU(2)$ we have $[T_i, T_j] = i\epsilon_{ijk} T_k$. Now

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \qquad \text{nucleon doublet.}$$

where ψ is the wave function of a single nucleon. So we have

$$\psi' = e^{-i\theta_i T_i} \psi \qquad \theta_i : 3 \ \text{ parameters} \tag{704}$$

Upgrade $\theta_i \to \theta_i$, then

$$\psi^{'}(x) = e^{-ig\vec{\theta}(\vec{x})\cdot\vec{T}}\psi \equiv U\psi \tag{705}$$

$$\mathcal{D}_\mu\psi \to (\mathcal{D}_\mu\psi)^{'} = U\mathcal{D}_\mu\psi \tag{706}$$

where $\mathcal{D}_\mu\psi = (\partial_\mu + ig\ \vec{A}_\mu \cdot \vec{T})\psi$. Then

$$\begin{aligned} \partial_\mu\psi^{'} + ig\ \vec{A}^{'}{}_\mu \cdot \vec{T}\psi^{'} &= U(\partial_\mu\psi + ig\ \vec{A}_\mu \cdot \vec{T}\psi) \\ (\partial_\mu U)\psi + U\partial_\mu\psi + ig\ \vec{A}^{'}{}_\mu \cdot \vec{T}U\psi &= U\partial_\mu\psi + ig\ U\vec{A}_\mu \cdot \vec{T}\psi \\ ig\ \vec{A}^{'}{}_\mu \cdot \vec{T}U &= -\partial_\mu U + ig\ U\vec{A}_\mu \cdot \vec{T} \end{aligned} \tag{707}$$

so we get

$$\vec{A}^{'}{}_\mu \cdot \vec{T} = U\vec{A}_\mu \cdot \vec{T}U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \tag{708}$$

Sometimes we define a Lie algebra valued value field

$$A_\mu \equiv \vec{A}_\mu \cdot \vec{T} \tag{709}$$

which then provides

$$A^{'}_\mu = UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \tag{710}$$

Let us now compute

$$\begin{aligned} ([\mathcal{D}_\mu, \mathcal{D}_\nu])\psi &= \mathcal{D}_\mu\mathcal{D}_\nu\psi - \mathcal{D}_\nu\mathcal{D}_\mu\psi \\ &= \mathcal{D}_\mu(\partial_\nu\psi + ig\vec{A}_\nu \cdot \vec{T}\psi) - \nu \leftrightarrow \mu \\ &= \partial_\mu(\partial_\nu\psi + ig\vec{A}_\nu \cdot \vec{T}\psi) + ig\vec{A}_\mu \cdot \vec{T}(\partial_\nu\psi + ig\vec{A}_\nu \cdot \vec{T}\psi) - \nu \leftrightarrow \mu \\ &= ig\,\partial_\mu\vec{A}_\nu \cdot \vec{T}\psi + ig\vec{A}_\nu \cdot \vec{T}\partial_\mu\psi + ig\vec{A}_\mu \cdot \vec{T}\partial_\nu\psi - g^2(\vec{A}_\mu \cdot \vec{T})(\vec{A}_\nu \cdot \vec{T})\psi - \nu \leftrightarrow \mu \end{aligned} \tag{711}$$

which can be written as

$$\begin{aligned} ([\mathcal{D}_\mu, \mathcal{D}_\nu])\psi = ig\Big(&(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T^a\psi + A^a_\nu T^a\partial_\mu\psi - A^a_\mu T^a\partial_\nu\psi \\ &+ A^a_\mu T^a\partial_\nu\psi - A^a_\nu T^a\partial_\mu\psi + igA^a_\mu A^a_\nu\underbrace{[T^a, T^b]}_{if^{abc}T^c}\psi\Big) \end{aligned} \tag{712}$$

So define

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu \qquad \Rightarrow \qquad F_{\mu\nu} = F^a_{\mu\nu}T^a \tag{713}$$

Thus we get

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi = igF_{\mu\nu}\psi \tag{714}$$

Work out how $F_{\mu\nu}$ and $F_{\mu\nu}F^{\mu\nu}$ transforms under gauge transformations. You will find

$$F^{'}_{\mu\nu} = UF_{\mu\nu}U^{-1} \quad \text{and} \quad F^{'}_{\mu\nu}F^{'\mu\nu} = UF_{\mu\nu}U^{-1}UF_{\mu\nu}U^{-1} \tag{715}$$

So

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) \quad \text{such that} \quad \mathrm{Tr}\,(T^aT^b) = \frac{1}{2}\delta^{ab} \tag{716}$$

Furthermore, the field equation can be written in the generator basis as

$$(\mathcal{D}_\mu F^{\mu\nu})^a = J^{\nu a} \tag{717}$$

Talk about the free and interacting parts. Also comment on the original YM paper and the current use of YM theory.

X. SYMMETRIES

$$S = \int_\Omega d^4x \; \mathcal{L}\Big(\phi_a(x), \partial_\mu \phi_a(x)\Big) \tag{718}$$

where $\phi_a(x)$ is a generic field. Let

$$x^\mu \to x^{'\mu} = x^{'\mu}(x) \quad \text{and} \quad \phi_a(x) \to \phi^{'}{}_a(x^{'}) = \phi^{'}_a(x, \phi_a(x)) \tag{719}$$

We say that a system is invariant under a transformation or has a symmetry if its action is *invariant up to a surface term*

$$S^{'} = \int_{\Omega^{'}} d^4x^{'} \; \mathcal{L}^{'}\Big(\phi^{'}{}_a(x^{'}), \partial^{'}{}_\mu \phi^{'}{}_a(x^{'})\Big) \tag{720}$$

Take $\Omega^{'} = \Omega$, then

$$\begin{aligned} \Delta S = S^{'} - S &= \int_\Omega d^4x \left[\mathcal{L}^{'}\Big(\phi^{'}{}_a(x^{'}), \partial^{'}{}_\mu \phi^{'}{}_a(x^{'})\Big) - \mathcal{L}\Big(\phi_a(x), \partial_\mu \phi_a(x)\Big)\right] \\ &= \int_\Omega d^4x \; \partial_\mu \Lambda^\mu\Big(x, \phi_a(x)\Big) \\ &= \int_{\partial\Omega} d\Sigma^\mu \; \Lambda_\mu\Big(x, \phi_a(x)\Big) \end{aligned} \tag{721}$$

Note that there is no $\partial\phi$ type term in right-hand side. So symmetry leads to

$$\mathcal{L}\Big(\phi^{'}{}_a(x^{'}), \partial^{'}{}_\mu \phi^{'}{}_a(x^{'})\Big) - \mathcal{L}\Big(\phi_a(x), \partial_\mu \phi_a(x)\Big) = \partial_\mu \Lambda^\mu \tag{722}$$

Note up to now, we have not used the equations of motion.

Continuous Symmetry: Consider the infinitesimal transformations

$$x^\mu \to x^{'\mu} = x^\mu + \delta x^\mu \quad ; \quad \delta x^\mu = O(\epsilon) \tag{723}$$

$$\phi_a(x) \to \phi^{'}_a(x^{'}) \equiv \phi_a(x) + \delta\phi_a(x) \quad ; \quad \delta\phi_a(x) = O(\epsilon) \tag{724}$$

Then

$$\phi^{'}_a(x^{'}) = \phi^{'}_a(x+\epsilon) = \phi^{'}_a(x) + \delta x^\mu \, \partial_\mu \phi_a(x) + O(\epsilon^2) \tag{725}$$

$$\phi_a(x) + \delta\phi_a(x) = \phi^{'}_a(x) + \delta x^\mu \, \partial_\mu \phi_a(x) \tag{726}$$

So we get

$$\phi^{'}_a(x) = \phi_a(x) + \delta\phi_a(x) - \delta x^\mu \, \partial_\mu \phi_a(x) \tag{727}$$

As before, define the Lie derivable as

$$\delta_L \phi_a(x) \equiv \phi^{'}_a(x) - \phi_a(x) \tag{728}$$

So this gives a functional or form change of the function. Define

$$\delta_L\phi_a(x) = \epsilon\, G_a(x,\phi_a(x)) = \delta\phi_a(x) - \delta x^\mu\, \partial_\mu\phi_a(x) \tag{729}$$

Symmetry of the action implies

$$\begin{aligned}\delta_L\mathcal{L} &= \mathcal{L}\Big(\phi^{'}, \partial^{'}{}_\mu\phi^{'}\Big) - \mathcal{L}\Big(\phi, \partial_\mu\phi\Big)\\ &= \partial_\mu\Lambda^\mu = \epsilon\,\partial_\mu X^\mu\end{aligned} \tag{730}$$

This is very important: this should be the case without using the equations of motion. So

$$\begin{aligned}\delta_L\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_a}\delta_L\phi_a(x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\delta_L\partial_\mu\phi_a(x)\\ &= \frac{\partial\mathcal{L}}{\partial\phi_a}\epsilon\, G_a(x,\phi) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\mu\Big(\epsilon\, G_a(x,\phi)\Big)\\ &= \Big(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\Big)\Big)\epsilon\, G_a(x,\phi + \epsilon\,\Big(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}G_a\Big)\\ &= \epsilon\,\partial_\mu X^\mu(x,\phi_a)\end{aligned} \tag{731}$$

So if Euler-Lagrange equations are satisfied, then we have the following conserved current

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}G_a - X^\mu \quad\Rightarrow\quad \partial_\mu J^\mu = 0 \tag{732}$$

Now for $\mu = 0$ component the conserved current is

$$J^0 = \pi_a G_a - X^a \tag{733}$$

Note that in X^μ there are no $\partial\phi_a$ terms. Then, the charge is

$$Q \equiv \int d^3x\; J^0(t,\vec{x}) \quad\Rightarrow\quad \frac{dQ}{dt} = \int d^3x\; \partial_0 J^0 \longrightarrow 0 \tag{734}$$

So we found

$$\text{Continuous symmetry} \longrightarrow \text{Conserved current} \tag{735}$$

<u>Note</u>: In Classical Mechanics, we have

$$\{\phi_a(x), Q\}_{Poisson} = G_a(\phi) \tag{736}$$

In Quantum Mechanics

$$\frac{1}{i}[\phi_a(x),\hat{Q}] = G_a(\phi) \tag{737}$$

That is commutator of the charge gives the generator of the symmetry.

Be careful. In point particle mechanics

$$\{A(x,p), B(x,p)\}_{Poisson} = \sum_a \Big(\frac{\partial A}{\partial x_a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x_a}\Big) \tag{738}$$

Here we should define it more carefully

$$\begin{aligned}\{\phi_a(x), Q\}_P &= \sum_b \int d^3x^{'} \left[\frac{\partial\phi_a(x)}{\partial\phi_b(x^{'})}\frac{\partial Q(t)}{\partial\pi_b(x)} - \frac{\partial\phi_a(x)}{\partial\pi_b(x)}\frac{\partial Q(t)}{\partial\phi_b(x^{'})}\right] \\ &= \sum_b \int d^3x^{'} \left[\delta_{ab}\delta^3(x-x^{'})\frac{\partial Q(t)}{\partial\pi_b(x^{'})}\right]\end{aligned} \tag{739}$$

Note that

$$\frac{\partial Q(t)}{\partial\pi_b(x^{'})} = \frac{\partial}{\partial\pi_b(x^{'})}\int d^3x^{''}(\pi_c G_c - X^0) = G_c(x^{'}, \phi(x^{'})) \tag{740}$$

So we arrive at

$$\{\phi_a(x), Q\}_{Poisson} = G_a(x, \phi) \tag{741}$$

So in general, actually we used

$$\{A, B\}_P = \sum_a \int d^3x^{'} \left[\frac{\partial A}{\partial\phi_a(x)}\frac{\partial B}{\partial\pi_a(x)} - \frac{\partial A}{\partial\pi_a(x)}\frac{\partial B}{\partial\phi_a(x)}\right] \tag{742}$$

A. Energy Momentum Tensors

The spacetime translation invariance

$$x^\mu \to x^{'\mu} = x^\mu - \epsilon^\mu \tag{743}$$

here ϵ^μ is a constant. We assume that the field is fixed

$$\phi_a(x) \to \phi^{'}_a(x^{'}) = \phi_a(x) \tag{744}$$

so

$$\phi^{'}_a(x-\epsilon) = \phi_a(x) \quad \Rightarrow \quad \phi^{'}_a(x) - \epsilon^\mu\partial_\mu\phi_a(x) \tag{745}$$

then

$$\delta_L\phi_a(x) = \phi^{'}_a(x) - \phi_a(x) = \epsilon^\mu\partial_\mu\phi_a(x) \equiv \epsilon\, G_a(x) \quad \Rightarrow \quad G^\mu_a = \partial^\mu\phi_a(x) \tag{746}$$

The Lagrangian is also a scalar under these transformations

$$\delta_L\mathcal{L} = \epsilon^\mu\,\partial_\nu\mathcal{L} \equiv \epsilon^\nu\partial_\mu X^\mu{}_\nu = \epsilon^\nu\,\partial_\mu(\delta_\nu{}^\mu\mathcal{L}) \tag{747}$$

So

$$X^\mu{}_\nu = \delta^\mu{}_\nu\mathcal{L} \tag{748}$$

The conserved quantity will be

$$J_\nu{}^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\nu\phi_a - \delta_\nu{}^\mu\mathcal{L} \equiv T_\nu{}^\mu \tag{749}$$

This is the canonical energy momenta tensor. In general it is not symmetric

$$\partial_\nu T^{\mu\nu} = 0 \tag{750}$$

To make it symmetric, one can use the so called Belinfante (1939) procedure and add an anti-symmetric $\partial_\alpha f^{\alpha\mu\nu}$. Define

$$p^\nu \equiv \int d^3x T^{\nu 0} \quad ; \quad \frac{dp^\nu}{dt} = 0 \tag{751}$$

B. Angular Momentum Density

Lorentz invariance $x^\mu \to x^{'\mu} = x^\mu + \epsilon^\mu{}_\nu x^\nu$, then

$$\phi_a(x) \to \phi_a^{'}(x^{'}) = \Big(\mathbb{I} - \frac{i}{2}\epsilon^{\rho\sigma}S_{\rho\sigma}\Big)_a{}^b\phi_b(x) \tag{752}$$

so

$$\begin{aligned} \phi_a^{'}(x^{'}) = \phi_a^{'}(x+\epsilon\, x) &= \phi_a^{'}(x) + (\epsilon x)^\nu \partial_\nu \phi_a(x) \\ &= \phi_a^{'}(x) + \epsilon^\nu{}_\nu x^\mu \partial_\nu \phi_a(x) \\ &= \phi_a^{'}(x) + \frac{1}{2}\epsilon^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi_a(x) \end{aligned} \tag{753}$$

which is

$$\phi_a(x) - \frac{i}{2}\epsilon^{\sigma\rho}(S_{\rho\sigma})_a{}^b\phi_b(x) = \phi_a^{'}(x) + \frac{1}{2}\epsilon^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi_a(x) \tag{754}$$

So

$$\delta_L\phi_a^{'}(x) = -\frac{i}{2}\epsilon^{\sigma\rho}\Big[(S_{\rho\sigma})_a{}^b + L_{\rho\sigma}\,S_a{}^b\Big]\phi_b(x) \tag{755}$$

where

$$L_{\rho\sigma} = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho) \tag{756}$$

Since

$$\delta_L\phi_a^{'}(x) = \epsilon G_a = \epsilon^{\rho\sigma}G_{(a)\rho\sigma} \tag{757}$$

Hence we get

$$G_{(a)\rho\sigma} \equiv -\frac{i}{2}\Big[(S_{\rho\sigma})_a{}^b + L_{\rho\sigma}\,S_a{}^b\Big]\phi_b(x) \tag{758}$$

In a Lorentz invariant theory, the Lagrangian itself is Lorentz invariant

$$\mathcal{L}\Big[\phi_a^{'}(x^{'}), \partial_\mu^{'}\phi_a^{'}(x^{'})\Big] = \mathcal{L}\Big[\phi_a(x), \partial_\mu\phi_a(x)\Big] \tag{759}$$

$$\mathcal{L}\Big[\phi_a^{'}(x+\epsilon x), \cdots\Big] = \mathcal{L}\Big[\phi_a(x), \cdots\Big] \tag{760}$$

then

$$\delta_L\mathcal{L} = -(\epsilon x)^\mu\partial_\mu\mathcal{L} = -\epsilon^\mu{}_\nu x^\nu\partial_\mu\mathcal{L} = -\frac{1}{2}\epsilon^{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\mathcal{L} = -\frac{i}{2}\epsilon^{\rho\sigma}L_{\rho\sigma}\mathcal{L} = \epsilon^{\rho\sigma}\partial_\mu X^\mu_{\rho\sigma} \tag{761}$$

then

$$\delta_L\mathcal{L} = \frac{\epsilon^{\rho\sigma}}{2}\Big[\partial_\sigma(x_\rho\mathcal{L}) - \partial_\rho(x_\sigma\mathcal{L})\Big] \tag{762}$$

So

$$X^\mu_{\rho\sigma} = \frac{1}{2}(x_\rho\delta^\mu{}_\sigma - x_\sigma\delta^\mu{}_\rho)\mathcal{L} \tag{763}$$

And so

$$M_{\rho\sigma}{}^{\mu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} G^a_{\rho\sigma} - X_{\rho\sigma}{}^{\mu} \tag{764}$$

Note that we dropped $\frac{1}{2}$ factors.

$$M_{\rho\sigma}{}^{\mu} = -i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a(x)} (S_{\rho\sigma})_a{}^b \, \phi_b(x) - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a(x)} (x_\rho \partial_\sigma - x_\sigma \partial_\rho) \, \phi_a(x) - (x_\rho \delta^\mu{}_\sigma - x_\sigma \delta^\mu{}_\rho) \mathcal{L} \tag{765}$$

Recall that we had

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_a} \partial^\mu \phi^a - g^{\mu\nu} \mathcal{L} \tag{766}$$

Then

$$M^{\rho\sigma\mu} = x^\rho \, T^{\sigma\mu} - x^\sigma \, T^{\rho\mu} - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a(x)} (S^{\rho\sigma})_a{}^b \, \phi_b(x) \tag{767}$$

So that

$$\partial_\mu M^{\rho\sigma\mu} = 0 \tag{768}$$

Define

$$M^{\rho\sigma} \equiv \int d^3x \, M^{\rho\sigma 0} = \int d^3x \left\{ x^\rho \, T^{\sigma 0} - x^\sigma \, T^{\rho 0} - i \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_a(x)} (S^{\rho\sigma})_a{}^b \, \phi_b(x) \right\} \tag{769}$$

And recall that

$$T^{\sigma 0} = \pi_a(x) \partial^\sigma \phi_a - g^{0\sigma} \mathcal{L} \tag{770}$$

<u>PHASE INVARIANCE</u>:

Suppose the action is invariant under

$$\phi_a(x) \quad \to \quad \phi^{'}(x) = e^{i\alpha} \phi_a(x) \quad \to \quad \phi_a(x) + i\epsilon \phi_a(x) \tag{771}$$

$$\phi_a^*(x) \quad \to \quad \phi^{'*}(x) = e^{-i\alpha} \phi_a^*(x) \quad \to \quad \phi_a^*(x) - i\epsilon \phi_a^*(x) \tag{772}$$

So we do not change the point x

$$\delta_L \phi_a(x) = \phi_a^{'}(x) - \phi_a(x) = i\epsilon \phi_a(x) \tag{773}$$

$$\delta_L \phi_a^*(x) = -i\epsilon \phi_a^*(x) \tag{774}$$

and so

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a(x)} i\phi_a - i\phi_a^* \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a^*(x)} \qquad \text{no } x^\mu \text{ term.} \tag{775}$$

hence

$$Q = i \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_a(x)} \phi_a - \phi_a^* \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_a^*(x)} \right\} \tag{776}$$

Consider

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \partial^0 \phi^* \tag{777}$$

then

$$Q = i \int d^3x \left\{ \phi \partial^0 \phi^* - \phi^* \partial^0 \phi \right\} \equiv \langle \phi \mid \phi \rangle = Q_{U(1)} \tag{778}$$

<u>EXAMPLES</u>: <u>Dirac Fields</u>

$$S = \int d^4x \, \bar{\psi} \, (i\gamma^\mu \partial_\mu - m)\psi \tag{779}$$

Take ψ and $\bar{\psi}$ to be independent

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial \partial_\nu \psi} \partial^\mu \psi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \bar{\psi}} \partial^\mu \bar{\psi} - \eta^{\mu\nu} \mathcal{L} \\ &= \bar{\psi} i \gamma^\nu \partial^\mu \psi - \eta^{\mu\nu} \mathcal{L} \neq T^{\nu\nu} \end{aligned} \tag{780}$$

Then

$$\begin{aligned} H = \int d^3x T^{00} &= \int d^3x \left\{ \bar{\psi} i \gamma^0 \partial^0 \psi - \bar{\psi} \, (i\gamma^\mu \partial_\mu - m)\psi \right\} \\ &= \int d^3x \, \bar{\psi}(-i\gamma^k \partial_k + m)\psi \end{aligned} \tag{781}$$

Recall that $\alpha^k = \gamma^0 \gamma^k$ and $\beta = \gamma^0$. And the Dirac Hamiltonian was $\hat{h} = -i \, \alpha^k \partial_k + m\beta$. So

$$H = \int d^3x \, \psi^+ (-i\alpha^k \partial_k + m\beta)\psi = \int d^3x \, \psi^+ \hat{h} \, \psi \tag{782}$$

$$p^k = \int d^3x \, T^{k0} = \int d^3x \, \bar{\psi} i \gamma^0 \partial^k \psi = -i \int d^3x \, \psi^+ \partial_k \psi = \int d^3x \, \psi^+ \hat{p}^k \psi \tag{783}$$

Note $p^i = i\partial^i = -i\partial_i$. Under Lorentz transformations

$$\psi^{'}(x^{'}) = \left(\mathbb{I} - \frac{i}{4} \epsilon^{\rho\sigma} \sigma_{\rho\sigma} \right) \psi(x) \tag{784}$$

$$\bar{\psi}^{'}(x^{'}) = \bar{\psi}(x) \left(\mathbb{I} + \frac{i}{4} \epsilon^{\rho\sigma} \sigma_{\rho\sigma} \right) \tag{785}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. So

$$S_{\rho\sigma} = \frac{1}{2} \sigma_{\rho\sigma} \quad \text{and} \quad \bar{S}_{\rho\sigma} = \frac{1}{2} \sigma_{\rho\sigma} \tag{786}$$

Then

$$\begin{aligned} M^{\rho\sigma\mu} &= x^\rho \, T^{\sigma\mu} - x^\sigma \, T^{\rho\mu} - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \frac{1}{2} \sigma^{\rho\sigma} \psi + i \bar{\psi} \frac{1}{2} \sigma^{\rho\sigma} \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} \\ &= x^\rho \, T^{\sigma\mu} - x^\sigma \, T^{\rho\mu} + \frac{1}{2} \bar{\psi} \gamma^\mu \sigma^{\rho\sigma} \psi \end{aligned} \tag{787}$$

So the angular momentum reads

$$\begin{aligned} M^{jk} &= \int d^3x M^{jk0} \\ &= \int d^3x \left\{ x^j\, T^{k0} - x^k\, T^{j0} + \frac{1}{2}\psi^+\sigma^{jk}\psi \right\} \\ &= \int d^3x\, \psi^+ \Big\{ -i(x^j\partial_k - x^k\partial_j) + \frac{1}{2}\sigma^{jk} \Big\}\psi \end{aligned} \tag{788}$$

In the standard representation

$$\sigma^{jk} = \epsilon^{jkl} \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}$$

Then we have

$$J^l = \frac{1}{2}\epsilon^{ljk} M^{jk} = \int d^3x\, \psi^+ \Big\{ -i\epsilon^{ljk} x^j \partial_k + \frac{1}{2}\Sigma^l \Big\}\psi \tag{789}$$

Phase Invariance:

$$\psi(x) \rightarrow \psi^{'}(x) \equiv e^{-i\alpha}\psi(x) \tag{790}$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}^{'}(x) \equiv e^{i\alpha}\bar{\psi}(x) \tag{791}$$

then

$$J^\mu(x) = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}(-i\psi) + i\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}} = \bar{\psi}\gamma^\mu\psi \tag{792}$$

which provides

$$Q = \int d^3x\, J^0 = \int d^3x\, \psi^+\psi \tag{793}$$

Real Scalar Field:

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \right) \tag{794}$$

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\partial^\mu\phi - \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} = T^{\nu\mu} \tag{795}$$

then

$$\begin{aligned} H = \int d^3x T^{00} &= \int d^3x\, (\partial^0\phi\partial^0\phi - \mathcal{L}) \\ &= \int d^3x \left(\frac{1}{2}(\partial^0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right) \end{aligned} \tag{796}$$

and

$$p^k = \int d^3x\, T^{k0} = \int d^3x\, \partial^0\phi\partial^k\phi = -i\int d^3x\, \partial_0 p^k\phi \tag{797}$$

and also

$$
\begin{aligned}
M^{\rho\sigma\mu} &= x^{\rho}T^{\rho\mu} - x^{\sigma}T^{\rho\mu} \\
M^{\rho\sigma} &= \int d^3x\, M^{\rho\sigma 0} = \int d^3x\, (x^{\rho}T^{\rho 0} - x^{\sigma}T^{\rho 0}) \\
M^{jk} &= \int d^3x\, \partial_0\phi\, (x^j\partial^k\phi - x^k\partial^j\phi) \\
M^{0k} &= \int d^3x\, (\partial_0\phi(x^0\partial^k - x^k\partial^0)\phi + x^k\mathcal{L}) \\
&= \int d^3x\, (tp^k - x^k\mathcal{H})
\end{aligned}
\tag{798}
$$

Important Note (Maggiore): Define the scalar product

$$
\langle \phi_1 \mid \phi_2 \rangle \equiv \frac{i}{2}\int d^3x\, \phi_1 \overleftrightarrow{\partial}_0 \phi_2 \tag{799}
$$

where

$$
\phi_1 \overleftrightarrow{\partial}_0 \phi_2 = \phi_1\partial_0\phi_2 - (\partial_0\phi_1)\,\phi_2 \tag{800}
$$

If ϕ_1 and ϕ_2 obey the KG equation, then this scalar product is time-independent. Check

$$
\begin{aligned}
\partial_0(\langle \phi_1 \mid \phi_2 \rangle) &= \frac{i}{2}\int d^3x\, (\partial_0\phi_1\, \partial_0\phi_2 + \phi_1\partial_0^2\phi_2 - (\partial_0^2\phi_1)\phi_2 - \partial_0\phi_1\partial_0\phi_2) \\
&= \frac{i}{2}\int d^3x\, (\phi_1\partial_0^2\phi_2 - \phi_2\partial_0^2\phi_1)
\end{aligned}
\tag{801}
$$

Since

$$
\partial_\mu\partial^\mu\phi_{1,2} + m^2\phi_{1,2} = 0 \quad \Rightarrow \quad (\partial_0^2 - \nabla^2)\phi_{1,2} + m^2\phi_{1,2} = 0 \tag{802}
$$

So

$$
\begin{aligned}
\partial_0(\langle \phi_1 \mid \phi_2 \rangle) &= \frac{i}{2}\int d^3x\, (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1) \\
&= \frac{i}{2}\int d^3x\, [\vec{\nabla}(\phi_1\vec{\nabla}\phi_2) - \vec{\nabla}(\phi_2\vec{\nabla}\phi_1)]
\end{aligned}
\tag{803}
$$

which is nothing but boundary term. Observe also that this scalar product is not positive definite. So what do we do with this? Recall that

$$
p^k = -i\int d^3x\, \partial_0\phi\, \hat{p}^k\phi = \frac{i}{2}\int d^3x\, (\phi\, \hat{p}^k\, \partial_0\phi - \partial_0\phi\, \hat{p}\, \phi) = \langle \phi \mid \hat{p}^k \mid \phi \rangle \tag{804}
$$

Since $\hat{p}^k = i\partial^k$, then we have

$$
p^\mu = \langle \phi \mid i\partial^\mu \mid \phi \rangle \tag{805}
$$

So the expectation value of the representation of the operator gives the corresponding Noether current

$$
M^{ij} = \langle \phi \mid L^{ij} \mid \phi \rangle \tag{806}
$$

Note again that

$$x^\mu \to x^{'\mu} = \lambda x^\mu \quad \text{and} \quad A^{'}_\mu(x^{'}) = l^{-\Delta} A_\mu(x) \tag{807}$$

Lets check

$$\int d^4x^{'}\, [\partial^{'}_\mu A^{'}_\nu(x^{'})]^2 = \int \lambda^4 d^4x \frac{1}{\lambda^2}\lambda^{-2\Delta}(\partial_\mu A_\nu)^2 \quad \text{so} \quad \Delta = 1 \tag{808}$$

Then $A^{'}_\mu(x^{'}) = \lambda^{-1}A_\mu(x)$. We better define a small scaling

$$x^{'\mu} = x^\mu + \tilde{\lambda}\, x^\mu \quad \text{where} \quad \lambda = 1 + \tilde{\lambda} \tag{809}$$

so

$$\begin{aligned} A^{'}_\mu(x^{'}) &= (1+\tilde{\lambda})^{-1} = (1-\tilde{\lambda})A_\mu(x) + O(\tilde{\lambda}^2) \\ A^{'}_\mu(x + \tilde{\lambda}\, x) &= A^{'}_\mu(x) + \tilde{\lambda}\, x^\alpha\, \partial_\alpha A_\mu(x) = A_\mu(x) - \tilde{\lambda} A_\mu(x) \end{aligned} \tag{810}$$

so we get

$$\delta_L A_\mu(x) = A^{'}_\mu - A_\mu(x) = -\tilde{\lambda}\,(1 + x^\alpha\partial_\alpha)A_\mu(x) = \tilde{\lambda} G_\mu(x) \tag{811}$$

So

$$G_\mu(x) = -(1 + x^\alpha\partial_\alpha)A_\mu(x) \tag{812}$$

How do we compute the X^μ term?

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{813}$$

then

$$\begin{aligned} \delta_L\mathcal{L} &= -\partial_\mu \delta_L A_\nu\, F^{\mu\nu} = \tilde{\lambda}\,\partial_\mu[(1 + x^\alpha\partial_\alpha)A_\nu]F^{\mu\nu} \\ &= \tilde{\lambda}\,(\partial_\mu A_\nu)F^{\mu\nu} + \tilde{\lambda}\,\partial_\mu(x^\alpha\partial_\alpha A_\nu)F^{\mu\nu} \\ &= \tilde{\lambda}\,(\partial_\mu A_\nu)F^{\mu\nu} + \tilde{\lambda}\,(\partial_\mu A_\nu)F^{\mu\nu} + \tilde{\lambda}\, x^\alpha\partial_\alpha\partial_\mu A_\nu\, F^{\mu\nu} \\ &= 2\tilde{\lambda}\,(\partial_\mu A_\nu)F^{\mu\nu} + \tilde{\lambda}\, x^\alpha\partial_\alpha\partial_\mu A_\nu\, F^{\mu\nu} \\ &= \tilde{\lambda}\, F^2_{\mu\nu} + \frac{\tilde{\lambda}}{2}x^\alpha(\partial_\alpha F_{\mu\nu})F^{\mu\nu} \\ &= -\tilde{\lambda}\,\partial_\alpha(x^\alpha\mathcal{L}) \qquad \Rightarrow \qquad X^\mu = -x^\mu\mathcal{L} \end{aligned} \tag{814}$$

Then we have the following conserved current

$$J^\mu = F^{\mu\nu}(1 + x^\alpha\partial_\alpha)A_\nu - \frac{x^\mu}{4}F^2_{\alpha\beta} \tag{815}$$

(Here check that upon use of the eom this yields $\partial J_\mu = 0.$). Then the total conformal charge is

$$\begin{aligned} Q = \int d^3x\, J^0 &= \int d^3x\, [F^{0i}(1 + x^\alpha\partial_\alpha)A_i - \frac{x^0}{4}F^2_{\alpha\beta}] \\ &= \int d^3x\, [\vec{r}\cdot(\vec{E}\times\vec{B}) - \frac{t}{2}(\vec{E}^2 + \vec{B}^2)] \\ &= \int d^3x\, [\vec{r}\cdot\vec{p} - tH] \\ &= -\int d^3x\, x^\mu p_\mu \end{aligned} \tag{816}$$

Questions: Do we get the same theory in YM theory? How about the broken theory?

Some details of the above computation:

$$\begin{aligned} F^{ji}(1+x^\alpha\partial_\alpha)A_i &= F^{ji}A_i + x^\alpha F^{ji}\partial_\alpha A_i \\ &= F^{ji}A_i + x^\alpha F^{ji}F_{\alpha i} + x^\alpha F^{ji}\partial_i A_\alpha \\ &= F^{ji}A_i + x^0 F^{ji}F_{0i} + x^i F^{ji}F_{ji} + \partial_i(x^\alpha F^{ji}A_i) - (\partial_i x^\alpha)F^{ji}A_\alpha - x^\alpha A_\alpha \partial_i(F^{ji}) \end{aligned} \tag{817}$$

Note that first $\partial_i(x^\alpha F^{ji}A_i)$ is boundary term and the last term on the right-hand side is zero upon use of equation of motion.

Digression: Abraham-Minkowski Controversy
In Maxwell's theory, using the Noether+Belinfante procedure, we get

$$T^{\mu\nu} = -F^\mu{}_\alpha F^{\nu\alpha} + \frac{1}{4}\eta^{\mu\nu}F^2_{\alpha\beta} \tag{818}$$

$$T^{00} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \qquad : \qquad \text{energy density} \tag{819}$$

$$T^{ji} = (\vec{E}\times\vec{B})^i \quad , \qquad \vec{p} = \vec{E}\times\vec{B} \qquad : \qquad \text{momentum density} \tag{820}$$

Restore now the usual units (Follow Miloni, Boyd 2010 Momentum of light in a Dielectric Medium Advances in optics and physics 2 519 – 553)

$$u = \frac{1}{2}(\epsilon_0\,\vec{E}^2 + \mu_0\,\vec{H}^2) \tag{821}$$

$$\vec{p} = \frac{1}{c^2}(\vec{E}\times\vec{H}) = \vec{D}\times\vec{B} \tag{822}$$

These are in a vacuum. For monochromatic plane waves

$$\vec{E}(\vec{r},t) = \hat{x}E_0\cos[w(\epsilon - \frac{z}{c})] \qquad , \qquad \vec{H}(\vec{r},t) = \hat{y}\sqrt{\frac{\epsilon_0}{\mu_0}}E_0\cos[w(\epsilon - \frac{z}{c})] \tag{823}$$

FIG 33 !!!!

Take $\cos^2(\;) = \frac{1}{2}$, then

$$u = \frac{1}{2}\epsilon_0 E^2 \quad \text{and} \quad \vec{p} = \frac{1}{2c}\hat{z}\epsilon_0 E^2 = \hat{z}\frac{u}{c} \tag{824}$$

In terms of photons we have $u = \frac{q\hbar w}{V}$ where a is the average number of photons in volume V. So

$$E_0^2 = \frac{2q\hbar w}{\epsilon_0 V} \tag{825}$$

Then

$$\vec{p} = \hat{z}\frac{q\hbar w}{cV} \tag{826}$$

2. More thoughts on the Noether's Theorem

Section 3.2 of Maggiore: Consider an infinitesimal transformation of the coordinates

$$x^\mu \to x^{'\mu} = x^\mu + \epsilon^a A_a^\mu \qquad\qquad a = 1, 2, \cdots N \tag{827}$$

which induces

$$\phi_i(x) \to \phi_i^{'}(x^{'}) = \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi) \tag{828}$$

So here we know A_a^μ and $F_{i,a}(\phi, \partial\phi)$. This transformation is called a symmetry of the theory of the action is *left-invariant*. Note that we do not use the equation of motion.

1. (a) global symmetry: if ϵ^a are constants.
   (b) Local symmetry: if ϵ^a depend on x. (Of course this implies i.)
2. (a) Internal symmetries $A_a^\mu(x) = 0$.
   (b) Space-time symmetries $A_a^\mu(x) \neq 0$.

   Note: For internal symmetries and for Poincare symmetries d^4x is intact so we can talk about the invariance of $\mathcal{L}$.

Case 1: Consider the *global* case. Suppose our action is invariant under the above transformations when ϵ^a is constant.. Then assume that ϵ^a is a slowly varying function of x

$$\mid \epsilon^a \mid \ll 1 \quad \text{and} \quad l \mid \partial_\mu \epsilon^a \mid \ll \mid \epsilon^a \mid \ll 1 \tag{829}$$

where l is some characteristic length. Now S will not be invariant, but we can expand it up to $O(\epsilon)$.

$$S(\phi^{'}) = S(\phi) + \int d^4x \, [\epsilon^a(x) K_a(\phi) - (\partial_\mu \epsilon^a) J_a^\mu(x) + O(\partial\partial\epsilon) + O(\epsilon^2) \cdots] \tag{830}$$

This equation is valid for any slowly varying ϵ's and in particular when ϵ does not vary! But when ϵ does not vary we know that $S(\phi^{'}) = S(\phi)$ so if it is clear that $K_a(\phi) = 0$ for any ϕ. Since $K(\phi)$ is independent of ϵ such a conclusion is valid even when ϵ varies with x. So then for any slowly varying $\epsilon(x)$, we have

$$S(\phi^{'}) = S(\phi) - \int d^4x \, (\partial_\mu \epsilon^a) J_a^\mu(\phi) + O(\epsilon^2) \;\; \text{etc.} \tag{831}$$

Now take ϵ to be a sufficiently vanishing function at infinity then integrate by parts

$$S(\phi^{'}) = S(\phi) + \int d^4x \, \epsilon^a(x) \partial_\mu J_a^\mu(\phi) \qquad \text{valid for any } \; \phi. \tag{832}$$

Here we have $\phi^{'}(x^{'})$ but we can rename $x^{'} = x$. S

$$\phi^{'}(x) = \phi^{'}(x^{'}) - \epsilon^a A_a^\mu \partial_\mu \phi^{'}(x^{'}) \tag{833}$$

$$\phi^{'}(x) = \phi(x) + \underbrace{\epsilon^a F_a(\phi, \partial\phi) - \epsilon^a A_a^\mu \partial_\mu \phi^{'}(x^{'})}_{\delta\phi} \tag{834}$$

So what we have done is to show that we have a generic variation of the action. For classical fields solutions$'$ generic first order variation of the action is zero. So then we have

$$\partial_\mu J^\mu_a(\phi^{cl}) = 0 \tag{835}$$

Important: We observe a global symmetry in the action. Then we make it a local one. This local symmetry works for classical solutions. But if it works generically, then it is a Bianchi identity and the symmetry is a gauge symmetry!!!

$$Q_a \equiv \int d^3x\, J^0_a(\vec{x}, t) \quad \text{such that} \quad \partial_0 Q_a = 0. \tag{836}$$

So how do we get the current

$$x^\mu \to x^{'\mu} = x^\mu + \epsilon^a A^\mu_a(x) \quad \text{so} \quad \phi_i(x) \to \phi^{'}(x^{'}) = \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi) \tag{837}$$

then

$$\delta_\epsilon S = \delta_\epsilon \int d^4x\, \mathcal{L} = \int [\delta_\epsilon(d^4x)\mathcal{L} + d^4x\, \delta_\epsilon \mathcal{L}] \tag{838}$$

now

$$\left| \frac{\partial x^{'\mu}}{\partial x^\nu} \right| = \det[\delta^\mu_\nu + \epsilon^a \partial_\nu A^\mu_a(x) + \partial_\nu \epsilon^a\, A^\mu_a] \tag{839}$$

$$\delta_\epsilon d^4x = [\epsilon^a \partial_\mu A^\mu_a(x) + \partial_\mu \epsilon^a\, A^\mu_a] d^4x \tag{840}$$

and

$$\delta_\epsilon \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_i} \delta_\epsilon \phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta_\epsilon \partial_\mu \phi_i \tag{841}$$

Note that δ_ϵ also changes the coordinates $\delta_\epsilon \partial_\mu \neq \partial_\mu \delta_\epsilon$, then

$$\begin{aligned} \delta_\epsilon(\partial_\mu \phi_i) &\equiv \frac{\partial \phi^{'}_i(x^{'})}{\partial x^{'\mu}} - \frac{\partial \phi_i(x)}{\partial x^\mu} \\ &= \frac{\partial x^\nu}{\partial x^{'\mu}} \frac{\partial}{\partial x^\nu}[\phi_i + \epsilon^a F_{i,a}(\phi, \partial\phi)] - \frac{\partial \phi_i(x)}{\partial x^\mu} \\ &= (\delta^\nu_\mu - \epsilon^a \partial^\mu A^\nu - \partial^\mu \epsilon\, A^\nu_a)[\frac{\partial \phi_i}{\partial x^\nu} + \frac{\partial(\epsilon^a F_a)}{\partial x^\nu}] - \frac{\partial \phi_i}{\partial x^\mu} \end{aligned} \tag{842}$$

which produces a term

$$-(\partial_\mu \epsilon^a)[A^\nu_a \partial_\nu \phi_i - F_{i,a}(\phi, \partial\phi)] \qquad \text{forget the rest.} \tag{843}$$

So all the $\partial_\mu \epsilon^a$ terms yield

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}[A^\nu_a \partial_\nu \phi_i - F_{i,a}(\phi, \partial\phi)] - A^\mu_a(x)\mathcal{L} \tag{844}$$

I think this is the most intuitive derivation. For internal symmetries $A^\mu_a = 0$, so

$$J^\mu_a = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} F_{i,a}(\phi, \partial\phi) \qquad : \qquad \text{internal symmetries} \tag{845}$$

Quite often one is interested in the linear transformation of the fields

$$F_{i,a}(\phi, \partial\phi) = (M_a)_i{}^j\, \phi_j \tag{846}$$

where $(M_a)_i{}^j$ are N constant matrices. Whta if the transformations

$$x^{'\mu} = x^\mu + \epsilon^a A_a^\mu \quad \text{and} \quad \phi^{'}(x^{'}) = \phi(x) + \epsilon^a F_a \tag{847}$$

is not global symmetry. (So it is not a symmetry !). Then

$$\partial_\mu J_a^\mu = -\delta_0 \mathcal{L}_{global} \tag{848}$$

where J_a^μ is as given before.

<u>The Energy Momentum Tensor</u>

<u>Translations</u>

$$\epsilon^a \rightarrow \epsilon^\mu \tag{849}$$

$$x^\mu \rightarrow x^{'\mu} = x^\mu + \epsilon^\mu = x^\mu + \epsilon^\nu \delta^\mu{}_\nu \tag{850}$$

$$\phi_i(x) \rightarrow \phi_i^{'}(x^{'}) = \phi_i(x) \tag{851}$$

all the fields are assumed to be scalars under translations. So then

$$A^\mu{}_\nu = \delta^\mu{}_\nu \quad , \quad F_{i,a} = 0 \tag{852}$$

Then

$$J^\mu_{(\nu)} = T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i} \partial^\mu \phi_i - \eta^{\mu\nu} \mathcal{L} \tag{853}$$

<u>Angular Momentum</u> (in fact general Lorentz invariance)

$$x^\mu \rightarrow x^{'\mu} = x^\mu + \epsilon^\mu{}_\nu \equiv x^\mu + \epsilon^a A_a^\mu \tag{854}$$

$$\phi_i(x) \rightarrow \phi_i^{'}(x^{'})(\mathbb{I} - \frac{i}{2}\epsilon^{\rho\sigma} S_{\rho\sigma})_i{}^j\, \phi_j(x) \equiv \phi_i(x) + \epsilon^a F_{i,a} \tag{855}$$

Observe that

$$\epsilon^\mu{}_\nu x^\nu = \epsilon^a A_a^\mu = \epsilon_\nu A^{\mu\nu} = \epsilon^\beta{}_\nu A^{\mu\nu}{}_\beta \tag{856}$$

So

$$\delta^\mu{}_\beta \epsilon^\beta{}_\nu x^\nu = \epsilon^\beta{}_\nu A^{\mu\nu}{}_\beta \tag{857}$$

Therefore keeping the $\beta \leftrightarrow \nu$ anti-symmetric in mind, we have

$$A^{\mu\nu}{}_\beta = \frac{1}{2}(\delta^\mu{}_\beta x^\nu - \eta^{\mu\nu} x_\beta) \tag{858}$$

Let us find $F_{i,a}$ now

$$-\frac{i}{2}\epsilon^{\rho\sigma}(S_{\rho\sigma})_i{}^j\,\phi_j(x) = \epsilon^a F_{i,a} = \epsilon^{\rho\sigma}F_{i,\rho\sigma} \qquad \Rightarrow \qquad F_{i,\rho\sigma} = -\frac{i}{2}(S_{\rho\sigma})_i{}^j\,\phi_j(x) \tag{859}$$

Then

$$J^\mu_a = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}(A^\nu_a\partial_\nu\phi_i - F_{i,a}) - A^\mu_a\mathcal{L} \tag{860}$$

which is

$$\begin{aligned} J^\mu{}_\lambda{}^\beta &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}(A^{\nu\beta}_\lambda\partial_\nu\phi_i - F_{i,\lambda}{}^\beta) - A^{\mu\beta}_\lambda\mathcal{L} \\ &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}[(\delta^\nu{}_\lambda x^\beta - \eta^{\nu\beta}x_\lambda)\partial_\nu\phi_i + i(S_\lambda{}^\beta)_i{}^j\,\phi_j] - (\delta^\nu{}_\lambda x^\beta - \eta^{\nu\beta}x_\lambda)\mathcal{L} \end{aligned} \tag{861}$$

or

$$J^{\mu\lambda\beta} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}[(x^\beta\partial^\lambda - x^\lambda\partial^\beta)\phi_i + i(S^{\lambda\beta})_i{}^j\,\phi_j] - (\eta^{\mu\lambda}x^\beta - \eta^{\mu\beta}x^\lambda)\mathcal{L} \tag{862}$$

where we dropped 1/2 factors.

SPINOR FIELDS

The Weyl equation; helicity ; Consider a theory with a single left-handed Weyl field ψ_L

$$\psi_L^+\bar{\sigma}^\mu\psi_L \tag{863}$$

is a four-vector $\bar{\sigma}^\mu(\mathbb{I}, -\sigma^i)$ which we found before. Then we can write a first-derivative Lorentz invariant Lagrangian

$$\mathcal{L}_L = i\psi_L^+\bar{\sigma}^\mu\partial_\mu\psi_L \qquad \text{where} \qquad \bar{\sigma}^{\mu+} = \bar{\sigma}^\mu \tag{864}$$

where "i" is introduced to make $\mathcal{L}$ real. Let us now check

$$\mathcal{L}_L = i\psi^*_{L\alpha}(\bar{\sigma}^\mu)_{\alpha\beta}\,\partial_\mu\psi_\beta \tag{865}$$

since

$$[(\bar{\sigma}^\mu)^T]^* = \bar{\sigma}^\mu \qquad \Rightarrow \qquad (\bar{\sigma}^\mu)^* = \bar{\sigma}^{\mu T} \tag{866}$$

then we have

$$\begin{aligned} \mathcal{L}^*_L &= -i\psi_{L\alpha}(\bar{\sigma}^\mu)^*_{\alpha\beta}\,\partial_\mu\psi_\beta \\ &= -i\psi_{L\alpha}\bar{\sigma}^\mu{}_{\beta\alpha}\,\partial_\mu\psi_\beta \\ &= \mathcal{L}_L \qquad\qquad \text{(up to a boundary term).} \end{aligned} \tag{867}$$

Now vary with respect to ψ^*_L and ψ_L to get

$$\bar{\sigma}^\mu\partial_\mu\psi_L = 0 \qquad \Rightarrow \qquad (\partial_0 - \sigma^i\partial_i)\psi_L = 0 \tag{868}$$

Since

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k, \tag{869}$$

we have

$$
\begin{aligned}
(\partial_0 + \sigma^j\partial_j)(\partial_0 - \sigma^i\partial_i)\psi_L &= 0 \\
(\partial_0^2 - \underbrace{\sigma^j\sigma^i\partial_j\partial_i}_{-\partial_i^2})\psi_L &= 0 \qquad \Rightarrow \qquad \Box\psi_L = 0
\end{aligned} \tag{870}
$$

which is massless Klein-Gordon equation. However, the first order equation has more information. Let us check: Consider a plane-wave solution of positive energy

$$
\psi_L(x) = u_L e^{-ip\cdot x} \tag{871}
$$

where u_L is a constant spinor. Then

$$
(\partial_0 - \sigma^i\partial_i)\psi_L = 0 \tag{872}
$$

$$
(-iE - i\sigma^i p^i)u_L = 0 \qquad \Rightarrow \qquad \frac{(\vec{p}\cdot\vec{\sigma})}{E}u_L = -u_L \tag{873}
$$

Of course $\Box\psi = 0$ gives $E = \mid \vec{p}\mid$. Since for a spin-$\frac{1}{2}$ field the angular momentum is $\vec{J} = \frac{\vec{\sigma}}{2}$ we have

$$
(\vec{p}\cdot\vec{J})u_L = -\frac{1}{2}u_L \qquad \text{where} \qquad \hat{p} = \frac{\vec{p}}{\mid\vec{p}\mid} \tag{874}
$$

Since helicity was defined as

$$
h = \hat{p}\cdot\vec{J} \tag{875}
$$

This equation shows that a left-handed massless Weyl spinor has helicity $h = -\frac{1}{2}$.

Energy-Momentum Tensor:
The general formula was

$$
\begin{aligned}
T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial\partial_\nu\phi_a}\partial^\mu\phi^a - \eta^{\mu\nu}\mathcal{L} \qquad\qquad (\partial_\nu T^{\mu\nu} = 0) \\
&= i\psi_L^+\bar{\sigma}^\nu\partial^\mu\psi_L - \eta^{\mu\nu}i\psi_L^+\bar{\sigma}^\alpha\partial_\alpha\psi_L
\end{aligned} \tag{876}
$$

on a classical solution $\bar{\sigma}^\mu\partial_\mu\psi_L = 0$, so the Lagragian vanishes and we have

$$
T^{\mu\nu} = i\psi_L^+\bar{\sigma}^\nu\partial^\mu\psi_L \qquad \Rightarrow \qquad T^{00} = i\psi_L^+\bar{\sigma}^0\partial^0\psi_L = i\psi_L^+\partial^0\psi_L \tag{877}
$$

so

$$
H = \int d^3x\, \psi_L^+ i\partial^0\psi_L \tag{878}
$$

Global $U(1)$ invariance: Clearly we have

$$
\psi_L \to e^{i\theta}\psi_L \tag{879}
$$

where θ is a constant. Then

$$
\delta\psi_L = i\theta\psi_L = \theta G_a \qquad \Rightarrow \qquad G_a = i\psi_L \tag{880}
$$

Then the Noether current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\, \epsilon G_a = i\psi_L^+ \sigma^\mu i\psi_L = -\psi_L^+ \sigma^\mu \psi_L \tag{881}$$

So

$$Q_{U(1)} = -\int d^3x\, \psi_L^+ \psi_L \tag{882}$$

Note that under parity, $\mathcal{L}$ is not invariant! Note also that I could have considered the following Lagrangian

$$\begin{aligned} \mathcal{L}' &= i\psi_L^+ \bar{\sigma}^\mu \partial_\mu \psi_L - \frac{i}{2}\partial_\mu(\psi_L^+ \sigma^\mu \psi_L) \\ &= \frac{i}{2}\psi_L^+ \bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu \psi_L \end{aligned} \tag{883}$$

which will give different currents but the changes will be the same.

<u>RIGHT-HANDED WEYL SPINORS</u>:

$$\mathcal{L}_R = i\psi_R^+ \sigma^\mu \partial_\mu \psi_R \qquad \text{where} \qquad \sigma^\mu = (\mathbb{I}, \sigma^i) \tag{884}$$

Then equation of motions are

$$\sigma^\mu \partial_\mu \psi_R = (\partial_0 + \sigma^i \partial_i)\psi_R = 0 \tag{885}$$

and the positive energy solution has helicity $h = +\frac{1}{2}$. Needless to say that neutrinos come in 3 flavors ν_e, ν_μ, ν_τ with spin-$\frac{1}{2}$

$$\Delta m^2 \sim 10^{-5} - 10^{-3}\ eV^2 \tag{886}$$

If $m = 0$, then we have left-handed massless Weyl spinors.

<u>Question</u>: Is it possible to describe a massive particle with a single Weyl spinor? Yes, that is called a *Majorana spinor*.

<u>THE DIRAC EQUATION</u>: (Our second visit)

Suppose at our disposal we have ψ_L and ψ_R, then we can construct *two* new Lorentz scalars

$$\psi_L^+ \psi_R \qquad \text{and} \qquad \psi_R^+ \psi_L \tag{887}$$

Recall

$$\psi_L \to \Lambda_L \psi_L = e^{(-i\vec{\theta} - \vec{\eta})\cdot \frac{\vec{\sigma}}{2}}\, \psi_L \quad \text{and} \quad \psi_R \to \Lambda_R \psi_R = e^{(-i\vec{\theta} + \vec{\eta})\cdot \frac{\vec{\sigma}}{2}}\, \psi_R \tag{888}$$

such that

$$\Lambda_L^+ \Lambda_R = \mathbb{I} = \Lambda_R^+ \Lambda_L \tag{889}$$

So we have *two* real contractions

1. $\psi_L^+ \psi_R + \psi_R^+ \psi_L$

2. $i(psi_L^+\psi_R - \psi_R^+\psi_L)$

   which, under parity transformation, transform as

$$\psi_L \to \psi_R \quad \text{and} \quad \psi_R \to \psi_L \tag{890}$$

So 1*st* is a scalar under parity and 2nd is a pseudoscalar . So if you want a parity invariant theory, then we have

$$\mathcal{L}_D = i\psi_L^+\bar{\sigma}^\mu\partial_\mu\psi_L + i\psi_R^+\sigma^\mu\partial_\mu\psi_R - m(\psi_L^+\psi_R + \psi_R^+\psi_L) \tag{891}$$

Under parity $\partial_i \to -\partial_i$ and since $\bar{\sigma}^\mu = (\mathbb{I}, -\sigma^i)$ and $\sigma^\mu = (\mathbb{I}, \sigma^i)$, we have

$$\bar{\sigma}^\mu\partial_\mu \leftrightarrow \sigma^\mu\partial_\mu \tag{892}$$

and thus, under parity, we get

$$\mathcal{L}_D^{'} \to \mathcal{L}_D \tag{893}$$

Euler-Lagrange equation:

Let (ψ_L^*, ψ_L) and (ψ_R^*, ψ_R) be independent variables, then

$$i\bar{\sigma}^\mu\partial_\mu\psi_L = m\psi_R \qquad ; \qquad i\sigma^\mu\partial_\mu\psi_R = m\psi_L \tag{894}$$

which can be written as

$$\sigma^\mu i\partial_\mu\bar{\sigma}^\alpha i\partial_\alpha\psi_L = m\sigma^\mu i\partial_\mu\psi_R \qquad \text{and} \qquad -\sigma^\mu\bar{\sigma}^\alpha\partial_\mu\partial_\alpha\psi_L = m^2\psi_L \tag{895}$$

Use the identity

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu} \tag{896}$$

Then

$$(\Box + m^2)\psi_L = 0 \qquad \text{and} \qquad (\Box + m^2)\psi_R = 0 \tag{897}$$

Let us rewrite everything in terms of a Dirac spinor

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad : \qquad \text{chiral representation}$$

We could of course choose

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix}$$

In the chiral representation we define

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad ; \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \gamma^k = -\gamma_i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

such that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Then we have

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{898}$$

whose Lagrangian density is

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{899}$$

Note that in the chiral representation

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

So we get

$$\frac{\mathbb{I} - \gamma^5}{2}\psi = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$

and

$$\frac{\mathbb{I} + \gamma^5}{2}\psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

Thus $\frac{\mathbb{I} \pm \gamma^5}{2}$ is a projection operator onto Weyl spinors. Therefore a left-handed neutrino in the Dirac representation is given as

$$\nu = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix} \qquad \text{such that} \qquad \frac{\mathbb{I} - \gamma^5}{2}\nu = \nu$$

Note that standard representation is good for non-relativistic limit whereas chiral representation is good for ultra-relativistic limit. Needless to say that chiral and standard representations are related by the following unitary transformations: Consider U to be a constant unitary matrix, then

$$\psi^{'} = U\psi \qquad \text{such that} \qquad U^+ = U^{-1} \tag{900}$$

is a new Dirac spinor. Then

$$\begin{aligned} \mathcal{L}_D &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\ &= \psi^+\gamma^0(i\gamma^\mu \partial_\mu - m)\psi \\ &= (U^{-1}\psi^{'})^+\gamma^0(i\gamma^\mu \partial_\mu - m)U^{-1}\psi^{'} \\ &= \psi^{'+}U\gamma^0(i\gamma^\mu \partial_\mu - m)U^+\psi^{'} \\ &= \bar{\psi}^{'}\gamma^{'0}U\gamma^0(i\gamma^\mu \partial_\mu - m)U^+\psi^{'} \\ &= \bar{\psi}^{'}(i\gamma^{'\mu} \partial_\mu - m)\psi^{'} \end{aligned} \tag{901}$$

where we used

$$\gamma^{'\mu} = \gamma^{0'}U\gamma^0\gamma^\mu U^+ \tag{902}$$

$$\gamma^{0'} = \gamma^{0'} U\gamma^0\gamma^0 U^+ = \mathbb{I} \tag{903}$$

$$\gamma^{0'}\gamma'^{\mu} = U\gamma^0\gamma^\mu U^+ \tag{904}$$

so we have

$$\gamma'^{\mu} = U\gamma^\mu U^+ \tag{905}$$

Since have

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & \mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{pmatrix}$$

then

$$\psi_{standard} = U\psi_{chiral} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & \mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} = -\begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

here

$$\gamma_s^\mu = U\gamma_c^\mu U^+ \tag{906}$$

then

$$\gamma_s^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad , \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

Solution of the Dirac Equation:

Consider plane-wave type solutions

$$\psi_1(x) = u(p)e^{-ip\cdot x} \qquad \text{and} \qquad \psi_2(x) = v(p)e^{ip\cdot x} \tag{907}$$

where $u(p)$ and $v(p)$ are four-component spinors. A general solution will be a superposition of these. ψ_1 in a classical theory corresponds to positive energy solutions and ψ_2 to negative energy solutions. Proper interpretation of the these solutions will come after quantization

$$(i\gamma^\mu\partial_\mu - m)\psi_1(x) = (\gamma^\mu p_\mu - m)u(p) = 0 \tag{908}$$

and

$$(i\gamma^\mu\partial_\mu - m)\psi_2(x) = (\gamma^\mu p_\mu + m)v(p) = 0 \tag{909}$$

Let us use the chiral representation and write

$$u(p) \equiv \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix} \quad \text{and consider } m \neq 0.$$

Then we can solve the equation in the rest frame

$$p^\mu = (m, 0, 0, 0) \tag{910}$$

which provides

$$(\gamma^0 p_0 - m)\begin{pmatrix} u_L \\ u_R \end{pmatrix} = m\left[\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} - \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}\right]\begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0$$

which is

$$\begin{pmatrix} -\mathbb{I} & \mathbb{I} \\ \mathbb{I} & -\mathbb{I} \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \qquad \Rightarrow \qquad u_L = u_R$$

Note that the Klein-Gordon equation imposes the mass condition $p^2 = m^2$ but 1^{st} order Dirac equation gives $u_L = u_R$.

I will actually go back to the standard representation and do the following computation, which is somewhat explicit

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{911}$$

let

$$\psi_+(x) \equiv e^{-ip\cdot x} u(p) \tag{912}$$

where

$$u(p) = \begin{pmatrix} u_1(p) \\ u_2(p) \\ u_3(p) \\ u_4(p) \end{pmatrix}$$

Note that (in the rest frame)

$$p^\mu = (m, \vec{0}) \qquad \text{and} \qquad \gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

Then

$$(\gamma^0 p_0 - m)u(0) = 0 \qquad \Rightarrow \qquad m\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = 0$$

So we have *two* solutions

$$u_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad u_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Consider the other case

$$\psi_-(x) = e^{+ip\cdot x} v(p) \qquad \Rightarrow \qquad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = 0$$

we get

$$v_1(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad v_2(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now let us obtain the finite $\vec{p}$ solutions out of these

$$(\not{p} - m)u(p) = 0 \qquad \Rightarrow \qquad u(p) = (\not{p} + m)u(0) \quad \text{is a solution} \tag{913}$$

since

$$(\not{p} + m)(\not{p} - m) = p^2 - m^2 = 0 \tag{914}$$

So we have

$$\psi_+(x) = e^{-ip\cdot x}(\not{p} + m)u(0) \qquad , \qquad \psi_-(x) = e^{+ip\cdot x}(\not{p} - m)v(0) \tag{915}$$

Let us find the solutions in a more direct approach: Consider the ansatz

$$\psi(x) \equiv e^{-ip\cdot x}u(p) \tag{916}$$

which leads

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = (\gamma^\mu p_\mu - m)u(p) = 0 \qquad \Rightarrow \qquad (\gamma^0 p_0 + \gamma^i p_i - m)u(p) = 0 \tag{917}$$

Take

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \qquad \text{and} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Let us

$$u(p) \equiv \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix} \qquad \text{such that} \qquad \phi(p) = \begin{pmatrix} \phi_1(p) \\ \phi_2(p) \end{pmatrix} \quad , \quad \chi(p) = \begin{pmatrix} \chi_1(p) \\ \chi_2(p) \end{pmatrix}$$

Then we have

$$\begin{pmatrix} p_0 - m & \sigma^i p_i \\ -\sigma^i p_i & -p_0 - m \end{pmatrix} \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix} = 0$$

since $\sigma^i p_i = -\vec{\sigma}\cdot\vec{p}$, we have

$$\begin{pmatrix} p_0 - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 - m \end{pmatrix} \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix} = 0$$

which gives

$$(p_0 - m)\phi(p) - (\vec{\sigma}\cdot\vec{p})\chi(p) = 0 \quad ; \quad \vec{\sigma}\cdot\vec{p}\phi(p) = (p_0 + m)\chi(p) \tag{918}$$

So

$$\chi(p) = \frac{\vec{\sigma}\cdot\vec{p}}{p_0 + m}\phi(p) \tag{919}$$

Then

$$(p_0^2 - m^2)\phi(p) = (\vec{\sigma}\cdot\vec{p})^2\phi(p) = \vec{p}^2\phi(p) \tag{920}$$

So

$$p_0 = \pm\sqrt{\vec{p}^2 + m^2} \tag{921}$$

1. Positive Energy solutions: $(p30 = E = \sqrt{\vec{p}^2 + m^2})$

$$\phi(p) = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\chi(p) \qquad ; \qquad \chi(p) = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi(p) \tag{922}$$

Observe that in the non-relativistic limit $E \sim m$ and $\phi \gg \chi$

Normalization: Write down a Normalization of the plane-wave which is Lorentz invariant

$$u(p) = \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix} \qquad \text{such that} \qquad \bar{u}(p)u(p) = u^{+}\gamma^{0}u(p) = \mathbb{I}$$

then

$$(\phi^{+}, \chi^{+})\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix} = \mathbb{I} \qquad \Rightarrow \qquad \phi^{+}(p)\phi(p) - \chi^{+}(p)\chi(p) = \mathbb{I}$$

Since

$$\chi(p) = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi(p) \tag{923}$$

we have

$$\phi^{+}(p)\phi(p) - \phi^{+}\frac{(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2}\phi(p) = \mathbb{I} \tag{924}$$

$$\phi^{+}\phi\Big(1 - \frac{\vec{p}^2}{(E+m)^2}\Big) = 1 \tag{925}$$

$$\phi^{+}\phi(E^2 + 2mE + m^2 - \vec{p}^2) = (E+m)^2 \tag{926}$$

so we get

$$\phi^{+}\phi = \frac{E+m}{2m} \tag{927}$$

Thus up to a phase we can choose

$$\phi(p) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \phi(p) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Define

$$w_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad w_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then

$$\phi(p) = \sqrt{\frac{E+m}{2m}}w_a \qquad ; \qquad \chi(p) = \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}w_a \tag{928}$$

where $a = 1, 2$.

Summary: Positive Energy solutions: ($p^0 = E = \sqrt{\vec{p}^2 + m^2}$)

$$\psi(x) = e^{-ip\cdot x} U_a(p) \tag{929}$$

where $a = 1, 2$ and

$$U_a(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} w_a \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}} w_a \end{pmatrix}$$

Note that Normalization is

$$\bar{u}_a(p) u_a(p) = \mathbb{I} \qquad \text{(no summations)} \tag{930}$$

2. Negative energy solutions: ($p^0 = -E = -\sqrt{\vec{p}^2 + m^2}$)

$$\psi(x) \equiv e^{ip\cdot x} v(p) \qquad \text{where} \qquad v(p) = \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix}$$

but now

$$\phi(p) = -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi(p) \qquad ; \qquad \chi(p) = -\frac{\vec{\sigma}\cdot\vec{p}}{E-m}\phi(p) \tag{931}$$

Now

$$\bar{v}(p) v(p) \stackrel{?}{=} -1 \tag{932}$$

OK this works.

Summary: Negative energy solutions

$$p^0 = -E = -\sqrt{\vec{p}^2 + m^2} \qquad \text{and} \qquad \psi(x) = e^{ip\cdot x} v(p) \tag{933}$$

where

$$U_a(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} w_a \\ \frac{-\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E-m)}} w_a \end{pmatrix}$$

Note that

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Choose $p^\mu = (p^0, 0, 0, p)$:

1. Positive energy solutions look like

$$\psi(x) = e^{-iEt + ipz} \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p}{E+m} \end{pmatrix}$$

where $E = \sqrt{p^2 + m^2}$.

2. Negative energy solutions look like

$$\psi(x) = e^{iEt-ipz}\sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}\ ,\ \begin{pmatrix} 0 \\ \frac{p}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

CHIRAL SYMMETRY

$$\mathcal{L}_D = i\psi_L^+\bar{\sigma}^\mu\partial_\mu\psi_L + i\psi_R^+\sigma^\mu\partial_\mu\psi_R - m(\psi_L^+\psi_R + \psi_R^+\psi_L) \tag{934}$$

for $m = 0$, we have the following global symmetry

$$\psi_L \to e^{i\theta_L}\psi_L \qquad ; \qquad \psi_R \to e^{i\theta_R}\psi_R \tag{935}$$

Since θ_L and θ_R we have a $U(1) \times U(1)$ symmetry

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \text{so when} \qquad \alpha = \theta_R = \theta_L$$

we have

$$\psi_D \to e^{i\alpha}\psi_D \tag{936}$$

And for $\theta_R = -\theta_L = \beta$ we have

$$\psi_D \to e^{i\beta\gamma^5}\psi_D \tag{937}$$

So these are transformations of $\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi$ theory. Check the second one

$$\begin{aligned} \bar{\psi}' i\gamma^\mu\partial_\mu\psi' &= \psi^{+'}\gamma^0 i\gamma^\mu\partial_\mu\psi' \qquad\qquad (\gamma^{5+} = \gamma^5) \\ &= \psi^+ e^{-i\beta\gamma^5}\gamma^0 i\gamma^\mu e^{i\beta\gamma^5}\partial_\mu\psi \\ &= \bar{\psi}e^{i\beta\gamma^5} i\gamma^\mu e^{i\beta\gamma^5}\partial_\mu\psi \\ &= \mathcal{L}_D \end{aligned} \tag{938}$$

where we used $\{\gamma^\mu, \gamma^5\} = 0$. Here $\psi \to e^{i\alpha}\psi$ will be promoted to a local $U(1)$. Also, we have

$$J^\mu_\nu = \bar{\psi}\gamma^\mu\psi \tag{939}$$

is the conserved charge. The other one is

$$J^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{940}$$

which is called *axial current*. If we turn m the mass, then we have

$$\partial_\mu J^\mu_A = 2im\bar{\psi}\gamma^5\psi \tag{941}$$

MAJORANA MASS: Is it possible to describe a massive particle with a single Weyl field? YES

Given a left-handed Weyl spinor ψ_L we can construct a right-handed one $\psi_R \equiv i\sigma^2\psi_L^*$. So we can write the Dirac equation as

$$\bar{\sigma}^\mu i\partial_\mu \psi_L = im\sigma^2\psi_L^* \qquad \Rightarrow \qquad (\Box + m^2)\psi_L = 0 \tag{942}$$

such a mass term is known a *Majorana mass*. And

$$\begin{aligned} \sigma^\mu i\partial_\mu(i\sigma^2\psi_L^*) &= m\psi_L \\ -\sigma^\mu\partial_\mu\sigma^2\psi_L^* &= m\psi_L \\ \sigma^2\sigma^\mu\sigma^2\partial_\mu\psi_L^* &= -m\sigma^2\psi_L \end{aligned} \tag{943}$$

Recall that

$$\sigma^2\sigma^i\sigma^2 = -\sigma^{i*} \quad , \quad \sigma^2\sigma^0\sigma^2 = \sigma^0 \quad \Rightarrow \quad \sigma^2\sigma^\mu\sigma^2 = \bar{\sigma}^{\mu*} \tag{944}$$

then we have

$$\bar{\sigma}^{\mu*}\partial_\mu\psi_L^* = -m\sigma^2\psi_L \tag{945}$$

Since

$$\psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix}$$

So then "$*''$ becomes

$$(i\not\partial - m)\psi_M = 0 \tag{946}$$

The interesting thing is that this equation cannot be derived from an action. Since

$$\begin{aligned} \bar{\psi}_M\psi_M &= \psi_M^+\gamma^0\psi_M \\ &= (\psi_L^{T*}, -i\psi_L^T\sigma^2)\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix} \\ &= (\psi_L^{T*}, -i\psi_L^T\sigma^2)\begin{pmatrix} i\sigma^2\psi_L^* \\ \psi_L \end{pmatrix} \\ &= \psi_L^{T*}i\sigma^2\psi_L^* - i\psi_L^T\sigma^2\psi_L \\ &= -i\psi_L^T\sigma^2\psi_L + \text{h.c.} \end{aligned}$$

Note that

$$\psi_L^T\sigma^2\psi_L = \psi_{L\alpha}\sigma^2_{\alpha\beta}\psi_\beta = 0 \qquad \text{since} \qquad \sigma^2_{\alpha\beta} = -\sigma^2_{\beta\alpha} \tag{947}$$

Note also that since $\psi_L \to e^{i\alpha}\psi_L$ implies $\psi_R \to e^{-i\alpha}\psi_R$, we cannot have global $U(1)$ symmetry for the Majorana mass. So this means that a spin-1/2 particle which carries a $U(1)$ charge cannot have a $U(1)$ conserved charge.

Similarly particles which are denoted by a Majorana mass violate lepton number.

Neutrino: It could be a right Dirac neutrino in which case we need a *sterile*(???) right-handed neutrino or Majorana neutrino in which case we need lepton-number violation.

Experiment aiming at neutrino-less β-decay aim at detecting these violations.

Double β-decay

$$2n \to 2p + 2e^- + 2\bar{\nu}_e \tag{948}$$

FIG 34 !!!!

Neutrinoless double β-decay

$$2n \to 2p + 2e^- \tag{949}$$

FIG 35 !!!!

3. *First quantization of the Relativistic Wave Equations*

Consider the non-relativistic limit and promote the Dirac field to the wave function

$$(i\gamma^\mu \mathcal{D}_\mu - m)\psi = 0 \qquad \text{where} \qquad \mathcal{D}_\mu = (\partial_\mu + iqA_\mu)\psi \quad (q = e < 0\text{for electron.}) \tag{950}$$

So

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi = 0 \tag{951}$$

To study the non-relativistic limit, consider the standard representation

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Define

$$\chi^{'}(\vec{x}, t) \equiv e^{imt}\chi(\vec{x}, t) \qquad \text{and} \qquad \phi^{'}(\vec{x}, t) \equiv e^{imt}\phi(\vec{x}, t) \tag{952}$$

Then

$$\Big(\gamma^0(i\partial_0 - eA_0) + \gamma^i(i\partial_i - eA_i) - m\Big)e^{-imt}\begin{pmatrix} \phi^{'} \\ \chi^{'} \end{pmatrix} = 0$$

Recall $\nabla^i = \frac{\partial}{\partial x^i} = \partial_i$ and $p^i = -i\partial_i$, then

$$\begin{pmatrix} i\partial_0 - eA_0 - m & \sigma^i(i\partial_i - eA_i) \\ -\sigma^i(i\partial_i - eA_i) & -i\partial_0 + eA_0 - m \end{pmatrix} e^{-imt}\begin{pmatrix} \phi^{'} \\ \chi^{'} \end{pmatrix} = 0$$

$$\begin{pmatrix} i\partial_0 - eA_0 & \vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A}) \\ -\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A}) & -i\partial_0 + eA_0 - 2m \end{pmatrix}\begin{pmatrix}\phi^{'} \\ \chi^{'}\end{pmatrix} = 0$$

so we get

$$(i\partial_0 - eA_0)\phi^{'} = -\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})\chi^{'} \qquad ; \qquad (i\partial_0 - eA_0 + 2m)\chi^{'} = -\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})\phi^{'} \tag{953}$$

Since $i\partial_0\chi^{'} \ll m\chi^{'}$ and $eA_0 \ll m$, from second equation, we have

$$\chi^{'} \simeq -\frac{1}{2m}\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})\phi^{'} \tag{954}$$

Insert back

$$\begin{aligned}(i\partial_0 - eA_0)\phi^{'} &= -\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})\phi^{'} \\ &= \frac{1}{2m}\sigma^i\sigma^j(i\nabla^i+eA^i)(i\nabla^j+eA^j)\phi^{'} \\ &= \frac{1}{2m}(\delta^{ij}+i\,\epsilon^{ijk}\sigma^k)(i\nabla^i+eA^i)(i\nabla^j+eA^j)\phi^{'} \\ &= \frac{1}{2m}[(i\vec{\nabla}+e\vec{A})^2 + i\,\epsilon^{ijk}\sigma^k(i\nabla^i+eA^i)(i\nabla^j+eA^j)]\phi^{'}\end{aligned} \tag{955}$$

Note that

$$i\,\epsilon^{ijk}(i\nabla^i+eA^i)(i\nabla^j+eA^j)\phi^{'} = -e\,\epsilon^{ijk}\nabla^iA^j\,\phi^{'} - e\,\epsilon^{ijk}A^j\nabla^i\phi^{'} - e\,\epsilon^{ijk}A^i\nabla^j\phi^{'} = -e\,B^k\phi^{'} \tag{956}$$

Thus, by noting that $\vec{p} = -i\vec{\nabla}$, we have

$$i\partial_0\phi^{'} \simeq \Big[\frac{(\vec{p}-e\,\vec{A})^2}{2m} - \underbrace{\frac{e}{2m}\,\vec{\sigma}\cdot\vec{B}}_{-\vec{\mu}\cdot\vec{B}\text{ term}} + e\,A_0\Big]\phi^{'} \tag{957}$$

Since

$$\vec{\mu} = \frac{e}{2m}\vec{\sigma} = \frac{e}{m}\vec{S} \tag{958}$$

XI. EXACT SOLUTION OF THE HYDROGEN ATOM IN DIRAC'S THEORY

Prelude: Consider first the non-relativistic case. Let us recall the dynamical symmetry of the hydrogen atom. n, l, m are quantum numbers. For given n, l takes values in $l \in (0, 1, 2, \cdots, n-1)$. Here

l : measure of the *magnitude* of the angular momentum
m : contains information about the *direction* of the angular momentum

$$m \in (-l, -l+1, -l+2, \cdots, 0, 1, 2, 3, \cdots, +l) \tag{959}$$

FIG 36 !!!!

Note in fact I am not considering the spin

Example: $n = 2$ states $\mid nlm\rangle$ are

$$\mid 200\rangle \quad , \quad \mid 2\,1-1\rangle \quad , \quad \mid 210\rangle \quad , \quad \mid 211\rangle \tag{960}$$

All have the same energy!

$$E_n = -\frac{mZ^2e^4}{2\hbar^2 n^2} \qquad\qquad n = 1, 2, 3, \cdots \tag{961}$$

It is easy to understand the degeneracy in m, since the direction of the angular momentum should not really matter, but it is really hard to understand the degeneracy in l!

Symmetry and Degeneracy: Consider a state $\mid \alpha\rangle$ with wave function $\psi_\alpha(\vec{r})$. Now consider a displaced state using the following prescription

$$\psi_{\alpha'}(\vec{r}) = \psi_\alpha(\vec{r} - \vec{r}_0) \tag{962}$$

FIG 37 !!!!

Look for a unitary transformation

$$\psi_{\alpha'}(\vec{r}) = U(\vec{r}_0)\psi_\alpha(\vec{r}) \tag{963}$$

Let us just concentrate on the x-axis

$$\begin{aligned} U(x_0)\psi_\alpha(x,y,z) &= \psi_\alpha(x - x_0, y, z) = \psi_{\alpha'}(x,y,z) \\ &= \psi_\alpha(x,y,z) - x_0\frac{\partial}{\partial x}\psi_\alpha(x,y,z) + \cdots \\ &= e^{-x_0\frac{\partial}{\partial x}}\psi_\alpha(x,y,z) \end{aligned} \tag{964}$$

So clearly we have

$$U(x_0) = e^{-ip_x x/\hbar} \tag{965}$$

And in $3D$

$$U(\vec{r}_0) = e^{-i\frac{\vec{p}\cdot\vec{r}_0}{\hbar}} \tag{966}$$

So $\vec{p}$ generates translations. Now in general $U(\vec{r}_0)\psi_\alpha(\vec{r})$ in general need not satisfy the same Hamiltonian, namely the translated state does not necessarily represent a possible motion of the system. So the quantization is under what condition does this new state satisfy the Schrodinger equation?

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{967}$$

Reverse the translation $U^+ \mid \alpha'\rangle = \mid \alpha\rangle$, then

$$i\hbar\frac{\partial}{\partial t}\mid \alpha'\rangle = H \mid \alpha'\rangle \tag{968}$$

$$i\hbar\frac{\partial}{\partial t}\mid \alpha\rangle = H \mid \alpha\rangle \tag{969}$$

U is time-independent. So if $[H, U] = 0$, then $\mid \alpha' \rangle$ satisfies the Schrodinger equation. In our example $[H, \vec{p}] =$ so $\vec{p}$ is constant, it also means that only a translationally invariant state can have a definite momentum.

Relation to degeneracy:
Suppose we have $[\Omega, H] = 0$ leads to degeneracy

$$H \mid \alpha\rangle = E_\alpha \mid \alpha\rangle \tag{970}$$

$$\Omega H \mid \alpha\rangle = H\Omega \mid \alpha\rangle = E_\alpha \Omega \mid \alpha\rangle \tag{971}$$

So $\Omega \mid \alpha\rangle$ and $\mid \alpha\rangle$ have the sames energies.

$$\text{symmetry} \quad \leftrightarrow \quad \text{degeneracy} \tag{972}$$

Extra degeneracy in the hydrogen atom

Energy of the hydrogen atom is degenerate in m and l. So there must be some symmetries. For m, this is easy to understand: we have spherical symmetry. So the *direction* of angular momentum does not change the result. So a rotationally symmetric potential should have *no* preferred direction in space. If we apply an external field, such as an electromagnetic field, degeneracy in m is lifted. BUT, how about the extra degeneracy? Why is the energy E_n independent of l? [The magnitude of $\vec{L}$]

Laplace-Runge-Lenz vector:

$$H = \frac{\vec{p}^2}{2\mu} - \frac{\kappa}{r} \tag{973}$$

where $\kappa = GMm$ or$= 2e^2$. Conserved quantities

$$E = -\frac{\kappa}{2a} \quad , \quad e = \sqrt{1 - \frac{b^2}{a^2}} \quad , \quad \vec{L}^2 = \mu\kappa a(1 - e^2) \tag{974}$$

where e is eccentricity. There is one more conserved quantity

$$\vec{M} = \frac{\vec{p} \times \vec{L}}{m} - \frac{\kappa}{r}\hat{r} \qquad \text{where} \qquad \mid \vec{M} \mid = \kappa e \tag{975}$$

classically $\vec{M}$ obeys two constraints

$$\vec{L} \cdot \vec{M} = 0 \qquad \text{and} \qquad \vec{M}^2 = \frac{2H}{\mu}\vec{L}^2 + \kappa^2 \tag{976}$$

Otherwise there could be conserved which is two much. There are only 4! In QM, we have

$$\vec{r}, \vec{p}, \vec{L} \to \hat{r}, \hat{p}, \hat{L} \quad \text{operators.} \tag{977}$$

$$\hat{\vec{M}} = \frac{1}{2m}\Big(\vec{p} \times \vec{L} - \hat{\vec{L}} \times \hat{\vec{L}} - \kappa\frac{\hat{r}}{r}\Big) \tag{978}$$

$$[\hat{M}, H] = 0 \qquad , \qquad \hat{\vec{L}} \cdot \hat{\vec{M}} = 0 \qquad , \qquad \hat{\vec{M}}^2 = \frac{2H}{\mu}(\vec{L}^2 + \hbar^2) + \kappa^2 \tag{979}$$

What is the algebra?

$$[L^i, L^j] = i\hbar\,\epsilon^{ijk} L^k \quad , \quad [M^i, L^j] = i\hbar\,\epsilon^{ijk} M^k \tag{980}$$

$$[M^i, M^j] = -2i\,\epsilon^{ijk} H L^k \qquad \text{(breaks the closed algebra)} \tag{981}$$

Let $\hat{H} \to E$ and scalar $\hat{M}' \equiv \sqrt{-\frac{m}{2E}}\,\hat{M}$. Then

$$\left.\begin{array}{l} ([\hat{M}'^i, \hat{L}^j]) = i\hbar\,\epsilon^{ijk}\,\hat{M}'^k \\ [\hat{M}^i, \hat{M}'^j] = i\hbar\,\epsilon^{ijk}\,\hat{L}^k \end{array}\right\} SO(4) \text{ algebra.} \tag{982}$$

Define

$$\hat{\vec{J}} \equiv \frac{1}{2}(\vec{L} + \vec{M}') \qquad ; \qquad \vec{K} = \frac{1}{2}(\vec{L} - \vec{M}') \tag{983}$$

they commute with H and constitute the *two* $SO(3)$s of $SO(4)$.

Casimir Operators

$$\hat{C}_1 = \vec{J}^2 + \vec{K}^2 \qquad \text{and} \qquad \hat{C}_2 = \vec{J}^2 - \hat{K}^2 = \vec{L} \cdot \hat{M}' = 0 \tag{984}$$

then

$$\hat{C}_1 \psi = 2\hat{K}^2 \psi = 2k(k+1)\hbar^2 \psi \tag{985}$$

$$\hat{C}_1 = \vec{J}^2 + \vec{K}^2 = \frac{1}{2}(L + M')^2 = -\frac{1}{2}\hbar^2 - \frac{\mu\kappa^2}{4E} \tag{986}$$

So

$$E_k = -\frac{\mu\kappa^2}{2\hbar^2 \underbrace{(2k+1)^2}_{n^2}} \qquad ; \qquad k = 0, \frac{1}{2}, \cdots \tag{987}$$

The fine structure of the hydrogen atom

$$\vec{A} = 0 \quad ; \quad A_0 = -\frac{Ze}{4\pi r} \quad ; \quad V(r) = eA_0 = -\frac{Ze}{r} \tag{988}$$

Dirac equation in the standard representation is

$$(i\partial_0 - V - m)\phi = -i\vec{\sigma} \cdot \vec{\nabla}\chi \qquad \text{and} \qquad (i\partial_0 - V + m)\chi = -i\vec{\sigma} \cdot \vec{\nabla}\phi \tag{989}$$

Insert the ansatz

$$\phi(\vec{x}, t) = e^{-iEt}\phi(\vec{x}) \qquad \text{and} \qquad \chi(\vec{x}, t) = e^{-iEt}\chi(\vec{x}) \tag{990}$$

Define $\varepsilon = E - m$, then

$$(\varepsilon - V)\phi = -i\vec{\sigma} \cdot \vec{\nabla}\chi \qquad \text{and} \qquad (2m + \varepsilon - V)\chi = -i\vec{\sigma} \cdot \vec{\nabla}\phi \tag{991}$$

We can solve this exactly, but we will make an approximation

$$\chi = -\frac{i}{2m+\varepsilon-V}\,\vec{\sigma}\cdot\vec{\nabla}\phi \simeq \frac{1}{2m}\Big(1-\frac{\varepsilon-V}{2m}\Big)\,\vec{\sigma}\cdot\vec{p}\phi \tag{992}$$

So $\frac{\varepsilon-V}{2m} \ll 1$.

Let us try to extract a Schrodinger wave function from the normalization condition

$$Q = \int d^3x\,\psi^+\psi = \int d^3x\,(\mid\phi\mid^2 + \mid\chi\mid^2) \equiv \int d^3x\ \mid\phi_S\mid^2 \tag{993}$$

where ϕ_S is Schrodinger wave function which is a two component spinor. Note that in field theory

$$Q \sim \#\text{electrons} - \#\text{positrons} \tag{994}$$

here it is different. So we have

$$\chi \simeq \frac{1}{2m}\vec{\sigma}\cdot\vec{p}\phi \tag{995}$$

Then

$$\begin{aligned}\int d^3x\ \mid\phi_S\mid^2 &= \int d^3x\Big[\mid\phi\mid^2 + \frac{1}{4m}(\vec{\sigma}\cdot\vec{\nabla}\phi^*)(\vec{\sigma}\cdot\vec{\nabla}\phi)\Big]\\ &= \int d^3x\,\phi^*\Big(1+\frac{p^2}{2m}\Big)\phi\end{aligned} \tag{996}$$

So

$$\phi_S = \Big(1+\frac{p^2}{8m^2}+O(\frac{p^4}{m^4})\Big)\phi \qquad \text{and} \qquad \phi = \Big(1-\frac{p^2}{8m^2}\Big)\phi_S \tag{997}$$

So then we have

$$\begin{aligned}\chi &\simeq \frac{1}{2m}\Big(1-\frac{\varepsilon-V}{2m}\Big)\vec{\sigma}\cdot\vec{p}\Big(1-\frac{p^2}{8m^2}\Big)\phi_S\\ &\simeq \frac{1}{2m}\Big[\vec{\sigma}\cdot\vec{p}\Big(1-\frac{p^2}{8m^2}\Big)-\frac{\varepsilon-V}{2m}\vec{\sigma}\cdot\vec{p}\Big]\phi_S\end{aligned} \tag{998}$$

Now

$$(\varepsilon-V)\Big(1-\frac{p^2}{8m^2}\Big)\phi_S = \vec{\sigma}\cdot\vec{p}\frac{1}{2m}\Big[\vec{\sigma}\cdot\vec{p}\Big(1-\frac{p^2}{8m^2}\Big)-\frac{\varepsilon-V}{2m}\vec{\sigma}\cdot\vec{p}\Big]\phi_S \tag{999}$$

which becomes

$$\Big[\varepsilon-\frac{p^2}{2m}-V+\frac{\varepsilon p^2}{8m^2}+\frac{p^4}{16m^3}+\frac{Vp^2}{8m^2}+\frac{1}{4m^2}\vec{\sigma}\cdot\vec{p}V\vec{\sigma}\cdot\vec{p}\Big]\phi_S = 0 \tag{1000}$$

To lowest order we have

$$\Big(\varepsilon-\frac{p^2}{2m}-V\Big)\phi_S = 0 \qquad ; \qquad \frac{\varepsilon p^2}{8m^2} = \frac{p^2}{8m^2}\Big(\frac{p^2}{2m}+V\Big) \tag{1001}$$

here ε is a c-number, then after a long journey we get

$$\varepsilon\phi_S = \Big[\frac{p^2}{2m}+V-\frac{p^4}{8m^3}+\frac{1}{2m^2}\frac{1}{r}\frac{dV}{dr}\vec{S}\cdot\vec{L}-\frac{e}{8m^2}\vec{\nabla}\cdot\vec{E}\Big]\phi_S \tag{1002}$$

where $\frac{p^4}{8m^3}$ comes from *relativistic expansion*; $\vec{S}\cdot\vec{L}$ term is *spin-orbit couplings* and $\vec{\nabla}\cdot\vec{E}$ term is *Darwin term.*

Note that

$$V(r) = -\frac{2\alpha}{r} \qquad ; \qquad \nabla^2\frac{1}{r} = -4\pi\delta^3(\vec{x}) \tag{1003}$$

So

$$\varepsilon\phi_S = (H_0 + H_{pert})\phi \qquad \text{where} \qquad H_0 = \frac{p^2}{2m} + V \tag{1004}$$

and

$$H_{pert} = -\frac{p^4}{8m^2} + \frac{Z\alpha}{2m^2r^3}\vec{S}\cdot\vec{L} + \frac{\pi Z\alpha}{2m^2}\delta^3(\vec{x}) \tag{1005}$$

Let $| \, njl\rangle$ be the unperturbed states of the hydrogen atom then

$$(\Delta E)_{njl} = \langle njl \mid H_{pert} \mid njl\rangle \tag{1006}$$

$$\Big(\frac{p^2}{2m} + V\Big)\psi_{njl} = \epsilon_n\psi_{njl} \qquad \text{where} \qquad \epsilon_n = -\frac{mZ^2\alpha^2}{2n^2} \tag{1007}$$

1^{st}: $\underline{\langle njl \mid p^4 \mid njl\rangle = ?}$

$$\frac{p^2}{2m}\psi_{njl} = \Big(\epsilon_n + \frac{Z\alpha}{r}\Big)\psi_{njl} \tag{1008}$$

So

$$\int d^3x\, \psi^*_{njl}\, p^4\psi_{njl} = 4m^2\langle njl \mid \Big(\epsilon_n + \frac{Z\alpha}{r}\Big)^2 \mid njl\rangle \tag{1009}$$

For Coulomb potential we have

$$\langle njl \mid \frac{1}{r} \mid njl\rangle = \frac{m\alpha Z}{n^2} \qquad \text{and} \qquad \langle njl \mid \frac{1}{r^2} \mid njl\rangle = \frac{(m\alpha Z)^2}{n^3(l+\frac{1}{2})} \tag{1010}$$

So

$$\langle njl \mid p^4 \mid njl\rangle = 4(mZ\alpha)^4\Big(-\frac{3}{4n^4} + \frac{1}{n^3(l+\frac{1}{2})}\Big) \tag{1011}$$

2^{nd}: $\underline{\langle njl \mid \frac{\vec{S}\cdot\vec{L}}{r^3} \mid njl\rangle = ?}$

Note that

$$\vec{J} = \vec{L} + \vec{S} \qquad \Rightarrow \qquad j(j+1) = l(l+1) + s(s+1) + 2\vec{S}\cdot\vec{L} \tag{1012}$$

For $s = \frac{1}{2}$, we have

$$\langle njl \mid \frac{1}{r^3} \mid njl\rangle = \frac{(m\alpha Z)^3}{n^3 l(l+\frac{1}{2})(l+1)} \quad \text{if } l \neq 0 \quad \text{OR} \quad = 0 \quad \text{if } l = 0 \tag{1013}$$

thus we get

$$\langle njl \mid \frac{\vec{S}\cdot\vec{L}}{r^3} \mid njl\rangle = (1-\delta_{l,0})\frac{(m\alpha Z)^3}{2n^3 l(l+\frac{1}{2})(l+1)}[j(j+1)-l(l+1)-\frac{3}{4}] \tag{1014}$$

3^{th}: $\underline{\langle njl \mid \delta^3(\vec{x}) \mid njl\rangle =?}$

$$\langle njl \mid \delta^3(\vec{x}) \mid njl\rangle = \mid \psi_{njl}(0) \mid^2 = \frac{(m\alpha Z)^3}{\pi n^3}\delta_{l,0} \tag{1015}$$

so that

$$(\Delta E)_{njl} = -\frac{m(Z\alpha)^4}{2n^3}\Big(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\Big) \tag{1016}$$

No separate dependence on l.

XII. QUANTIZATION

1. Canonical Quantization: resembles the development of quantum mechanics. Time is singled out so Lorentz invariance is lost. No ghosts, unitary but cumbersome.

2. Path Integral: Powerful tool but the path integral may not even exist. Very hard to solve some problems in single Quantum Mechanics.

3. Gubta-Bleuler: Covariant quantization scheme. Ghosts propagate, only removed at the end.

4. Becchi-Rouet-Stora-Tyutin (BRST): Powerful, ghosts propagate.

5. Butalin-Vilkovsky (BV): Powerful but cumbersome.

6. Covariant Canonical Quantization: Using symplectic techniques.

7. Stochastic Quantization: introduces a fifth coordinate.

Let us start to look over some of the Important ones:

CANONICAL QUANTIZATION OF POINT PARTICLES:

$$S = \int_{t_1}^{t_2} dt\, L(q^i, \dot{q}^i) \qquad \text{where} \qquad p_i \equiv \frac{\partial L}{\partial \dot{q}^i} \tag{1017}$$

So the Hamiltonian is

$$H = \sum_i p_i \dot{q}^i - L \tag{1018}$$

so that

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} = \{q^i, H\}_P \qquad ; \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} = \{p_i, H\}_P \tag{1019}$$

In general

$$\frac{dF(q,p)}{dt} = \{F, H\}_P \tag{1020}$$

$$\{q^i, p_j\} = \delta^i_j \quad \xrightarrow[\text{canonical quantization}]{} \quad [q^i, p_j] = i\hbar\delta^i_j \tag{1021}$$

The commutation relations are imposed at *equal* times in the Heisenberg picture. Since a *time* is chosen, it is not clear if Lorentz invariance will survive quantization. In general in the canonical quantization the Poisson bracket is upgraded to a commutator with the following prescription :

$$\{A, B\}_P \quad \rightarrow \quad \frac{1}{i\hbar}[A, B], \tag{1022}$$

while the change of a generic function follows as

$$\frac{dF(q,p)}{dt} = \frac{1}{i\hbar}[F, H]. \qquad \text{(Heisenberg equation.)} \tag{1023}$$

QUANTIZATION OF FIELDS:

$$\begin{aligned} S = \int dt\, L(q^i, \dot{q}^i) \quad &\longrightarrow \quad S = \int d^4x\, \mathcal{L}(\phi_a, \partial_\mu\phi_a) \\ t \quad &\longrightarrow \quad t \\ i \quad &\longrightarrow \quad \vec{x}\,,\, a \\ q^i(t) \quad &\longrightarrow \quad \phi_a(t, \vec{x}) \\ \dot{q}^i(t) \quad &\longrightarrow \quad \partial_0\phi_a(t, \vec{x}) \\ p_i = \frac{\partial L}{\partial \dot{q}^i} \quad &\longrightarrow \quad \Pi_a(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \partial_0\phi_a(t, \vec{x})} \end{aligned} \tag{1024}$$

Now

$$[q^i, p_j] = i\hbar\delta^i_j \qquad \text{so} \qquad \sum_j [q^i, p_j]A^j = i\hbar A^i \quad \text{where} \quad A \quad \text{is arbitrary.} \tag{1025}$$

which leads

$$\sum_b \int d^3y\, [\phi_a(t, \vec{x}),\, \Pi_b(t, \vec{y})] f_b(\vec{y}) = i\hbar f_a(\vec{x}) \tag{1026}$$

Hence we propose the quantization rules as

$$\begin{aligned} &[\phi_a(t, \vec{x}),\, \Pi_b(t, \vec{y})] = i\hbar\, \delta_{ab}\, \delta^3(\vec{x} - \vec{y}) \\ &[\phi_a(t, \vec{x}),\, \phi_b(t, \vec{y})] = 0 \\ &[\Pi_a(t, \vec{x}),\, \Pi_b(t, \vec{y})] = 0 \end{aligned} \tag{1027}$$

these are equal-time commutation relations (ETC) relations; they are postulates of the theory, there is no way to derive them.

Choosing commutators or anti-commutators for quantization is crucial. For bosonic fields, we choose commutators, for fermionic fields we choose anti-commutators. This is related to the spin statistics convention and microscopic causality, which we will get back to.

The Hamiltonian reads

$$H = \sum_a \int d^3x \, \Big(\Pi_a(t,\vec{x}) \dot{\phi}_a(t,\vec{x}) - \mathcal{L} \Big) = H[\phi, \Pi] \tag{1028}$$

Canonical Heisenberg equations are

$$\dot{\phi}_a(t,\vec{x}) = \frac{1}{i\hbar} [\phi_a(t,\vec{x}),\, H] \qquad ; \qquad \dot{\Pi}_a(t,\vec{x}) = \frac{1}{i\hbar} [\Pi_a(t,\vec{x}),\, H] \tag{1029}$$

which are valid for all t. Let us show this. Suppose we have at time $t = t_0$

$$[\phi_a(t,\vec{x}),\, \Pi_b(t,\vec{y})]_{t=t_0} = i\hbar\, \delta_{ab}\, \delta^3(\vec{x} - \vec{y}) \tag{1030}$$

Now look at

$$\begin{aligned} \frac{d}{dt} [\phi_a(t,\vec{x}),\, \Pi_b(t,\vec{y})] &= [\dot{\phi}_a,\, \Pi_b] + [\phi_a,\, \dot{\Pi}_b] \\ &= \frac{1}{i\hbar} \Big[[\phi_a,\, H],\, \Pi_b \Big] + \frac{1}{i\hbar} \Big[\phi_a, [\Pi_b,\, H] \Big] \end{aligned} \tag{1031}$$

Recall the Jacobi identity

$$\Big[[A,\, B],\, C \Big] + \Big[[B,\, C],\, A \Big] + \Big[[C,\, A],\, B \Big] = 0 \tag{1032}$$

So we get

$$\frac{d}{dt} [\phi_a,\, \Pi_b] = -\frac{1}{i\hbar} \Big[[\Pi_b,\, \phi_a],\, H \Big] \tag{1033}$$

we use this below.

$$\begin{aligned} &[\phi_a(t_0 + \delta t, \vec{x}),\, \Pi_b(t_0 + \delta t, \vec{y})] = [\phi_a(t_0, \vec{x}),\, \Pi_b(t_0, \vec{y})] + \frac{d}{dt} [\phi_a,\, \Pi_b]_{t_0} \delta t \\ &= i\hbar\, \delta_{ab}\, \delta^3(\vec{x} - \vec{y}) + \delta t\, \frac{1}{i\hbar} [i\hbar\, \delta_{ab}\, \delta^3(\vec{x} - \vec{y}),\, H] \\ &= i\hbar\, \delta_{ab}\, \delta^3(\vec{x} - \vec{y}) \end{aligned} \tag{1034}$$

So the commutation relation remains intact for time.

<u>IMPORTANT QUESTION</u>: Why do we deal with fields? Why not quantize single particles?

QFT demands many particles, but what goes wrong if we just blindly try to quantize a relativistic single particle? (We will see that causality will be lost!)

Let us consider the following amplitude (See Peskin)

$$U(t) = \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle \tag{1035}$$

1. Compute this for $H = \frac{\vec{p}^2}{2m}$

2. Compute it for $H = \sqrt{\vec{p}^2 + m^2}$ We shall use the completeness relation $\int d^3p\, |\vec{p}\rangle\langle\vec{p}| = 1$.

i) Now

$$\begin{aligned} U(t) &= \int d^3p\, \langle\vec{x}|e^{-iHt}|\vec{p}\rangle\langle\vec{p}|\vec{x}_0\rangle \\ &= \int d^3p\, e^{-iE_pt}\, \langle\vec{x}|\vec{p}\rangle\langle\vec{p}|\vec{x}_0\rangle \end{aligned} \tag{1036}$$

Note that

$$\langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{p}\cdot\vec{x}} \tag{1037}$$

So

$$\begin{aligned} U(t) &= \int \frac{d^3p}{(2\pi)^3}\; e^{-iE_pt}\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \\ &= \int_0^\infty \frac{p^2dp}{(2\pi)^2}\; e^{-iE_pt} \underbrace{\int_{-1}^1 dz\, e^{ip|\vec{x}-\vec{x}_0|z}}_{\frac{2\sin(p|\vec{x}-\vec{x}_0|)}{p|\vec{x}-\vec{x}_0|}} \\ &= \frac{1}{2\pi^2|\vec{x}-\vec{x}_0|}\int_0^\infty p\,dp\, \sin(p|\vec{x}-\vec{x}_0|)\, e^{-iE_pt} \end{aligned} \tag{1038}$$

which in non-relativistic limit becomes

$$\begin{aligned} U_{NR}(t) &= \frac{1}{2\pi^2|\vec{x}-\vec{x}_0|}\int_0^\infty p\,dp\, \sin(p|\vec{x}-\vec{x}_0|)\, e^{-i\frac{p^2}{2m}t} \\ &= -\frac{i}{\pi^2|\vec{x}-\vec{x}_0|}\int_{-\infty}^\infty p\,dp\; e^{ip|\vec{x}-\vec{x}_0|-i\frac{p^2}{2m}t} \end{aligned} \tag{1039}$$

Thus we obtain

$$U_{NR}(t) = \left(\frac{m}{2\pi it}\right)^{3/2} e^{\frac{im|\vec{x}-\vec{x}_0|}{2t}} \tag{1040}$$

When $|\vec{x}-\vec{x}_0| > t$, that is when we have a space-like separation, then $U(t)$ is non-zero. So we have a break down of causality, faster than light propagation is allowed. In the relativistic case

$$\begin{aligned} U(t) &= \frac{1}{2\pi^2|\vec{x}-\vec{x}_0|}\int_0^\infty p\,dp\, \sin(p|\vec{x}-\vec{x}_0|)\, e^{-i\sqrt{p^2+m^2}t} \\ &= \frac{it}{2\pi^2|\vec{x}-\vec{x}_0|}\;\frac{\partial}{\partial|\vec{x}-\vec{x}_0|}\left\{\frac{K_1(m\sqrt{-t^2+(\vec{x}-\vec{x}_0)^2})}{\sqrt{-t^2+(\vec{x}-\vec{x}_0)^2}}\right\} \end{aligned} \tag{1041}$$

where $K_1(m_1)$ is *modified Bessel function*. When $|\vec{x}-\vec{x}_0| \gg t$, we have

$$U(t) \sim e^{-m|\vec{x}-\vec{x}_0|} \tag{1042}$$

so the *causality is broken*.

A. QUANTIZATION OF THE FREE SCALAR FIELD

Consider

$$S = \int d^4x \, \mathcal{L} \qquad \text{where} \qquad \mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 \tag{1043}$$

Here φ is real. Then the canonical momentum is

$$\Pi(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}(t, \vec{x}) \tag{1044}$$

and so

$$\mathcal{H} = \Pi\dot{\varphi} - \mathcal{L} = \Pi^2 - \frac{1}{2}\Big(\Pi^2 - (\nabla\varphi)^2\Big) + \frac{m^2\varphi^2}{2} \tag{1045}$$

therefore

$$H = \int d^3x \, \mathcal{H} = \frac{1}{2}\int d^3x \, \Big(\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\Big) \geq 0 \tag{1046}$$

The non-trivial commutation relation is

$$[\varphi(t, \vec{x}), \, \Pi(t, \vec{y})] = i\hbar \, \delta^3(\vec{x} - \vec{y}) \tag{1047}$$

and the others are zero. Take $\hbar = 1$ from now on. Equation of motions (or field equations) are

$$\begin{aligned} \dot{\varphi}(t, \vec{x}) &= i[H, \varphi(t, \vec{x})] \\ &= \frac{i}{2}\Big[\int d^3y \, (\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2), \, \varphi(t, \vec{x})\Big] \end{aligned} \tag{1048}$$

Recall that

$$[AB, \, C] = A[B, \, C] + [A, C]B \tag{1049}$$

Then

$$[\Pi^2, \, \varphi] = \Pi[\Pi, \, \varphi] + [\Pi, \, \varphi]\Pi = -2i \, \delta^3(\vec{x} - \vec{y}) \, \Pi(t, \vec{y}) \tag{1050}$$

Then we have

$$\dot{\varphi}(t, \vec{x}) = \Pi(t, \vec{x}) \tag{1051}$$

which is just like the classical field equation above. Furthermore

$$\begin{aligned} \dot{\Pi}(t, \vec{x}) &= i[H, \Pi(t, \vec{x})] \\ &= \frac{i}{2}\Big[\int d^3y \, (\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2), \, \Pi(t, \vec{x})\Big] \end{aligned} \tag{1052}$$

Since

$$[(\nabla\varphi)^2, \, \Pi] = \nabla\varphi[\nabla\varphi, \Pi] + [\nabla\varphi, \, \Pi]\nabla\varphi = \nabla\varphi(t, \vec{y}) i \nabla_y \delta^3(\vec{x} - \vec{y}) + \text{same} \tag{1053}$$

So

$$\begin{aligned} \dot{\Pi}(t, \vec{x}) &= \frac{i}{2}\int d^3y \, [2i\vec{\nabla}_y \delta^3(\vec{x} - \vec{y}) \cdot \vec{\nabla}_y \, \varphi(t, \vec{y}) + 2m^2 i \, \varphi(t, \vec{y})\delta^3(\vec{x} - \vec{y})] \\ &= \vec{\nabla}^2{}_x \, \varphi(t, \vec{x}) - m^2\varphi(t, \vec{x}) \end{aligned} \tag{1054}$$

Again, this is the same as Euler-Lagrange equation, but the important difference is that now this is an operator equation

$$(\partial^2 + m^2)\varphi(t, \vec{x}) = 0 \tag{1055}$$

B. MODE EXPANSION

Put the scalar field in a box and use the periodic boundary conditions. (Note that we can also use non-Cartesian coordinates but Cartesian coordinates are easier.)

$$\varphi(t, x+L, y+L, z+L) = \varphi(t, x, y, z) \tag{1056}$$

We will take $L \to \infty$ at the end. Since φ is periodic in space, we can expand it in Fourier series

$$\varphi(t, \vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} c(t, \vec{p})\, e^{i\vec{p}\cdot\vec{x}} \tag{1057}$$

$\varphi(t, \vec{x}+L) = \varphi(t, \vec{x})$ gives $p_i = \frac{2\pi}{L}\ n_i$ with $n_i = 0, \pm 1, \pm 2, \cdots$ [29] Using

$$\int d^3x\, e^{i(\vec{p}-\vec{q})\cdot\vec{x}} = \delta_{\vec{p},\vec{q}}\, V \tag{1060}$$

Then

$$\int d^3x\ \varphi(t, \vec{x})\, e^{-i\vec{q}\cdot\vec{x}} = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \int d^3x\ e^{i(\vec{p}-\vec{q})\cdot\vec{x}}\, c(t, \vec{p}) \tag{1061}$$

So

$$c(t, \vec{p}) = \frac{1}{\sqrt{V}} \int d^3x\ e^{-i\vec{p}\cdot\vec{x}}\, \varphi(t, \vec{x}) \tag{1062}$$

Since $\varphi(t, \vec{x})$ is a Hermitian operator

$$\varphi(t, \vec{x}) = \varphi^\dagger(t, \vec{x}), \tag{1063}$$

we have

$$c^\dagger(t, \vec{p}) = \frac{1}{\sqrt{V}} \int d^3x\ e^{i\vec{p}\cdot\vec{x}}\, \varphi^\dagger(t, \vec{x}) = c(t, -\vec{p}) \tag{1064}$$

therefore we obtain

$$c^\dagger(t, \vec{p}) = c(t, -\vec{p}) \tag{1065}$$

The equation of motion $(\partial^2 + m^2)\varphi(t, \vec{x}) = 0$ gives

$$\frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \Big(\frac{\partial^2}{\partial t^2} + \vec{p}^2 + m^2\Big) c(t, \vec{p}) = 0 \tag{1066}$$

which implies

$$\Big(\frac{\partial^2}{\partial t^2} + \vec{p}^2 + m^2\Big) c(t, \vec{p}) = 0 \tag{1067}$$

[29] Here I am using Peskin's conventions

$$\int d^4x\, e^{-i\vec{p}\cdot\vec{x}} = V\, \delta_{p,0} \tag{1058}$$

$$\int \frac{d^4k}{(2\pi)^4} e^{-ikx} = \delta^{(4)}(x). \tag{1059}$$

Define $w_p \equiv \sqrt{\vec{p}^2 + m^2}$, then

$$\Big(\frac{\partial^2}{\partial t^2} + w_p^2\Big) c(t, \vec{p}) = 0 \tag{1068}$$

So we have

$$c(t, \vec{p}) = c_1(\vec{p})\, e^{-iw_p t} + c_2(\vec{p})\, e^{iw_p t} \tag{1069}$$

$$\begin{aligned} c^\dagger(t, \vec{p}) &= c_1^\dagger(\vec{p})\, e^{iw_p t} + c_2^\dagger(\vec{p})\, e^{-iw_p t} \\ &= c_1(-\vec{p})\, e^{-iw_p t} + c_2(-\vec{p})\, e^{iw_p t} \end{aligned} \tag{1070}$$

So we get

$$c_1(-\vec{p}) = c_2^\dagger(\vec{p}) \qquad \text{and} \qquad c_2(-\vec{p}) = c_1^\dagger(\vec{p}) \tag{1071}$$

or

$$c_2(\vec{p}) = c_1^\dagger(-\vec{p}) \tag{1072}$$

Define

$$c_1(\vec{p}) \equiv \frac{1}{\sqrt{2w_p}}\ a(\vec{p}) \tag{1073}$$

Note that the coefficient will be required for Lorentz invariance. Then we have

$$\begin{aligned} \varphi(t, \vec{x}) &= \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2w_p}} \Big(a(\vec{p})\ e^{-iw_p t} + a^\dagger(-\vec{p})\ e^{iw_p t} \Big) \\ &= \sum_{\vec{p}} \frac{1}{\sqrt{2w_p V}} \Big(a(\vec{p})\ e^{-ip\cdot x} + a^\dagger(\vec{p})\ e^{ip\cdot x} \Big) \end{aligned} \tag{1074}$$

Note also that we could have arrived this from a much simpler route of considering arbitrary linear combinations of $e^{-ip\cdot x}$ and $e^{ip\cdot x}$ as solutions to the Klein-Gordon equation.

Furthermore

$$\Pi(t, \vec{x}) = \dot{\varphi}(t, \vec{x}) = -i \sum_p \sqrt{\frac{w_p}{2V}} \Big(a(\vec{p})\ e^{-ip\cdot x} - a^\dagger(p)\ e^{ip\cdot x} \Big) \tag{1075}$$

Let us determine $[a(\vec{p}),\, a^\dagger(\vec{q})] =?$: To be able to do this let us find $a(\vec{p})$ in terms of the field and the momentum conjugate [30]

$$\begin{aligned} \int d^3x\ e^{-i\vec{q}\cdot\vec{x}}\, \varphi(t, \vec{x}) &= \sum_{\vec{p}} \frac{1}{\sqrt{2w_p V}} \int d^3x\, \Big(a(\vec{p})\ e^{-iw_p t + i(\vec{p}-\vec{q})\cdot\vec{x}} + a^\dagger(\vec{p})\ e^{iw_p t - i(\vec{p}+\vec{q})\cdot\vec{x}} \Big) \\ &= \sqrt{\frac{V}{2w_q}} \Big(a(\vec{q})\ e^{-iw_q t} + a^\dagger(-\vec{q})\ e^{iw_q t} \Big) \end{aligned} \tag{1077}$$

[30] Note

$$\int d^4p\ \theta(p_0)\ \delta(p_0^2 - \vec{p}^2 - m^2) = \int d^3p \int_0^\infty dp_0\ \delta[(p_0 - w_p)(p_0 + w_p)] = \int \frac{d^3p}{2w_p} \tag{1076}$$

Similarly

$$i\int d^3x\; e^{-i\vec{q}\cdot\vec{x}}\,\Pi(t,\vec{x}) = \sqrt{\frac{Vw_q}{2}}\Big(a(\vec{q})\; e^{-iw_qt} - a^\dagger(-\vec{q})\; e^{iw_qt}\Big) \tag{1078}$$

Thus we arrive at

$$a(\vec{q})\; e^{-iw_qt} = \frac{1}{\sqrt{2Vw_q}}\int d^3x\; e^{-i\vec{q}\cdot\vec{x}}\Big(w_q\;\varphi(t,\vec{x}) + i\,\Pi(t,\vec{x})\Big) \tag{1079}$$

$$a^\dagger(-\vec{q})\; e^{iw_qt} = \frac{1}{\sqrt{2Vw_q}}\int d^3x\; e^{-i\vec{q}\cdot\vec{x}}\Big(w_q\;\varphi(t,\vec{x}) - i\,\Pi(t,\vec{x})\Big) \tag{1080}$$

Let us now evaluate the commutation relation

$$\begin{aligned}
[a(\vec{p}),\, a^\dagger(\vec{q})] &= \frac{e^{i(w_p-w_q)t}}{2V\sqrt{w_pw_q}}\int d^3x\;\int d^3y\; e^{-i\vec{p}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}\Big[w_q\,\varphi(t,\vec{x}) + i\,\Pi(t,\vec{x}),\; w_q\,\varphi(t,\vec{y}) - i\,\Pi(t,\vec{y})\Big]\\
&= \frac{e^{i(w_p-w_q)t}}{2V\sqrt{w_pw_q}}\int\int d^3x\; d^3y\; e^{-i\vec{p}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}\Big(w_p\,\delta^3(\vec{x}-\vec{y}) + w_q\,\delta^3(\vec{x}-\vec{y})\Big)\\
&= \frac{e^{i(w_p-w_q)t}}{2V\sqrt{w_pw_q}}\int d^3x\;(w_p+w_q)\; e^{-i\vec{x}\cdot(\vec{p}-\vec{q})}\\
&= \delta_{\vec{p},\vec{q}}
\end{aligned} \tag{1081}$$

So we obtain

$$[a(\vec{p}),\, a^\dagger(\vec{q})] = \delta_{\vec{p},\vec{q}} \tag{1082}$$

Also check that

$$[a(\vec{p}),\, a(\vec{q})] = 0 \qquad , \qquad [a^\dagger(\vec{p}),\, a^\dagger(\vec{q})] = 0 \tag{1083}$$

What is the energy and the momentum of the vacuum: a.k.a the lowest energy state ?

$$H = \frac{1}{2}\int d^3x\;\Big(\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\Big) = \frac{1}{2}\int d^3x\;\Big(\Pi^2 + \varphi(-\nabla^2+m^2)\varphi\Big) + \text{B.T.} \tag{1084}$$

Now

$$(-\nabla^2+m^2)\varphi = \sum_{\vec{p}}\frac{(\vec{p}^2+m^2)}{\sqrt{2w_pV}}\Big(a(\vec{p})\; e^{-i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(\vec{p})\; e^{i\mathbf{p}\cdot\mathbf{x}}\Big) \tag{1085}$$

Recall that $w_p^2 = \vec{p}^2 + m^2$, so the

$$
\begin{aligned}
\hat{H} = \frac{1}{2}\int d^3x \, \Big\{ & - \sum_{\vec{p}}\sum_{\vec{q}} \sqrt{\frac{w_p w_q}{2V2V}} \Big(a(\vec{p}) \; e^{-i\mathbf{p}\cdot\mathbf{x}} - a^\dagger(\vec{p}) \; e^{i\mathbf{p}\cdot\mathbf{x}}\Big)\Big(a(\vec{q}) \; e^{-i\mathbf{q}\cdot\mathbf{x}} - a^\dagger(\vec{q}) \; e^{i\mathbf{q}\cdot\mathbf{x}}\Big) \\
& + \sum_{\vec{p}}\sum_{\vec{q}} \frac{w_q}{\sqrt{2w_p V}} \sqrt{\frac{w_q}{2V}} \Big(a(\vec{p}) \; e^{-i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(\vec{p}) \; e^{i\mathbf{p}\cdot\mathbf{x}}\Big)\Big(a(\vec{q}) \; e^{-i\mathbf{q}\cdot\mathbf{x}} + a^\dagger(\vec{q}) \; e^{i\mathbf{q}\cdot\mathbf{x}}\Big)\Big\} \\
= \frac{1}{2}\int \frac{d^3x}{2V} & \sum_{\vec{p}}\sum_{\vec{q}} \Big\{ \sqrt{w_p w_q}\Big(- a(\vec{p})a(\vec{q})e^{-i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{x}} + a(\vec{p})a^\dagger(\vec{q})e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{x}} \\
& + a^\dagger(\vec{p})a(\vec{q})e^{i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{x}} - a^\dagger(\vec{p})a^\dagger(\vec{q})e^{i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{x}}\Big) \\
& + \sqrt{\frac{w_q}{w_p}} w_q \Big(a(\vec{p})a^\dagger(\vec{q})e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{x}} + a^\dagger(\vec{p})a(\vec{q})e^{i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{x}} \\
& + a(\vec{p})a(\vec{q})e^{-i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{x}} + a^\dagger(\vec{p})a^\dagger(\vec{q})e^{i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{x}}\Big)\Big\} \\
= \frac{1}{4}\sum_{p} w_{\vec{p}} \Big\{ & - a(\vec{p})a(-\vec{p}) \, e^{-2iw_p t} + a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) \\
& - a^\dagger(\vec{p})a^\dagger(-\vec{p}) \, e^{2iw_p t} + a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) \\
& + a(\vec{p})a(-\vec{p}) \, e^{-2iw_p t} + a^\dagger(\vec{p})a^\dagger(-\vec{p}) \, e^{2iw_p t}\Big\}
\end{aligned}
\tag{1086}
$$

Therefore, by using the commutation relations, we will get

$$
\hat{H} = \frac{1}{2}\sum_{\vec{p}} w_p \Big(a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p})\Big) \tag{1087}
$$

Furthermore, using $[a(\vec{p}), \, a^\dagger(\vec{q})] = \delta_{\vec{p},\vec{q}}$ finally gives

$$
\hat{H} = \sum_{\vec{p}} \sqrt{\vec{p}^2 + m^2}\Big(a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}\Big) \tag{1088}
$$

Recall the simple harmonic oscillator

$$
\hat{H} = \frac{p^2}{2m} + \frac{1}{2}mw^2x^2 \qquad \Rightarrow \qquad \hat{H} = w(a^\dagger \, a + \frac{1}{2}) \tag{1089}
$$

In QFT, this divergence is not really a big problem, we always measure the difference $\hat{H} - \hat{H}_0$ finite for a physical system. [But when a QFT is coupled to gravity, this divergence gives a big problem.]

Take a very conservative upper limit of $H_0 = 1 \;\; TeV$ the experiments measure $H_0 = 10^{-3} \;\; eV$. So

$$
\frac{H_0^{theory}}{H_0^{experiment}} \sim 10^{15} \tag{1090}
$$

Actually this is more than

$$
\frac{10^{28}}{10^{-3}} \sim 10^{31} = \frac{\text{Planck Mass}}{10^{-3}} \tag{1091}
$$

In terms of the action we have

$$\int d^4x \; \mathcal{L} \quad \Rightarrow \quad [\mathcal{L}] \sim \text{Energy}^4 \tag{1092}$$

Then the so called Cosmological Constant problem is

$$\frac{\Lambda_{theory}}{\Lambda_0} \sim 10^{124} \tag{1093}$$

One possible resolution is SUSY. For fermions this vacuum energy is negative. So in terms of exact SUSY one might have a vanishing vacuum energy. But we know that SUSY is broken. So this "vacuum energy " is really a great problem!

Let us consider the continuum limit

$$\delta^3(\vec{p} - \vec{q}) = \frac{V}{(2\pi)^3} \delta_{\vec{p},\vec{q}} \tag{1094}$$

which can be obtained as follows

$$p^i = \frac{2\pi}{L} n^i \quad ; \quad \int d^3p \rightarrow (\frac{2\pi}{L})^3 \sum_{\vec{n}} \tag{1095}$$

and so

$$\int d^3p \, \delta^3(\vec{p} - \vec{q}) = 1 \tag{1096}$$

which gives

$$\delta^3(\vec{p} - \vec{q}) = \frac{V}{(2\pi)^3} \delta_{\vec{p},\vec{q}} \qquad \text{which gives} \qquad \delta^3(\vec{p} = 0) = \frac{V}{(2\pi)^3} \tag{1097}$$

Recall that $[a_{\vec{p}}, \, a^\dagger_{\vec{q}}] = \delta_{\vec{p},\vec{q}}$, the continuum version will be

$$[a(\vec{p}), \, a^\dagger(\vec{q})] = (2\pi)^3 \; \delta^3(\vec{p} - \vec{q}) \tag{1098}$$

where I have actually recast $a(\vec{p})\sqrt{V} = \tilde{a}(\vec{p})$ to get rid of the volume factors. Then

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(a(\vec{p}) \; e^{-ip\cdot x} + a^\dagger(\vec{p}) \; e^{ip\cdot x} \Big) \tag{1099}$$

Recall that

$$[\varphi(t, \vec{x}), \, \Pi(t, \vec{y})] = i \, \delta^3(\vec{x} - \vec{y}) \tag{1100}$$

which gives

$$[a(\vec{p}), \, a^\dagger(\vec{q})] = (2\pi)^3 \, \delta^3(\vec{p} - \vec{q}) \tag{1101}$$

and the unevenly distributed Fourier transform

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \; e^{i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k}) \qquad \text{and} \qquad f(\vec{k}) = \int d^3k \; e^{-i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{x}) \tag{1102}$$

where

$$\int d^n x \; e^{ikx} = (2\pi)^n \; \delta^{(n)}(k). \tag{1103}$$

$|0\rangle$: vacuum state defined as

$$a(\vec{p})|0\rangle = 0 \qquad\qquad \langle 0|0\rangle = 1 \tag{1104}$$

Generic state is obtained as

$$|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle \equiv (2E_{\vec{p}_1})^{1/2}\,(2E_{\vec{p}_2})^{1/2}\cdots(2E_{\vec{p}_n})^{1/2}\,a^\dagger(\vec{p}_1)\,a^\dagger(\vec{p}_2)\cdots a^\dagger(\vec{p}_n)|0\rangle \tag{1105}$$

In particle 1 particle state is normalized as

$$|\vec{p}\rangle \equiv (2E_{\vec{p}})^{1/2}\,a^\dagger(\vec{p})|0\rangle \tag{1106}$$

Here the coefficient $(2E_{\vec{p}})^{1/2}$ is required for Lorentz invariant momentum. Let us check

$$\begin{aligned}\langle\vec{p}_1|\vec{p}_2\rangle &= (2E_{\vec{p}_1})^{1/2}\,(2E_{\vec{p}_2})^{1/2}\langle 0|a(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle \\ &= (2E_{\vec{p}_1})^{1/2}\,(2E_{\vec{p}_2})^{1/2}\langle 0|[a(\vec{p}_1),\,a^\dagger(\vec{p}_2)]|0\rangle \\ &= 2E_{\vec{p}_1}\,(2\pi)^3\,\delta^3(\vec{p}_1 - \vec{p}_2)\end{aligned} \tag{1107}$$

Let us show that $E_{\vec{p}_1}\,\delta^3(\vec{p}_1 - \vec{p}_2)$ is Lorentz invariant. In fact let us make a Lorentz transformation as

$$p_3^{'} = \gamma(p_3 + \beta E) \quad , \quad E^{'} = \gamma(E + \beta p_3) \tag{1108}$$

Now

$$\begin{aligned}\delta^3(\vec{p}^{\,'} - \vec{q}^{\,'}) &= \delta^{(2)}(\quad)\,\delta(p^{3'} - q^{3'}) \\ &= \delta^{(2)}(\quad)\,\frac{\delta(p^3 - q^3)}{\gamma + \gamma\beta(dE/dp_3)}\end{aligned} \tag{1109}$$

where we used $\delta(f) = \frac{\delta(x)}{df/dx}$, so

$$\delta^3(\vec{p} - \vec{q}) = \gamma\Big(1 + \beta\,\frac{dE}{dp_3}\Big)\delta^3(\vec{p}^{\,'} - \vec{q}^{\,'}) \tag{1110}$$

Note that $\frac{dE}{dp_3} = \frac{p_3}{E}$, so

$$\delta^3(\vec{p} - \vec{q}) = \gamma\Big(1 + \beta\,\frac{p_3}{E}\Big)\delta^3(\vec{p}^{\,'} - \vec{q}^{\,'}) \tag{1111}$$

Hence

$$E\,\delta^3(\vec{p} - \vec{q}) = E^{'}\,\delta^3(\vec{p}^{\,'} - \vec{q}^{\,'}) \tag{1112}$$

How does the Lorentz transformation act on the states? We can assume that vacuum is Lorentz invariant

$$U(\Lambda)|0\rangle = |0\rangle \tag{1113}$$

and

$$U(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle \tag{1114}$$

where $U(\Lambda)$ is a unitary operator. Then

$$U(\Lambda)a^\dagger(\vec{p})\sqrt{2E_{\vec{p}}}\,|0\rangle = a^\dagger(\Lambda\vec{p})\sqrt{2E_{\Lambda\vec{p}}}\,|0\rangle \tag{1115}$$

which can also be written as

$$U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1}U(\Lambda)\sqrt{2E_{\vec{p}}}|0\rangle = a^\dagger(\Lambda\vec{p})\sqrt{2E_{\Lambda\vec{p}}}\,|0\rangle \tag{1116}$$

since vacuum is Lorentz invariant

$$U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1}\sqrt{2E_{\vec{p}}}|0\rangle = a^\dagger(\Lambda\vec{p})\sqrt{2E_{\Lambda\vec{p}}}\,|0\rangle \tag{1117}$$

So we get

$$U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1} = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\,a^\dagger(\Lambda\vec{p}) \tag{1118}$$

Completeness relation for the 1-particle states reads

$$(1)_{1-particle} = \int \frac{d^3p}{(2\pi)^3\,2E_{\vec{p}}}|\vec{p}\rangle\langle\vec{p}| \tag{1119}$$

Important Question: What is the physical meaning of $\varphi(x)|0\rangle$?

$$\begin{aligned}\varphi(x)|0\rangle &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\ a^\dagger(\vec{p})\,e^{ip\cdot x}|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2E_{\vec{p}}}\ e^{ip\cdot x}|\vec{p}\rangle\end{aligned} \tag{1120}$$

the integral over all possible states means that it is the superposition of 1 particle states with definite momentum. Except for the $\frac{1}{2E_{\vec{p}}}$ factor. This is like the eigenstates of $\hat{x}$, that is $|\vec{x}\rangle$. Look at the $t = 0$ case

$$\varphi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2E_{\vec{p}}}\ e^{-i\vec{p}\cdot\vec{x}}|\vec{p}\rangle \tag{1121}$$

Let us consider the non-relativistic theory

$$|\vec{x}\rangle = \int \frac{d^3p}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}|\vec{x}\rangle = \int d^3p\ e^{-i\vec{p}\cdot\vec{x}}|\vec{p}\rangle \tag{1122}$$

So clearly for small $\vec{p}$, $E_{\vec{p}}$ is nearly constant, therefore $\varphi(\vec{x})$ acting on the vacuum creates a particle at position $\vec{x}$.

The following part is really a repetition of what we have done above in the continuum limit.

XIII. QUANTIZATION OF FREE FIELDS

(CHAPTER-4 of Maggiore)

A. QUANTIZATION OF FREE FIELDS

1. Quantization of free Fields

   (a) Scalar Fields

      i. Real Scalar Fields. (Fock Space)

$$[q^i,\, p^j] = i\,\delta^{ij}, \qquad [q^i,\, q^j] = 0, \qquad [p^i,\, p^j] = 0 \tag{1123}$$

In the Heisenberg picture then, the commutation relations are imposed at equal time.

In the Heisenberg picture in QFT for a *real* scalar field, which is promote to a *Hermitian* operator, we have

$$[\phi(t,\vec{x}),\, \Pi(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1124}$$

and the rest are zero

$$S = \frac{1}{2}\int d^4x\,(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2) \tag{1125}$$

whose field equation is

$$(\Box + m^2)\phi = 0 \tag{1126}$$

assuming the plane-wave solutions are of the form $e^{\pm ip\cdot x}$ yields

$$-p^2 + m^2 = 0 \quad \Rightarrow \quad (p^0)^2 - \vec{p}^2 = m^2 \tag{1127}$$

Since ϕ is real in classical field theory and Hermitian in QFT we have

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\,\left(a(\vec{p})\,e^{-ip\cdot x} + a^\dagger(\vec{p})\,e^{ip\cdot x}\right)\Big|_{p^0=E_p} \tag{1128}$$

where $E_p \equiv +\sqrt{\vec{p}^2+m^2}$. So we have both positive and negative frequency modes. Recall our definition of Fourier transform

$$f(x) = \int \frac{d^4p}{(2\pi)^4}\,e^{-ikx}\,\tilde{f}(k) \quad \leftrightarrow \quad \tilde{f}(k) = \int d^4x\,e^{ikx}\,f(x) \tag{1129}$$

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{x}}\,\tilde{f}(\vec{k}) \quad \leftrightarrow \quad \tilde{f}(\vec{k}) = \int d^3x\,e^{-i\vec{k}\cdot\vec{x}}\,f(\vec{x}) \tag{1130}$$

such that

$$\int d^nx\,e^{ik\cdot x} = (2\pi)^n\,\delta^{(n)}(k) \tag{1131}$$

Note that

$$\int d^4p\;\theta(p_0)\,\delta(p^2-m^2) = \int d^3p\int_0^\infty dp_0\,\delta[(p_0-E_p)(p_0+E_p)] = \int \frac{d^3p}{2E_p} \tag{1132}$$

So this is our relativistically invariant measure in the momentum space.

Now let us check the conjugate momentum operator

$$
\begin{aligned}
\Pi_\phi(t,\vec{x}) &= \frac{\partial\mathcal{L}}{\partial\partial_0\phi} = \partial_0\phi \\
&= -i\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\,\left(a(\vec{p})\,e^{-ip\cdot x} - a^\dagger(\vec{p})\,e^{ip\cdot x}\right)
\end{aligned}
\tag{1133}
$$

Now

$$
\begin{aligned}
\int d^3x\;\phi(t,\vec{x})\,e^{-i\vec{q}\cdot\vec{x}} &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int d^3x\,\left(a(\vec{p})\,e^{-iE_pt+i(\vec{p}-\vec{q})\cdot\vec{x}} + a^\dagger(\vec{p})\,e^{iE_pt-i(\vec{p}+\vec{q})\cdot\vec{x}}\right) \\
&= \int\frac{d^3p}{\sqrt{2E_p}}\left(a(\vec{p})\,e^{-iE_pt}\,\delta^{(3)}(\vec{p}-\vec{q}) + a^\dagger(\vec{p})\,e^{iE_pt}\;\delta^{(3)}(\vec{p}+\vec{q})\right)
\end{aligned}
\tag{1134}
$$

So we get

$$
\int d^3x\;\phi(t,\vec{x})\,e^{-i\vec{q}\cdot\vec{x}} = \frac{1}{\sqrt{2E_q}}\left(a(\vec{q})\,e^{-iE_qt} + a^\dagger(-\vec{q})\,e^{iE_qt}\right)
\tag{1135}
$$

Similarly we have

$$
i\int d^3x\;\Pi(t,\vec{x})\,e^{-i\vec{q}\cdot\vec{x}} = \sqrt{\frac{E_q}{2}}\left(a(\vec{q})\,e^{-iE_qt} - a^\dagger(-\vec{q})\,e^{iE_qt}\right)
\tag{1136}
$$

So then

$$
a(\vec{q})\,e^{-iE_qt} = \int d^3x\;e^{-i\vec{q}\cdot\vec{x}}\left(\sqrt{\frac{E_q}{2}}\phi(t,\vec{x}) + i\frac{\Pi(t,\vec{x})}{\sqrt{2E_q}}\right)
\tag{1137}
$$

$$
a^\dagger(-\vec{q})\,e^{iE_qt} = \int d^3x\;e^{-i\vec{q}\cdot\vec{x}}\left(\sqrt{\frac{E_q}{2}}\phi(t,\vec{x}) - i\frac{\Pi(t,\vec{x})}{\sqrt{2E_q}}\right)
\tag{1138}
$$

or I could write this as

$$
a^\dagger(\vec{q})\,e^{iE_qt} = \int d^3x\;e^{i\vec{q}\cdot\vec{x}}\left(\sqrt{\frac{E_q}{2}}\phi(t,\vec{x}) - i\frac{\Pi(t,\vec{x})}{\sqrt{2E_q}}\right)
\tag{1139}
$$

Then let us check

$$\begin{aligned}
[a(\vec{p}),\, a^\dagger(\vec{q})] &= e^{i(E_p-E_q)t} \int d^3x\, d^3y\ e^{i\vec{q}\cdot\vec{y}-i\vec{p}\cdot\vec{x}} \\
&\quad\times \Big[\sqrt{\frac{E_p}{2}}\phi(t,\vec{x}) + i\frac{\Pi(t,\vec{x})}{\sqrt{2E_p}}\,,\ \sqrt{\frac{E_q}{2}}\phi(t,\vec{y}) - i\frac{\Pi(t,\vec{y})}{\sqrt{2E_q}}\ \Big] \\
&= e^{i(E_p-E_q)t} \int d^3x\, d^3y\ e^{i\vec{q}\cdot\vec{y}-i\vec{p}\cdot\vec{x}} \\
&\quad\times \Big\{ -\frac{i}{2}\sqrt{\frac{E_p}{E_q}}\,[\phi(t,\vec{x}),\, \Pi(t,\vec{y})] + \frac{i}{2}\sqrt{\frac{E_q}{E_p}}\,[\Pi(t,\vec{x}),\, \phi(t,\vec{y})] \Big\} \\
&= e^{i(E_p-E_q)t} \int d^3x\, d^3y\ e^{i\vec{q}\cdot\vec{y}-i\vec{p}\cdot\vec{x}} \\
&\quad\times \Big\{ \frac{1}{2}\sqrt{\frac{E_p}{E_q}}\,\delta^3(\vec{x}-\vec{y}) + \frac{1}{2}\sqrt{\frac{E_q}{E_p}}\,\delta^3(\vec{x}-\vec{y}) \Big\} \\
&= e^{i(E_p-E_q)t} \int d^3y\ e^{i(\vec{q}-\vec{p})\cdot\vec{x}} \times \Big\{ \frac{1}{2}\sqrt{\frac{E_p}{E_q}} + \frac{1}{2}\sqrt{\frac{E_q}{E_p}} \Big\} \\
&= (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})
\end{aligned} \tag{1140}$$

So we get

$$[a(\vec{p}),\, a^\dagger(\vec{q})] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}) \tag{1141}$$

The rest are zero.

Note that it is sometimes useful to put the system in a finite box of volume $V = L^3$. One then imposes periodic Boundary conditions to get

$$p^i = \frac{2\pi}{L} n^i \qquad n^i = 0, \pm 1, \pm 2, \cdots \tag{1142}$$

Then

$$\int d^3p \quad \rightarrow \quad \Big(\frac{2\pi}{L}\Big)^3 \sum_{n_x}\sum_{n_y}\sum_{n_z} \tag{1143}$$

$$\int d^3p\, \delta^{(3)}(\vec{p}-\vec{q}) = 1 \quad \rightarrow \quad \Big(\frac{2\pi}{L}\Big)^3 \sum_{\vec{p}} \delta^{(3)}(\vec{p}-\vec{q}) = \delta_{\vec{p},\vec{q}} \tag{1144}$$

So

$$\delta^{(3)}(\vec{p}-\vec{q}) = \Big(\frac{L}{2\pi}\Big)^3 \delta_{\vec{p},\vec{q}} \qquad \Rightarrow \qquad \delta^{(3)}(\vec{p}=0) = \frac{V}{(2\pi)^3} \tag{1145}$$

So then

$$[a(\vec{p}),\, a^\dagger(\vec{q})] = V\, \delta_{\vec{p},\vec{q}} \tag{1146}$$

So apart from a $\frac{1}{\sqrt{V}}$ normalization, this is like a collection of harmonic oscillators. That means we can build the *Fock Space*

$$a(\vec{p}) \qquad : \qquad \text{destruction operator} \tag{1147}$$

$$a^\dagger(\vec{p}) \qquad : \qquad \text{creation operator} \tag{1148}$$

Furthermore, the *Vacuum* is defined as

$$a(\vec{p})|0\rangle = 0 \tag{1149}$$

such that

$$\langle 0|0\rangle = 1 \qquad \text{is our normalization.} \tag{1150}$$

The generic state in the Fock space is obtained as

$$|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle \equiv (2E_{\vec{p}_1})^{1/2}\,(2E_{\vec{p}_2})^{1/2}\cdots(2E_{\vec{p}_n})^{1/2}\,a^\dagger(\vec{p}_1)\,a^\dagger(\vec{p}_2)\cdots a^\dagger(\vec{p}_n)|0\rangle \tag{1151}$$

Here the coefficient $(2E_{\vec{p}})^{1/2}$ is convenient for normalization. Let us check look at the one-particle state

$$|\vec{p}\rangle \equiv (2E_{\vec{p}})^{1/2}\,a^\dagger(\vec{p})|0\rangle \tag{1152}$$

then

$$\begin{aligned} \langle \vec{p}_1|\vec{p}_2\rangle &= (2E_{\vec{p}_1})^{1/2}\,(2E_{\vec{p}_2})^{1/2}\langle 0|a(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle \\ &= (2E_{\vec{p}_1})^{1/2}\,(2E_{\vec{p}_2})^{1/2}\langle 0|[a(\vec{p}_1),\,a^\dagger(\vec{p}_2)]|0\rangle \\ &= 2E_{p_1}\,(2\pi)^3\,\delta^3(\vec{p}_1-\vec{p}_2) \end{aligned} \tag{1153}$$

This is Lorentz invariant. Let us check. Let us perform a Lorentz transformation in the p^3 direction, then

$$\delta^3(\vec{p}-\vec{q}) \quad \longrightarrow \quad \delta^3(\vec{p}^{\,'}-\vec{q}^{\,'}) = \delta^{(2)}(\vec{p}_\perp-\vec{q}_\perp)\,\delta(p^{3'}-q^{3'}) \tag{1154}$$

since

$$p^{3'} = \gamma(p^3+\beta E) \quad , \quad E^{'} = \gamma(E+\beta p^3) \tag{1155}$$

By using $\delta(f) = \frac{\delta(x)}{df/dx}$, we have

$$\delta(p^{3'}-q^{3'}) = \frac{\delta(p^3-q^3)}{\left|\frac{dp^{3'}}{dp^3}\right|} = \frac{\delta(p^3-q^3)}{\gamma+\gamma\beta(dE/dp_3)} \tag{1156}$$

And since

$$E = \sqrt{(p^3)^2+p_\perp^2+m^2} \quad \Rightarrow \quad \frac{dE}{dp^3} = \frac{p^3}{E} \tag{1157}$$

we obtain

$$\delta(p^{3'}-q^{3'}) = \frac{\delta(p^3-q^3)}{\gamma+\beta\gamma\,\frac{p^3}{E}} = \frac{E}{E'}\delta(p^3-q^3) \tag{1158}$$

How does the Lorentz transformation act on the states? We can assume that vacuum is Lorentz invariant

$$U(\Lambda)|0\rangle = |0\rangle \tag{1159}$$

and

$$U(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle \tag{1160}$$

where $U(\Lambda)$ is an unitary operator. Then

$$U(\Lambda)a^\dagger(\vec{p})\sqrt{2E_{\vec{p}}}|0\rangle = a^\dagger(\Lambda p)\sqrt{2E_{\Lambda p}}|0\rangle \tag{1161}$$

which can also be written as

$$U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1}U(\Lambda)\sqrt{2E_{\vec{p}}}|0\rangle = a^\dagger(\Lambda p)\sqrt{2E_{\Lambda p}}|0\rangle \tag{1162}$$

since vacuum is Lorentz invariant

$$U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1}\sqrt{2E_{\vec{p}}}|0\rangle = a^\dagger(\Lambda p)\sqrt{2E_{\Lambda p}}|0\rangle \tag{1163}$$

So we get

$$U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1} = \sqrt{\frac{E_{\Lambda p}}{E_p}}\, a^\dagger(\Lambda p) \tag{1164}$$

Completeness relation for the 1-particle states reads

$$(1)_{1-particle} = \int \frac{d^3p}{(2\pi)^3\, 2E_{\vec{p}}}|\vec{p}\rangle\langle\vec{p}| \tag{1165}$$

Let us now compute the energy of these states. Recall that the Hamiltonian density is

$$\mathcal{H} = \Pi_\phi\partial_0\phi - \mathcal{L} = \frac{1}{2}[\Pi_\phi^2 + (\nabla\phi)^2 + m^2\phi^2] \tag{1166}$$

$\mathcal{H}$ is somewhat lengthy but after integrations one gets

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3}\frac{E_p}{2}\left(a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p})\right) \\ &= \int \frac{d^3p}{(2\pi)^3}E_p\left(a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}[a(\vec{p}),\, a^\dagger(\vec{p})]\right) \end{aligned} \tag{1167}$$

Note that the commutator is $[a(\vec{p}),\, a^\dagger(\vec{p})] = (2\pi)^3\delta^3(0)$ which is *divergent*. In a finite volume we have

$$(2\pi)^3\delta^3(0) = V \tag{1168}$$

Then the zero-point energy is

$$E_{vac} = \frac{V}{2}\int \frac{d^3p}{(2\pi)^3}\, E_p \tag{1169}$$

and the energy density is

$$\rho = \frac{E_{vac}}{V} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, E_p = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{\vec{p}^2 + m^2} \quad \longrightarrow \quad \infty \tag{1170}$$

To regulate the integral, we can put a cut-off Λ

$$\rho_{vac} \;\sim\; \int^{\Lambda} p^3 dp \;\sim\; \Lambda^4 \tag{1171}$$

This is an ultraviolet (UV) divergence. Observed value of $\Lambda_0 \sim 10^{-3}\ eV$, we can take $\Lambda_{theory} = 10^{19}\ GeV = 10^{28}\ eV$ so

$$\frac{theory}{experiment} \sim \Big(\frac{10^{28}}{10^{-3}}\Big)^4 = 10^{124} \tag{1172}$$

This is the famous *cosmological constant problem.* But in QFT, this zero-point energy does not cause a problem since we always measure the energy differences. Thus we take

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_p\ a^\dagger(\vec{p})\, a(\vec{p}) \tag{1173}$$

So we can actually formalize this by declaring that we have *normal* ordering

$$:\ a(\vec{p})\, a^\dagger(\vec{p})\ :\ = a^\dagger(\vec{p})\, a(\vec{p}) \tag{1174}$$

So

$$H = \frac{1}{2}\int d^3x\ :\ [\Pi_\phi^2 + (\nabla\phi)^2 + m^2\phi^2]\ : \tag{1175}$$

Note that

$$H|0\rangle = 0 \tag{1176}$$

Then

$$\begin{aligned} H|\vec{p}\rangle &= \int \frac{d^3q}{(2\pi)^3}\ E_p\ a^\dagger(\vec{q})\, a(\vec{q})\, \sqrt{2E_p}\, a^\dagger(\vec{p})|0\rangle \\ &= E_p|\vec{p}\rangle \end{aligned} \tag{1177}$$

So in general we have

$$H|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle = (E_{p_1} + E_{p_2} + \cdots + E_{p_n})|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle \tag{1178}$$

<u>SPATIAL MOMENTUM</u>: Recall that

$$\begin{aligned} p^i &= \int d^3x\ : T^{0i} : \\ &= \int d^3x\ : \partial_0\phi\, \partial^i\phi : \\ &= \int \frac{d^3p}{(2\pi)^3}\ p^i\, a^\dagger(\vec{p})\, a(\vec{p}) \end{aligned} \tag{1179}$$

So

$$\begin{aligned} p^i|\vec{q}\rangle &= \sqrt{2E_q}\int \frac{d^3p}{(2\pi)^3}\ p^i\, a^\dagger(\vec{p})\, a(\vec{p})\, a^\dagger(\vec{q})|0\rangle \\ &= q^i|\vec{q}\rangle \end{aligned} \tag{1180}$$

So one-particle state has momentum $\vec{p}$ and energy $E = \sqrt{\vec{p}^2 + m^2}$. Therefore, the generic state

$$|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle = (2E_{p_1})^{1/2}(2E_{p_2})^{1/2} \cdots (2E_{p_n})^{1/2}\, a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2) \cdots a^\dagger(\vec{p}_n)|0\rangle \tag{1181}$$

has momentum $\sum\limits_{i=1}^{n} p_i$ and energy $\sum\limits_{i=1}^{n} E_i$.

Observe that since the creation operators commute among themselves, the generic state is symmetric under the exchange of *two* or any particles. So they obey Bose-Einstein statistics.

Summary and Some Remarks

1. $\phi(x)$ is a Hilbert space operator which creates and destroys particles that are quanta of the field.

2. Dual particle and wave interpretations of the Quantum Mechanics are present here. Waves are given by $e^{\pm ip\cdot x}$ such that

$$e^{-ip_0 t} \quad \Rightarrow \quad \text{positive frequency} \tag{1182}$$

$$e^{ip_0 t} \quad \Rightarrow \quad \text{negative frequency} \tag{1183}$$

The coefficients of *negative* frequency solution creates particles.

B. CAUSALITY

Let us return to the issue of causality:

y. ________________ . x

The amplitude (sometimes called the Wightman function) for a particle to propagate from y to x is

$$\langle 0|\phi(x)\phi(y)|0\rangle \equiv \mathcal{D}(x-y) \tag{1184}$$

Since the only term that survives is

$$\langle 0|a(\vec{p})\, a^\dagger(\vec{q})|0\rangle = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}), \tag{1185}$$

we have

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\, e^{-ip\cdot(x-y)} \tag{1186}$$

Save the factor $\frac{1}{2E_p}$ factor, this expression is exactly what we had in the relativistic point-particle. Then

$$\mathcal{D}(x-y) = \frac{1}{(2\pi)^2} \int_0^\infty \int_{-1}^{1} p^2 dp\, dz \frac{1}{2\sqrt{\vec{p}^2+m^2}}\, e^{-i\sqrt{\vec{p}^2+m^2}(x^0-y^0)+i|\vec{p}||\vec{x}-\vec{y}|z} \tag{1187}$$

Before I evaluate this, let us make some assumptions: Consider a time-like separation

$$(x-y)^2 = (x^0-y^0)^2 - (\vec{x}-\vec{y})^2 > 0 \tag{1188}$$

Then find a frame in which $\vec{x} - \vec{y} = 0$ and let $x^0 - y^0 = t$, then[31]

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2}\int_0^\infty \frac{p^2\,dp}{\sqrt{p^2+m^2}}\; e^{-i\sqrt{p^2+m^2}t} \quad \xrightarrow[t\to\infty]{} \quad e^{-imt} \tag{1190}$$

Consider the case for which $(x-y)$ is purely spatial, $(x^0 - y^0) = 0$. And let $\vec{x} - \vec{y} \equiv \vec{r}$

$$\begin{aligned}\mathcal{D}(x-y) &= \frac{1}{4\pi^2}\int_0^\infty p^2\,dp\int_{-1}^1 \frac{dz}{2\sqrt{p^2+m^2}}\; e^{iprz} \\ &= -\frac{i}{8\pi^2 r}\int_0^\infty \frac{p\,dp}{\sqrt{p^2+m^2}}\,(e^{ipr} - e^{-ipr}) \\ &= -\frac{i}{8\pi^2 r}\int_{-\infty}^\infty \frac{p\,dp\,e^{ipr}}{\sqrt{p^2+m^2}}\end{aligned} \tag{1191}$$

FIG 39 !!!!

$$\mathcal{D}(x-y) \overrightarrow{=} e^{-mr} \quad \text{as} \quad r \to \infty \tag{1192}$$

So again outside the light-cone the propagation amplitude is exponentially vanishing but still not zero. However to really discuss causality we should actually consider whether a *measurement* performed at one point can affect a measurement *at another point* whose separation from the first point is space-like.

In this theory, as a Hermitian operator, the simplest thing we can measure is $\phi(x)$, we should compute

$$[\phi(x),\, \phi(y)] \tag{1193}$$

For space-like separations, if $[\phi(x),\, \phi(y)] = 0$, then one measurement cannot affect the other, they can be simultaneously measured. Now we know that for equal times

$$[\phi(t,\vec{x}),\, \phi(t,\vec{y})] = 0 \tag{1194}$$

So we need to find the general covariant commutator

$$\begin{aligned}&[\phi(x),\, \phi(y)] = \int\!\!\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\Big[a(\vec{p})\,e^{-ip\cdot x} + a^\dagger(\vec{p})\,e^{ip\cdot x}\,,\; a(\vec{q})\,e^{-iq\cdot y} + a^\dagger(\vec{q})\,e^{iq\cdot y}\Big] \\ &= \int\!\!\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\Big(e^{-ip\cdot x+iq\cdot y}\,(2\pi)^3\,\delta^3(\vec{p}-\vec{q}) - e^{ip\cdot x-iq\cdot y}\,(2\pi)^3\,\delta^3(\vec{p}-\vec{q})\Big) \\ &= \int \frac{d^3p}{(2\pi)^3 2E_p}\Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\Big) \\ &\equiv i\Delta(x-y) \qquad\qquad \text{(Pauli-Jordan function)}\end{aligned} \tag{1195}$$

Observe that in terms of the propagation amplitude the covariant commutator reads as

$$[\phi(x)\,,\, \phi(y)] = \mathcal{D}(x-y) - \mathcal{D}(y-x) \tag{1196}$$

[31] Recall

$$\Delta t^{'} = \gamma(\Delta t + \frac{\Delta x}{v}) \qquad \Delta x^{'} = \gamma(\Delta x + v\Delta t) = 0 \tag{1189}$$

Since

$$\int \frac{d^3p}{2E_p} = \int d^4p\, \theta(p_0)\, \delta(p^2 - m^2) \tag{1197}$$

$$i\Delta(x-y) = \int \frac{d^4p}{(2\pi)^3}\, \theta(p_0)\, \delta(p^2-m^2)\Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\Big) \tag{1198}$$

1. So the commutator is Lorentz invariant.

2. Let us show that for space-like separation

$$\Delta(x-y) = 0 \tag{1199}$$

   it will be sufficient to show this for $x^0 = y^0 = 0$. Then

$$i\Delta(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3 2E_p}\Big(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big) \tag{1200}$$

   just let $\vec{p} \to -\vec{p}$ in the second integral, then $i\Delta(\vec{x}-\vec{y}) = 0$. So for space-like separation $[\phi(x)\,,\,\phi(y)] = 0$ and no measurement at $(t,\vec{x})$ can affect the measurement at $(t',\vec{y})$ if they are space-like separated. Note that all the other commutators will be related to this commutator. [NOTE: If we used anti-commutators, it would fail. This is called *microscopic causality.*]

3. Show that

$$(\partial^2 + m^2)\Delta(x) = 0 \quad\Rightarrow\quad (\partial_t^2 - \nabla^2 + m^2)\Delta(x) = 0 \tag{1201}$$

4.

$$\frac{\partial \Delta(x)}{\partial x_0}\bigg|_{x_0=0} = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} = -\delta^{(3)}(\vec{x}) \tag{1202}$$

5.

$$\Big[\frac{\partial\phi(x_0,\vec{x})}{\partial x_0}\,,\,\phi(y_0,\vec{y})\Big]\bigg|_{x_0=y_0} = i\frac{\partial\Delta(x-y)}{\partial x_0}\bigg|_{x_0=y_0} = -i\delta^{(3)}(\vec{x}-\vec{y}) \tag{1203}$$

   so

$$[\phi(t,\vec{x})\,,\,\Pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{1204}$$

THE KLEIN-GORDON PROPAGATOR:

Since it is a c-number, we can insert it inside the state-vectors

$$\begin{aligned}
[\phi(x),\, \phi(y)] &= \langle 0|[\phi(x),\,\phi(y)]|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\Big) \\
&\text{Assume } x^0 > y^0 \\
&= \int \frac{d^3p}{(2\pi)^3}\left\{ \frac{1}{2E_p}\, e^{-ip\cdot(x-y)}\bigg|_{p^0=E_p} - \frac{1}{2E_p}\, e^{-ip\cdot(x-y)}\bigg|_{p^0=-E_p}\right\} \\
&= \int \frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\,\frac{-1}{p^2-m^2}\, e^{-ip\cdot(x-y)}
\end{aligned} \tag{1205}$$

FIG 40 !!!!

For $x^0 < y^0$, we can close the contour from above to get 0. Then we can define the *retarded Green's function* as

$$\mathcal{D}_R(x-y) \equiv \theta(x^0-y^0)\,\langle 0|[\phi(x),\,\phi(y)]|0\rangle \tag{1206}$$

Let us compute

$$\begin{aligned}(\partial_x^2+m^2)\mathcal{D}_R(x-y) &= \partial_x^2\theta(x^0-y^0)\ \langle 0|[\phi(x),\,\phi(y)]|0\rangle \\ &+ 2\partial_\mu\theta(x^0-y^0)\ \partial^\mu\Big(\langle 0|[\phi(x),\,\phi(y)]|0\rangle\Big) \\ &+ \theta(x^0-y^0)\,(\partial_x^2+m^2)\,\langle 0|[\phi(x),\,\phi(y)]|0\rangle \\ &= -\delta(x^0-y^0)\,\langle 0|[\Pi(x),\,\phi(y)]|0\rangle \\ &+ 2\delta(x^0-y^0)\,\langle 0|[\Pi(x),\,\phi(y)]|0\rangle + 0 \\ &= -i\delta^{(4)}(x-y)\end{aligned} \tag{1207}$$

So $\mathcal{D}_R(x-y)$ is a Green function of the Klein-Gordon equation. Since it vanishes for $x^0 < y^0$, it is the retarded Green's function. O course given the definition

$$(\partial_x^2+m^2)\mathcal{D}_R(x-y) \equiv -i\delta^{(4)}(x-y) \tag{1208}$$

We can also find it using the Fourier transform

$$\mathcal{D}_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\ \tilde{\mathcal{D}}_R(p) \tag{1209}$$

such that

$$(-p^2+m^2)\tilde{\mathcal{D}}_R(p) = -i \tag{1210}$$

So we get

$$\mathcal{D}_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\ \frac{i}{p^2-m^2}\ e^{-ip\cdot(x-y)} \tag{1211}$$

Observe that the p^0 integral can be evaluated according to four different contours. One such prescription is the *Feynman prescription*

FIG 41 !!!!

$$\mathcal{D}_R(x-y) \equiv \int \frac{d^4p}{(2\pi)^4}\ \frac{i}{p^2-m^2+i\epsilon}\ e^{-ip\cdot(x-y)} \tag{1212}$$

where

$$p^0 = \pm(E_p - i\epsilon) \tag{1213}$$

which comes with the prescription

$$\text{“when } x^0 > y^0, \text{ close the contour from below''} \tag{1214}$$

$$\text{“when } x^0 < y^0, \text{ close the contour from above''} \tag{1215}$$

Then

$$\mathcal{D}_F(x-y) = \begin{cases} \mathcal{D}(x-y) & \text{for } \ x^0 > y^0 \\ \mathcal{D}(y-x) & \text{for } \ x^0 < y^0 \end{cases}$$

Then the Feynman propagator (so this is the propagation amplitude)

$$\begin{aligned} \mathcal{D}_F(x-y) &= \theta(x^0-y^0)\ \langle 0|\phi(x)\,\phi(y)|0\rangle + \theta(y^0-x^0)\ \langle 0|\phi(y)\,\phi(x)|0\rangle \\ &= \langle 0|T\{\phi(x)\,\phi(y)\}|0\rangle \end{aligned} \tag{1216}$$

where T stands for *time-ordering* symbol which puts the operators that come later on the left. $\mathcal{D}_F(x-y)$ is a Green's function of the Klein-Gordon equation.

Particle creation by a classical source:

Consider the massive Klein-Gordon equation in the presence of a source

$$(\partial^2 + m^2)\phi(x) = j(x) \tag{1217}$$

where $j(x)$ is a fixed known function. Also it is non-zero only for a finite time interval.

Question: Start with the vacuum state turn on $j(x)$ and turn it off, what will we find after that?

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + j(x)\phi(x) \tag{1218}$$

Before $j(x)$ is turned on, we have

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\,\left(a(\vec{p})\,e^{-ip\cdot x} + a^\dagger(\vec{p})\,e^{ip\cdot x}\right) \tag{1219}$$

When the source is turned on we have

$$\phi(x) = \phi_0(x) + i\int d^4y\,\mathcal{D}_R(x-y)\,j(y) \tag{1220}$$

Recall that

$$(\partial_x^2 + m^2)\,\mathcal{D}_R(x-y) = -i\,\delta^{(4)}(x-y) \tag{1221}$$

then

$$\phi(x) = \phi_0(x) + i\int d^4y\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,\theta(x^0-y^0)\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)j(y) \tag{1222}$$

Say $x^0 > y^0$

$$\phi(x) = \phi_0(x) + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(i\tilde{j}(p)\, e^{-ip\cdot x} + \text{h.c.} \right) \tag{1223}$$

where

$$\tilde{j} \equiv \int d^4y \; e^{ip\cdot y}\, j(y) \tag{1224}$$

Then we have

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \left(a(\vec{p}) + \frac{i}{\sqrt{2E_p}} \tilde{j}(p) \right) e^{-ip\cdot x} + \text{h.c.} \right\} \tag{1225}$$

So the Hamiltonian after $j(x)$ is turned on is

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(a^\dagger(\vec{p}) - \frac{i}{\sqrt{2E_p}} \tilde{j}^*(p) \right) \left(a(\vec{p}) + \frac{i}{\sqrt{2E_p}} \tilde{j}(p) \right) \tag{1226}$$

so that

$$\langle 0|H|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} |\tilde{j}(p)|^2 \tag{1227}$$

If we define $\frac{|\tilde{j}(p)|^2}{2E_p}$ as the probability density of creating a particle in the mode p, then the total number of particles produced is

$$\int dN = N = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 \tag{1228}$$

C. Canonical Quantization of the Complex Scalar Field

$$\mathcal{L} = \partial_\mu \phi^* \, \partial^\mu \phi - m^2 \phi^* \phi - V(\phi^* \phi) \tag{1229}$$

where ϕ is complex and also $V^* = V$. Actually for now assume that $V = 0$. We have the following global symmetry

$$\phi \to \phi^{'} = e^{i\alpha}\, \phi \qquad ; \qquad \phi^* \to \phi^{'*} = e^{-i\alpha}\, \phi^* \tag{1230}$$

where α is constant. This symmetry gives the conserved current given as

$$j^\mu = (j^\mu)^* = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \tag{1231}$$

Since $\phi(x)$ is a complex classical field at this level, when quantized it need not be a Hermitian field. So then the mode expansion reads

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p})\, e^{-ip\cdot x} + b^\dagger(\vec{p})\, e^{ip\cdot x} \right) \tag{1232}$$

on the other hand $\phi^*(x)$ becomes the Hermitian conjugate of $\phi(x)$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\,\left(a^\dagger(\vec{p})\,e^{ip\cdot x} + b(\vec{p})\,e^{-ip\cdot x}\right) \tag{1233}$$

then

$$\Pi_\phi = \frac{\partial\mathcal{L}}{\partial\partial_0\phi} = \partial_0\phi^* \qquad ; \qquad \Pi_{\phi^*} = \partial_0\phi \tag{1234}$$

So let us impose

$$[\phi(t,\vec{x}),\, \Pi_\phi(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1235}$$

$$[\phi^*(t,\vec{x}),\, \Pi_{\phi^*}(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1236}$$

These then lead to

$$[a(\vec{p}),\, a^\dagger(\vec{q})] = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{1237}$$

$$[b(\vec{p}),\, b^\dagger(\vec{q})] = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{1238}$$

and the rest are zero. Let us just check the consistency

$$\begin{aligned}
&[\phi(t,\vec{x}),\, \partial_0\phi^*(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \\
&= \int\!\!\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\,\frac{d^3q}{(2\pi)^3}\,\frac{i\,q_0}{\sqrt{2E_q}} \\
&\qquad \times \left[\left(a(\vec{p})\,e^{-ip\cdot x} + b^\dagger(\vec{p})\,e^{ip\cdot x}\right),\, \left(a^\dagger(\vec{q})\,e^{iq\cdot y} - b(\vec{q})\,e^{-iq\cdot y}\right)\right] \\
&= \int\!\!\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\,\frac{d^3q}{(2\pi)^3}\,\frac{i\,q_0}{\sqrt{2E_q}} \\
&\qquad \times \left\{[a(\vec{p}),\, a^\dagger(\vec{q})]\,e^{-ip\cdot x+iq\cdot y} + [b(\vec{q}),\, b^\dagger(\vec{p})]\,e^{ip\cdot x-iq\cdot y}\right\} \\
&= \int\!\!\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\,\frac{d^3q}{\sqrt{2E_q}}\,i\,q_0 \\
&\qquad \times \left\{\,\delta^{(3)}(\vec{p}-\vec{q})\,e^{-ip\cdot x+iq\cdot y} +\, \delta^{(3)}(\vec{p}-\vec{q})\,e^{ip\cdot x-iq\cdot y}\right\} \\
&= i\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right) \\
&= i\,\delta^{(3)}(\vec{x}-\vec{y})
\end{aligned} \tag{1239}$$

1. Fock Space

Vacuum is defined by

$$a(\vec{p})|0\rangle = 0 \qquad ; \qquad b(\vec{p})|0\rangle = 0 \tag{1240}$$

The rest of the states are created by acting on the vacuum by $a^\dagger(\vec{p})\, b^\dagger(\vec{q}) \cdots$ types operators. After normal ordering we have

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_p \left(a^\dagger(\vec{p})\, a(\vec{p}) \; + b^\dagger(\vec{p})\, b(\vec{p}) \right) \tag{1241}$$

$$p^i = \int \frac{d^3p}{(2\pi)^3}\, p^i \left(a^\dagger(\vec{p})\, a(\vec{p}) \; + b^\dagger(\vec{p})\, b(\vec{p}) \right) \tag{1242}$$

So the quanta of the complex scalar field is given by two different particles with the same mass. Let us calculate the $U(1)$ charge

$$\begin{aligned} Q &= i \int d^3x \, \left(\phi^* \, \partial^0 \phi - \phi \, \partial^0 \phi^* \right) \\ &= i \int d^3x \, \left(\phi^* \, \Pi_{\phi^*} - \phi \, \Pi_{\phi} \right) \end{aligned} \tag{1243}$$

This can be easily done as we have been doing for the Hamiltonian etc. computation. One gets

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(a^\dagger(\vec{p})\, a(\vec{p}) \; - b(\vec{p})\, b^\dagger(\vec{p}) \right) \tag{1244}$$

which after normal ordering gives

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(a^\dagger(\vec{p})\, a(\vec{p}) \; - b^\dagger(\vec{p})\, b(\vec{p}) \right) \tag{1245}$$

Note: Otherwise we would get the charge of the vacuum to be $-\infty$! Normal ordering comes of careless behavior when jumping from the classical expression to the quantum expression. Recall that $a^\dagger\, a$ was number operator. So here we have

$$Q \sim \#\ \text{particle} - \#\ \text{anti-particle} \tag{1246}$$

So

$$a^\dagger(\vec{p})|0\rangle \quad : \text{ represents a particle with momentum } \vec{p}\text{, mass } m \text{ and say charge } +1. \tag{1247}$$

$$\begin{aligned} b^\dagger(\vec{p})|0\rangle \quad &: \text{represents a particle with momentum } \vec{p}\text{, mass } m \text{ and say charge } -1. \\ &\quad \text{(This is aptly called an anti-particle.)} \end{aligned} \tag{1248}$$

Thus

$$\phi \quad \sim \quad \underbrace{a\, e^{-ip\cdot x}}_{\text{destruction of a particle}} \quad + \quad \underbrace{b^\dagger\, e^{-ip\cdot x}}_{\text{creation of an anti-particle}} \tag{1249}$$

For the real scalar field since $a(\vec{p}) = b(\vec{p})$, *the particle is its own anti-particle.*

D. Canonical Quantization of the Massive Dirac Field

$$\mathcal{L} = \bar{\psi}_a \, (i\gamma^\mu{}_{ab} \, \partial_\mu - m \, \delta_{ab})\psi_b \qquad a, b = 1, 2, 3, 4 \tag{1250}$$

Canonical momentum reads

$$\Pi_c(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \partial_0 \psi_c(t, \vec{x})} = i\bar{\psi}_a \gamma^0{}_{ac} = i(\bar{\psi}\gamma^0)_c = i\psi_c^\dagger \tag{1251}$$

A posteriori we know that if we quantize with commutators, we will have problems with causality. Hence we quantize with the anti-commutators

$$\{\psi_a(t, \vec{x}) \, , \, \Pi_b(t, \vec{y})\} = i \, \delta_{ab} \, \delta^{(3)}(\vec{x} - \vec{y}) \tag{1252}$$

as usual the others are zero $\{\psi_a, \psi_b\} = 0$ and $\{\Pi_a, \, \Pi_b\} = 0$. So then we have

$$\{\psi_a(t, \vec{x}) \, , \, \psi_b^\dagger(t, \vec{y})\} = \delta_{ab} \, \delta^{(3)}(\vec{x} - \vec{y}) \tag{1253}$$

Good thing the "i" dropped on the right-hand side since the left-hand side is Hermitian.

To be able to do the mode expansion of the Dirac field, let us recall the free solutions (plane-wave solutions) of the Dirac equation. Since we have done this before, this will be a brief recap.

The general solution will be a sum of $u(p) \, e^{-ip \, , x}$ and $v(p) \, e^{ip \, , x}$ solutions where $u(p)$ and $v(p)$ are four-component spinors. Then

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{1254}$$

gives

$$(p\!\!\!/ - m)u(p) = 0 \qquad \text{and} \qquad (p\!\!\!/ + m)v(p) = 0 \tag{1255}$$

Let

$$u(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}$$

and go to the rest frame of the particle, assuming of course $m \neq 0$. Then

$$p^\mu = (m, 0) \tag{1256}$$

So

$$m(\gamma^0 - \mathbb{I})u(p) = 0 \tag{1257}$$

let us choose the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

Then

$$\begin{pmatrix} -\mathbb{I} & \mathbb{I} \\ \mathbb{I} & -\mathbb{I} \end{pmatrix} \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix} = 0 \qquad \Rightarrow \qquad u_L = u_R$$

So Klein-Gordon equation imposes only $p^2 - m^2 = 0$, but the Dirac equation imposes an other constraint reducing the number of Degree of Freedom.

Let us choose the following normalization

$$u_L = u_R = \sqrt{m}\,\xi \tag{1258}$$

such that ξ is a two component spinor $\xi^+\xi = \mathbb{I}$.

How do we find the general solution? As I did earlier in class

$$u(p) = (\not{p} + m)\,u(0) \tag{1259}$$

will be a general solution as is clear. But here let us use another approach: Let us *boost the rest frame solution.* Under Lorentz transformations, recall that

$$u_L \longrightarrow \Lambda_L u_L = e^{(-i\vec{\theta}-\vec{\eta})\cdot\frac{\vec{\sigma}}{2}} u_L \tag{1260}$$

$$u_R \longrightarrow \Lambda_R u_R = e^{(-i\vec{\theta}+\vec{\eta})\cdot\frac{\vec{\sigma}}{2}} u_R \tag{1261}$$

Let us just concentrate on boosts (no rotations) in the z-direction. Then

$$u'_L(p^3) = e^{-\frac{\eta_3\sigma_3}{2}} u_L \qquad \text{and} \qquad u'_R(p^3) = e^{\frac{\eta_3\sigma_3}{2}} u_R \tag{1262}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

by setting $\eta_3 = \eta$, we have

$$u'_L(p^3) = \Big(\mathbb{I}\cosh(\frac{\eta}{2}) - \sigma^3\sinh(\frac{\eta}{2})\Big) u_L \tag{1263}$$

$$u'_R(p^3) = \Big(\mathbb{I}\cosh(\frac{\eta}{2}) + \sigma^3\sinh(\frac{\eta}{2})\Big) u_R \tag{1264}$$

Note that $\cosh\eta \equiv \gamma$ and η is the *rapidity parameter* which is addition (?) where

$$-\infty < \eta < \infty \qquad \text{and} \qquad \gamma = \frac{1}{(1 - v^2/c^2)^{1/2}} \tag{1265}$$

Recall that

$$\begin{aligned} t' &= \gamma(t + vx) \qquad & E' &= \gamma(E + vp^3) \\ x' &= \gamma(x + vt) \qquad & p'^3 &= \gamma(p^3 + vE) \end{aligned} \tag{1266}$$

Or by noting that $v = \tanh\eta$, we have

$$t' = t\cosh\eta + x\sinh\eta \qquad , \qquad x' = t\sinh\eta + x\cosh\eta \tag{1267}$$

So that we obtain

$$E' = E\cosh\eta + p^3\sinh\eta \qquad , \qquad p'^3 = E\sinh\eta + p^3\cosh\eta \tag{1268}$$

Before going further, let us first check: In the infinitesimal version for our problem, we have

$$\begin{pmatrix} E' \\ p'^3 \end{pmatrix} = \left[\mathbb{I} + \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix}$$

Since unlike the speeds, the rapidity (η) can be computed, then finite transformation is

$$\begin{pmatrix} E' \\ p'^3 \end{pmatrix} = exp\left[\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}$$

as is consistent.

Anyway, after that, then we have

$$u_L'(p^3) = \left(\mathbb{I}\cosh(\frac{\eta}{2}) - \sigma^3\sinh(\frac{\eta}{2})\right)u_L(0) = \begin{pmatrix} \cosh(\frac{\eta}{2}) - \sinh(\frac{\eta}{2}) & 0 \\ 0 & \cosh(\frac{\eta}{2}) + \sinh(\frac{\eta}{2}) \end{pmatrix} \sqrt{m}\,\xi$$

In fact, I better write it as

$$u_L'(p^3) = \left[e^{\eta/2}\Big(\frac{\mathbb{I} - \sigma^3}{2}\Big) + \, e^{-\eta/2}\Big(\frac{\mathbb{I} + \sigma^3}{2}\Big)\right] \sqrt{m}\,\xi \tag{1269}$$

Note that

$$E' + p^{3'} = m\,(\cosh\eta + \sinh\eta) = m\,e^{\eta} \tag{1270}$$

So we get

$$e^{\eta/2} = \sqrt{\frac{E' + p^{3'}}{m}} \tag{1271}$$

and similarly

$$e^{-\eta/2} = \sqrt{\frac{E' - p^{3'}}{m}} \tag{1272}$$

So then, dropping primes

$$u_L(p^3) = \left[\sqrt{E + p^3}\,\Big(\frac{\mathbb{I} - \sigma^3}{2}\Big) + \sqrt{E - p^3}\,\Big(\frac{\mathbb{I} + \sigma^3}{2}\Big)\right]\xi \tag{1273}$$

Similarly the right-handed spinor yields

$$u_R(p^3) = \left[\sqrt{E + p^3}\,\Big(\frac{\mathbb{I} + \sigma^3}{2}\Big) + \sqrt{E - p^3}\,\Big(\frac{\mathbb{I} - \sigma^3}{2}\Big)\right]\xi \tag{1274}$$

Collecting these we have

$$u^s(p) = \begin{pmatrix} \left[\sqrt{E + p^3}\,\frac{\mathbb{I} - \sigma^3}{2} + \sqrt{E - p^3}\,\frac{\mathbb{I} + \sigma^3}{2}\right]\xi^s \\ \\ \left[\sqrt{E + p^3}\,\frac{\mathbb{I} + \sigma^3}{2} + \sqrt{E - p^3}\,\frac{\mathbb{I} - \sigma^3}{2}\right]\xi^s \end{pmatrix}$$

Note that $\xi\xi^\dagger = \mathbb{I}$ has two solutions

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So we have taken both of them.

In fact, in the ultra-relativistic limit $p^\mu \longrightarrow (E, 0, 0, E)$, we have

$$u^1(p) \longrightarrow \begin{pmatrix} \sqrt{2E}\ \frac{\mathbb{I}-\sigma^3}{2}\ \xi^1 \\ \\ \sqrt{2E}\ \frac{\mathbb{I}+\sigma^3}{2}\ \xi^1 \end{pmatrix} = \sqrt{2E} \begin{pmatrix} \mathbf{0} \\ \xi^1 \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and also

$$u^2(p) \longrightarrow = \sqrt{2E} \begin{pmatrix} \xi^2 \\ \mathbf{0} \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Similar computation for $v(p)$ yields

$$v^s(p) = \begin{pmatrix} \left[\sqrt{E+p^3}\ \frac{\mathbb{I}-\sigma^3}{2} + \sqrt{E-p^3}\ \frac{\mathbb{I}+\sigma^3}{2}\right] \eta^s \\ \\ -\left[\sqrt{E+p^3}\ \frac{\mathbb{I}+\sigma^3}{2} + \sqrt{E-p^3}\ \frac{\mathbb{I}-\sigma^3}{2}\right] \eta^s \end{pmatrix}$$

Note that $\eta\eta^\dagger = \mathbb{I}$ has two solutions

$$\eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So the sign of the lower component is different compared to $u(p)$.

Let us find some useful identities obeyed by the four-spinors $u(p)$ and $v(p)$. Clearly we have

$$\xi^{r+}\xi^s = \delta^{rs} \qquad , \qquad \eta^{r+}\eta^s = \delta^{rs} \tag{1275}$$

Define

$$\bar{u}^s(p) = u^{s+}(p)\gamma^0 \qquad ; \qquad \bar{v}^s(p) = v^{s+}(p)\gamma^0 \tag{1276}$$

Then we have

$$\bar{u}^r(p)\, u^s(p) = 2m\, \delta^{rs} \qquad \text{and} \qquad \bar{v}^r(p)\, v^s(p) = -2m\, \delta^{rs} \tag{1277}$$

(Just use the explicit forms or even the ultra-relativistic or $p^3 = 0$ limits.)

By the way our final expressions above are somewhat cumbersome. We can certainly simplify them as follows

$$\sqrt{E+p^3}\ \Big(\frac{\mathbb{I}-\sigma^3}{2}\Big) + \sqrt{E-p^3}\ \Big(\frac{\mathbb{I}+\sigma^3}{2}\Big) = \sqrt{p\cdot\sigma} \tag{1278}$$

where $\sigma^\mu = (\mathbb{I}\,,\,\vec{\sigma})$. So

$$p_\mu \sigma^\mu = p_0\sigma^0 + p_3\sigma^3 = E\sigma^0 - p^3\sigma^3 = \begin{pmatrix} E-p^3 & \\ & E+p^3 \end{pmatrix}$$

So we arrive at

$$\sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p^3} & \\ & \sqrt{E+p^3} \end{pmatrix}$$

Compute the left-hand side and observe that it is the same thing: So defining

$$\bar{\sigma}^\mu = (\mathbb{I}\,,\,-\vec{\sigma}) \tag{1279}$$

we then have

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi^s \\ \\ \sqrt{p\cdot\bar{\sigma}}\;\xi^s \end{pmatrix} \qquad ; \qquad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\eta^s \\ \\ -\sqrt{p\cdot\bar{\sigma}}\;\eta^s \end{pmatrix}$$

Clearly

$$\begin{aligned}(p\cdot\sigma)(p\cdot\bar{\sigma}) &= p_\mu p_\nu\,\sigma^\mu\bar{\sigma}^\nu \\ &= (\mathbb{I}\,E - \vec{p}\cdot\vec{\sigma})(\mathbb{I}\,E + \vec{p}\cdot\vec{\sigma}) \\ &= E^2 - \vec{p}^2 = p^2 = m^2\end{aligned} \tag{1280}$$

Note that

$$\bar{u}^r(p)\,v^s(p) = 0 \quad , \quad \bar{v}^r(p)\,u^s(p) = 0 \tag{1281}$$

But

$$u^{r+}(p)\,v^s(p) \neq 0 \quad , \quad v^{r+}(p)\,u^s(p) \neq 0 \tag{1282}$$

However

$$u^{r+}(\vec{p})\,v^s(-\vec{p}) = v^{r+}(-\vec{p})\,u^s(\vec{p}) = 0 \tag{1283}$$

1. *SPIN SUMS*

Later we will see that we often sum over the polarization states of the fermions in a scattering experiment. So we need the following completeness relations

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_{s=1,2} \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi^s \\ \\ \sqrt{p\cdot\bar{\sigma}}\;\xi^s \end{pmatrix} \left(\xi^{s+}\sqrt{p\cdot\bar{\sigma}}\,,\,\xi^{s+}\sqrt{p\cdot\sigma}\right)$$

Here the matrices are in outer product. Note that

$$\bar{u}^s = u^{s+}\gamma^0 = \left(\xi^{s+}(\sqrt{p\cdot\sigma})^+\,,\,\xi^{s+}(\sqrt{p\cdot\bar{\sigma}})^+\right)\gamma^0 \tag{1284}$$

By using

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and noting that

$$\sigma = (\mathbb{I}, \vec{\sigma}) \quad \Rightarrow \quad \sigma^\dagger = \sigma \quad \Rightarrow \quad \bar{\sigma}^+ = \bar{\sigma} \tag{1285}$$

So then

$$\bar{u}^s = \left(\xi^{s+} \sqrt{p \cdot \bar{\sigma}} \, , \, \xi^{s+} \sqrt{p \cdot \sigma} \right) \tag{1286}$$

Therefore

$$\sum_{s=1,2} u^s(p) \, \bar{u}^s(p) = \sum_{s=1,2} \begin{pmatrix} \sqrt{p \cdot \sigma} \; \xi^s \, \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \; \xi^s \, \xi^{s+} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \; \xi^s \, \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \bar{\sigma}} \; \xi^s \, \xi^{s+} \sqrt{p \cdot \sigma} \end{pmatrix}$$

Note that

$$\begin{aligned} \sum_{s=1,2} \xi^s \, \xi^{s+} = \xi^1 \, \xi^{1+} + \xi^2 \, \xi^{2+} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(1 \, , \, 0 \right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(0 \, , \, 1 \right) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

So the we get

$$\begin{aligned} \sum_{s=1,2} u^s(p) \, \bar{u}^s(p) &= \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} \\ &= p \cdot \begin{pmatrix} 0 & \sigma \\ \bar{\sigma} & 0 \end{pmatrix} + m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Recall that

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Thus we get

$$\sum_{s=1,2} u^s(p) \, \bar{u}^s(p) = p \cdot \gamma + m \tag{1287}$$

Similarly it is easy to show that

$$\sum_{s=1,2} v^s(p) \, \bar{v}^s(p) = \gamma \cdot p - m \tag{1288}$$

Observe that these are constant with

$$(\not{p} - m) u^s = 0 \quad ; \quad \bar{u}^s(\not{p} + m) = 0 \tag{1289}$$

Now we are ready to write the quantized (free) Dirac field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(a_s(\vec{p})\, u^s(p)\, e^{-ip\cdot x} + b_s^\dagger(\vec{p})\, v^s(p)\, e^{ip\cdot x} \Big) \tag{1290}$$

$$\bar{\psi}(x) = \psi^\dagger \gamma^0 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(b_s(\vec{p})\, \bar{v}^s(p)\, e^{-ip\cdot x} + a_s^\dagger(\vec{p})\, \bar{u}^s(p)\, e^{ip\cdot x} \Big) \tag{1291}$$

Again using the Fourier transform we have

$$\{a_r(\vec{p})\, ,\, a_s^\dagger(\vec{q})\} = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})\, \delta_{rs} \tag{1292}$$

$$\{b_r(\vec{p})\, ,\, b_s^\dagger(\vec{q})\} = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})\, \delta_{rs} \tag{1293}$$

and the rest anti-commute!

$$\{a, b\} = 0 \quad , \quad \{a_r, a_s\} = \{b_r, b_s\} = 0 \tag{1294}$$

2. FOCK SPACE

First we define the vacuum as

$$a_r(\vec{p})|0\rangle = 0 \quad , \quad b_r(\vec{p})|0\rangle = 0 \tag{1295}$$

Multi-particle states are obtained by acting on the vacuum with the creation operators $a^\dagger$ and $b^\dagger$. But since these operators anti-commute then the resulting state is anti-symmetric under a change of labels of two particles. Hence we have spin-1/2 particles obeying Fermi-Dirac statistics.

Consider the 1-particle states

$$|\vec{p}, s\rangle = (2E_p)^{1/2}\, a_s^\dagger(\vec{p})|0\rangle \tag{1296}$$

$$|\vec{p}, s\rangle = (2E_p)^{1/2}\, b_s^\dagger(\vec{p})|0\rangle \tag{1297}$$

Let us calculate the Hamiltonian. Recall that the Hamiltonian density was

$$\begin{aligned} \mathcal{H} &= \Pi_\psi\, \partial_0 \psi - \mathcal{L} \\ &= i\, \psi^\dagger \partial_0 \psi - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\ &= \bar{\psi}(-i\gamma^i \partial_i + m)\psi \end{aligned} \tag{1298}$$

where $(\vec{\nabla})^i \equiv \partial_i$. So then

$$H = \int d^3x\ \bar{\psi}(-i\gamma^i \partial_i + m)\psi = \int d^3x\ \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi \tag{1299}$$

We must insert the mode expansion and carry out the integrals. We also perform the normal ordering which in this case is a little different from the bosonic case

$$: a_s(\vec{p})\, a_s^\dagger(\vec{p}) : = -a_s^\dagger(\vec{p})\, a_s(\vec{p}) \tag{1300}$$

At the end we obtain

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} E_p \left(a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right) \tag{1301}$$

Note that quantizing the Dirac field with commutators would yield a "$-$" sign in front of the above term. That would be a disaster giving an unstable vacuum for example. Meanwhile, momentum operator yields

$$\vec{p} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \vec{p} \left(a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right) \tag{1302}$$

How about the spin of the field? Consider the spin part of the total angular momentum

$$\vec{S} = \frac{1}{2} \int d^3x \;\psi^\dagger \vec{\Sigma}\, \psi \tag{1303}$$

where

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

Let us evaluate the spin part explicitly

$$\begin{aligned} \psi^\dagger \vec{\Sigma} \psi = & \int\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{s=1,2} \sum_{r=1,2} \\ & \times \left(a_s^\dagger(\vec{p}) u_s^+(p) e^{ip\cdot x} + b_s(\vec{p}) v_s^+(p) e^{-ip\cdot x} \right) \vec{\Sigma} \left(a_r(\vec{q}) u_r(q)\, e^{-iq\cdot x} + b_r^\dagger(\vec{q})\, v_r(q) e^{iq\cdot x} \right) \end{aligned} \tag{1304}$$

The x and $\vec{q}$-integration yield

$$\begin{aligned} \vec{S} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} \Big[& a_s^\dagger(\vec{p}) a_r(\vec{p}) u_s^+(p) \vec{\Sigma}\, u_r(p) + a_s^\dagger(\vec{p}) b_r^\dagger(-\vec{p}) u_s^+(p) \vec{\Sigma}\, v_r(-p) e^{2iE_p t} \\ & + b_s(\vec{p}) a_r(\vec{p}) v_s^+(p) \vec{\Sigma}\, u_r(-p) e^{-2iE_p t} + b_s(\vec{p}) b_r^\dagger(\vec{p}) v_s^+(p) \vec{\Sigma}\, v_r(p) \Big] \end{aligned} \tag{1305}$$

Note that

$$\begin{aligned} u_s^+(p_0, \vec{p}) \vec{\Sigma}\, v_r(p_0, -\vec{p}) &= \left(\xi^{s+} \sqrt{p\cdot\sigma}\,,\, \xi^{s+} \sqrt{p\cdot\bar{\sigma}} \right) \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\, \eta^r \\ -\sqrt{p\cdot\bar{\sigma}}\, \eta^r \end{pmatrix} \\ &= \left(\xi^{s+} \sqrt{p\cdot\sigma}\,,\, \xi^{s+} \sqrt{p\cdot\bar{\sigma}} \right) \begin{pmatrix} \vec{\sigma} \sqrt{p\cdot\bar{\sigma}}\, \eta^r \\ -\vec{\sigma} \sqrt{p\cdot\bar{\sigma}}\, \eta^r \end{pmatrix} \\ &= \xi^{s+} \sqrt{p\cdot\sigma}\;\; \vec{\sigma}\;\; \sqrt{p\cdot\bar{\sigma}}\;\eta^r - \xi^{s+} \sqrt{p\cdot\bar{\sigma}}\;\; \vec{\sigma}\;\; \sqrt{p\cdot\bar{\sigma}}\;\eta^r \end{aligned}$$

in the rest frame, this term is zero, as a vector it must be zero elsewhere. So this term is gone. This will be also true for the other, $v_s^+ \Sigma u_r$, term. This argument is correct as I checked it explicitly. So then we have

$$\vec{S} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} \left[a_s^\dagger(\vec{p}) a_r(\vec{p}) u_s^+(p) \vec{\Sigma}\, u_r(p) + b_s(\vec{p}) b_r^\dagger(\vec{p}) v_s^+(p) \vec{\Sigma}\, v_r(p) \right] \tag{1306}$$

Consider S_z and consider 1-particle zero momentum state

$$a_s^\dagger(0) \sqrt{2m} |0\rangle \tag{1307}$$

By assuming normal ordering in S_z, then

$$S_z|0\rangle = 0 \tag{1308}$$

Hence

$$\begin{aligned}
S_z a_s^\dagger(0)|0\rangle &= [S_z\,,\, a_s^\dagger(0)]|0\rangle \\
&\sim [a_r^\dagger(\vec{p})\, a_{r'}(\vec{p})\,,\, a_s^\dagger(0)]|0\rangle \\
&= a_r^\dagger(\vec{p})\, [a_{r'}(\vec{p})\,,\, a_s^\dagger(0)]|0\rangle \\
\Big(\text{since } \{a_{r'}(\vec{p})\,,\, a_s^\dagger(\vec{q})\} &= (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})\Big) \\
&= \delta^{r' s}\, a_r^\dagger(\vec{p})\, (2\pi)^3\, \delta^{(3)}(\vec{p})|0\rangle
\end{aligned} \tag{1309}$$

Then we have

$$S_z a_s^\dagger(0)|0\rangle = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_{r,r'}(2\pi)^3\,\delta^{(3)}(\vec{p})\,\delta^{r' s}\, a_r^\dagger(\vec{p})|0\rangle \tag{1310}$$

which is

$$S_z a_s^\dagger(0)|0\rangle = \frac{1}{4m}\sum_r u_r^+(0)\,\Sigma^z\, u_r(0)\, a_r^\dagger(0)|0\rangle \tag{1311}$$

Recall that

$$u^+ u(0) = 2m\,\xi^+\xi \tag{1312}$$

So choose ξ to be an eigenvector of σ^3 then more explicitly we have

$$S_z a_s^\dagger(0)|0\rangle = \sum_r \xi^{r+}\,\frac{\sigma^3}{2}\,\xi^s\, a_r^\dagger(0)|0\rangle \tag{1313}$$

For

$$\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we get

$$S_z a_s^\dagger(0)|0\rangle = \frac{1}{2}\, a_s^\dagger(0)|0\rangle \tag{1314}$$

For

$$\xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we get

$$S_z a_s^\dagger(0)|0\rangle = -\frac{1}{2}\, a_s^\dagger(0)|0\rangle \tag{1315}$$

The same calculation works for the $b_s^\dagger(0)|0\rangle$ state with one difference:

$$S_z b_s^\dagger(0)|0\rangle = \mp\frac{1}{2}\, b_s^\dagger(0)|0\rangle \tag{1316}$$

Namely for

$$\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we get negative!

NOTE: this is consistent with Dirac's hole theory: absence of a negative-energy electron is a positron. So if the missing electron has positive J_z, then its absence gives a negative J_z !

Finally, we can compute the conserved charge, which after normal ordering gives

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_s^\dagger(\vec{p})\, a_s(\vec{p}) - b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right) \tag{1317}$$

So

$$a_s^\dagger \text{ creates fermions with charge } +1 \tag{1318}$$

$$b_s^\dagger \text{ creates anti-fermions with charge } -1 \tag{1319}$$

Therefore let us make a table. Electric charge is in units of electrons charge $e < 0$.

state	j_z	U(1) charge
$a_1^\dagger(\vec{p})\|0\rangle$	$+\frac{1}{2}$	+1
$a_2^\dagger(\vec{p})\|0\rangle$	$-\frac{1}{2}$	+1
$b_1^\dagger(\vec{p})\|0\rangle$	$-\frac{1}{2}$	−1
$b_2^\dagger(\vec{p})\|0\rangle$	$+\frac{1}{2}$	−1

Table I. Fock Space for Dirac Field.

3. *COVARIANT ANTI-COMMUTATORS*

Recall that

$$\psi_a(t,\vec{x}),\, \psi_a(t,\vec{y})\} = \delta_{ab}\, \delta^{(3)}(\psi_a(t,\vec{x}) - \psi_a(t,\vec{y})) \tag{1320}$$

while all the others are zero. Now consider the generic *covariant anti-commutators*:

$$\{\psi_a(x),\, \bar{\psi}_b(y)\} \equiv F_{ab}(x,y) \quad ; \quad \{\psi_a(x),\, \psi_b(y)\} \equiv G_{ab}(x,y) \quad ; \quad \{\bar{\psi}_a(x),\, \bar{\psi}_b(y)\} \equiv H_{ab}(x,y) \tag{1321}$$

We aslo know that these operators satisfy the Dirac equation and its conjugate

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \qquad , \qquad \bar{\psi}(i\gamma^\mu\partial_\mu + m) = 0 \tag{1322}$$

Then applying the Dirac operator $\mathcal{D} \equiv i\gamma^\mu\partial_\mu - m$ to the first anti-commutator, we have

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)_{a'a} \{\psi_a(x),\, \bar{\psi}_b(y)\} = (i\gamma^\mu\partial_\mu - m)_{a'a} F_{ab}(x,y) = 0 \tag{1323}$$

which provides

$$(i\gamma^\mu \partial_\mu^x - m)_{a'a} F_{ab}(x,y) = 0 \tag{1324}$$

$$F_{ab}(x,y)(i\gamma^\mu \partial_\mu^y + m)_{b'b} = 0 \tag{1325}$$

So we get

$$F_{ab}(x,y) = F_{ab}(x-y) \tag{1326}$$

From this we can easily deduce that

$$F_{ab}(x-y) = i\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + m\right)_{ab} \Delta(x-y) \tag{1327}$$

where $\Delta(x-y)$ Pauli-Jordan function. This works since $(\partial^2 + m^2)\Delta(x) = 0$.

Actually it is somewhat a better exercise to find the covariant anti-commutators and the Dirac Propagator in the following way: Let us compute

$$\begin{aligned} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \langle 0| \sum_s a_s(\vec{p})\, u_a^s(p)\, e^{-ip\cdot x} \\ &\times \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_r a_r^\dagger(\vec{q})\, \bar{u}_b^r(q)\, e^{iq\cdot y}|0\rangle \end{aligned} \tag{1328}$$

Recall the following

$$\{a_s(\vec{p}),\, a_r^\dagger(\vec{q})\} = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})\, \delta_{sr} \tag{1329}$$

$$\sum_{s=1,2} u_a^s(p)\bar{u}_b^s(p) = (p^\mu\gamma_\mu + m)_{ab} \quad ; \quad \langle 0|0\rangle = 1 \quad ; \quad p^\mu = i\partial^\mu \tag{1330}$$

Then

$$\begin{aligned} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s u_a^s(p)\bar{u}_b^s(p)\, e^{-ip\,(x-y)} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s (i\gamma^\mu\partial_\mu^x + m)_{ab}\, e^{-ip\,(x-y)} \\ &= (i\gamma^\mu\partial_\mu^x + m)_{ab} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\, e^{-ip\,(x-y)}}_{\mathcal{D}(x-y)} \end{aligned} \tag{1331}$$

So we get

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = (i\gamma^\mu\partial_\mu^x + m)_{ab}\, \mathcal{D}(x-y) \tag{1332}$$

Similarly

$$\begin{aligned} \langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s v_a^s(p)\bar{v}_b^s(p)\, e^{-ip\,(y-x)} \\ &= -(i\not{\partial}_x + m)_{ab}\, \mathcal{D}(y-x) \end{aligned} \tag{1333}$$

Retarded Green's function can be defined as (exactly in the same spirit as we more explicitly did in the Klein-Gordon case)

$$\begin{aligned} S_R^{ab} &\equiv \theta(x^0 - y^0)\,\langle 0|\{\psi_a(x)\,,\,\bar{\psi}_b(y)\}|0\rangle \\ &= \theta(x^0 - y^0)\,(i\partial\!\!\!/_x + m)_{ab}\left(\mathcal{D}(x-y) - \mathcal{D}(y-x)\right) \end{aligned} \tag{1334}$$

Recall that

$$\mathcal{D}_R(x-y) = \theta(x^0 - y^0)\left(\mathcal{D}(x-y) - \mathcal{D}(y-x)\right) \tag{1335}$$

So

$$S_R^{ab}(x-y) = (i\partial\!\!\!/_x + m)_{ab}\,\mathcal{D}_R(x-y) \tag{1336}$$

So S_R is a retarded Green's function, satisfying

$$(i\partial\!\!\!/_x - m)\,S_R(x-y) = i\,\delta^{(4)}(x-y)\,\mathbb{I}_{4\times 4} \tag{1337}$$

Of course as in the Klein-Gordon case, we could start from this equaiton and do a Fourier transform to get

$$S_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\tilde{S}_R(p)\,e^{-ip\cdot(x-y)} \tag{1338}$$

So

$$(i\partial\!\!\!/_x - m)\;S_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\,(p\!\!\!/ - m)\;\tilde{S}_R(p)\,e^{-ip\cdot(x-y)} = i\int \frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)} \tag{1339}$$

So we get

$$\tilde{S}_R(p) = \frac{i}{p\!\!\!/ - m} \tag{1340}$$

For the retarded Green's function, we again use our earlier contour

FIG 42 !!!!

for

$$x^0 > y^0 \;\text{ we close the contour from below to get zero.} \tag{1341}$$

$$x^0 < y^0 \;\text{ we close it from above to get zero.} \tag{1342}$$

Feynman's Propagator (or Green's function):

Recall that Feynman's resulted from the following choice

FIG 43 !!!!

$$S_F = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{1343}$$

$$\begin{aligned} S_F &= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \\ &= \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle & \text{for } x^0 > y^0 \text{ close contour from below} \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle & \text{for } x^0 < y^0 \text{ close contour from above} \end{cases} \\ &\equiv \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle \end{aligned}$$

Note that *time-ordered product T introduces a minus sign for fermions.*

E. ELECTROMAGNETIC FIELD

Quantization of the Abelian gauge field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi \tag{1344}$$

$\mathcal{L}$ is invariant under

$$\psi^{'}(x) = e^{ie\Lambda(x)}\,\psi(x) \tag{1345}$$

such that

$$A^{'}_\mu = A_\mu(x) + \partial_\mu\Lambda(x) \tag{1346}$$

Therefore by defining *gauge covariant derivative* as $\mathcal{D}_\mu \equiv \partial_\mu - ieA_\mu$, we have

$$(\mathcal{D}_\mu\psi)^{'} = e^{ie\Lambda(x)}\,\mathcal{D}_\mu\psi(x) \tag{1347}$$

So there is a redundancy in the describtion. The question is how do we quantize such a theory? We can gauge-fix and work with only the physical DOF or we quantize covariantly. First let us consider gauge-fixing procedures. (We will come back to covariant quantization at the end)

Coulomb gauge: Not that the equation of motion is

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{1348}$$

Let first assume $\nu = 0$, then we get

$$\begin{aligned} \partial_i F^{i0} &= 0 \\ \partial_i\partial^i A^0 - \partial_i\partial^0 A^i &= J^0 \\ -\nabla^2 A^0 - \partial^0\partial_i A^i &= J^0 \end{aligned} \tag{1349}$$

that is to say, we get

$$-\nabla^2 A^0 - \partial^0\vec{\nabla}\cdot\vec{A} = J^0 \tag{1350}$$

Subsequently let us set $\nu = j$

$$\begin{aligned} \partial_0 F^{0j} + \partial_i F^{ij} &= J^j \\ \partial_0\partial^0 A^j - \partial_0\partial^j A^0 + \partial_i\partial^i A^j - \partial_i\partial^j A^i &= J^j \end{aligned} \tag{1351}$$

so we have

$$\partial_0^2 A^j - \partial^j\partial_0 A^0 - \nabla^2 A^j - \partial^j\,\vec{\nabla}\cdot\vec{A} = J^j \tag{1352}$$

Consider the following gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \qquad \text{such that} \qquad \partial_k A'^k = \vec{\nabla}\cdot\vec{A}'' = 0 \tag{1353}$$

Starting from $\vec{\nabla}\cdot\vec{A} \neq 0$, is this possible? Check that it is

$$\begin{aligned} \partial_k(A^k + \partial^k\Lambda) &= 0 \\ \partial_k A^k + \partial_k\partial^k\Lambda &= 0 \\ \partial_k A^k - \nabla^2\Lambda &= 0 \end{aligned} \tag{1354}$$

So we have

$$\Lambda(x) = \frac{1}{\nabla^2}\,\vec{\nabla}\cdot\vec{A} \tag{1355}$$

Well recall that

$$\nabla_x^2\,\frac{1}{|\vec{x}-\vec{y}|} = -4\pi\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1356}$$

Therefore, we get

$$\Lambda(x) = -\frac{1}{4\pi}\int d^3y\,\frac{1}{|\vec{x}-\vec{y}|}\,\frac{\partial}{\partial y^k}A^k(t,\vec{y}) \tag{1357}$$

So such a gauge exists if $\Lambda(x)$ is given as above. Then in this gauge, and drop the primes, we have

$$-\vec{\nabla}^2\cdot\vec{A}^0 = J^0 \qquad ; \qquad \partial_0^2 A^j + \partial_j\partial_0 A_0 - \vec{\nabla}^2 A^j = J^j \tag{1358}$$

Note that there is still a residual gauge transformation: $\partial_k A^k = 0$ is our gauge condition which remains intact under

$$A^k \longrightarrow A'^k = A^k + \partial^k\Lambda_2(t,\vec{x}) \tag{1359}$$

so that

$$\partial_k A'^k = 0 = \underbrace{\partial_k A^k}_{0} + \partial_k\partial^k\Lambda_2(t,\vec{x}) = 0 \tag{1360}$$

Therefore

$$\nabla^2\Lambda_2(t,\vec{x}) = 0 \tag{1361}$$

This type of transformations keep us in the $\vec{\nabla}\cdot\vec{A} = 0$ gauge. The most general solution of $\nabla^2\Lambda_2(t,\vec{x}) = 0$ is of the form $\Lambda_2(t,\vec{x}) = f(t) + h(t)\vec{a}\cdot\vec{x}$ where $\vec{a}$ is some constant 3-vector. The requirement that Λ_2 be finite as $|\vec{x}| \to \infty$ yields $h(t) = 0$. In the Coulomb gauge we can solve for $A^0(x)$ as

$$A^0(x) = -\frac{1}{4\pi}\int d^3y\, \frac{1}{|\vec{x}-\vec{y}|}\, J^0(t,\vec{y}) \tag{1362}$$

We can use this in the A^j equaiton

$$\begin{aligned}(\partial_0^2 - \vec{\nabla}^2)A^j &= J^j - \partial_j \partial_0 A_0 \\ &= J^j + \frac{1}{4\pi}\partial_j^x \int d^3y\, \frac{1}{|\vec{x}-\vec{y}|}\, \partial_0 J^0\end{aligned} \tag{1363}$$

Use the conservation law $\partial_0 J^0 + \partial_i J^i = 0$ to get

$$(\partial_0^2 - \vec{\nabla}^2)A^j = J^j - \frac{1}{4\pi}\partial_j^x \int d^3y\, \frac{1}{|\vec{x}-\vec{y}|}\, \partial_i J^i \tag{1364}$$

So once the current is given we have the solution. Note that we can use the Green's function to solve the above equation, but we really don't need that. So let us consider the *source-free* case. Then in addition to $\vec{\nabla}\cdot\vec{A} = 0$, we have $A^0 = 0$. (Together sometimes this gauge is called the *radiation gauge*). So in the radiation gauge

$$A^0 = 0 \quad , \quad \vec{\nabla}\cdot\vec{A} = 0 \tag{1365}$$

which gives

$$\partial^2 A^j = 0 \tag{1366}$$

So we have 3 dynamical equation and 1 constraint, that means we have 2 DOF, which obey the massless Klein-Gordon equation

$$(\partial_0^2 - \vec{\nabla}^2)A^j = 0 \quad , \quad \vec{\nabla}\cdot\vec{A} = 0 \tag{1367}$$

Plane-wave solutions will be of the form $A^j = \epsilon^j(p)e^{\pm ip\cdot x}$

1.

$$(p^0)^2 - \vec{p} = 0 \quad \Rightarrow \quad p^0 = \pm|\vec{p}| \quad \text{define} \quad w_p \equiv |\vec{p}| \tag{1368}$$

2.

$$\partial_j A^j = 0 \quad \Rightarrow \quad \epsilon^j p_j = 0 \quad \text{or} \quad \vec{\epsilon}\cdot\vec{p} = 0 \tag{1369}$$

So if we choose $\vec{p} = (0,0,p)$ then $\vec{\epsilon}$ is transverse to them omentum. Say

$$\vec{\epsilon}_1 = (1,0,0) \quad \text{and} \quad \vec{\epsilon}_2 = (0,1,0) \tag{1370}$$

These are two independent linear polarization

FIG 44 !!!!

We can also use circular (or elliptic) polarizations

$$\vec{\epsilon}_\pm \equiv \vec{\epsilon}_1 + i\vec{\epsilon}_2 \tag{1371}$$

Recall that the representations of the Poincare group for massless particles are classified according to their helicity. (helicity $\sim$ projection of the angular omentum/spin in the direction of propagation). Under rotations

$$\vec{\epsilon}^{\,\prime}_{\pm} = e^{\pm i\theta}\,\vec{\epsilon}_{\pm} \tag{1372}$$

Recall that $U(\theta) = e^{-ih\theta}$ where h is the helicity. Thus, here, $\vec{\epsilon}_{+}$ has helicity -1 (left circularly polarized) and $\vec{\epsilon}_{-}$ has helicity $+1$ (right circularly polarized).

Let us look at the Lagrangian in the radiation gauge

$$\partial^2 A^j = 0 \quad , \quad \partial_j A^j = 0 \quad , \quad A^0 = 0 \tag{1373}$$

then up to a boundary term

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} \\ &= -\frac{1}{2}F_{0i}F^{0i} - \frac{1}{4}F_{ij}F^{ij} \\ &= \frac{1}{2}\partial_\mu A^i\,\partial^\mu A^i \end{aligned} \tag{1374}$$

Compare this with the Klein-Gordon Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$. Let us get back to the mode expansion. In Cartesian coordinates the most general solution of $\partial^2 A^i = 0$ is

$$\vec{A}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2w_p}}\sum_{\lambda=1,2}\Big(\vec{\epsilon}(\vec{p},\lambda)a(\vec{p},\lambda)\,e^{-ip\cdot x} + \vec{\epsilon}^{\,*}(\vec{p},\lambda)a^*(\vec{p},\lambda)\,e^{ip\cdot x}\Big) \tag{1375}$$

where $\vec{A}(x)$ is real field. In QFT, we will upgrade $\vec{A}$ to be a Hermitian operator

$$\vec{A}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2w_p}}\sum_{\lambda=1,2}\Big(\vec{\epsilon}(\vec{p},\lambda)a(\vec{p},\lambda)\,e^{-ip\cdot x} + \vec{\epsilon}^{\,*}(\vec{p},\lambda)a^\dagger(\vec{p},\lambda)\,e^{ip\cdot x}\Big) \tag{1376}$$

How do we proceed to quantization or canonical commutation relations? Very naively we could impose

$$[A_i(t,\vec{x})\,,\,\Pi_j(t,\vec{y})] \stackrel{?}{=} i\,\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1377}$$

but as we shall see in a moment, this does not work.

$$\Pi^j(t,\vec{x}) = \frac{\partial\mathcal{L}}{\partial\partial_0 A_j} = \frac{\partial(-\frac{1}{4}F_{0i}F^{0i})}{\partial\partial_0 A_j} = -F^{0j} = E^j \tag{1378}$$

So then we naively have

$$[A^i(t,\vec{x})\,,\,E^j(t,\vec{y})] \stackrel{?}{=} -i\,\delta^{ij}\,\delta^{(3)}(\vec{x}-\vec{y} \tag{1379}$$

But let us take the derivative of this expansion ∂_i^x, the left-hand side is zero since $\partial_i^x A_i = 0$, but the right-hand side is non-zero! This means we need a new delta function whose derivative is zero. Or we need the so-called *transverse* δ-function

$$\delta^{ij}_{\text{tr}}(\vec{x}-\vec{y}) \qquad \text{such that} \qquad \partial_i\delta^{ij}_{\text{tr}}(\vec{x}-\vec{y}) = 0 \tag{1380}$$

This is actually easy to guess

$$\delta^{ij}_{\rm tr}(\vec{x}-\vec{y}) = \Big(\delta^{ij} - \frac{\partial^i\partial^j}{\partial^2}\Big)\delta^{(3)}(\vec{x}-\vec{y}) \tag{1381}$$

Check $\partial_i\delta^{ij}_{\rm tr}(\vec{x}-\vec{y}) = 0$. Same for the j-component of course. But what is $\frac{1}{\partial^2}$?

$$\frac{1}{\partial^2}A(\vec{x}) \equiv \int d^3x^{'}\, G(\vec{x},\vec{x}^{'})A(\vec{x}^{'}) \tag{1382}$$

OK then we can go back to the canonical commutators. Π^0 and A^0 are not dynamical; i.e., there is no $\partial_0 A_0$ in the Lagrangian. So we have

$$[A^i(t,\vec{x})\,,\, E^j(t,\vec{y})] = -i\,\delta^{ij}_{\rm tr}(\vec{x}-\vec{y}) \tag{1383}$$

Note that in some papers $\hat{\partial}^i \equiv \frac{\partial^i}{\sqrt{-\partial^2}}$ is used so

$$\delta^{ij}_{\rm tr}(\vec{x}-\vec{y}) = (\delta^{ij} + \hat{\partial}^i\hat{\partial}^j)\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1384}$$

Let us convert this to the momentum space as it is often needed. Recall that

$$\delta^{(3)}(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \tag{1385}$$

So

$$\frac{\partial^i\partial^j}{\partial^2_x}\delta^{(3)}(\vec{x}-\vec{y}) = -\frac{1}{\partial^2_x}\int \frac{d^3p}{(2\pi)^3}\, p_i p_j e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \tag{1386}$$

Recall that

$$\partial^2 G(\vec{x},\vec{x}^{'}) = \delta^{(3)}(\vec{x}-\vec{x}^{'}) \qquad \Rightarrow \qquad G(\vec{x},\vec{x}^{'}) = -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}^{'}|} \tag{1387}$$

So

$$\frac{1}{\partial^2_x}f(\vec{x}) \equiv -\int d^3x^{'}\,\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}^{'}|}f(\vec{x}^{'}) \tag{1388}$$

Hence

$$\begin{aligned}\frac{\partial^i\partial^j}{\partial^2}\delta^{(3)}(\vec{x}-\vec{y}) &= -\int \frac{d^3p}{(2\pi)^3}\, p_i p_j e^{i\vec{p}\cdot\vec{y}}\frac{1}{\partial^2_x}e^{-i\vec{p}\cdot\vec{x}} \\ &= \int \frac{d^3p}{(2\pi)^3}\, p_i p_j e^{i\vec{p}\cdot\vec{y}}\underbrace{\int \frac{d^3x^{'}}{4\pi}\,\frac{e^{-i\vec{p}\cdot\vec{x}^{'}}}{|\vec{x}-\vec{x}^{'}|}}_{I}\end{aligned} \tag{1389}$$

Let us now evaluate I

$$\begin{aligned} I &\equiv \frac{1}{4\pi}\int d^3x^{'}\,\frac{e^{-i\vec{p}\cdot\vec{x}^{'}}}{|\vec{x}-\vec{x}^{'}|} \\ &= \frac{e^{-i\vec{p}\cdot\vec{x}}}{4\pi}\int d^3x^{'}\,\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{x}^{'})}}{|\vec{x}-\vec{x}^{'}|}\end{aligned} \tag{1390}$$

Let us choose

$$\vec{p}\cdot(\vec{x}-\vec{x}^{'}) = p|\vec{x}-\vec{x}^{'}| \qquad \text{and let} \qquad \vec{x}-\vec{x}^{'} \equiv \vec{r} \tag{1391}$$

Then

$$\begin{aligned} I &= \frac{e^{-i\vec{p}\cdot\vec{x}}}{4\pi} 2\pi \int_0^\infty \frac{r^2 dr}{r} \int_{-1}^{1} dz\, e^{+iprz} \\ &= \frac{e^{-i\vec{p}\cdot\vec{x}}}{2} \int_0^\infty \frac{dr}{+ip} \Big(e^{+ipr} - e^{-ipr} \Big) \\ &= \frac{e^{-i\vec{p}\cdot\vec{x}}}{p} \int_0^\infty dr \sin(pr) \\ &= \frac{e^{-i\vec{p}\cdot\vec{x}}}{p} \lim_{a\to 0} \underbrace{\int_0^\infty dr \sin(pr)\, e^{-ar}}_{\mathcal{L}\{\sin(pr)\}} \\ &= \frac{e^{-i\vec{p}\cdot\vec{x}}}{p^2} \end{aligned} \tag{1392}$$

Then

$$\frac{\partial^i\partial^j}{\partial^2}\delta^{(3)}(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3} \frac{p^i p^j}{p^2}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \qquad (\text{Here } p^2 \equiv \vec{p}^2) \tag{1393}$$

Then we have

$$\delta^{ij}_{\text{tr}}(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3} \Big(\delta^{ij} - \frac{p^i p^j}{\vec{p}^2} \Big)\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \tag{1394}$$

So the momentum space version, as expected, is

$$\tilde{\delta}^{ij}_{\text{tr}}(p) = \delta^{ij} - \frac{p^i p^j}{\vec{p}^2} \tag{1395}$$

OK. Let us go back to our canonical commutation relation

$$E^j = -F^{0j} = -\partial^0 A^j = \int \frac{d^3p}{(2\pi)^3}\, i\sqrt{\frac{w_p}{2}} \sum_{\lambda=1,2} \Big(\epsilon^j(\vec{p},\lambda) a(\vec{p},\lambda) e^{-ip\cdot x} - \epsilon^{*j}(\vec{p},\lambda) a^\dagger(\vec{p},\lambda) e^{ip\cdot x} \Big) \tag{1396}$$

Then

$$\begin{aligned} [A^i(t,\vec{x}),\, E^j(t,\vec{y})] = &\int\int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2w_q}} \frac{d^3p}{(2\pi)^3}\, i\sqrt{\frac{w_p}{2}} \sum_r \sum_\lambda \\ &\times \Big[\epsilon^i(\vec{q},r) a(\vec{q},r) e^{-iq\cdot x} + \epsilon^{*i}(\vec{q},r) a^\dagger(\vec{q},r) e^{iq\cdot x}\,, \\ &\epsilon^j(\vec{p},\lambda) a(\vec{p},\lambda) e^{-ip\cdot y} - \epsilon^{*j}(\vec{p},\lambda) a^\dagger(\vec{p},\lambda) e^{ip\cdot y} \Big] \end{aligned} \tag{1397}$$

Actually, it would have benn a lot better if I used inverse Fourier transform and express a and $a^\dagger$ in terms of A^i and E^i. But this version is also OK. At the end we should have

$$[a(\vec{p},\lambda),\, a^\dagger(\vec{q},\lambda^{'})] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})\delta_{\lambda\lambda'} \tag{1398}$$

and the rest are zero.

In fact this expression would be a good starting point to quantize the Electromagnetic field before going into the transverse delta function discussion. Apply(?) let us use this

$$\begin{aligned}
[A^i(t,\vec{x}),\, E^j(t,\vec{y})] &= \int\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2w_q}}\frac{d^3p}{(2\pi)^3}\, i\sqrt{\frac{w_p}{2}}\sum_r\sum_\lambda \\
&\times \Big\{ -\epsilon^i(\vec{q},r)\epsilon^{*j}(\vec{p},\lambda)e^{-iq\cdot x+ip\cdot y}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{r\lambda} \\
&\quad -\epsilon^{*i}(\vec{q},r)\epsilon^{j}(\vec{p},\lambda)e^{iq\cdot x-ip\cdot y}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{r\lambda}\Big\} \\
&= -\int \frac{d^3p}{(2\pi)^3}\frac{i}{2}\sum_\lambda\Big\{\epsilon^i(\vec{p},\lambda)\epsilon^{*j}(\vec{p},\lambda)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+\epsilon^{*i}(\vec{p},\lambda)\epsilon^{j}(\vec{p},\lambda)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big\} \\
&= -\int \frac{d^3p}{(2\pi)^3}e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\frac{i}{2}\sum_\lambda\Big\{\epsilon^i(-\vec{p},\lambda)\epsilon^{*j}(-\vec{p},\lambda)+\epsilon^{*i}(\vec{p},\lambda)\epsilon^{j}(\vec{p},\lambda)\Big\}
\end{aligned} \tag{1399}$$

Recall that we had

$$[A^i(t,\vec{x}),\, E^j(t,\vec{y})] = -i\,\delta^{ij}_{\text{tr}}(\vec{x}-\vec{y}) = -\int\frac{d^3p}{(2\pi)^3}\Big(\delta^{ij}-\frac{p^ip^j}{p^2}\Big)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \tag{1400}$$

So then we have

$$\frac{1}{2}\sum_\lambda\Big(\epsilon^i(-\vec{p},\lambda)\epsilon^{*j}(-\vec{p},\lambda)+\epsilon^{*i}(\vec{p},\lambda)\epsilon^{j}(\vec{p},\lambda)\Big) = \delta^{ij}-\frac{p^ip^j}{p^2} \tag{1401}$$

Note that this relation is essentially a consistency check.

Let us check this for

$$\vec{\epsilon}(p,1) = (1,0,0) \quad , \quad \vec{\epsilon}(p,2) = (0,1,0) \quad , \quad \vec{p} = (0,0,1) \tag{1402}$$

then we have

$$\epsilon^1(p,1)\epsilon^1(p,1)+\epsilon^1(p,2)\epsilon^1(p,2) = \delta^{11}-\frac{p^1p^1}{p^2} = \delta^{11} \qquad \Rightarrow \qquad 1=1 \quad \text{(OK)} \tag{1403}$$

$$\epsilon^3(p,1)\epsilon^3(p,1)+\epsilon^3(p,2)\epsilon^3(p,2) = \delta^{33}-\frac{p^3p^3}{p^2} = 0 \qquad \text{(OK)} \tag{1404}$$

Recall that in the radiation gauge we ended up with the commutator

$$[A^i(t,\vec{x}),\, E^j(t,\vec{y})] = -i\,\delta^{ij}_{\text{tr}}(\vec{x}-\vec{y}) \tag{1405}$$

where

$$\delta^{ij}_{\text{tr}}(\vec{x}-\vec{y}) = \int\frac{d^3p}{(2\pi)^3}e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big(\delta^{ij}-\frac{p^ip^j}{p^2}\Big) \tag{1406}$$

We have

$$[\partial_iA^i,\, E^j] = 0 \qquad \text{and} \qquad [A^i,\, \partial_jE^j] = 0 \tag{1407}$$

as is clear from the $i \leftrightarrow j$ symmetry. This equation says thet $\vec{\nabla}\cdot\vec{E}$ commutes with all the operators in the theory, therefore it allows us to impose

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{1408}$$

as an operator equation. (Remember that in the classical Maxwell theory, this equation. says that there are no sources.) In QFT, this equation is a constraint on possible $\vec{E}$-fields.

Just as before, we define the vacuum as

$$a_\lambda(\vec{p})|0\rangle = 0 \quad , \quad \lambda = 1,2 \tag{1409}$$

Then 1-particle state is defined as

$$|\vec{p},\lambda\rangle \sim a^\dagger_\lambda(\vec{p})|0\rangle \tag{1410}$$

What are the properties of the vacuum and say 1-particle states? Hamiltonian is

$$H = \frac{1}{2}\int d^3x\,(\vec{E}^2 + \vec{B}^2) \tag{1411}$$

which after *normal ordering* and the insertion of the fields yield

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=1,2} w_{\vec{p}}\, a^\dagger_\lambda(\vec{p})\, a_\lambda(\vec{p}) \tag{1412}$$

where $w_{\vec{p}} = |\vec{p}|$. Recall that

$$E^i = -F^{0i} = -\partial^0 A^i \quad \text{and} \quad B^i = \epsilon^{ijk}\,\partial^j A^k \tag{1413}$$

Clearly, as expected, and as designed the vacuum has zero energy

$$H|0\rangle = 0 \tag{1414}$$

Let us check the 1-particle state

$$\begin{aligned} H a^\dagger_\lambda(\vec{p})|0\rangle &= \int \frac{d^3q}{(2\pi)^3} \sum_{\lambda'=1}^{2} w_{\vec{q}}\, a^\dagger_{\lambda'}(\vec{q})\, a_{\lambda'}(\vec{q}) a^\dagger_\lambda(\vec{p})|0\rangle \\ &= w_{\vec{p}}\, a^\dagger_{\lambda'}(\vec{q})\,|0\rangle \end{aligned} \tag{1415}$$

So 1-particle state has energy $w_{\vec{p}}$. What is the momentum of this state?

$$\begin{aligned} \vec{\mathcal{P}} &= \int d^3x\ :\vec{E}\times\vec{B}\ : \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=1}^{2} \vec{p}\, a^\dagger_\lambda(\vec{p})\, a_\lambda(\vec{p}) \end{aligned} \tag{1416}$$

It is easy to see that

$$\vec{\mathcal{P}} a^\dagger_\lambda(\vec{p})|0\rangle = \vec{p} a^\dagger_\lambda(\vec{p})|0\rangle \tag{1417}$$

So $a^\dagger_\lambda(\vec{p})$ creates out of the vacuum a particle with momentum $\vec{p}$ and energy $w_{\vec{p}} = |\vec{p}|$, namely a massless particle.

Note that symbolically, we have

$$\vec{A} \sim a + a^\dagger \quad \text{and} \quad \vec{E} \sim a - a^\dagger \tag{1418}$$

So

$$\vec{A}|0\rangle \neq 0 \quad \text{and} \vec{E}|0\rangle \neq 0 \tag{1419}$$

But

$$\langle 0|\vec{E}|0\rangle = 0 \tag{1420}$$

of course this is expected.

2. Total Angular Momentum of the $\vec{A}$-field

Recall that

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \tag{1421}$$

yields $\partial_\mu T^{\nu\mu} = 0$. where

$$T^{\nu\mu} = \frac{\partial \mathcal{L}}{\partial\partial_\mu A_\alpha}\partial^\nu A^\alpha - \eta^{\nu\mu}\mathcal{L} \tag{1422}$$

which gives

$$T^{\nu\mu} = -F^\mu{}_\alpha \partial^\nu A^\alpha + \frac{1}{4}\eta^{\mu\nu}F^2 \tag{1423}$$

which of course is not symmetric. But we can do the following trick

$$T^{\nu\mu} = -F^\mu{}_\alpha F^{\nu\alpha} - F^\mu{}_\alpha \partial^\alpha A^\nu + \frac{1}{4}\eta^{\mu\nu}F^2 \tag{1424}$$

I can drop the second term since

$$\partial_\mu (F^\mu{}_\alpha \partial^\alpha A^\nu) = 0 \tag{1425}$$

So then

$$T^{\nu\mu} = -F^\mu{}_\alpha F^{\nu\alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{1426}$$

Then the *energy density* is

$$T^{00} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \tag{1427}$$

and the *momentum density* is

$$T^{0i} = (\vec{E}\times\vec{B})^i \quad \text{or} \quad \vec{p} = \vec{E}\times\vec{B} \tag{1428}$$

We can get the *angular momentum density* from

$$M^{\rho\sigma 0} = x^\rho T^{\sigma 0} - x^\sigma T^{\rho 0} - i\,\frac{\partial \mathcal{L}}{\partial\partial_0 A_\mu}\,(S^{\rho\sigma})_\mu{}^\nu\, A_\nu \tag{1429}$$

Recall that for vectors we have

$$(S_{\rho\sigma})_\mu{}^\nu = i\,(\eta_{\rho\mu}S_\sigma{}^\nu - \eta_{\sigma\mu}S_\rho{}^\nu) \tag{1430}$$

then

$$M^{ij} = x^i(\vec{E}\times\vec{B})^j - x^j(\vec{E}\times\vec{B})^i + E^iA^j - E^jA^i \tag{1431}$$

$$J^k \equiv \frac{1}{2}\epsilon^{kij}M^{ij} = \left(\vec{r}\times(\vec{E}\times\vec{B})\right)^k + (\vec{E}\times\vec{A})^k \tag{1432}$$

Then the *total angular momentum* will be

$$\vec{J} = \int d^3x \quad \Big\{ \underbrace{\vec{r} \times (\vec{E} \times \vec{B})}_{\text{orbital angular part}} \quad + \quad \underbrace{\vec{E} \times \vec{A}}_{\text{spin part}} \Big\} \tag{1433}$$

Let us calculate S^z in the quantized theory

$$\begin{aligned}
S^z &=: \int d^3x \, \epsilon^{zij} \, E^i A^j \; : \\
&= -\epsilon^{ij} \;\; : \int d^3x \; (\partial^0 A^i) \; A^j \; : \\
&= -\epsilon^{ij} \;\; : \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{(-iw_p)}{\sqrt{2w_p}} \sum_\lambda \Big(\epsilon^i(\vec{p},\lambda) a_\lambda(\vec{p}) \, e^{-ip\cdot x} - \epsilon^{*i}(\vec{p},\lambda) a^\dagger_\lambda(\vec{p}) \, e^{ip\cdot x} \Big) \\
&\qquad \times \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2w_q}} \sum_{\lambda'} \Big(\epsilon^j(\vec{q},\lambda') a_{\lambda'}(\vec{q}) \, e^{-iq\cdot x} + \epsilon^{*j}(\vec{q},\lambda') a^\dagger_{\lambda'}(\vec{q}) \, e^{iq\cdot x} \Big) \; :
\end{aligned} \tag{1434}$$

Integrate over x to get $\delta^{(3)}(\vec{p} \pm \vec{q})$ type delta functions and then integrate over them to get

$$\begin{aligned}
S^z = i\,\epsilon^{ij} \;\; : \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_\lambda \sum_{\lambda'} \Big(& \epsilon^i(\vec{p},\lambda) \epsilon^j(-\vec{p},\lambda') a_\lambda(\vec{p}) a_{\lambda'}(-\vec{p}) \\
& - \epsilon^{*i}(\vec{p},\lambda) \epsilon^{*j}(-\vec{p},\lambda') a^\dagger_\lambda(\vec{p}) a_{\lambda'}(-\vec{p}) \\
& + \epsilon^i(\vec{p},\lambda) \epsilon^{*j}(\vec{p},\lambda') a_\lambda(\vec{p}) a^\dagger_{\lambda'}(\vec{p}) \\
& - \epsilon^{*i}(\vec{p},\lambda) \epsilon^j(\vec{p},\lambda') a^\dagger_\lambda(\vec{p}) a_{\lambda'}(\vec{p}) \;\; \Big) \; :
\end{aligned} \tag{1435}$$

The first two terms die because of their symmetry in $i \leftrightarrow j$. Then normal ordering combines the last two terms as

$$S^z = i\,\epsilon^{ij} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_\lambda \sum_{\lambda'} \Big\{ \epsilon^i(\vec{p},\lambda) \epsilon^{*j}(\vec{p},\lambda') - \epsilon^{*i}(\vec{p},\lambda') \epsilon^j(\vec{p},\lambda) \Big\} a^\dagger_{\lambda'}(\vec{p}) a_\lambda(\vec{p}) \tag{1436}$$

Let us choose

$$\vec{k} = (0,0,k) \quad \text{and} \quad \vec{\epsilon}(\vec{k},1) = (1,0,0) \quad ; \quad \vec{\epsilon}(\vec{k},2) = (0,1,0) \tag{1437}$$

or in components we have $\epsilon^i(\vec{k},\lambda) = \delta^i{}_\lambda$. Then

$$\begin{aligned}
S^z a^\dagger(\vec{k},\lambda'')|0\rangle &= \int \frac{d^3p}{(2\pi)^3} i\,\epsilon^{ij} \, \frac{1}{2} \sum_{\lambda,\lambda'} \Big\{ \epsilon^i(\vec{p},\lambda) \epsilon^j(\vec{p},\lambda') - \epsilon^i(\vec{p},\lambda') \epsilon^j(\vec{p},\lambda) \Big\} \\
&\qquad\qquad \delta_{\lambda'\lambda''} (2\pi)^3 a^\dagger_\lambda(\vec{p}) \delta^{(3)}(\vec{p} - \vec{k})|0\rangle \\
&= \frac{i}{2}\,\epsilon^{ij} \sum_\lambda \Big(\epsilon^i(\vec{k},\lambda) \epsilon^j(\vec{k},\lambda'') - \epsilon^i(\vec{k},\lambda'') \epsilon^j(\vec{k},\lambda) \Big) a^\dagger_\lambda(\vec{k})|0\rangle \\
&= \frac{i}{2}\,\epsilon^{ij} \sum_\lambda \Big(\delta^i_\lambda \delta^j{}_{\lambda''} - \delta^i{}_{\lambda''} \delta^j_\lambda \Big) a^\dagger_\lambda(\vec{k})|0\rangle
\end{aligned} \tag{1438}$$

So one can easiy find that

$$S^z a^\dagger(\vec{k},1)|0\rangle = -i a^\dagger_2|0\rangle \tag{1439}$$

I seem to have a sign difference with the book! But in any case, we have found that linear polarizations are not eigenstates of S^z. Keeping the books sign convention, we have found

$$S^z a_1^\dagger(\vec{k})|0\rangle = i a_2^\dagger(\vec{k})|0\rangle \tag{1440}$$

$$S^z a_2^\dagger(\vec{k})|0\rangle = -i a_1^\dagger(\vec{k})|0\rangle \tag{1441}$$

with $\vec{k} = (0, 0, k)$. We then define the circular polarizations

$$a_\pm^\dagger(\vec{k}) = \frac{1}{\sqrt{2}}\Big(a_1^\dagger(\vec{k}) \pm i a_2^\dagger(\vec{k})\Big) \tag{1442}$$

which are eigenstates of S^z. Hence

$$S^z a_+^\dagger(\vec{k})|0\rangle = +a_+^\dagger(\vec{k})|0\rangle \tag{1443}$$

$$S^z a_-^\dagger(\vec{k})|0\rangle = -a_-^\dagger(\vec{k})|0\rangle \tag{1444}$$

So $a_\pm^\dagger(\vec{k})|0\rangle$ describe particles with momentum $\vec{k}$, energy $w_k = |\vec{k}|$, mass zero and helicity ± 1. These quanta are *photon*.

In the above quantization procedure, we have lost explicit Lorentz invariance. This must be checked at the end. We must show that Lorentz algebra is closed and we must show that the multi-particle states transform properly.

Let us now consider the effects of discrete symmetries on the photon field. [Actually, we shold turn back and do the same thing for the spinor field.]

$$\vec{A}(t, \vec{x}) \tag{1445}$$

is a true vector, hence under *parity* transforms as

$$\vec{A}(t, \vec{x}) \longrightarrow -\vec{A}(t, -\vec{x}) \tag{1446}$$

In terms of the photon states

$$\mathcal{P}|\gamma; \vec{k}, \vec{s}\rangle = -|\gamma; -\vec{k}, \vec{s}\rangle \tag{1447}$$

So intrinsic parity of a physical photon is -1.

Charge Conjugation: As expected for fermionic fields

$$c\bar{\psi}\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\psi \tag{1448}$$

So we can demand that

$$cA^\mu c = -A^\mu \tag{1449}$$

to keep QED symmetric under charge conjugation. This then gives

$$c\, a_\lambda(\vec{p})c = -a_\lambda(\vec{p}) \tag{1450}$$

We define $c|0\rangle = |0\rangle$ and $c^2 = 1$. So

$$c\, a_\lambda(\vec{p})|0\rangle = -a_\lambda(\vec{p})|0\rangle \tag{1451}$$

So photon has charge conjugation number -1.

A little useful Digression: Finite temperature, finite density (condensed matter system or gravity systems, neutron starts, nuclear matter etc.) Consider a piece of metal.

FIG 45 !!!!

The notion of the vacuum with no particle does not work. Say there are N electrons in this box with volume V. What is the *ground* state ? It is easy to see that electrons fill all states with a momentum $|\vec{p}| \leq \rho_F$ (Fermi momentum)

FIG 46 !!!!

$$p_i = \frac{2\pi n_i}{L} \qquad , \qquad \frac{4}{3}\pi\rho_F^3 \ : \ \text{volume of the Fermi sphere} \tag{1452}$$

So

$$2\frac{\frac{4}{3}\pi\rho_F^3}{(\frac{2\pi}{L})^3} \tag{1453}$$

where $\frac{2\pi}{L})^3$ is the volume of each "cell ". Furthermore, the coefficient 2 stands for either spin-up and spin-down. So

$$N = \frac{V\,\rho_F^3}{3\pi^2} \tag{1454}$$

This ground state is referred to as Fermi vacuum $|0\rangle_F$ which is obviously different then the no particle state $|0\rangle$. Define

$$A_s(\vec{p}) = \theta(|\vec{p}| - \rho_F)\ a_s(\vec{p}) + \theta(\rho_F - |\vec{p}|)\ a^\dagger_{-s}(-\vec{p}) \tag{1455}$$

$$A^\dagger_s(\vec{p}) = \theta(|\vec{p}| - \rho_F)\ a^\dagger_s(\vec{p}) + \theta(\rho_F - |\vec{p}|)\ a_{-s}(-\vec{p}) \tag{1456}$$

Show that

$$A^\dagger_s(\vec{p})|0\rangle_F = \Big(\theta(|\vec{p}| - \rho_F)\ a^\dagger_s(\vec{p}) + \theta(\rho_F - |\vec{p}|)\ a_{-s}(-\vec{p})\Big)|0\rangle_F \tag{1457}$$

Say $|\vec{p}| > \rho_F$ then we have $A^\dagger_s(\vec{p})|0\rangle_F = a_s(\vec{p})|0\rangle_F = 0$ since there is no such $\vec{p}$ state in $|0\rangle_F$. Now, say $|\vec{p}| < \rho_F$ then $A^\dagger_s(\vec{p})|0\rangle_F = a^\dagger_{-s}(-\vec{p})|0\rangle_F = 0$ since there is already a state with $-\vec{p}$ there.

Show that

$$\begin{aligned}
\{A_s(\vec{p})\,,\ A^\dagger_{s'}(\vec{q})\} &= (2\pi)^3\delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{q}) \\
&= \{\theta(|\vec{p}| - \rho_F)\ a_s(\vec{p}) + \theta(\rho_F - |\vec{p}|)\ a^\dagger_{-s}(-\vec{p})\,, \\
&\quad\ \theta(|\vec{q}| - \rho_F)\ a^\dagger_s(\vec{q}) + \theta(\rho_F - |\vec{q}|)\ a_{-s}(-\vec{q})\} \\
&= \theta(|\vec{p}| - \rho_F)\,\theta(|\vec{q}| - \rho_F)\,\{a_s(\vec{p})\,,\ a^\dagger_{s'}(\vec{q})\} \\
&+ \theta(\rho_F - |\vec{p}|)\,\theta(\rho_F - |\vec{q}|)\,\{a^\dagger_{-s}(-\vec{p})\,,\ a_{s'}(-\vec{q})\} \\
&= (2\pi)^3\delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{q})
\end{aligned} \tag{1458}$$

the second line is zero. What does $A_r^\dagger(\vec{q})$ do on $|0\rangle_F$?

$$A_r^\dagger(\vec{q})|0\rangle_F = \Big(\theta(|\vec{q}| - \rho_F)\ a_r^\dagger(\vec{q}) + \theta(\rho_F - |\vec{q}|)\ a_{-r}(-\vec{q})\Big)|0\rangle_F \tag{1459}$$

Say $||\vec{q}| > \rho_F$ then $A_r^\dagger(\vec{q})|0\rangle_F = a_r^\dagger(\vec{q})|0\rangle_F = |\vec{q}, r\rangle_F$. So $A_r^\dagger(\vec{q})$ adds a particle (electron) on the Fermi vacuum if $|\vec{q}| > \rho_F$ Say $||\vec{q}| < \rho_F$, then

$$A_r^\dagger(\vec{q})|0\rangle_F = a_{-r}(-\vec{q})|0\rangle_F \tag{1460}$$

destroys an electron in the Fermi vacuum and creates a *hole* in the Fermi vacuum.

Generically when re-defining the vacuum as above we make a Bogoliubov transformation (both on Fermions and Bosons) as

$$A_s(\vec{p}) = \alpha_{\vec{p}} a_s(\vec{p}) - \beta_{\vec{p}} a_{-s}^\dagger(-\vec{p}) \tag{1461}$$

$$A_s^\dagger(\vec{p}) = \alpha_{\vec{p}}^* a_s^\dagger(\vec{p}) - \beta_{\vec{p}}^* a_{-s}(-\vec{p}) \tag{1462}$$

In the bosonic case we can show that

$$|\alpha_{\vec{p}}|^2 - |\beta_{\vec{p}}|^2 = 1 \tag{1463}$$

$$\alpha_{\vec{p}}\beta_{-\vec{p}} - \alpha_{-\vec{p}}\beta_{\vec{p}} = 0 \tag{1464}$$

and for Fermions we have

$$|\alpha_{\vec{p}}|^2 + |\beta_{\vec{p}}|^2 = 1 \tag{1465}$$

$$\alpha_{\vec{p}}\beta_{-\vec{p}} + \alpha_{-\vec{p}}\beta_{\vec{p}} = 0 \tag{1466}$$

Example: Let us consider the spin-0 case. Let

$$a(\vec{p})|0, a\rangle = 0 \quad \text{and} \quad A(\vec{p})|0, A\rangle = 0 \tag{1467}$$

Let

$$n_{\vec{p}} = a^\dagger(\vec{p})a(\vec{p}) \quad \text{and} \quad n_{\vec{p}}|n_{\vec{p}}\rangle = n_p|n_{\vec{p}}\rangle \tag{1468}$$

Define

$$N_{\vec{p}} \equiv \langle n_{\vec{p}}|A^\dagger(\vec{p})A(\vec{p})|n_{\vec{p}}\rangle \tag{1469}$$

Show that

$$N_{\vec{p}} = n_{\vec{p}} + |\beta_{\vec{p}}|^2(n_{\vec{p}} + \frac{1}{2}) \tag{1470}$$

$$\begin{aligned} N_{\vec{p}} &= \langle n_{\vec{p}}|\Big(\alpha_{\vec{p}}^* a^\dagger(\vec{p}) - \beta_{\vec{p}}^* a(-\vec{p})\Big) \times \Big(\alpha_{\vec{p}} a(\vec{p}) - \beta_{\vec{p}} a^\dagger(-\vec{p})\Big)|n_{\vec{p}}\rangle \\ &= |\alpha_{\vec{p}}^*|^2\, n_p + |\beta_p|^2 \underbrace{\langle n_{\vec{p}}|a(-\vec{p})\, a^\dagger(-\vec{p})|n_{\vec{p}}\rangle}_{n_p + (2\pi)^3\delta^3(0)} \\ &= n_p + 2n_p|\beta_p|^2 + (2\pi)^3\delta^3(0)|\beta_p|^2 \end{aligned} \tag{1471}$$

In the problem volume V is normalized to 1. Note that

$$\int d^3x\, e^{i\vec{p}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{p}) \qquad \Rightarrow \qquad \int d^3x = (2\pi)^3\delta^3(0) = V = 1 \tag{1472}$$

So it is proven. Note the $|0\rangle$ a vacuum corresponds to $N_{\vec{p}} = |\beta_p|^2$.

A mixed state in QM is described by a density matrix ρ. [For example 27% of up states + 73% of down states would correspond to a mixed state. Clearly a mixed state cannot be written as a superposition state such as

$$|\alpha\rangle = c_1|+\rangle + c_2|-\rangle \tag{1473}$$

This is a pure state.] Expectation value of any operator is given by

$$\langle\theta\rangle_{mixed} = \mathrm{Tr}(\rho\theta) \quad \text{such that} \quad \mathrm{Tr}\rho = 1 \tag{1474}$$

On a thermal state with state with temperature $T = \frac{1}{\beta}$ (set $k_\beta = 1$) the density matrix is

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} \tag{1475}$$

Note that trace is over the Fock space. So thermal expectation values are

$$\langle\theta\rangle_\beta = \frac{\mathrm{Tr}\ \theta e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} \tag{1476}$$

1. Take H to be the Hamiltonian of a second quantized free scalar field

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}}\, a^\dagger(\vec{p})a(\vec{p}) \tag{1477}$$

where $E_{\vec{p}} = \sqrt{p^2 + m^2}$, then show that

$$e^{-\beta H} a^\dagger(\vec{p}) = a^\dagger(\vec{p})\, e^{-\beta(H+E_p)} \tag{1478}$$

So that

$$\begin{aligned} e^{-\beta H} a^\dagger(\vec{p})|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle &= e^{-\beta H}\ \underbrace{|\vec{p}, \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle}_{n+1 \text{ particle states}} \\ &= e^{-\beta(E_p + E_{p_1} + E_{p_2} + \cdots + E_{p_n})}|\vec{p}, \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle \\ &= a^\dagger(\vec{p})e^{-\beta(E_p + E_{p_1} + E_{p_2} + \cdots + E_{p_n})}|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle \\ &= a^\dagger(\vec{p})e^{-\beta(E_p + \vec{H})}|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle \end{aligned} \tag{1479}$$

So we have

$$e^{-\beta H} a^\dagger(\vec{p}) = a^\dagger(\vec{p})\, e^{-\beta(H+E_p)} \tag{1480}$$

Now let us get the Bose-Einstein statistics: Show that

$$\langle a^\dagger(\vec{p})a(\vec{q})\rangle_\beta = \frac{(2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{q})}{e^{\beta E_p} - 1} \tag{1481}$$

Recall from above that

$$\begin{aligned} \langle a^\dagger(\vec{p})a(\vec{q})\rangle_\beta &= \frac{\mathrm{Tr}\Big(e^{-\beta H}a^\dagger(\vec{p})a(\vec{q})\Big)}{\mathrm{Tr}(e^{-\beta H})} \\ &= \frac{\mathrm{Tr}\Big(a^\dagger(\vec{p})e^{-\beta(H+E_p)}a(\vec{q})\Big)}{\mathrm{Tr}(e^{-\beta H})} \end{aligned} \tag{1482}$$

Then we use the fact that the Trace is cyclic

$$\begin{aligned}\langle a^\dagger(\vec{p})a(\vec{q})\rangle_\beta &= \frac{\mathrm{Tr}\Big(e^{-\beta(H+E_p)}a(\vec{q})a^\dagger(\vec{p})\Big)}{\mathrm{Tr}(e^{-\beta H})} \\ &= \frac{\mathrm{Tr}\Big(e^{-\beta(H+E_p)}a^\dagger(\vec{p})a(\vec{q})\Big)}{\mathrm{Tr}(e^{-\beta H})} + (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,e^{-\beta E_p} \\ &= e^{-\beta E_p}\langle a^\dagger(\vec{p})a(\vec{q})\rangle_\beta + e^{-\beta E_p}(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\end{aligned} \tag{1483}$$

So we have

$$\langle a^\dagger(\vec{p})a(\vec{q})\rangle_\beta = \frac{(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})}{e^{\beta E_p}-1} \tag{1484}$$

In a finite volume V we have

$$\langle a^\dagger(\vec{p})a(\vec{p})\rangle_\beta = \frac{V}{e^{\beta E_p}-1} \tag{1485}$$

So

$$\frac{a^\dagger(\vec{p})}{\sqrt{V}} \tag{1486}$$

creates a particle that obeys the Bose-Einstein statistics.

2. Let us compute the analogous result for the Fermi-Dirac case. The only difference will be at the point we used the commutator in the Trace, we now should use the anti-commutator, which introduces a minus sign. That is it. So we gets

$$\langle a^\dagger(\vec{p})a(\vec{p})\rangle_\beta = \frac{V}{e^{\beta E_p}+1} \tag{1487}$$

for fermions. Can we get Maxwell-Boltzmann? Nope! This is QFT, not your grandmother's supermarket :) We do not have mixed statistics we have either fermions or bosons, (at least in 4-dimensions)

F. Covariant Quantization of the Maxwell Theory

Recall that we quantized the free Maxwell theory in the radiation gauge: $A^0 = 0,\;\; \vec{p}\cdot\vec{A}$, which allowed us to work with only the physics DOF. But this quantization is not explicitly Lorentz invariant. After quantization, one must make sure that Lorentz invariance survived (i.e., it is not anomalous). Happily Lorentz invariance survives quantization in thi theory.

Let us now quantize the theory in a covariant way (Gubta-Bleuler 1950)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1488}$$

it is clear that a naive approach would not work since

$$\Pi^0(x) = \frac{\partial\mathcal{L}}{\partial\partial_0 A_0} = 0\,! \tag{1489}$$

Let us modify our Lagrangian in such a wat that we have a non-zero $\partial_0 A^0$ term

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2}(\partial_\mu A^\mu)^2 \tag{1490}$$

where ξ is a constant parameter. Then

$$\Pi^0(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \partial_0 A_0} = \xi \partial_\mu A^\mu \tag{1491}$$

$$\Pi^i(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \partial_0 A_i} = -F^{0i} = E^i \tag{1492}$$

Note again: momentum conjugate to A_i is Π^i. We then impose

$$[A^0(t,\vec{x})\,,\,\Pi^0(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1493}$$

$$[A^i(t,\vec{x})\,,\,\Pi^j(t,\vec{y})] = -i\,\delta^{ij}\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1494}$$

We can combine these two commutation relations to

$$[A^\mu(t,\vec{x})\,,\,\Pi^\nu(t,\vec{y})] = i\,\eta^{\mu\nu}\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1495}$$

Recall that $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. To be able to do mode expansions, we need to go back to the field equaitons

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{1496}$$

which is

$$\begin{aligned} -\partial_\mu F^{\mu\nu} + \xi \partial^\nu \partial_\alpha A^\alpha = 0 \\ -\partial_\mu \partial^\mu A^\nu + (1+\xi)\partial^\nu \partial_\mu A^\mu = 0 \end{aligned} \tag{1497}$$

thus we get

$$\partial^2 A^\nu - (1+\xi)\partial^\nu \partial_\mu A^\mu = 0 \tag{1498}$$

Fermi's choice $\xi = -1$, then $\partial^2 A^\nu = 0$. Note that there is no condition on A^ν, namely we did not choose $\partial_\mu A^\mu$. So we can write the solution to $\partial^2 A^\mu = 0$ as

$$A^\mu(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2w_{\vec{p}}}}\sum_{\lambda=0}^{3}\left(\epsilon^\mu(\vec{p},\lambda)a(\vec{p},\lambda)\,e^{-ip\cdot x} + \epsilon^{\mu *}(\vec{p},\lambda)a^\dagger(\vec{p},\lambda)\,e^{ip\cdot x}\right) \tag{1499}$$

We have now all the four polarizations! Of course in order for this A^μ to be a solution, we must have $p^2 = 0$. But no condition on ϵ^μ. In any case, we have a gauge-non-invariant action, which together with Fermi's choice reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{1500}$$

Let us choose a frame $p^\mu = (p,0,0,p)$, we can choose the following polarizations:

$$\begin{aligned} \epsilon^\mu(\vec{p},0) &= (1,0,0,0) \qquad & \epsilon^\mu(\vec{p},1) &= (0,1,0,0) \\ \epsilon^\mu(\vec{p},2) &= (0,0,1,0) \qquad & \epsilon^\mu(\vec{p},3) &= (0,0,0,1) \end{aligned} \tag{1501}$$

or in a compact notation

$$\epsilon^\mu(\vec{p},\lambda) = \delta^\mu{}_\lambda \tag{1502}$$

We can of course boost this to any other Lorentz frame. Note that

$$\left.\begin{aligned} \epsilon^\mu(\vec{p},1)p_\mu = 0 \\ \epsilon^\mu(\vec{p},2)p_\mu = 0 \end{aligned}\right\} \text{These are transverse.} \tag{1503}$$

But

$$\epsilon^\mu(\vec{p},0)p_\mu \neq 0 \qquad ; \qquad \epsilon^\mu(\vec{p},3)p_\mu \neq 0 \tag{1504}$$

So $\epsilon^\mu(\vec{p},0)$ and $\epsilon^\mu(\vec{p},3)$ are not transverse, they are longitidual polarizations. Since they are not physical, at the end of the day, they should ????... To construct the Fock space, let us find the commutation relations between the a and $a^\dagger$ operators. Recall that with Fermi's choice of $\xi = -1$, we have

$$\Pi^0(t,\vec{x}) = -\partial_\mu A^\mu = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2w_{\vec{p}}}}\, i \sum_{\lambda=0}^{3}\Big(p\cdot\epsilon(\vec{p},\lambda)a(\vec{p},\lambda)\,e^{-ip\cdot x} - p\cdot\epsilon^*(\vec{p},\lambda)a^\dagger(\vec{p},\lambda)\,e^{ip\cdot x}\Big) \tag{1505}$$

We also have

$$\Pi^i(t,\vec{x}) = -\partial_0 A^i = \int \frac{d^3p}{(2\pi)^3}\, i\sqrt{\frac{w_p}{2}} \sum_{\lambda=0}^{3}\Big(\epsilon^i(\vec{p},\lambda)a(\vec{p},\lambda)\,e^{-ip\cdot x} - \epsilon^{*i}(\vec{p},\lambda)a^\dagger(\vec{p},\lambda)\,e^{ip\cdot x}\Big) \tag{1506}$$

Using the Fourier transform and commutation relations, we get

$$[a(\vec{p},\lambda)\,,\,a^\dagger(\vec{q},\lambda^{'})] = -(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\eta_{\lambda\lambda'} \tag{1507}$$

and the rest commute.

Observe that for $\lambda,\ \lambda^{'} \sim i,j$ we got what we expected, but when $\lambda = \lambda^{'} = 0$ we have the wrong sign! This wrong sign leads to *ghosts*, namely negative normed states.

Let us show this ghost behavior explicitly on 1-particle states

$$|\vec{p},\lambda\rangle = (2w_p)^{1/2}\,a^\dagger(\vec{p},\lambda)|0\rangle \tag{1508}$$

so that

$$\begin{aligned} \langle\vec{p},\lambda|\vec{p},\lambda\rangle &= 2w_p\,\langle 0|a(\vec{p},\lambda)\,a^\dagger(\vec{p},\lambda)|0\rangle \\ &= 2w_p\,\langle 0|[a(\vec{p},\lambda)\,,\,a^\dagger(\vec{p},\lambda)]|0\rangle \\ &= -2w_p\,(2\pi)^3\eta_{\lambda\lambda}\,\delta^{(3)}(0) \end{aligned} \tag{1509}$$

where there is no summation on the $\eta_{\lambda\lambda}$. Since $\eta_{00} = 1$ we have a problem for

$$\langle\vec{p},0|\vec{p},0\rangle = -2w_p\,(2\pi)^3\eta_{\lambda\lambda}\,\delta^{(3)}(0) \tag{1510}$$

So the probabilistic interpretation of the scalar product seems to fail.

We can remedy this by imposing a restriction on the Fock space. Remember that we have added $(\partial_\mu A^\mu)^2$ to the action, which in fact should be zero. So we impose

$$_1\langle\text{physical state}|\partial_\mu A^\mu|\text{physical state}\rangle_2 = 0 \tag{1511}$$

let us decompose $\partial_\mu A^\mu$ into *positive frequency* part $e^{-ip\cdot x}$ and *negative frequency* part $e^{ip\cdot x}$

$$\partial_\mu A^\mu = (\partial_\mu A^\mu)^+ + (\partial_\mu A^\mu)^- \tag{1512}$$

Since $(\partial_\mu A^\mu)^+ = (\partial_\mu A^\mu)^-$ we can define the physical states as

$$(\partial_\mu A^\mu)^+ |\text{phys}\rangle = 0 \qquad \text{or} \qquad \langle\text{phys}|(\partial_\mu A^\mu)^- = 0 \tag{1513}$$

which are *physical state condition.* Note that

$$(\partial_\mu A^\mu)^+ = -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} \sum_{\lambda=0}^{3} p_\mu \epsilon^\mu a\, e^{-ip\cdot x} \tag{1514}$$

Let us see what the physical state condition tell us? Consider a generic 1-particle state

$$|\psi\rangle = \sum_\lambda c_\lambda\, a^\dagger(\vec{p},\lambda)|0\rangle \tag{1515}$$

Choose $p^\mu = (p, 0, 0, p)$. Then

$$(\partial_\mu A^\mu)^+|\psi\rangle = -i \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2w_k}} \sum_{\lambda'=0}^{3} k_\mu \epsilon^\mu(\vec{k},\lambda')\, e^{-ik\cdot x} a(\vec{k},\lambda') \times \sum_{\lambda=0}^{3} c_\lambda\, a^\dagger(\vec{p},\lambda)|0\rangle = 0$$

Use the commutation relation and carry out the integral (1516)

$$= i \sum_{\lambda=0}^{3} p_\mu \epsilon^\mu(\vec{p},\lambda) \frac{e^{-ip\cdot x}}{\sqrt{2w_p}}\, c_\lambda|0\rangle$$

So we have

$$c_0 + c_3 = 0 \tag{1517}$$

There is no condition on $\lambda = 1$ and $\lambda = 2$ polarizations but there is a condition on the other two. So we have

$$\left.\begin{array}{l} a^\dagger(\vec{p},1)|0\rangle \\ a^\dagger(\vec{p},2)|0\rangle \end{array}\right\} \text{physical states.} \tag{1518}$$

but

$$\left.\begin{array}{l} a^\dagger(\vec{p},0)|0\rangle \\ a^\dagger(\vec{p},3)|0\rangle \end{array}\right\} \text{unphysical states.} \tag{1519}$$

Especially that fact $a^\dagger(\vec{p},0)|0\rangle$ is an unphysical state is good because its norm was -1. The last longitidual polarization also is not physical, which is also good. But note that

$$|\phi\rangle = \Big(a^\dagger(\vec{p},0) - a^\dagger(\vec{p},3)\Big)|0\rangle \tag{1520}$$

is a physical state. Note that $\langle\phi|\phi\rangle = 0$. Let us check

$$\begin{aligned} \langle\phi|\phi\rangle &= \langle 0|\Big(a^\dagger(\vec{p},0) - a^\dagger(\vec{p},3)\Big)\Big(a^\dagger(\vec{p},0) - a^\dagger(\vec{p},3)\Big)|0\rangle \\ &= \langle 0|[a(0),\, a^\dagger(0)]|0\rangle + \langle 0|[a(3),\, a^\dagger(3)]|0\rangle = 0 \end{aligned} \tag{1521}$$

Consider a generic state

$$|\psi_{\text{Transverse}}\rangle + c|\phi\rangle \tag{1522}$$

Since

$$\langle\phi|\psi_{\text{Transverse}}\rangle = 0 \tag{1523}$$

$|\phi\rangle$ plays no role at all. Physical state is defined up to a state with a vanishing norm. So we have managed to *keep Lorentz invariance*!

How about the energy and momentum in this quantization scheme? It is not difficult to show that we get (after normal ordering)

$$H = \int \frac{d^3p}{(2\pi)^3} w_{\vec{p}}(-\eta^{\lambda\lambda'})a^\dagger(\vec{p},\lambda)\, a(\vec{p},\lambda) \tag{1524}$$

Note the minus sign! For the momentum we have

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p}(-\eta^{\lambda\lambda'})a^\dagger(\vec{p},\lambda)\, a(\vec{p},\lambda) \tag{1525}$$

Between physical states the 0^{th} and 3^{rd} components will luckly cancel.

G. Massive Maxwell Theory (Proca Theory)

Experimentally the limit on the photon mass is

$$m_\gamma < 10^{-62}\ g \quad , \quad q_\gamma < 10^{-33}\ C \tag{1526}$$

The Lagragian density for the Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\gamma^2 A_\mu A^\mu \tag{1527}$$

Obviously this is not a gauge invariant theory! Field equaitons yield

$$-\partial_\mu F^{\mu\nu} - m_\gamma^2 A^\nu = 0 \qquad \Rightarrow \qquad -\partial^2 A^\nu + \partial_\mu\partial^\nu A^\mu - m_\gamma^2 A^\nu = 0 \tag{1528}$$

If $m_\gamma^2 \neq 0$ take the derivative of this equaiton with respect to ∂_ν

$$-\partial^2\partial_\nu A^\nu + \partial_\nu A^\nu - m_\gamma^2 \partial_\nu A^\nu = 0 \qquad \Rightarrow \qquad \partial_\nu A^\nu = 0 \tag{1529}$$

This is not a gauge choice! It is imposed!

$$(\partial^2 + m_\gamma^2)A^\nu = 0 \qquad \text{with} \qquad \partial_\mu A^\mu = 0 \tag{1530}$$

So we have 3 DOF. These are massive. So massive photon has a longitidual part in addition to the transverse polarizations. Let us go back to the Lagrangian and play with it a little bit.

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\gamma^2 A_\mu A^\mu \\ &= -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2}m_\gamma^2(A_0^2 - A_i A^i) \\ &= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{2}m_\gamma^2(A_0^2 - \vec{A}^2) \end{aligned} \tag{1531}$$

Take the middle term

$$\partial_\mu A_\nu \partial^\nu A^\mu = \partial^\nu(\partial_\mu A_\nu \ A^\mu) - (\partial^\nu \partial_\mu A_\nu) A^\mu = B.T. - \partial_\mu \underbrace{(\partial^\nu A_\nu)}_{0} A^\mu = 0 \tag{1532}$$

So we can drop the middle term

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}m_\gamma^2(A_0^2 - \vec{A}^2) \\ &= -\frac{1}{2}\partial_\mu A_0 \, \partial^0 A^0 + \frac{1}{2}\partial_\mu A^i \, \partial^\mu A^i + \frac{1}{2}m_\gamma^2(A_0^2 - \vec{A}^2) \end{aligned} \tag{1533}$$

Compare with

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 \tag{1534}$$

A_0 part seems to be problematic!

DIGRESSION: Quantization of massive/gauge non-invariant vector theories is tricky.In fact Higgs mechanism was invented for this purpose. Remember that $W^\pm$, Z^0 particles are highly massive!

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathcal{D}_\mu \phi)^* \mathcal{D}_\mu \phi - \frac{\lambda}{4}(\phi\phi^* - \nu^2)^2 \tag{1535}$$

where $\mathcal{D}_\mu \phi = \partial_\mu \phi - ieA_\mu \phi$. Here ϕ is complex scalar field. This theory is called the Abelian Higgs model. It works fine for super-conductivity. (ϕ refers to a Cooper-pair and $e \to 2e$ in that case.) At the level of the Lagrangian the theory has

$$\phi(x) \longrightarrow \phi^{'}(x) = e^{i\alpha(x)}\phi(x) \tag{1536}$$

$$A_\mu \longrightarrow A^{'}_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x) \tag{1537}$$

$U(1)$ symmetry. But the vacuum of the theory is *not* $U(1)$ invariant. Look at the field equaitons and above that

$$A^\mu = 0 \qquad \text{and} \qquad \phi = \nu e^{i\theta} = \text{constant} \tag{1538}$$

Now if we expand our fields about this vacuum solution, we will get sort of the small oscillations Lagrangian. That Lagrangian will have a mass term

$$e^2\nu^2 \, A_\mu^{small} A^\mu_{small} \tag{1539}$$

for the photon. This theory can be quantized!

1. *Some Additional Material*

Coherent states: As we stated before $\langle n|\vec{E}|n\rangle$, so for a given photon state $|n\rangle$ we do not get classical EM fields even if the photon number is huge! Then the natural question is how do we get the $\vec{E}$, $\vec{B}$ (classical ones) from the quantum field theory?

We can resort to the Coherent (or Glauber) states. Define a Coherent state as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \qquad \text{where} \qquad c_n = \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2}|\alpha|^2} \tag{1540}$$

where α is complex. Note that $|c_n|^2$ is the probability of finding n photons. So $\sum_{n=0}^{\infty} |c_n|^2 = 1$, as is clear. H is easy to show that

$$\langle \alpha | a | \alpha \rangle = \alpha \quad , \quad \langle \alpha | a^\dagger | \alpha \rangle = \alpha^* \tag{1541}$$

OK. Let us check $\langle \alpha | \vec{E} | \alpha \rangle$ in the radiation gauge

$$E^i = -\partial_0 A^i = i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} \sum_{\lambda=1}^{2} \left(\epsilon^i a e^{-ip\cdot x} - \epsilon^{*i} a^\dagger e^{ip\cdot x} \right) \tag{1542}$$

Let us consider a single mode, then we will have

$$\langle \alpha | \vec{E} | \alpha \rangle = i\, \epsilon^i e^{-ip\cdot x} \alpha - i\, \epsilon^{*i} \alpha^* e^{ip\cdot x} \tag{1543}$$

Choose linear polarization $\epsilon^i = \epsilon^{*i}$ and $\alpha = |\alpha| e^{i\delta}$ then

$$\langle \alpha | \vec{E} | \alpha \rangle = i\, \epsilon^i |\alpha| e^{-ip\cdot x + i\delta} - i\, \epsilon^i |\alpha| e^{ip\cdot x - i\delta} \tag{1544}$$

XIV. PERTURBATION THEORY AND FEYNMAN DIAGRAMS

We will assume that there is a small (dimensions) parameter that defines the interactions. In QED

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137} \qquad\qquad \text{at small energies} \tag{1545}$$

In Weak theory

$$\alpha_W = \frac{g_W^2}{4\pi} \simeq \frac{1}{40} \tag{1546}$$

In Strong interaction

$$\alpha_S = \frac{g_S^2}{4\pi} \simeq 1 \tag{1547}$$

so you cannot do perturbation theory at small energies.

Perturbation Theory: To preserve causality we can only allow local interactions such as $[\phi(x)]^4$ in the scalar theory. We cannot allow $\phi(x)\phi(y)$ type interactions.

So the interaction Hamiltonian will be

$$H_{int} = \int d^3x\, \mathcal{H}_{int}[\phi(x)] = -\int d^3x\, \mathcal{L}_{int}[\phi(x)] \tag{1548}$$

So we have also ruled out the derivative terms in the interactions. For example in QED we have

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e A_\mu \bar{\psi} \gamma^\mu \psi \tag{1549}$$

FIG 47 !!!!

So

$$\mathcal{L}_{int} = -eA_\mu \bar{\psi}\gamma^\mu \psi \tag{1550}$$

In a typical experiment we actually observe something like
FIG 48 !!!!

So e^2 comes out not the "e" itself. Another example of an interesting field theory is the ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1551}$$

whose classical field equation read

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \tag{1552}$$

Note that unlike the $\lambda = 0$ theory, we cannot solve this equaiton by Fourier analysis.

To canonically quantize the theory we impose the ETCR

$$[\phi(t,\vec{x})\,,\,\Pi(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1553}$$

which does not change because of $\mathcal{L} = -\frac{\lambda}{4!}\phi^4$.

Now to be able to develop perturbative techniques to study the interacting field theories, let us briefly recap the 3 different picture in QM.

In fact our main question is this:

Find a picture which allows us to evolve the operators with the free Hamiltonian (H_0) and the states with the Interacting Hamiltonian H_1

$$H = H_0 + H_1 \tag{1554}$$

where H_0 is free Hamiltonian that one can solve.

1. *DIRAC-TOMANAGA (OR THE INTERACTION) PICTURE*

Recall that

In the *Schrodinger picture*: $|\alpha, t\rangle^S$ the state ket is time-dependent but the operators are time-independent $\hat{\theta}^S(\vec{x})$.

In the *Heisenberg picture*: It is the other way around $|\alpha\rangle^H$ is time-independent and $\hat{\theta}^H(t,\vec{x})$ is time-dependent.

Note that the base kets in the Schrodinger picture are time-independent but they are time-dependent in the Heisenberg picture. This point is often missed. Time dependence of the base kets in the Heisenberg picture follows opposite to the Schrodinger picture state ket time-dependence.

Heisenberg picture is more suitable in QFT and in fact it is also more close to the classical physics.

Assume no time-dependence of the total Hamiltonian, then it is picture independent.

$$i\,\partial_t\hat{\theta}^H(t) = [\hat{\theta}^H(t), \hat{H}] \tag{1555}$$

or we can solve this equaiton as

$$\hat{\theta}^H(t) = e^{i\hat{H}t}\hat{\theta}^H(0)e^{-i\hat{H}t} \tag{1556}$$

$$|\alpha, t\rangle^H = |\alpha, 0\rangle^H \equiv |\alpha\rangle^H \tag{1557}$$

We can take $\hat{\theta}^S = \hat{\theta}^H(0)$, therefore

$$\hat{\theta}^S = e^{-i\hat{H}t}\hat{\theta}^H(t)e^{i\hat{H}t} \qquad \Rightarrow \qquad |\alpha, t\rangle^S = e^{-iHt}|\alpha\rangle^H \tag{1558}$$

Note that the change/transformation between the Schrodinger and the Heisenberg pictures is a canonical transformation that keeps the commutation relations intact

$$\text{If} \quad [A^S,\, B^S]_\pm = c^S \quad \text{then} \quad [A^H,\, B^H]_\pm = c^H \tag{1559}$$

Dirac-Tomanaga Picture:

$H = H_0 + H_1 \rightarrow$ time-independent. But $H_0,\ H_1$ seperately need not be so! $|\alpha, t\rangle^I,\ ,\ \hat{\theta}^I(t)$ are defined as follows

$$\hat{\theta}^I(t) = e^{iH_0t}\hat{\theta}^S e^{-iH_0t} \tag{1560}$$

so

$$|\alpha, t\rangle^I = e^{iH_0t}|\alpha, t\rangle^S = e^{iH_0t}e^{-iHt}|\alpha\rangle^H \tag{1561}$$

Note that H_1 and H_0 may not commute.

Observations:

1. At $t = 0$, all 3 pictures agree.
2. When $\hat{H}_1 = 0$, Heisenberg picture=Interaction picture.

$$\begin{aligned} i\partial_t|\alpha, t\rangle^I &= -H_0|\alpha, t\rangle^I + e^{iH_0t}i\,\partial_t|\alpha, t\rangle^S \\ &= -H_0|\alpha, t\rangle^I + \underbrace{e^{iH_0t}He^{-iH_0t}}_{H^I}e^{iH_0t}|\alpha, t\rangle^S \\ &= -H_0|\alpha, t\rangle^I + H^I|\alpha, t\rangle^I \end{aligned} \tag{1562}$$

So we get

$$i\partial_t|\alpha, t\rangle^I = H_1^I|\alpha, t\rangle^I \tag{1563}$$

$$i\partial_t\hat{\theta}^I(t) = [\hat{\theta}^I(t)\,,\, H_0] \tag{1564}$$

Take a state at T_i in the Schrodinger picture $|a, T^i\rangle$, where a is a collective index for the set of all commuting observables. This state evolves to

$$|a,t\rangle = e^{-iH(t-T_i)}|a,T_i\rangle \tag{1565}$$

The amplitude for this state to evolve to $|b, T_f\rangle$ is

$$\langle b,T_f|e^{-iH(T_f-T_i)}|a,T_i\rangle \tag{1566}$$

This quantity for $T_f - T_i \to \infty$ and H be a second quantized Hamiltonian is called the *S-matrix*. Then we have an operator S that maps the initial state $|a, T_i\rangle$ to $S|a, T_i\rangle$. Let us show that S is unitary. Say $\langle a|a\rangle = 1$ (I dropped T_i) and say $|n\rangle$ is a complete set of states. Then

$$\sum_n \Big|\langle n|S|a\rangle\Big|^2 = 1 = \sum_n \langle n|S|a\rangle\langle a|S|n\rangle = \langle a|S^+S|a\rangle = 1 \tag{1567}$$

so $S^+S = 1$. So the unitarity of the S-matrix is related to the conservation of probability. Let us define the T-matrix

$$S = 1 + iT \quad ; \quad SS^+ = 1 \tag{1568}$$

yields

$$-i(T - T^+) = TT^+ \tag{1569}$$

Let $T_{ba} = \langle b|T|a\rangle$, so

$$-i(T_{ba} - T^*_{ab}) = \sum_n T_{bn}T^*_{an} \tag{1570}$$

If $a = b$, then

$$2\mathrm{Im}T_{aa} = \sum_n |T_{an}|^2 \tag{1571}$$

which is a direct consequence of unitarity.

LSZ Reduction Formula (Lehmann-Symenzik-Zimmerman 1955)

$$S_{ab} = \langle b|e^{-iH(T_f-T_i)}|a\rangle \tag{1572}$$

use the elements of the S-matrix in the Schrodinger picture. In the Heisenberg picture, we just have

$$S_{ab} = {}^H\langle b,T_f|a,T_i\rangle^H \tag{1573}$$

Note $|a, T_i\rangle^H$ are time-independent but we must still denote τ_i, since that refers to the operator to which $|a, \tau^i\rangle^H$ could be an eigenfunctions.

Let us consider massive scalar fields for notational simplicity

$$S_{fi} = \langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f|\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_m; T_i\rangle \tag{1574}$$

We would like to write this as

$$\langle 0|\text{operator}|0\rangle \tag{1575}$$

Recall the free scalar field

$$\phi^{free}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p}) e^{-ip\cdot x} + a^\dagger(\vec{p}) e^{ip\cdot x} \right) \tag{1576}$$

Note that

$$[a(\vec{p})\,,\, a^\dagger(\vec{q})] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}) \tag{1577}$$

As we found before

$$(2E_p)^{1/2} a(\vec{p}) = i \int d^3x\, e^{ip\cdot x} \overleftrightarrow{\partial}_0 \phi^{free}(x) \tag{1578}$$

$$(2E_p)^{1/2} a^\dagger(\vec{p}) = -i \int d^3x\, e^{-ip\cdot x} \overleftrightarrow{\partial}_0 \phi^{free}(x) \tag{1579}$$

where $f\overleftrightarrow{\partial}g = f\partial g - (\partial f)g$. Note that the integrands depend on time but the integral is time-independent.

FIG 49 !!!!

As $t \to -\infty$ we expect

$$\phi(x) \longrightarrow Z^{1/2} \phi_{in}(x) \tag{1580}$$

where ϕ_{in} is a free field and Z is a c number, called wave function renormalization.We also assume that as $t \to \infty$

$$\phi(x) \longrightarrow Z^{1/2} \phi_{out}(x) \tag{1581}$$

We will see how Z will be computed. Note also that the limits are correct within matrix elements, not as operators.

Since the a and $a^\dagger$ are time-independent, we will compute them as follows

$$\begin{aligned} (2E_p)^{1/2} a^{+in}(\vec{p}) &= -i \int_{t\to-\infty} d^3x \;\; e^{-ip\cdot x} \overleftrightarrow{\partial}_0 \phi_{in} \\ &= -iZ^{-1/2} \lim_{t\to-\infty} \int d^3x\, e^{-ip\cdot x} \overleftrightarrow{\partial}_0 \phi_{in} \end{aligned} \tag{1582}$$

Note that $a^{+in}(\vec{p})$ acts on the space of initial states at $T_i = -\infty$. Similarly we have

$$(2E_p)^{1/2} a^{+out}(\vec{p}) = -iZ^{-1/2} \lim_{t\to+\infty} \int d^3x\, e^{-ip\cdot x} \overleftrightarrow{\partial}_0 \phi_{in} \tag{1583}$$

So we actually have the same integral evaluated on the right-hand side. The integrand and the integral is time dependent, in conrast to the free field case.

$$\begin{aligned} &\langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f | \vec{k}_1, \vec{k}_2, \cdots, \vec{k}_m; T_i \rangle \\ &= (2E_{\vec{k}_1})^{1/2} \langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f | a^{+in}(\vec{k}_1) | \vec{k}_2, \cdots, \vec{k}_m; T_i \rangle \\ &= -iZ^{-1/2} \lim_{t\to-\infty} \int d^3x\, e^{-ik_1\cdot x} \langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f | \overleftrightarrow{\partial}_0 \phi | \vec{k}_2, \cdots, \vec{k}_m; T_i \rangle \end{aligned} \tag{1584}$$

Let us use

$$(\lim_{t\to\infty} - \lim_{t\to-\infty}) \int d^3x f(t,\vec{x}) = \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} \int d^3x f(t,\vec{x}) \tag{1585}$$

Let us apply this to

$$f(t,\vec{x}) = -iZ^{-1/2} e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \phi \tag{1586}$$

then

$$\begin{aligned}
(2E_k)^{1/2}\Big(a^{+in}(\vec{k}) - a^{+out}(\vec{k})\Big) &= (\lim_{t\to\infty} - \lim_{t\to-\infty})\, iZ^{-1/2} \int d^3x\, e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \phi \\
&= iZ^{-1/2} \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} \int d^3x\, e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \phi \\
&= iZ^{-1/2} \int d^4x\, \partial_0 \Big(e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \phi\Big) \\
&= iZ^{-1/2} \int d^4x\, \partial_0 \Big(e^{-ik\cdot x} \partial_0 \phi - \phi \partial_0 e^{-ik\cdot x}\Big) \\
&= iZ^{-1/2} \int d^4x\, \Big(e^{-ik\cdot x} \partial_0^2 \phi + \phi\, k_0^2\, e^{-ik\cdot x}\Big) \\
&= iZ^{-1/2} \int d^4x\, \Big(e^{-ik\cdot x} \partial_0^2 \phi - \phi (\nabla^2 - m^2)\, e^{-ik\cdot x}\Big)
\end{aligned} \tag{1587}$$

Note that

$$k_0^2 - \vec{k}^2 = m^2 \quad \Rightarrow \quad k_0^2 = \vec{k}^2 + m^2 = -\vec{\nabla}^2 + m^2 \tag{1588}$$

ϕ is localized in 3 space but not time. So integrate $\phi\nabla^2$ term twice to get

$$(2E_k)^{1/2}\Big(a^{+in}(\vec{k}) - a^{+out}(\vec{k})\Big) = iZ^{-1/2} \int d^4x\, e^{-ik\cdot x} \Big(\Box + m^2\Big) \phi(x) \tag{1589}$$

So we have

$$\begin{aligned}
&(2E_{k_1})^{1/2} \langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f| \Big(a^{+in}(\vec{k}_1) - a^{+out}(\vec{k}_1)\Big) |\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle \\
&= iZ^{-1/2} \int d^4x\, e^{-ik\cdot x} \Big(\Box + m^2\Big) \langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f| \phi(x) |\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle
\end{aligned} \tag{1590}$$

Note that

$$a^{+out}(\vec{k}_1) \qquad \text{acts on} \qquad \langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f| \tag{1591}$$

eliminating a particle with momentum $\vec{k}_1$. We assume that there is no such particle; namely there is no spectator in the process. We eleminate spectator.

FIG 50 !!!!

In the language of Feynmann diagrams, we are only looking at the *connected* diagrams. So then here is what we have gotten.

$$\begin{aligned}
&\langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f| \vec{k}_1, \vec{k}_2, \cdots, \vec{k}_m; T_i\rangle \\
&= iZ^{-1/2} \int d^4x\, e^{-ik\cdot x} \Big(\Box + m^2\Big) \langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f| \phi(x) |\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle
\end{aligned} \tag{1592}$$

Now let us try to remove on of the final particles

$$\begin{aligned}&\langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f|\phi(x)|\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle \\ &= (2E_{p_1})^{1/2}\langle \vec{p}_2, \vec{p}_3, \cdots, \vec{p}_n; T_f|a^{out}(\vec{p}_1)\phi(x)|\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle\end{aligned} \tag{1593}$$

Recalling the time ordering operator

$$T\{\phi(x)\phi(y)\} = \begin{cases}\phi(y)\phi(x) & y^0 > x^0 \\ \phi(x)\phi(y) & \text{if } x^0 > y^0\end{cases}$$

And the fact that

$$T\{a^{in}(\vec{p}_1)\phi(x)\} = \phi(x)a^{in}(\vec{p}_1) \tag{1594}$$

$$T\{a^{out}(\vec{p}_1)\phi(x)\} = a^{out}(\vec{p}_1)\phi(x) \tag{1595}$$

We have

$$\begin{aligned}&(2E_{p_1})^{1/2}\langle \vec{p}_2, \cdots, \vec{p}_n; T_f|a^{out}(\vec{p}_1)\phi(x)|\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle \\ &= (2E_{p_1})^{1/2}\langle \vec{p}_2, \cdots, \vec{p}_n; T_f|T\{a^{out}(\vec{p}_1) - a^{in}(\vec{p}_1)\}\phi(x)|\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle\end{aligned} \tag{1596}$$

We can now write

$$(2E_{p_1})^{1/2}\Big(a^{out}(\vec{p}_1) - a^{in}(\vec{p}_1)\Big) = iZ^{-1/2}\int d^4y\, e^{ip_1\cdot y}(\Box_y + m^2)\phi(y) \tag{1597}$$

So we have shown that

$$\begin{aligned}&\langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f|\phi(x)|\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle \\ &= iZ^{-1/2}\int d^4y\, e^{ip_1\cdot y}(\Box_y + m^2)\langle \vec{p}_2, \cdots, \vec{p}_n; T_f|T\{\phi(x)\phi(y)\}|\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle\end{aligned} \tag{1598}$$

Note that we have made a mistake in pulling out $\Box_y$ out of T product. But this does not cost as much.

So after removing 1-particle from the incoming and 1-particle from the outgoing beam, we have arrive at

$$\begin{aligned}\langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f|\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_m; T_i\rangle &= (iZ^{-1/2})^2\int d^4x\, e^{-ik_1\cdot x}(\Box_x + m^2)\int d^4y\, e^{ip_1\cdot y}(\Box_y + m^2) \\ &\times \langle \vec{p}_2, \cdots, \vec{p}_n; T_f|T\{\phi(y)\phi(x)\}|\vec{k}_2, \cdots, \vec{k}_m; T_i\rangle\end{aligned} \tag{1599}$$

We can continoue this procedure to remove all the particles to gets

$$\begin{aligned}\langle \vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n; T_f|\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_m; T_i\rangle &= (iZ^{-1/2})^{n+m}\int \prod_{i=1}^{m} d^4x_i \prod_{j=1}^{n} d^4y_j\, e^{\left[i\sum\limits_{j=1}^{n} p_j y_j - i\sum\limits_{i=1}^{m} k_i x_i\right]} \\ &\times (\Box_{x_1} + m^2)\cdots(\Box_{y_n} + m^2)\,\langle 0|T\{\underbrace{\phi(x_1)\cdots\phi(y_n)}_{m+n \text{ fields}}\}|0\rangle\end{aligned} \tag{1600}$$

In the Heisenberg picture, we have

$$\langle \vec{p}_1, \cdots, \vec{p}_n; T_f|\vec{k}_1, \cdots, \vec{k}_m; T_i\rangle \tag{1601}$$

In the Schrodinger picture

$$\langle \vec{p}_1, \cdots, \vec{p}_n | S | \vec{k}_1, \cdots, \vec{k}_m \rangle \tag{1602}$$

Recall $S \equiv 1 + iT$. Since

$$\langle \vec{p}_1, \cdots, \vec{p}_n | \vec{k}_1, \cdots, \vec{k}_m \rangle = 0 \tag{1603}$$

Non-trivial matrix element of the S-matrix is

$$\begin{aligned} \langle \vec{p}_1, \cdots, \vec{p}_n | iT | \vec{k}_1, \cdots, \vec{k}_m \rangle = (iZ^{-1/2})^{n+m} \int \prod_{i=1}^{m} d^4 x_i \prod_{j=1}^{n} d^4 y_j \, e^{\left[i \sum\limits_{j=1}^{n} p_j y_j - i \sum\limits_{i=1}^{m} k_i x_i \right]} \\ \times (\Box_{x_1} + m^2) \cdots (\Box_{y_n} + m^2) \, \langle 0 | T\{\phi(x_1) \cdots \phi(y_n)\} | 0 \rangle \end{aligned} \tag{1604}$$

N-point Green's function is defined as

$$G(x_1 \cdots x_N) \equiv \langle 0 | T\{\phi(x_1) \cdots \phi(x_N)\} | 0 \rangle \tag{1605}$$

We can define Fourier transform of it as

$$G(x_1 \cdots x_N) = \int \prod_{i=1}^{N} \frac{d^4 k_i}{(2\pi)^4} \, e^{-i \sum\limits_{i=1}^{N} x_i k_i} \, \tilde{G}(k_1 \cdots k_N) \tag{1606}$$

Then

$$(\Box_{x_j} + m^2) G(x_1 \cdots x_N) = - \int \prod_{i=1}^{N} \frac{d^4 k_i}{(2\pi)^4} \, (k_j^2 - m^2) \, e^{-i \sum\limits_{i=1}^{N} x_i k_i} \, \tilde{G}(k_1 \cdots k_N) \tag{1607}$$

Then we can summarize the LSZ reduction formula as

$$\begin{aligned} \prod_{i=1}^{m} \int d^4 x_i e^{-ik_i x_i} \prod_{j=1}^{n} \int d^4 y_j e^{ip_j y_j} \langle 0 | T\{\phi(x_1) \cdots \phi(x_m) \phi(y_1) \cdots \phi(y_n)\} | 0 \rangle \\ = \left(\prod_{i=1}^{m} \frac{i\sqrt{Z}}{k_i^2 - m^2} \right) \left(\prod_{j=1}^{n} \frac{i\sqrt{Z}}{p_j^2 - m^2} \right) \langle \vec{p}_1, \cdots, \vec{p}_n | iT | \vec{k}_1, \cdots, \vec{k}_m \rangle \end{aligned} \tag{1608}$$

We have managed to write the scattering amplitude in terms of vacuum expectation values of the fields. On mass-shell, there are poles on the right-hand side, but the left-hand side will develop similar poles and they will cancel.

3. *PERTURBATIVE EXPANSION*

We need to find a way to compute the vacuum expectation value of the time-ordered products of the quantum fields. This gives us the elements of the scattering matrix a la the LSZ formula.

The full Heisenberg quantum field $\phi(t, \vec{x})$ is in general not of the plane wave form in the interacting theory. So we define the interacting picture field $\phi_I(t, \vec{x})$ which evolves with H_0

$$\phi_I(t, \vec{x}) = e^{iH_0(t-t_0)} \phi_I(t_0, \vec{x}) e^{-iH_0(t-t_0)} \tag{1609}$$

It is free field, so it can be expanded as

$$\phi_I(t, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(a(\vec{p}) e^{-ip \cdot x} + a^\dagger(\vec{p}) e^{ip \cdot x} \Big) \tag{1610}$$

We can express ϕ in terms of $\phi_I(t,\vec{x})$

$$\begin{aligned}\phi(t,\vec{x}) &= e^{iH(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\end{aligned} \tag{1611}$$

Say we have $\phi(t_0,\vec{x}) = \phi_I(t_0,\vec{x})$, then

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(t,\vec{x})e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{1612}$$

Define the time evolution operator

$$U(t,t_0) \equiv e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{1613}$$

then

$$\phi(t,\vec{x}) = U^+(t,t_0)\phi_I(t,\vec{x})U(t,t_0) \tag{1614}$$

Note: Since $[H_0,H] \neq 0$ in general we cannot combine the exponentials in $U(t,t_0)$. Now

$$\begin{aligned}i\frac{\partial U}{\partial t} &= e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{int}e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}U(t,t_0)\end{aligned} \tag{1615}$$

Define the interaction picture Hamiltonian as

$$H_I(t) \equiv e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)} \tag{1616}$$

Then

$$i\frac{\partial U}{\partial t} = H_I(t)U(t,t_0) \tag{1617}$$

Given the initial condition $U(t_0,t_0) = 1$, we have

$$U(t,t_0) = T\Big\{exp[-i\int_{t_0}^{t} dt' H_I(t')]\Big\} \tag{1618}$$

See the appendix of these notes on how to get thi properly. Recall that the n-point Green's function was

$$G(x_1,x_2,\cdots x_n) = \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle \tag{1619}$$

Let us assume that $t_1 > t_2 > \cdots > t_{n-1} > t_n$, then

$$\begin{aligned}&\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \\ &= \langle 0|U^+(t_1,t_0)\phi_I(x_1)U(t_1,t_0)U^+(t_2,t_0)\phi_I(x_2)U(t_2,t_0)\cdots U^+(t_n,t_0)\phi_I(x_n)U(t_n,t_0)|0\rangle\end{aligned} \tag{1620}$$

Note that $U^+(t_2,t_0) = U(t_0,t_2)$ and U's can be compossed as

$$U(t_1,t_0)U(t_0,t_2) = U(t_1,t_2) \qquad \text{etc.} \tag{1621}$$

Then we have

$$\langle 0|U^{+}(t_1,t_0)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)U(t_2,t_3)\cdots U(t_{n-1},t_n)\phi_I(x_n)U(t_n,t_0)|0\rangle \tag{1622}$$

Introduce a very large t such that

$$t \gg t_1 \quad \text{and} \quad t_n \gg -t \tag{1623}$$

Then we have

$$\langle 0|U^{+}(t,t_0)U(t,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)\cdots U(t_{n-1},t_n)\phi_I(x_n)U(t_n,-t)U(-t,t_0)|0\rangle \tag{1624}$$

I combine the term which are already time-ordered as

$$= \langle 0|U^{+}(t,t_0)\, T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\, e^{[-i\int_{-t}^{t} H_I(t^{'})dt^{'}]}\}\, U(-t,t_0)|0\rangle \tag{1625}$$

Now let $t_0 = -t$ and $t \to -\infty$. Then $U(-\infty,-\infty) = 1$ but we also have $U^{+}(\infty,-\infty)$. What is the meaning of

$$\langle 0|U^{+}(\infty,-\infty)\ ? \tag{1626}$$

This is the conjugate of

$$U(\infty,-\infty)|0\rangle \tag{1627}$$

That is the physical evolution of the vacuum. If the vacuum is stable, then this evolution will lead to a simple phase change

$$U(\infty,-\infty)|0\rangle = e^{i\alpha}|0\rangle \tag{1628}$$

which provides

$$\begin{aligned} e^{i\alpha} &= \langle 0|U(\infty,-\infty)|0\rangle \\ &= \langle 0|T\{e^{[-i\int_{-\infty}^{\infty} dt^{'}\, H_I(t^{'})]}\}|0\rangle \end{aligned} \tag{1629}$$

And

$$\langle 0|U^{+}(\infty,-\infty) = e^{-i\alpha}\langle 0| \tag{1630}$$

Thus we get for the n-point Green's function

$$\langle 0|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|0\rangle = \frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\, e^{-i\int d^4x\,\mathcal{H}_I}\}|0\rangle}{\langle 0|T\{e^{[-i\int d^4x\,\mathcal{H}_I}\}|0\rangle} \tag{1631}$$

So we have found a way to compute the n-point Green's function, which enters the LSZ reduction formula, in terms of *the free field* ϕ_I.

Example: Take $H_{int} = \frac{\lambda}{4!}\phi^4$ then

$$\begin{aligned} H_I(t) &= \underbrace{e^{iH_0(t-t_0)}}_{U_0}\frac{\lambda}{4!}\phi^4\underbrace{e^{-iH_0(t-t_0)}}_{U_0^{-1}} \\ &= \frac{\lambda}{4!}U_0\phi U_0^{-1}U_0\phi U_0^{-1}U_0\phi U_0^{-1}U_0\phi U_0^{-1} = \frac{\lambda}{4!}\phi_I^4 \end{aligned} \tag{1632}$$

Using this we will we will develop a perturbation theory.

XV. APPENDIX

Let us give a somewhat detailed derivation of the Dyson's time-ordered exponential. We would like to solve

$$i\partial_t U(t,t_0) = H_I(t)\,U(t,t_0) \quad \text{where} \quad U(t_0,t_0) = 1 \tag{1633}$$

Clearly, we can convert into an integral equaiton

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t^{'})\,U(t^{'},t_0)dt^{'} \tag{1634}$$

This is a Voltera type 1^{st} kind integral equaiton. (Voltera: limit of the integral is not fixed. 1^{st} kind means unknown is also out of the integral sign. It is also a non-homogenous equaiton because of the "1". It is a linear equaiton.)

Assuming that $H_I(t)$ is in some sense small, or there is a small parameter, we can use the Neuman-Livouille series.

The first correction is

$$U^{(1)}(t,t_0) \simeq 1 - i\int_{t_0}^{t} H_I(t^{'})\,dt^{'} \tag{1635}$$

which is the Born-approximation. The second correction is

$$U^{(2)} \simeq 1 - i\int_{t_0}^{t} H_I(t^{'})\,dt^{'} + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} H_I(t_1)H_I(t_2)\,dt_2 \tag{1636}$$

So n^{th} order approximation is

$$U^{(n)}(t,t_0) \simeq 1 - i\int_{t_0}^{t} H_I(t^{'})\,dt^{'} + (-i)^n\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots H_I(t_1)H_I(t_2)\cdots H_I(t_n) \tag{1637}$$

This is the formal solution, if it gives a convergent result, then we really have the solution. If it does not converge as $n \to \infty$, then we are in trouble.

There is a better way to write this messy looking solution: consider the second term

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) \tag{1638}$$

FIG 51 !!!!

We can cover the same region of integration as

$$\int_{t_0}^{t} dt_2 \int_{t_1}^{t} dt_2 H_I(t_1)H_I(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I(t_2)H_I(t_1) \tag{1639}$$

where we let $t_2 \to t_1$ and $t_1 \to t_2$. Then we have

$$\begin{aligned} 2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I(t_2)H_I(t_1) \\ &= \int_{t_0}^{t} dt_1 \Big\{ \underbrace{\int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)}_{t_1>t_2} + \underbrace{\int_{t_1}^{t} dt_2 H_I(t_2)H_I(t_1)}_{t_2>t_1} \Big\} \\ &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\{H_I(t_1)H_I(t_2)\} \end{aligned} \tag{1640}$$

Let us check that this is indeed correct

$$\begin{aligned}&\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left\{\theta(t_1-t_2)H_I(t_1)H_I(t_2)+\theta(t_2-t_1)H_I(t_2)H_I(t_1)\right\}\\&=\int_{t_0}^{t} dt_1 \Big\{ \int_{t_0}^{t_1} dt_2\, H_I(t_1)H_I(t_2)+\int_{t_1}^{t} dt_2\, H_I(t_2)H_I(t_1)\Big\}\end{aligned} \tag{1641}$$

In general we have

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1)\cdots H_I(t_n)=\frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\, T\{H_I(t_1)\cdots H_I(t_n)\} \tag{1642}$$

Then we have

$$\begin{aligned}U(t,t_0)&=\sum_{n=0}^{\infty}\frac{1}{n!}(-i)^n\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t} dt_n\, T\{H_I(t_1)\cdots H_I(t_n)\}\\&\equiv Te^{-i\int_{t_0}^{t} dt'\, H_I(t')}\end{aligned} \tag{1643}$$

Thus we get

$$U(t,t_0)=Te^{-i\int_{t_0}^{t} d^4x'\, \mathcal{H}_I(x')} \tag{1644}$$

where $\mathcal{H}_I(x')$ is Hamiltonian density. In particular elements of S-matrix one

$$S_{fi}\equiv \lim_{\substack{t_2\to\infty\\ t_1\to-\infty}}\langle\phi_f|U(t_2,t_1)|\phi_i\rangle \tag{1645}$$

So these are amplitudes for a process in which the system makes a transition from an initial state to a final state. Thus

$$S=U(\infty,-\infty)=Te^{-i\int_{t_0}^{t} d^4x'\, \mathcal{H}_I(x')} \tag{1646}$$

A. Instantons and Tunneling:

Many field theories have solutions that vanish at infinity, which are not just $\phi=0$ everywhere.
a. Example:

$$S_E=\int d^Dx\left[\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi+V(\varphi)\right] \tag{1647}$$

with

$$V(\varphi)=\frac{m^2}{2}\varphi^2\left(1-\frac{\varphi}{v}\right)^2 \tag{1648}$$

so when expanded we have φ^2, φ^3, φ^4 terms. For $D=1$ we have

$$S_E=\int dt\left[\frac{1}{2}(\partial_t\varphi)^2+V(\varphi)\right] \tag{1649}$$

which is Euclidean quantum mechanics $\varphi(t)=q(t)$. $V(\varphi)$ has two perturbative minima

$$\varphi=0,\, V(0)=0 \tag{1650}$$

$$\varphi=v,\, V(v)=0. \tag{1651}$$

We can define the perturbation theory as an expansion around any one of these.

Let us choose $\varphi = 0$ as our vacuum and consider field configurations $\varphi(t)$ in such a way that

$$\varphi(t \to \pm\infty) = 0. \tag{1652}$$

Q: Are there classical solutions with these boundary conditions? Clearly there is the obvious one $\varphi(t) = 0$ for all t. Minkowski version of the theory is we have

$$\varphi(t \to -\infty) = 0 \tag{1653}$$
$$\varphi(t \to \infty) = v \tag{1654}$$

solutions which just need

$$\frac{d\varphi}{dt}|_{t\to-\infty} > 0. \tag{1655}$$

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Since $\varphi = 0$, $\varphi = v$ are the two degenerate vacua, these solutions correspond to tunneling between vacua. These are called instantons.

But remember that we need $\varphi(t \to \infty) = 0$ so we actually need an instanton (that goes from $\varphi = 0$ to $\varphi = v$) and an anti-instanton (that goes from $\varphi = v$ to $\varphi = 0$) pair $(I\,\bar{I})$.

Then we can aproximate the PI as a sum of two terms: the contributions of the small fluctuations around the perturbative vacuum $\varphi = 0$, which gives the perturbative expansion. And the contribution of the small fluctuations about the $I\,\bar{I}$ pair, which gives a non-perturbative contribution.

Say $\varphi_0(t)$ be the $I\,\bar{I}$ solution. They consider a small fluctuation about $\varphi_0(t)$.

$$\varphi = \varphi_0(t) + \delta\varphi, \tag{1656}$$

$$S[\varphi] = S[\varphi_0] + S_2[\delta\varphi] + S_3[\delta\varphi] \tag{1657}$$

where $S_2[\delta\varphi]$ is quadratic in $\delta\varphi$. Since φ_0 satisfies the classical equations of motion, there is no linear term in $\delta\varphi$.

So we have

$$e^{-S[\varphi_0]} \int D\delta\varphi\, e^{-(S_2[\delta\varphi]+S_3[\delta\varphi])} \tag{1658}$$

where $S[\varphi_0]$ is the action of $I\,\bar{I}$. In our example $S_{I\,\bar{I}} = S_I + S_{\bar{I}} = 2S_I$.

$$\ddot{\varphi}_0 = m^2\varphi\left(1 - \frac{\varphi}{v}\right)\left(1 - 2\frac{\varphi}{v}\right) \tag{1659}$$

has the instanton solution

$$\varphi(t) = \frac{v}{1 + \exp\{-mt\}} \tag{1660}$$

where

$$\varphi(t \to -\infty) = 0 \tag{1661}$$
$$\varphi(t \to \infty) = v. \tag{1662}$$

Then we have

$$S_I = \frac{mv^2}{6}. \quad (1663)$$

So we have

$$e^{-2S_I} = \exp\left\{-\frac{4m^3}{\lambda}\right\} \quad (1664)$$

where

$$\frac{\lambda}{4!} = \frac{m^2}{2v^2}. \quad (1665)$$

Clearly $\exp\left\{-\frac{4m^3}{\lambda}\right\}$ will never show up in perturbation theory.

B. Vaccum Energy:

Let us apply the PI formalism for the free scalar field in the absence of sources.

$$W = \int D\varphi e^{i\int d^4x \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right]} \quad (1666)$$

$$= \int D\varphi e^{-\frac{i}{2}\int d^4x \varphi\left[\partial^2 + m^2 - i\epsilon\right]\varphi} \quad (1667)$$

$$= C\left(\det\left[\partial^2 + m^2 - i\epsilon\right]\right) \quad (1668)$$

where all the inessential constants are absorbed into C. Since $\det M = e^{\mathrm{Tr}\log M}$ we have

$$W = Ce^{-\frac{1}{2}\mathrm{Tr}\log\left[\partial^2 + m^2 - i\epsilon\right]}$$
$$= < 0|e^{-iHT}|0 >$$
$$= e^{-iHT} \quad (1669)$$

where $T \to \infty$. Hence we have

$$iET = \frac{1}{2}\mathrm{Tr}\ \log\left[\partial^2 + m^2 - i\epsilon\right] - \ln C. \quad (1670)$$

Recall that

$$\mathrm{Tr}\theta = \int d^4x < x|\theta|x >$$
$$= \int d^4x \int d^4k \int d^4q < x|k >< k|\theta|q >< q|x >$$
$$= \int d^4x \int d^4k \int d^4q \frac{1}{(2\pi)^2} e^{-ik.x} \frac{1}{(2\pi)^2} e^{iq.x} < k|\theta|q > \quad (1671)$$

$$\mathrm{Tr}\ \log\left[\partial^2 + m^2 - i\epsilon\right] = \int d^4x \int d^4k \int d^4q \frac{1}{(2\pi)^4} e^{-i(k-q).x} < k|\log\left[\partial^2 + m^2 - i\epsilon\right]|q >$$
$$= \int d^4x \int d^4k \int d^4q \frac{1}{(2\pi)^4} e^{-i(k-q).x} < k|\log\left[-q^2 + m^2 - i\epsilon\right]|q >$$
$$= \int d^4x \int d^4k \frac{1}{(2\pi)^4} \log\left[-k^2 + m^2 - i\epsilon\right] \quad (1672)$$

where we have used $< k|q >= \delta^{(4)}\left(\vec{k}-\vec{q}\right)$. Then we havem

$$iET = \frac{1}{2}VT\int\frac{d^4k}{(2\pi)^4}\log\left[-k^2+m^2-i\epsilon\right]-\ln C, \tag{1673}$$

observe that $\log\left(\text{mass}^2\right)$ does not make sense. Choose $\ln C$ in such a way that we get

$$iE = \frac{1}{2}V\int\frac{d^4k}{(2\pi)^4}\log\left[\frac{k^2-m^2+i\epsilon}{k^2-M^2+i\epsilon}\right], \tag{1674}$$

which actually makes sense since we are now comparing the vacuum energy of two different mass particles; m and M.

$$\frac{E}{V} = \frac{-i}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\log\left[\frac{\omega^2-\omega_m^2+i\epsilon}{\omega^2-\omega_M^2+i\epsilon}\right], \tag{1675}$$

where $\omega_m \equiv +\sqrt{\vec{k}^2+m^2}$ and $\omega_M \equiv +\sqrt{\vec{k}^2+M^2}$.

Recall that earlier (on page 146) we derived

$$\int\frac{d^Dp}{(2\pi)^D}\log\left[\frac{p^2+2p.q+\Delta}{\mu^2}\right] = \frac{2}{D}\frac{1}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\left(\Delta-q^2\right)^{\frac{D}{2}}. \tag{1676}$$

Let use this for the ω integral

$$\int\frac{d\omega}{2\pi}\log\left[\frac{\omega^2-\omega_m^2}{\mu}\right] = \frac{2}{\sqrt{4\pi}}\Gamma\left(\frac{1}{2}\right)\left(-\omega_m^2\right)^{\frac{1}{2}} \tag{1677}$$

where $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, so

$$\int\frac{d\omega}{2\pi}\log\left[\frac{\omega^2-\omega_m^2}{\mu}\right] = i\omega_m. \tag{1678}$$

Then we get

$$\frac{E}{V} = \int\frac{d^3k}{(2\pi)^3}\left(\frac{1}{2}\hbar\omega_m-\frac{1}{2}\hbar\omega_M\right), \tag{1679}$$

where we have restored $\hbar$ and of course this result is consistent with the canonical quantization result.

From canonical quantization, we also recall that the vacuum energy density for fermions is opposite to that of bosons. Since the sign merely is reluted to the Gaussian integral

$$\int\limits_{-\infty}^{\infty}dxe^{-\frac{1}{2}\alpha x^2} = \sqrt{\frac{2\pi}{\alpha}} = \sqrt{2\pi}e^{-\frac{1}{2}\log\alpha}, \tag{1680}$$

what kind of integrals or variables should we use for fermions?

C. Grassmann Path Integral:

It is clear that PI for fermions can not be defined with the usual functions. We must incorporate the spin-statistics connection, the exclusion principle. In the computation of the vacuum energy, using canonical techniques, we saw that fermions had negative vacuum energy, albeit an infinite one. So some how we must have an integral which gives

$$e^{+\frac{1}{2}\log a} \tag{1681}$$

as opposed to

$$e^{-\frac{1}{2}\log a}. \tag{1682}$$

This is done via anti-commuting numbers called Grassmann numbers.

$$\eta\xi = -\xi\eta \quad \text{s.t.} \quad \eta^2 = 0. \tag{1683}$$

So take any function of η we have $f(\eta) = a + b\eta$.

Integration:

$$\int_{-\infty}^{\infty} dx\, f(x+c) = \int_{-\infty}^{\infty} dx\, f(x) \tag{1684}$$

is valid for ordinary functions. Assume it works for Grassmann numbers

$$\begin{aligned}\int d\eta\, f(\eta+\xi) &= \int d\eta\, f(\eta)\\ &= \int d\eta\,[a + b(\eta+\xi)]\\ &= \int d\eta\,(a+b\eta),\end{aligned}$$

where

$$\int d\eta\, b\xi = 0 \Rightarrow \int d\eta\, b = 0, \tag{1685}$$

since b is ordinary number. So

$$\int d\eta = 0. \tag{1686}$$

$\chi(\eta\xi) = (\eta\xi)\chi$ so $(\eta\xi)$ is an ordinary number since $d\eta$ and η are two Grassmannians then

$$\int d\eta\, \eta = \text{ordinary number we choose it to be } 1 \tag{1687}$$

so

$$\int d\eta = 0, \qquad \int d\eta\, \eta = 1 \tag{1688}$$

and

$$\int d\eta\, f(\eta) = \int d\eta\,[a + b\eta] = b. \tag{1689}$$

Range of integration does not exist for these numbers. Since $\frac{\partial\eta}{\partial\eta}=1$ integration and differentiation pretty much gives the same result

$$\frac{\partial f(\eta)}{\partial\eta}=b. \tag{1690}$$

Say η and $\bar{\eta}$ be two independent Grassmann numbers and a an ordinary number

$$\begin{aligned}\int d\eta\int d\bar{\eta}\, e^{\bar{\eta}a\eta}&=\int d\eta\int d\bar{\eta}\,[1+\bar{\eta}a\eta]\\ &=a=e^{+\log a},\end{aligned} \tag{1691}$$

here the order of integrands are important. So this will work for the vacuum energy considerations. In general say

$$\eta\equiv(\eta_1,\,\eta_2,\,...,\,\eta_N) \tag{1692}$$

$$\bar{\eta}\equiv(\bar{\eta}_1,\,\bar{\eta}_2,\,...,\,\bar{\eta}_N) \tag{1693}$$

$$\int d\eta\int d\bar{\eta}\, e^{\bar{\eta}a\eta}=\det A. \tag{1694}$$

1. *Grassmann PI:*

We can easily extend the scalar field PI to the fermion field PI.

$$\begin{aligned}W&=\int D\psi D\bar{\psi}e^{iS(\psi,\,\bar{\psi})}\\ &=\int D\psi D\bar{\psi}e^{i\int d^4x\,\bar{\psi}(i\gamma^\mu\partial_\mu-m+i\epsilon)\psi}\\ &=C'\det\left(i\gamma^\mu\partial_\mu-m+i\epsilon\right)\\ &=C'e^{\mathrm{tr}\,\log(i\gamma^\mu\partial_\mu-m+i\epsilon)}.\end{aligned} \tag{1695}$$

$$\begin{aligned}\mathrm{tr}\,\log\left(i\gamma^\mu\partial_\mu-m\right)=&\mathrm{tr}\,\log\gamma^5\left(i\gamma^\mu\partial_\mu-m\right)\gamma^5\\ =&\mathrm{tr}\,\log\left(-i\gamma^\mu\partial_\mu-m\right)\\ =&\frac{1}{2}\left[\mathrm{tr}\,\log\left(i\gamma^\mu\partial_\mu-m\right)+\mathrm{tr}\,\log\left(-i\gamma^\mu\partial_\mu-m\right)\right]\\ =&\frac{1}{2}\mathrm{tr}\,\log\left(\partial^2+m^2\right),\end{aligned} \tag{1696}$$

where we have used $\{\gamma^\mu,\,\gamma^5\}=0$. So then we have

$$W=e^{+\frac{1}{2}\mathrm{tr}\,\log\left(\partial^2+m^2\right)}, \tag{1697}$$

where tr contains a factor 4 due to the 4×4 structure of the γ-matrices.

2. *Dirac Propagator:*

Now let us introduce sources

$$W[\eta,\,\bar{\eta}]=\int D\psi D\bar{\psi}e^{i\int d^4x\left[\bar{\psi}(i\gamma^\mu\partial_\mu-m)\psi+\bar{\eta}\psi+\bar{\psi}\eta\right]}. \tag{1698}$$

We use the background field method (or complete the square, same thing)

$$\psi K \bar{\psi} + \bar{\eta}\psi + \bar{\psi}\eta = \left(\bar{\psi} + \bar{\eta}K^{-1}\right) K \left(\psi + K^{-1}\eta\right) - \bar{\eta}K^{-1}\eta \tag{1699}$$

where

$$K\psi = \eta \quad \text{is the field equation,} \tag{1700}$$

$$\psi_0 = K^{-1}\eta \quad \text{is the solution.} \tag{1701}$$

$$\bar{\psi}K = \bar{\eta}, \qquad \bar{\psi} = \bar{\eta}K^{-1}, \tag{1702}$$

so then

$$W[\eta, \bar{\eta}] = C'' e^{-i\int d^4x \left[\bar{\eta}(i\gamma^\mu \partial_\mu - m)^{-1}\eta\right]}. \tag{1703}$$

The Feynman propagator is the inverse of $i\gamma^\mu \partial_\mu - m$. So

$$(i\gamma^\mu \partial_\mu - m) S_F(x-y) = i\delta^{(4)}(x-y), \tag{1704}$$

where

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\gamma^\mu p_\mu - m} e^{-ip.(x-y)} \tag{1705}$$

as we discussed at length before.

To define the QED path integral, next we must study the PI for gauge fields. We will see that it has the added problem of gauge invariance, which we have to deal with to get rid of unwanted over counting.

D. Path Integral for Abelian Gauge Fields:

Up to now we have seen that, symbolically we have

$$\int D\varphi\, e^{-\frac{1}{2}\varphi K \varphi - V(\varphi) + J.\varphi} = e^{-V\left(\frac{\delta}{\delta J}\right)} e^{\frac{1}{2}J.K^{-1}K} \tag{1706}$$

(Zee calls this central identity of QFT). Note that φ could represent a vector containing all the fields in it.

The BIG question is what if K^{-1} does not exist? Well actually in gauge theories K^{-1}, naively, does not exist. We have discussed this many things before.

Take the Maxwell theory

$$S(A) = \int d^4x \mathcal{L} = \int d^4x \left[\frac{1}{2}A_\mu \left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\nu + A_\mu J^\mu\right] \tag{1707}$$

and define

$$K^{\mu\nu} \equiv \eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \tag{1708}$$

which does not have an inverse. Why? Because it has zero eigenvalue.

$$\left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\nu = 0 \tag{1709}$$

whenever $A_\nu = \partial_\nu \Lambda$ (a pure gauge)

$$\left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\partial_\nu\Lambda = 0 \tag{1710}$$

clearly. Since in finding the inverse of an operator, one needs the determinant of the operator, the inverse does not exist when the determinant vanishes. And the determinant vanishes whenever you have a zero eigenvalue. Hence the problem. Pure gauge fields are annihilated by the operator. What do we do?

Observe that the field equations are

$$K^{\mu\nu}A_\nu = J^\mu, \tag{1711}$$

and

$$A^\nu = \left(K^{-1}\right)^{\nu\mu} J_\mu \tag{1712}$$

but K^{-1} does not exist. Of course we need to fix the gauge. That is what we learned before. How do we that in the PI.

1. *A Simple Example:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} da\, db\, e^{-a^2} \tag{1713}$$

is divergent. Written in the form $A.K.A$ we have $A = \begin{pmatrix} a \\ b \end{pmatrix}$ and $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So

$$\int_{-\infty}^{\infty} dA\, e^{-A.K.A} \tag{1714}$$

but K^{-1} does not exist. That is the end of story as far as math is concerned. But we want to make sense of integral. Then insert a δ-function

$$\delta\left(b - \xi\right) \tag{1715}$$

to the integral which picks up a single b. Or we could insert $\delta\left[f\left(b\right)\right]$ with a generic

$$\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db\, \delta\left[f\left(b\right)\right]\, e^{-a^2}. \tag{1716}$$

This restricts the integral of b to there satisfying

$$f\left(b\right) = 0. \tag{1717}$$

2. Restricting the PI a la Faddeev-Popov:

$$I=\int DA\,e^{iS(A)} \tag{1718}$$

say we have this integral which could be a path integral ($A\to A(x)$) or an ordinary integral. Let us suppose that under the transformation

$$A\to A_g,\ S(A_g)=S(A),\ DA_g=DA \tag{1719}$$

both the measure and the integrand are invariant. The transformation forms a group. Our task is to write the PI in such a way that

$$I=\left(\int Dg\right)J \tag{1720}$$

where $\int Dg$ is the integral over the group volume and J is g-independent integral.

a. Example:

$$I=\int dx\,dy\,e^{iS(x,\,y)} \tag{1721}$$

say $S(x,\,y)=f(x^2+y^2)$. Then we go to the polar coordinates

$$x=r\cos\theta \tag{1722}$$
$$y=r\sin\theta \tag{1723}$$
$$dx\,dy=d\theta\,r\,dr \tag{1724}$$

to have

$$I=\int d\theta\int r\,dr\,e^{iS(r)} \tag{1725}$$

where $\int d\theta$ is the volume of rotations in two-dimensions and it is 2π and the rest is $J=\int r\,dr\,e^{iS(r)}$.

Faddeev-Popov basically extended this procedure to gauge theories. Let

$$1\equiv\Delta(A)\int Dg\,\delta\left[f(A_g)\right] \tag{1726}$$

which defines $\Delta(A)$. f is a function we choose.

$\Delta(A)$ is Faddeev-Popov determinant which depends on our choice f.

Here is an important point: Dg is an invariant volume element in the group space. Namely,

$$Dg=Dg'. \tag{1727}$$

Then let us first show that FD determinant is gauge invariant.

$$\begin{aligned}[\Delta(A)]^{-1}&=\int Dg\,\delta\left[f(A_g)\right]\\&=\int Dg'\,\delta\left[f(A_{g'})\right]\\&=\int Dg\,\delta\left[f(A_{g'})\right]=\left[\Delta(A_{g''})\right]^{-1}\end{aligned} \tag{1728}$$

where $g' = g''g$. QED

Then

$$I = \int DA\, e^{iS(A)} = \int DA\, e^{iS(A)} \Delta(A) \int Dg\, \delta[f(A_g)] = \int Dg \int DA\, e^{iS(A)} \Delta(A)\, \delta[f(A_g)] \tag{1729}$$

where $\Delta(A) \int Dg\, \delta[f(A_g)] = 1$. Let

$$A \to A_{g^{-1}} \tag{1730}$$

then

$$I = \left(\int Dg\right) \int DA\, e^{iS(A)} \Delta(A)\, \delta[f(A_g)] \tag{1731}$$

where $\int DA\, e^{iS(A)} \Delta(A)$ is gauge invariant. So the group integration (which is divergent in QFT) is factored out. [Note: The volume of a compact group is finite but we have a compact group at each point in space and that is divergent.]

Physically what we have done is the following, because of gauge invariance we were integrating over the gauge copies of the same physical "path". We had to fix the gauge to take one representative of the copies. We are hoping that $\delta[f(A)]$ insertion will completly fix the gauge. It turns out in Abelian theories this is no problem; the gauge can be completly fixed. In non-Abelian gauge theories, many gauge choices can not totally fix the gauge. Actually there is no choice of gauge which can eliminate all but one element of a given orbit. [Singer, not the machine but the man.]

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It so happens that in non-Abelian theories, we have the Gribov copies such as

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3. *Gauge-Fixing in Maxwell Theory:*

$$S(A) = \int d^4x \left[\frac{1}{2} A_\mu \left(\partial^2 \eta^{\mu\nu} - g^\mu g^\nu\right) A_\nu + A_\mu J^\mu\right] \tag{1732}$$

is invariant under

$$A_\mu \to A_\mu - \partial_\mu \Lambda. \tag{1733}$$

So

$$A_g = A - \partial\Lambda \tag{1734}$$

in our notation. Choose

$$f(A) = \partial.A - \sigma(x). \tag{1735}$$

Note that since $I = \int DA\, e^{iS(A)}$ is independent of f

$$I = \int Dg \int DA\, e^{iS(A)} \Delta(A)\, \delta[f(A)] \tag{1736}$$

is still independent of f. So I can choose any f I like in principle. That means when I choose $f(A) = \partial.A - \sigma(x)$, I can integrate this with any functional of $\sigma(x)$ as I like

$$[\Delta(A)]^{-1} = \int Dg\,\delta[f(A)] = \int D\Lambda\,\delta\left[\partial.A - \partial^2\Lambda - \sigma\right]. \tag{1737}$$

Since in the PI we have $\Delta(A)\,\delta[f(A)]$ which basically acts $f(A) = 0$, then

$$[\Delta(A)]^{-1} = \int D\Lambda\,\delta\left[\partial^2\Lambda\right] \tag{1738}$$

which is independent of, throw it out! So then

$$I = \int D\sigma\, e^{-\frac{i}{2\xi}\int d^4x\,\sigma(x)^2} \int DA\, e^{iS(A)}\delta(\partial A - \sigma) \tag{1739}$$

and

$$I = \int DA\, e^{iS(A) - \frac{i}{2\xi}\int d^4x\,(\partial A)^2}. \tag{1740}$$

So we basically added the action with a gauge-fixing term

$$\begin{aligned} S_{\text{eff}}(A) &= S(A) - \frac{1}{2\xi}\int d^4x\,(\partial A)^2 \\ &= \int d^4x \left\{\frac{1}{2}A_\mu K^{\mu\nu}A_\nu + A_\mu J^\mu\right\} \end{aligned} \tag{1741}$$

where

$$K^{\mu\nu} = \partial^2\eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu \tag{1742}$$

which now has an inverse and we found the inverse before!

Note: Since A_μ and $A_\mu - \partial_\mu\Lambda$ correspond to the same $\vec{E}$ and $\vec{B}$, we had to get rid of this redundant over counting. Zee thinks that "a totally deep advance in theoretical physics would involve writing down QED without A_μ's." I agree with this remark.

E. Additional Material:

The following material is a brief introduction to certain topics in QFT. Due to time constraints we shall only talk a bit out of these topics.

1. *Field Theory Without Relativity (Zee Chapters 111.5):*

Let us consider one example of a non-relativistic limit of QFT. Condensed matter applications of this procedure is obvious.

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \tag{1743}$$

where $\lambda > 0$. $\mathcal{L}$ describes interacting bosons.

$$\left(\partial^2 + m^2\right)\Phi = 0, \quad \left(\partial^2 + m^2\right)\Phi^\dagger = 0 \tag{1744}$$

are the corresponding KG equations.

Since we have discussed taking the non-relativistic limit of the free KG equation before (in 541), let us use use that. Let

$$\Phi\left(\vec{x},\,t\right)\equiv\frac{1}{\sqrt{2m}}e^{-mt}\varphi\left(\vec{x},\,t\right) \tag{1745}$$

(our logic is that $\mathcal{L}$ above should also describe slow moving bosons). Then

$$\mathcal{L}=i\varphi^{\dagger}\partial_{0}\varphi-\frac{1}{2m}\partial_{i}\varphi^{\dagger}\partial_{i}\varphi-g^{2}\left(\varphi^{\dagger}\varphi\right)^{2} \tag{1746}$$

where $g^{2}=\frac{\lambda}{4m^{2}}$.

$$\Pi=\frac{\partial\mathcal{L}}{\partial\partial_{0}\varphi}=i\varphi^{\dagger} \tag{1747}$$

so

$$\left[\varphi\left(\vec{x},\,t\right),\,\Pi\left(\vec{x}',\,t\right)\right]=i\delta^{(D)}\left(\vec{x}-\vec{x}'\right) \tag{1748}$$

$$\left[\varphi\left(\vec{x},\,t\right),\,\varphi^{\dagger}\left(\vec{x}',\,t\right)\right]=\delta^{(D)}\left(\vec{x}-\vec{x}'\right). \tag{1749}$$

Let

$$\varphi\left(\vec{x},\,t\right)=\sqrt{\rho\left(\vec{x},\,t\right)}\,e^{i\theta\left(\vec{x},\,t\right)} \tag{1750}$$

where $\rho=\varphi^{\dagger}\varphi$. Then

$$i\varphi^{\dagger}\partial_{0}\varphi=\frac{i}{2}\partial_{0}\rho-\rho\partial_{0}\theta. \tag{1751}$$

So we have

$$\mathcal{L}=\frac{i}{2}\partial_{0}\rho-\rho\partial_{0}\theta-\frac{1}{2m}\left[\rho\left(\partial_{i}\theta\right)^{2}+\frac{1}{4\rho}\left(\partial_{i}\rho\right)^{2}\right]-g^{2}\rho^{2} \tag{1752}$$

where $\frac{i}{2}\partial_{0}\rho$ is a boundary term and

$$\Pi_{\theta}=\frac{\partial\mathcal{L}}{\partial\partial_{0}\theta}=-\rho. \tag{1753}$$

Then we get

$$\left[\theta\left(\vec{x},\,t\right),\,\Pi_{\theta}\left(\vec{x}',\,t\right)\right]=i\delta^{(D)}\left(\vec{x}-\vec{x}'\right) \tag{1754}$$

$$\left[\theta\left(\vec{x},\,t\right),\,\rho\left(\vec{x}',\,t\right)\right]=-i\delta^{(D)}\left(\vec{x}-\vec{x}'\right). \tag{1755}$$

Note that $\theta\left(\vec{x},\,t\right)$ phase field is conjugate to $\rho\left(\vec{x},\,t\right)$ number density.

$$N=-\int d^{D}x'\rho\left(\vec{x},\,t\right) \tag{1756}$$

so then

$$\left[N,\,\theta\right]=i \tag{1757}$$

“number is conjugate to phase angle”.

2. The Sign of Repulsion:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 \qquad \text{relativistic} \tag{1758}$$

$$\mathcal{L} = -\rho \partial_0 \theta - \frac{1}{2m}\left[\rho \left(\partial_i \theta\right)^2 + \frac{1}{4\rho}\left(\partial_i \rho\right)^2\right] - g^2 \rho^2 \qquad \text{non-relativistic} \tag{1759}$$

in the non-relativistic case, since $\mathcal{H} \sim +g^2\rho^2$, it is clear to get a high density region, large ρ^2, we must pay the price, so these bosons are repulsive! (for $g^2 > 0$).

For the relativistic theory, again $\mathcal{H} \sim \lambda \left(\phi^\dagger \phi\right)^2$. To have a bounded potential, we have $\lambda > 0$. And we know that a free boson gas wants condense and clump, namely it wants $\phi^\dagger \phi$ to be large, that means $\lambda > 0$ is repulsive. [It sounds weired indeed but think about it three times and you will get it.] Or let us actually prove it.

Using an auxilary field $\sigma(x)$, we can rewrite the PI of (1758) as

$$W = \int D\phi\, D\sigma\, e^{i\int d^4x\left[\partial\phi^\dagger \partial\phi - m^2\phi^\dagger\phi + 2\sigma\phi^\dagger\phi + \frac{1}{\lambda}\sigma^2\right]}. \tag{1760}$$

In condensed matter physics this transformation is called the Habbard-Stratonovich transformation.

Observe that the scalar field $\sigma(x)$ does not have a kinetic term: the poor guy is frozen like the 6-0 score of FB over GS. If it did have a kinetic term it would read

$$\frac{1}{2}\left(\partial\sigma\right)^2 - \frac{1}{2}m^2\sigma^2 \tag{1761}$$

and have the propagator

$$\frac{i}{k^2 - m^2 + i\epsilon}. \tag{1762}$$

The Yukawa term is

$$2\sigma\phi^\dagger\phi \tag{1763}$$

so σ is exchanged between the two bosons. Since $\sigma(x)$ does not have a kinetic term, its propagator is

$$\frac{i}{\left(\frac{1}{\lambda}\right)} = i\lambda$$

and for $\lambda > 0$ it has the opposite sign for low momentum propagator

$$\frac{i}{k^2 - m^2 + i\epsilon} \sim -\frac{i}{m^2}. \tag{1764}$$

So then σ-exchange leads to a repulsive force. Hence, $\lambda\left(\phi^\dagger\phi\right)^2$ gives a repulsive interaction between the Bose fields.

Also observe that the repulsion is infinitely since $\frac{1}{k^2 - m^2}$ which leads to $-\frac{1}{r}e^{-mr}$ attractive would be infinitely short ranged (δ-function) as $m \to \infty$ same for $i\lambda$ interaction: hard-core repulsion.

F. Non-Abelian Gauge Theories:

First let us recap group theory. (Zee Appendix B is sufficient.)

1. Group Theory:

$SO(N)$: Special orthogonal group; $N \times N$ real matrices $\mathcal{O}$ that are orthogonal

$$\mathcal{O}^T\mathcal{O} = 1, \qquad \det \mathcal{O} = 1 = \varepsilon^{i_1,i_2\ldots i_N}\theta^{i_1 1}\theta^{i_2 2}\ldots\theta^{i_N N}, \tag{1765}$$

$$\left(\theta^{ij}\right) = \begin{pmatrix} \theta^{11} & \theta^{12} & \ldots & \theta^{1N} \\ \theta^{21} & \ldots & \ldots & \theta^{2N} \\ \vdots & & & \vdots \\ \theta^{N1} & \ldots & \ldots & \theta^{NN} \end{pmatrix}. \tag{1766}$$

Fundamental (or defining) represantation of the group is given by the N-component vector

$$\vec{v} = \left\{v^j,\, j = 1, \ldots N\right\}, \tag{1767}$$

$$v^i \to v'^i = \theta^{ij}v^j \tag{1768}$$

(rotations in N-dimensions). So the represantion is the vector space that the group acts.

Tensors are defined as objects that transform as the products of vectors. Example

$$T^{ijk} = T'^{ijk} = \theta^{il}\theta^{jm}\theta^{kn}T^{lmn}. \tag{1769}$$

So tensors are also represantations of the group. Unlike vectors, they are reducible, in the sense that nat all elements are scrambled together under the group action. Example; T^{ij}

$$T^{ij} \to T'^{ij} = \theta^{il}\theta^{jm}T^{lm}. \tag{1770}$$

We can define a symmetric and an anti-symmetric tensor from T^{ij} as follows

$$S^{ij} \equiv \frac{1}{2}\left(T^{ij} + T^{ji}\right); \quad A^{ij} \equiv \frac{1}{2}\left(T^{ij} - T^{ji}\right) \tag{1771}$$

such that

$$S^{ij} \to \theta^{il}\theta^{jm}S^{lm} \qquad \text{which is symmetric,} \tag{1772}$$

$$A^{ij} \to \theta^{il}\theta^{jm}S^{lm} \qquad \text{which is anti-symmetric,} \tag{1773}$$

so

$$\begin{aligned} T^{ij} &= S^{ij} + A^{ij} \\ &= \frac{1}{2}\left(T^{ij} + T^{ji}\right) + \frac{1}{2}\left(T^{ij} - T^{ji}\right) \\ N^2 \text{ objects} &= \frac{N(N+1)}{2} + \frac{N(N-1)}{2} \text{ objects} \end{aligned} \tag{1774}$$

and these objects transform among each other. Observe that

$$S \equiv \delta^{ij} S^{ij}, \tag{1775}$$

that is the trace does not transform

$$\begin{aligned} S' &= \delta^{ij} \theta^{il} \theta^{jm} S^{lm} \\ &= \theta^{il} \theta^{im} S^{lm} \\ &= \left(\theta^T\right)^{il} \theta^{im} S^{lm} = S. \end{aligned} \tag{1776}$$

So then it is clear that

$$\tilde{S}^{ij} = S^{ij} - \frac{1}{N} \delta^{ij} S \tag{1777}$$

is the traceless part of S^{ij}. ???? traceless under the $SO(N)$ transformation. So we have

$$N^2 = \frac{N(N+1)}{2} - 1 + 1 + \frac{N(N-1)}{2} \tag{1778}$$

where $\frac{N(N+1)}{2} - 1$ is the traceless symmetric part, $+1$ is the trace part and $\frac{N(N-1)}{2}$ is anti-symmetric part. It is common to write this as

$$N \otimes N = \left[\frac{N}{2}(N+1) - 1\right] \oplus 1 \oplus \left[\frac{N}{2}(N-1)\right]. \tag{1779}$$

For $SO(N)$ we have

$$3 \otimes 3 = 5 \oplus 1 \oplus 3. \tag{1780}$$

Any orthogonal matrix can be written as

$$\theta = e^A \tag{1781}$$

where A is real and anti-symmetric. Let us check quickly for small A.

$$\theta^T \theta = 1 \tag{1782}$$

$$= \left(1 + A^T\right)(1 + A) = 1 \tag{1783}$$

where $A^T + A = 0$ since A is anti-symmetric. Since A_{ij} is anti-symmetric, it can be expanded as a linear combination of $\frac{N(N-1)}{2}$ anti-symmetric matrices J^{ij}

$$\theta = e^{i\alpha^{ij} J^{ij}} \tag{1784}$$

where $J^\dagger = J$. Recall that in QFT-1 we found

$$\left[J^{ij}, J^{kl}\right] = i\left(\delta^{ik} J^{jl} - \delta^{jk} J^{il} + \delta^{jl} J^{ik} - \delta^{il} J^{jk}\right). \tag{1785}$$

For $SO(3)$, we have just need one index and the algebra simplifies to

$$\left[J^i, J^j\right] = i\epsilon^{ijk} J^k, \tag{1786}$$

and

$$J^i \equiv \frac{1}{2}\epsilon^{ijk}J^{jk}. \tag{1787}$$

For $SO(4)$ we can also simplify the algebra as follows: we have $\frac{4}{2}(4-1) = 6$ anti-symmetric matrices J^{12}, J^{13}, J^{14},J^{23}, J^{24}, J^{34}. Let

$$J^1_{\pm} \equiv \frac{1}{2}\left(J^{23} \pm J^{14}\right) \tag{1788}$$

$$J^2_{\pm} \equiv \frac{1}{2}\left(J^{13} \pm J^{24}\right) \tag{1789}$$

$$J^3_{\pm} \equiv \frac{1}{2}\left(J^{12} \pm J^{34}\right). \tag{1790}$$

Then we get

$$\left[J^i_+,\, J^j_+\right] = i\epsilon^{ijk}J^k_+ \tag{1791}$$

$$\left[J^i_-,\, J^j_-\right] = i\epsilon^{ijk}J^k_- \tag{1792}$$

$$\left[J^i_+,\, J^j_-\right] = 0. \tag{1793}$$

So $SO(4)$ is locally isomorphic to $SO(3) \times SO(3)$. Recall also that $SU(2) \approx SO(3)$ locally. And so the Lorentz group $SO(1,\, 3) \approx SU(2) \times SU(2)$ locally (we studied these before). Remember that $SU(N)$ is special unitary group that satisfy below identities

$$U^\dagger U = 1, \quad \det U = 1. \tag{1794}$$

Defining as fundamental represantation consits of N objects $\varphi^i, \quad i = 1, ...N$

$$\varphi^i \rightarrow \varphi'^i = U^i_j \varphi^j. \tag{1795}$$

Take the complex conjugate

$$\varphi{*}^i \rightarrow \varphi{*}'^i = U^{i*}_j \varphi{*}^j\ . \tag{1796}$$

Define an object φ_i which transforms like $\varphi{*}^i$

$$\varphi_i \rightarrow \varphi'_i = \left(U^\dagger\right)^j_i \varphi'_j. \tag{1797}$$

We can also have tensor representations φ^{ij}_k etc.

$$\varphi^{ij}_k \rightarrow \varphi'^{ij}_k = U^i_l U^j_m \left(U^\dagger\right)^n_k \varphi^{lm}_n \tag{1798}$$

where φ^i is covariant vector and φ_i is contravariant vector.

$$U^\dagger U = 1 \Rightarrow \left(U^\dagger\right)^k_i U^j_k = \delta^j_i. \tag{1799}$$

Trace is taken as

$$\varphi^{ij}_j \equiv \delta^k_j \varphi^{ij}_k. \tag{1800}$$

Just like the case of $SO(N)$, the symmetry properties of the tensors are intact.

So say we have $\varphi_k^{ij} = \varphi_k^{ji}$ and say that it is traceless

$$\delta_j^k \varphi_k^{ji} = 0. \tag{1801}$$

Then φ_k^{ij} is an

$$\frac{N^2}{2}(N+1) - N \tag{1802}$$

dimensional represantation of $SU(N)$. So irreducibale represantations of $SU(N)$ are traceless tensors with definite symmetry properties.

$SU(N)$ represantations:

- 1 trace,
- N vectors (more properly spinors),
- $\frac{N(N-1)}{2}$ anti-symmetric tensors $\varphi^{ij} = -\varphi^{ji}$,
- $\frac{N(N+1)}{2} - 1$ symmetric traceless $\varphi^{ij} = \varphi^{ji}$,
- $N^2 - 1$ traceless,
- $\frac{N^2}{2}(N-1) - N$, $\varphi_k^{ij} = -\varphi_k^{ji}$.

2. *Adjoint Represantation:*

Traceless φ_j^i is called the adjoint representation.

$$\varphi_j^i \to \varphi_j'^i = U_l^i \left(U^\dagger\right)_j^n \varphi_n^l = U_l^i \varphi_n^l \left(U^\dagger\right)_j^n. \tag{1803}$$

So consider φ as a matrix transformation

$$\varphi \to \varphi' = U\varphi U^\dagger. \tag{1804}$$

Say φ is Hermitian $\varphi = \varphi^\dagger$ then

$$\varphi'^\dagger = \left(U\varphi U^\dagger\right)^\dagger = \varphi \tag{1805}$$

so we can take φ to be a traceless Hermitian matrix.

$$\det U = 1 = \epsilon_{i_1 i_2 \ldots i_N} U_1^{i_1} U_2^{i_2} \ldots U_N^{i_N} \tag{1806}$$
$$= \epsilon^{i_1 i_2 \ldots i_N} U_{i_1}^1 U_{i_2}^2 \ldots U_{i_N}^N. \tag{1807}$$

$U = e^{iH}$ H is Hermitian and traceless. Check again for small H

$$U^\dagger U = \left(1 - iH^\dagger\right)(1 + iH) = 1 \tag{1808}$$

so $H = H^\dagger$ ok. Since we have $N \times N$ matrices there are $N^2 - 1$ such matrices: T^a where $a = 1, \ldots N^2 - 1$. Then

$$U = e^{i\theta_a T_a} \tag{1809}$$

and

$$\left[T^{a},\, T^{b}\right]^{\dagger} = -\left[T^{a},\, T^{b}\right] = -i\, f^{abc}T^{c}. \tag{1810}$$

How do generators act on the represantations?

$$U \simeq 1 + i\theta^{a}T^{a} \tag{1811}$$

1. Defining represantation:

$$\begin{aligned}\varphi^{i} \to U^{i}_{j}\varphi^{j} &= \left[\delta^{i}_{j} + i\theta^{a}\left(T^{a}\right)^{i}_{j}\right]\varphi^{j}\\ &= \varphi^{i} + i\theta^{a}T^{a}\varphi.\end{aligned} \tag{1812}$$

2. Adjoint Represantation:

$$\begin{aligned}&\varphi^{i}_{j} \to U^{i}_{l}\left(U^{\dagger}\right)^{k}_{j}\varphi^{l}_{k}\\ &= \left(\delta^{i}_{l} + i\theta^{a}\left(T^{a}\right)^{i}_{l}\right)\left(\delta^{k}_{j} - i\theta^{b}\left(T^{b}\right)^{k}_{j}\right)\varphi^{l}_{k}\\ &=\varphi^{i}_{j} + i\theta^{a}\left(T^{a}\right)^{i}_{l}\varphi^{l}_{j} - i\theta^{b}\left(T^{b}\right)^{k}_{j}\varphi^{i}_{k}\\ &=\varphi^{i}_{j} + i\theta^{a}\left[T^{a},\, \varphi\right]^{i}_{j},\end{aligned} \tag{1813}$$

$$\varphi \to \varphi' = \varphi + i\theta^{a}\left[T^{a},\, \varphi\right]. \tag{1814}$$

Now, since the adjoint represantation is Hermitian and traceless, we can write it as a combo of generators

$$\varphi \equiv \varphi^{a}T^{a}, \tag{1815}$$

then

$$\begin{aligned}\varphi \to \varphi' &= \varphi + i\theta^{a}\left[T^{a},\, T^{b}\right]\varphi^{b}.\\ &= \varphi - f^{abc}\theta^{a}\varphi^{b}.\end{aligned} \tag{1816}$$

And for $SU\left(2\right)$ $f^{abc} = \epsilon^{abc}$.

Important: $\varphi^{i} \neq \varphi^{a}$ for example for $SU\left(2\right)$ $i = 1,\, 2$ but $a = 1,\, 2,\, 3$.

G. QFT and Critical Phenomena:

Consider the Euclidean space PI

$$\frac{\int D\phi\,\phi\left(x_{1}\right)...\phi\left(x_{n}\right)e^{-S}}{\int D\phi\, e^{-S}}. \tag{1817}$$

The integral is over all configurations that vanish at $t_{E} \to \pm\infty$. In the language of statistical mechanics (1817) is the corellation function in four dimensions.

Q: What kind of a statistical system can reproduce the properties of the QFT?

Let us discretize the four dimensional Euclidean space with lattice spacing a. Then

$$\phi_i = \phi(x_i) \tag{1818}$$

are the (finite) number of DOF where x_i is a lattice site. So the PI is a sum of finite number of terms. But at the end we must get back QFT in the $a \to 0$ limit.

Consider the Euclidean propagator (two point correlation function)

$$< \phi(x)\phi(0) >= \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip.x}}{p^2 + m^2} \tag{1819}$$

($p^\mu p_\mu = \delta_{\mu\nu} p^\mu p^\nu$ etc.) for $m|x| \gg 1$ we have

$$< \phi_n \phi_0 >\sim e^{-amn} \tag{1820}$$

where $x = na$ and $n \gg 1$ (exact computation can also be carried out).

In statistical mechanics

$$< \phi_n \phi_0 >\sim e^{-\frac{n}{\xi}} \tag{1821}$$

defines the correlation lengt , $\xi \equiv \frac{1}{am}$ for our case, which is dimensionless. So correlation length diverges. ξ is a function of the coupling constants of the statistical system. To get QFT, we must take it ∞.

Removing the cut-off in QFT (that is setting $a \to 0$) corresponds to tuning the statistical system to its critical point, which characterized by a diverging correlation length.

So QFT's can be considered as critical statistical systems.

Example: $2D$ Ising Model:

$S_i = \pm 1$ is the spin variable and i refers to the site number.

$$H = -J \sum_{i,\,j} S_i S_j \tag{1822}$$

where i, j runs after nearest neighbours.

Figure-a

Take $J > 0$ so the interaction tends to align the spins, it is a ferromagnetic coupling.

The partition function $Z = e^{-\beta F} = \mathrm{tr} e^{-\beta H}$ where $\beta = \frac{1}{k_\beta T}$ has all the information. Define a dimensionless parameter

$$K \equiv \frac{J}{k_\beta T}. \tag{1823}$$

At a fixed temperature T and with a fixed number of particles the equilibrium state is given by minimizing the free energy

$$F = E - TS \tag{1824}$$

where E is minimized by spin alignment and S is maximized by anti-alignment.

- At small T, energy minimization is more important,
- At large T, entropy maximization is more important.

So there is a competition between these two effects. In the Ising model, below a critical T_c (or $\frac{k_\beta T_c}{j}$) the system develops a spontaneous magnetization

$$M \equiv < S_i > \neq 0. \tag{1825}$$

Above T_c $M = 0$.

$$\underset{T \to T_c^-}{M} \to (T_c - T)^\beta \tag{1826}$$

where β is the critical index and it is $\frac{1}{8}$ for the Ising model.

Figure-105

In the magnetized phase, $M =< S_i > \neq 0$, there is long-range order. That means given two far seperated spins, the probability of finding them aligned is higher than finding them anti-aligned. In fact since $\xi \to \infty$, there is no exponential decay of the correlation function.

In the disordered phase $T > T_c$, ξ is finite.

Important: As $T \to T_c^+$ (from above) $\xi \to \frac{1}{(T-T_c)^\nu}$ where $\nu = 1$, is the critical index.

Figure-106

XVI. APPENDIX-I: NOTES ON SCATTERING

First some simple stuff. Consider the simplest one-dimensional scattering problem

$$V(x) = \begin{cases} 0, & x < -a \\ -V_0, & -a < x < a \\ 0, & x > a. \end{cases} \tag{1827}$$

So bound states have $E < 0$ and scattering states $E > 0$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - V_0\right)\varphi_E(x) = E\varphi_E(x) \qquad |x| < a, \tag{1828}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_E(x) = E\varphi_E(x) \qquad |x| > a. \tag{1829}$$

For scattering states $E > 0$ define

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}, \quad k_2 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}. \tag{1830}$$

Then

$$\ddot{\varphi}_E = -k_2^2\varphi_E(x) \qquad |x| < a \tag{1831}$$

$$\ddot{\varphi}_E = -k_1^2\varphi_E(x) \qquad |x| > a. \tag{1832}$$

So then

$$\varphi_E(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x}, & x < -a \\ Ce^{ik_2x} + De^{-ik_2x}, & -a < x < a \\ Fe^{ik_1x} + Ge^{-ik_1x}, & x > a. \end{cases} \tag{1833}$$

Seven unknowns to be determined. But the continuity of the wave function and its derivative give four equations. For scattering states, we do not have normalization or the decay of the wave function.

Absence of normalization $\to$ energy is not quantized.

- Scattering states have a continuum energy spectrum. So energy E is an initial given parameter.
- Assuming that the particle is incident from the left that means $G = 0$. So now from seven unknowns we have reduced to five.
- Clearly the amplitude of the incident way must be an initial condition, if can not be predicted! Hence, we have four unknowns and four equations.

Then we get

$$\frac{F}{A} = \frac{e^{-2ik_1 a}}{\cos(2k_2 a) - i\frac{\left(k_1^2+k_2^2\right)}{2k_1 k_2}\sin(2k_2 a)}, \tag{1834}$$

and

$$\frac{B}{A} = i\frac{F}{A}\frac{\left(k_2^2-k_1^2\right)}{2k_1 k_2}\sin(2k_2 a), \tag{1835}$$

then the transmission coefficient can be written as

$$T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\frac{\left(k_1^2-k_2^2\right)}{4k_1 k_2}\sin^2(2k_2 a)} \tag{1836}$$

which is the probability that an incident particle is transmitted. The reflection coefficient is

$$T = \frac{|B|^2}{|A|^2} = \frac{1}{1+\frac{4k_1 k_2}{\left(k_1^2-k_2^2\right)\sin^2(2k_2 a)}}, \tag{1837}$$

and

$$T + R = 1. \tag{1838}$$

Observe the following:

Of course as $\frac{E}{V_0} \to$ large, $T \to 1$. But there are also some resonances. At $2k_2 a = n\pi$ $T = 1$ where $n = 0,\ 1,\ 2...$

Figure-108

Let $\lambda_2 = \frac{2\pi}{k_2}$ then at $2a = n\frac{\lambda_2}{2}$ we have resonances. In terms of energy resonance condition reeds

$$E = -V_0 + \frac{n^2\pi^2\hbar^2}{2m\,(2a)^2} > 0. \tag{1839}$$

Funny but these energies with respect to the $p2n = n\hbar$ bottom of the well are just ∞-well energies $\frac{p^2}{2m} = \frac{n^2\hbar^2}{2m(2a)^2}$ (Ramsauer-Townsend effect). To get the barrier Figure-b case, just let $V_0 \to -V_0$.

A. Analitic Properties of the Transmission Coefficient:

Let us consider the properties of $T(E)$ in the complex E-plane.

$$\cos(2k_2 a) - i\frac{\left(k_1^2+k_2^2\right)}{2k_1 k_2}\sin(2k_2 a) = 0 \tag{1840}$$

gives the location of the poles.

$$\cot(2k_2a) = i\frac{(k_1^2 + k_2^2)}{2k_1k_2} \tag{1841}$$

and

$$\cot(k_2a) - \tan(k_2a) = i\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right). \tag{1842}$$

Note that this equation is like $x - \frac{1}{x} = y - y^{-1}$ whose solutions are

$$i)\ \tan k_2a = -i\frac{k_1}{k_2},\ \text{or} \tag{1843}$$

$$ii)\ \cot k_2a = i\frac{k_1}{k_2}. \tag{1844}$$

i) For $E > 0$, that is for real k_1 and k_2 these equations have no solutions.
ii) For $E < -V_0$ again these equations have no solutions, since both k_1 and k_2 are imaginary.
iii) For $-V_0 < E < 0$ k_1 is imaginary, k_2 is real, we might have solutions.
Let $E = |E|e^{i\varphi}$ and $\sqrt{E} = |E|^{\frac{1}{2}}e^{i\frac{\varphi}{2}}$. Choose $\varphi = \pi$ then

$$k_1 = i\left(\frac{2m|E|}{\hbar^2}\right)^{\frac{1}{2}}. \tag{1845}$$

Then i) gives

$$\tan(k_2a) = \frac{(2m|E|)^{\frac{1}{2}}}{\hbar k_2} \tag{1846}$$

which is energy eigenvalues of even bound states and ii) gives

$$\cot(k_2a) = -\frac{(2m|E|)^{\frac{1}{2}}}{\hbar k_2} \tag{1847}$$

that is energy eigenvalues of odd bound states in the potential well.

So $T(E)$ has poles at the positions (energies) of the bound states. So $T(E_{\text{bound}}) \to \infty$. Let us explain this a bit. Since k_1 is purely imaginary $A \to 0$. But since $T(E) \to \infty$, there is a reflected wave at $x < -a$ (even though there is no incident wave). The transmitted wave also folls of exponentially.

Figure-109

Let us expand $T(E)$ near a sesonance E_R, Taylor series expansion gives

$$T(E)e^{2ik_1a} = (-1)^n \frac{i\frac{\Gamma}{2}}{E - E_R + i\frac{\Gamma}{2}} \tag{1848}$$

where

$$\begin{aligned}\frac{2}{\Gamma} &= \left[\frac{1}{2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)\frac{d(2k_1a)}{dE}\right]_{E_R} \\ &= \frac{1}{2}\frac{\sqrt{2m}}{\hbar}a\frac{2E_R + V_0}{\sqrt{E_R}(E_R + V_0)}.\end{aligned} \tag{1849}$$

For $V_0 \gg E_R$ (deep well)

$$\frac{2}{\Gamma} \simeq \frac{1}{2}\frac{\sqrt{2mV_0}}{\hbar}a\frac{1}{\sqrt{V_0 E_R}} = \frac{a}{v_R \hbar} \tag{1850}$$

where $v_R = \sqrt{\frac{2E_R}{m}}$ is incident velocity at the resonance energy. So $T(E)$ has poles at the complex energy values

$$E = E_R - i\frac{\Gamma}{2}. \tag{1851}$$

Figure-110

$$|T(E)|^2 = \frac{\left(\frac{\Gamma}{2}\right)^2}{(E - E_R)^2 + \left(\frac{\Gamma}{2}\right)^2} \qquad \text{(Breit-Wigner)} \tag{1852}$$

Figure-111
Then we can write

$$T(E) = |T(E)|e^{i\delta(E) - 2ik_1 a} \tag{1853}$$

where

$$\tan\delta(E) = \frac{\mathrm{Im}\left(T(E)\,e^{2ik_1 a}\right)}{\mathrm{Re}\left(T(E)\,e^{2ik_1 a}\right)} = \frac{1}{2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)\tan(2k_2 a). \tag{1854}$$

So near the resonance

$$\tan\delta(E) = \frac{2}{\Gamma}(E - E_R) \tag{1855}$$

which is the phase shift of the transmitted wave.

$$\psi_{\text{transmitted}}(a) = |T(E)|e^{i\delta(E)}\psi_{\text{incident}}(\leftarrow a). \tag{1856}$$

Figure-112

B. Wavepacket Near a Resonance:

$$\psi_{\text{in}}(\vec{x}, t) = \int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}} g(p) \exp\left\{\frac{i}{\hbar}(px - E(p)t)\right\} \tag{1857}$$

where $E(p) = \frac{p^2}{2m}$. Here only right moving noders are taken into account. Assume $g(p)$ has a maximum value at p_0 with $E(p_0) \simeq E_R$. Then $v_0 = \frac{p_0}{m}$ and $x(t) = v_0 t$ which are the velocity and the position of the maximum. Then the transmitted wave has

$$\psi_t(x, t) = \int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}} \exp\left\{\frac{i}{\hbar}(px - E(p)t - 2pa + \delta(E)\hbar)\right\}|T(E)|. \tag{1858}$$

Center of the wave packet comes from the stationarity of the phase.

$$x(t) - \frac{dE}{dp}t - 2a + \frac{d\delta(E)}{dp}\hbar = 0, \tag{1859}$$

$$x(t) = v_0 t + 2a - \hbar\frac{d\delta(E)}{dE}\frac{dE}{dp}|_{p_0} \tag{1860}$$

where $v_0 = \frac{dE}{dp}|_{p_0}$. Then we get

$$x(t) = v_0 t + 2a - \frac{2\frac{\hbar}{\Gamma}}{1 + \left[\frac{2}{\Gamma}\left(E(p_0) - E_R\right)\right]^2} v_0 \tag{1861}$$

where the first term denotes the free motion without a potential and the second one says ∞ fast transmission which is correct for a potential in an ∞ deep potential well. The third term is the time spot near the well. For $E(p_0) = E_R$, $2\frac{\hbar}{\Gamma}$ is the time spot.

Clearly for sharp resonances Γ is small and this time is large. So the resonance becomes a bound state. Recall that

$$v_i = \sqrt{\frac{2(E_R + V_0)}{m}} \tag{1862}$$

so for the time interval $2\frac{\hbar}{\Gamma}$

$$2\frac{\hbar}{\Gamma}\frac{v_i}{4a} \simeq \frac{1}{4}\sqrt{1 + \frac{V_0}{E_R}} \tag{1863}$$

where the particle makes this many back and forth oscillations.

Examples:

1. Pb^{206} nuclei bombarded with α-particles of energy $E_\alpha = 5.4\,\mathrm{MeV}$. Po^{210} is formed which decays by α-emission with a half-life of 138 days ($\Gamma = 10^{-18}\,\mathrm{eV}$).

2. $\pi +$ nucleon $\to$ nucleon$* \to \pi + N$, $\Gamma \simeq 120\,\mathrm{MeV}$ and $\tau \simeq 0.5 \times 10^{-23}$ sec. formed.

3. $\frac{J}{\psi}$ in 1974 a very sharp resonance ($c\bar{c}$) with mass 3.1 GeV

$$p + p \to e^+ + e^- + X \qquad \text{Ting} \tag{1864}$$

$$e^+ + e^- \to \text{hadron} \qquad \text{(1974resolution)}. \tag{1865}$$

Figure-113

Finally let us note the following: consider a sharp resonance. Then $g(p) \sim$ constant near the resonance region. Then we can find (after some integrations and approximations)

$$P_{\text{transmitted}}(t) = \int_a^\infty dx\, |\psi_t(x, t)|^2$$
$$= |g(p_{\text{tr}})|^2 \frac{\frac{\Gamma}{\hbar}}{4v_R}\left(1 - \exp\left(-\frac{\Gamma}{\hbar}(t - t_0)\right)\right). \tag{1866}$$

So the decay rate of the resonantly bound state is $\frac{\hbar}{\Gamma} =$ life-time

$$\Delta E \sim \frac{\Gamma}{2}. \tag{1867}$$

XVII. APPENDIX-II: DESCRETE SYMMETRIES OF THE DIRAC THEORY (FROM PESKIN)

Parity,

$$P:\,(t,\,\vec{x})\rightarrow(t,\,-\vec{x})\quad\text{(reversing the handedness of spacetime).}\tag{1868}$$

Time reversal,

$$T:\,(t,\,\vec{x})\rightarrow(-t,\,\vec{x})\quad\text{(interchanging forward and backward light comes).}\tag{1869}$$

Charge-conjugation,

$$C:\,\text{(charge-conjugation which leads to an interchange of particles and anti-particles).}\tag{1870}$$

- Any relativistic field theory must be invariant under the proper outochronous Lorentz transformations.
- But it may not be invariant under P, T and C.
- What is the experimental situation? Gravity, electrodynamics and strong interactions are symmetric with respect to P, C and T.
- Weak interactions violate C and P maximally. Neutral K-mesons also show CP and T violations. Physical origin is not understood yet.
- CPT is a perfect symmetry of Nature.

A. Parity:

P reverses the momentum of the particle, but does not flip its spin. So

$$P:\,a_s\left(\vec{p}\right)|0>\rightarrow a_s\left(-\vec{p}\right)|0>,\tag{1871}$$

$$P\,a_s\left(\vec{p}\right)PP|0>=P\,a_s\left(\vec{p}\right)P|0>=\eta_a a_s\left(-\vec{p}\right)|0>,\tag{1872}$$

(where P and $P^2=1$ and $P^\dagger P=1$, a unitary operator $P^\dagger=P^{-1}=P$) so

$$P\,a_s\left(\vec{p}\right)P=\eta_a a_s\left(-\vec{p}\right)\tag{1873}$$

where $\eta_a^2=\pm1$ and

$$P\,b_s\left(\vec{p}\right)P=\eta_b b_s\left(-\vec{p}\right)\tag{1874}$$

where $\eta_b^2=\pm1$.

$$P\psi\left(x\right)P=\int\frac{d^3p}{\left(2\pi\right)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s\left(\eta_a a_s\left(-\vec{p}\right)u^s\left(p\right)e^{-ip.x}+\eta_b^* b_s^\dagger\left(-\vec{p}\right)v^s\left(p\right)e^{ip.x}\right).\tag{1875}$$

Let $\tilde{p}\equiv\left(p^0,\,-\vec{p}\right)$ then $p.x=\tilde{p}.\left(t,\,-\vec{x}\right)$, $\tilde{p}.\sigma=p.\bar{\sigma}$ and $\tilde{p}.\bar{\sigma}=p.\sigma$. Then

$$u\left(p\right)=\begin{pmatrix}\sqrt{p.\sigma}\,\xi\\ \sqrt{p.\sigma}\,\xi\end{pmatrix}=\begin{pmatrix}\sqrt{\tilde{p}.\bar{\sigma}}\,\xi\\ \sqrt{\tilde{p}.\sigma}\,\xi\end{pmatrix}=\gamma^0u\left(\tilde{p}\right)\tag{1876}$$

where $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and

$$v\left(p\right) = \begin{pmatrix} \sqrt{p.\sigma}\,\xi \\ -\sqrt{p.\bar{\sigma}}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}.\bar{\sigma}}\,\xi \\ -\sqrt{\tilde{p}.\sigma}\,\xi \end{pmatrix} = -\gamma^0 v\left(\tilde{p}\right). \tag{1877}$$

So then

$$P\psi\left(x\right)P = \int \frac{d^3\tilde{p}}{\left(2\pi\right)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s \left(\eta_a a_s\left(\tilde{p}\right)\gamma^0 u^s\left(\tilde{p}\right)e^{-i\tilde{p}.\left(t,\,-\vec{x}\right)} - \eta_b^* b_s^\dagger\left(\tilde{p}\right)\gamma^0 v^s\left(\tilde{p}\right)e^{i\tilde{p}.\left(t,\,-\vec{x}\right)}\right) \tag{1878}$$

$$= \eta_a\gamma^0\psi\left(t,\,-\vec{x}\right) \qquad \text{if } \eta_a = -\eta_b, \tag{1879}$$

$$P\bar{\psi}\left(t,\,\vec{x}\right)P = P\psi^\dagger\left(t,\,\vec{x}\right)\gamma^0 P = P\psi^\dagger\left(t,\,\vec{x}\right)P\gamma^0 \tag{1880}$$

$$= \left(P\psi\left(x\right)P\right)^\dagger\gamma^0 = \eta_a^*\psi^\dagger\left(t,\,-\vec{x}\right) \tag{1881}$$

$$= \eta_a^*\bar{\psi}\left(t,\,-\vec{x}\right)\gamma^0. \tag{1882}$$

1. Then the scalar bilinear transforms as

$$\bar{\psi}\psi \to P\bar{\psi}\psi P = \left|\eta_a\right|^2\bar{\psi}\left(t,\,-\vec{x}\right)\psi\left(t,\,-\vec{x}\right) = \bar{\psi}\left(t,\,-\vec{x}\right)\psi\left(t,\,-\vec{x}\right). \tag{1883}$$

2. The vector bilinear transforms as a vector

$$\bar{\psi}\gamma^\mu\psi \to P\bar{\psi}\gamma^\mu\psi P = \bar{\psi}\left(t,\,-\vec{x}\right)\gamma^0\gamma^\mu\gamma^0\psi\left(t,\,-\vec{x}\right) = \begin{cases} +\bar{\psi}\left(t,\,-\vec{x}\right)\gamma^\mu\psi\left(t,\,-\vec{x}\right), & \mu = 0 \\ -\bar{\psi}\left(t,\,-\vec{x}\right)\gamma^\mu\psi\left(t,\,-\vec{x}\right), & \mu = i \end{cases}. \tag{1884}$$

3. The Hermitian pseudoscalar bilinear transforms as

$$P\,i\bar{\psi}\gamma^5\psi P = i\bar{\psi}\left(t,\,-\vec{x}\right)\gamma^0\gamma^5\gamma^0\psi\left(t,\,-\vec{x}\right) = -i\bar{\psi}\left(t,\,-\vec{x}\right)\gamma^5\psi\left(t,\,-\vec{x}\right). \tag{1885}$$

4. The pseudoscalar bilinear transforms as

$$P\bar{\psi}\gamma^\mu\gamma^5\psi P = \bar{\psi}\left(t,\,-\vec{x}\right)\gamma^0\gamma^\mu\gamma^5\gamma^0\psi\left(t,\,-\vec{x}\right) = \begin{cases} -\bar{\psi}\left(t,\,-\vec{x}\right)\gamma^\mu\gamma^5\psi\left(t,\,-\vec{x}\right), & \mu = 0 \\ +\bar{\psi}\left(t,\,-\vec{x}\right)\gamma^\mu\gamma^5\psi\left(t,\,-\vec{x}\right), & \mu = i \end{cases}. \tag{1886}$$

5. Tensor bilinear transforms as

$$P\,i\bar{\psi}\left[\gamma^\mu,\,\gamma^\nu\right]\psi P = P2\bar{\psi}\sigma^{\mu\nu}\psi P = \left(-1\right)^\mu\left(-1\right)^\nu 2\bar{\psi}\left(t,\,-\vec{x}\right)\sigma^{\mu\nu}\psi\left(t,\,-\vec{x}\right). \tag{1887}$$

B. Time Reversal:

We would like T to be a unitary operator that sends $a_s\left(\vec{p}\right) \to a_s\left(-\vec{p}\right)$ and $b_s\left(\vec{p}\right) \to b_s\left(-\vec{p}\right)$ and $\psi\left(t,\,\vec{x}\right)$ to $\psi\left(t,\,-\vec{x}\right)$ times some constant matrix. But actually this is different since sending $a\left(\vec{p}\right) \to a\left(-\vec{p}\right)$ is related to sending $\left(t,\,\vec{x}\right) \to \left(t,\,-\vec{x}\right)$.

Say we want free Dirac theory to be invariant under time-reversal ,

$$\left[T,\,H\right] = 0. \tag{1888}$$

$$\psi\left(t,\,\vec{x}\right)=e^{iHt}\psi\left(\vec{x}\right)e^{-iHt}, \tag{1889}$$

$$T\psi\left(t,\,\vec{x}\right)T=e^{iHt}T\psi\left(\vec{x}\right)Te^{-iHt}, \tag{1890}$$
$$T\psi\left(t,\,\vec{x}\right)T|0>=e^{iHt}T\psi\left(\vec{x}\right)T|0>, \qquad H|0>=0 \tag{1891}$$
$$\text{constant matrix}\,\psi\left(-t,\,\vec{x}\right)|0>=e^{iHt}T\psi\left(\vec{x}\right)T|0>, \tag{1892}$$
$$c\,e^{-iHt}\psi\left(\vec{x}\right)|0>=e^{iHt}T\psi\left(\vec{x}\right)T|0>, \qquad T|0>=0. \tag{1893}$$

This is only possible if

$$Te^{iHt}=e^{-iHt}T, \tag{1894}$$

with (1888). Then

$$T\text{ (c-number)}=\text{(c-number)}^{*}\,T \tag{1895}$$

so T is anti-linear or anti-unitary.

In addition to reversing the momentum of a particle, T should also flip th spin. [Reversing the momentum only would be parity transformation.]

Figure-114

Consider

$$\xi\left(\uparrow\right)=\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{pmatrix},\quad \xi\left(\downarrow\right)=\begin{pmatrix}-e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)\end{pmatrix}, \tag{1896}$$

so

$$\xi^{s}=\left(\xi\left(\uparrow\right),\,\xi\left(\downarrow\right)\right) \tag{1897}$$

where $s=1,\,2$. Define

$$\begin{aligned}\xi^{-s}\equiv-i\sigma^{2}\left(\xi^{s}\right)^{*}&=-i\begin{pmatrix}0&-i\\ i&0\end{pmatrix}\left(\xi^{s}\right)^{*}\\ &=\begin{pmatrix}0&-1\\ 1&0\end{pmatrix}\left(\xi^{s}\right)^{*}=\left(\xi\left(\downarrow\right),\,\xi\left(\uparrow\right)\right)\end{aligned} \tag{1898}$$

flipped spinor. Observe that

$$-i\sigma^{2}\left(\xi^{-s}\right)^{*}=\left(-i\sigma^{2}\right)\left(-i\sigma^{2}\right)\left(\left(\xi^{s}\right)^{*}\right)^{*}=-\xi^{s} \tag{1899}$$

so twice spin-flipped gives (-1) times the original spinor.

The electron annihilation operator $a^{s}\left(\vec{p}\right)$ destroys an electron where spinor $u^{s}\left(p\right)$ contains ξ^{s}. The positron annihilation operator $b^{s}\left(\vec{p}\right)$ destroys a positron whose spinor $v^{s}\left(p\right)$ contains ξ^{-s}

$$v^{s}\left(p\right)=\begin{pmatrix}\sqrt{p.\sigma}\,\xi^{-s}\\ -\sqrt{p.\bar{\sigma}}\,\xi^{-s}\end{pmatrix} \tag{1900}$$

recall that $\xi^{1}=\begin{pmatrix}1\\ 0\end{pmatrix}$ would correspond to spin $+\frac{1}{2}$ electron hit a spin $-\frac{1}{2}$ positron. This is the reason for ξ^{-s} in $v^{s}\left(p\right)$.

Define

$$a^{-s}(\vec{p}) \equiv \left(a^2(\vec{p}),\, -a^1(\vec{p})\right) \tag{1901}$$

$$b^{-s}(\vec{p}) \equiv \left(b^2(\vec{p}),\, -b^1(\vec{p})\right). \tag{1902}$$

Again define

$$\tilde{p} \equiv \left(p^0,\, \vec{p}\right). \tag{1903}$$

Then

$$\sqrt{\tilde{p}.\sigma}\,\sigma^2 = \sigma^2\sqrt{p.\sigma^*}. \tag{1904}$$

This follows easily if first $\sqrt{p.\sigma} = \frac{p.\sigma+m}{\sqrt{2(p^0+m)}}$ is realized. Check

$$\begin{aligned}
\sqrt{p.\sigma}\sqrt{p.\sigma} =& \frac{(p.\sigma)^2 + m^2 + 2mp.\sigma}{2\left(p^0+m\right)} \\
p.\sigma 2\left(p^0+m\right) =& \left(p^0\right)^2 - 2p^0\vec{p}.\vec{\sigma} + \vec{p}^2 + m^2 + 2mp.\sigma \\
2p.\sigma p^0 =& 2\left(p^0\right)^2 - 2p^0\vec{p}.\vec{\sigma} \\
2\left(p^0\right)^2 - 2p^0\vec{p}.\sigma =& 2\left(p^0\right)^2 - 2p^0\vec{p}.\vec{\sigma} \quad \text{OK.}
\end{aligned} \tag{1905}$$

works for $p^2 = m^2$. With this result (1904) follows in a line.

Given $u^s(p)$, $u^{-s}(p)$ will be the spin and momentum flipped version.

$$u^{-s}(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}.\sigma}\ \left(-i\sigma^2\xi^s\right)^* \\ \sqrt{\tilde{p}.\bar{\sigma}}\ \left(-i\sigma^2\xi^s\right)^* \end{pmatrix}. \tag{1906}$$

Note that since $-i\sigma^2$ is real $*$ is actually on $(\xi^s)^*$. Then use our result $\sqrt{\tilde{p}.\sigma}\sigma^2 = \sigma^2\sqrt{p.\sigma^*}$

$$\begin{aligned}
u^{-s}(\tilde{p}) &= \begin{pmatrix} -i\sigma^2\sqrt{\tilde{p}.\sigma^*}\ (\xi^s)^* \\ -i\sigma^2\sqrt{\tilde{p}.\bar{\sigma}^*}\ (\xi^s)^* \end{pmatrix} \\
&= -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\left[u^s(p)\right]^*,
\end{aligned} \tag{1907}$$

and

$$u^{-s}(\tilde{p}) = -\gamma^1\gamma^3\left[u^s(p)\right]^* \tag{1908}$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \tag{1909}$$

and

$$\gamma^1\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix} = i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}. \tag{1910}$$

Similarly we get

$$v^{-s}(\tilde{p}) = -\gamma^1\gamma^3 [v^s(p)]^* . \tag{1911}$$

Define

$$T\, a_s(\vec{p})\, T = a_{-s}(-\vec{p}) \tag{1912}$$

$$T\, b_s(\vec{p})\, T = b_{-s}(-\vec{p}) \tag{1913}$$

(one can of course put an additional over all phase). Then

$$\begin{aligned} T\,\psi(t,\, \vec{x})\, T &= \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{1}{\sqrt{2E_{\vec{p}}}} T \left(a_s(\vec{p})\, u^s(p)\, e^{-ip.x} + b_s^\dagger(\vec{p})\, v^s(p)\, e^{ip.x} \right) \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a^{-s}(-\vec{p})\, [u^s(p)]^*\, e^{ip.x} + (b^{-s})^\dagger(-\vec{p})\, [v^s(p)]^*\, e^{-ip.x} \right) \\ &= -\gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a^{-s}(-\vec{p})\, \gamma^1\gamma^3 [u^s(p)]^*\, e^{ip.x} + (b^{-s})^\dagger(-\vec{p})\, \gamma^1\gamma^3 [v^s(p)]^*\, e^{-ip.x} \right) \\ &= \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a^{-s}(-\vec{p})\, u^{-s}(\tilde{p})\, e^{ip.x} + (b^{-s})^\dagger(-\vec{p})\, v^{-s}(\tilde{p})\, e^{-ip.x} \right), \end{aligned}$$

let $\vec{p} \to -\vec{p}$ and $s \to -s$,

$$= \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a^s(\vec{p})\, u^s(p)\, e^{ip_0t + i\vec{p}.\vec{x}} + (b^s)^\dagger(\vec{p})\, v^s(p)\, e^{-ip_0t + -i\vec{p}.\vec{x}} \right),$$

$$T\,\psi(t,\, \vec{x})\, T = \gamma^1\gamma^3\psi(-t,\, \vec{x}) . \tag{1914}$$

(Peskin has a $-$ sign problem I think).

C. Charge Conjugation for Fermions:

Charge conjugation takes a fermion with a spin orientation into an anti-fermion with the same spin orientation

$$C\, a^s(\vec{p})\, C = b^s(\vec{p}) \tag{1915}$$

$$C\, b^s(\vec{p})\, C = a^s(\vec{p}) \tag{1916}$$

we define them this way.

$$\begin{aligned} [v^s(p)]^* &= \begin{pmatrix} \sqrt{p.\sigma}\left(-i\sigma^2\xi^s\right)^* \\ -\sqrt{p.\sigma}\left(-i\sigma^2\xi^s\right)^* \end{pmatrix}^* \\ &= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\bar{\sigma}}\xi^s \end{pmatrix}, \end{aligned} \tag{1917}$$

so

$$u^s(p) = -i\gamma^2 [v^s(p)]^* , \tag{1918}$$

$$v^s(p) = -i\gamma^2 [u^s(p)]^* . \tag{1919}$$

$$C\,\psi\,(x)\,C = -i\gamma^2\psi^* = -i\left(\bar{\psi}\gamma^0\gamma^2\right)^T \tag{1920}$$

so C takes $\psi \to \psi^*$

$$\begin{aligned} C\,\bar{\psi}\,C &= C\,\psi^\dagger C\gamma^0 = (C\,\psi\,C)^\dagger\,\gamma^0 \\ &= \left(-i\gamma^2\psi^*\right)^{*T}\gamma^0 = \left(-i\gamma^2\psi\right)^T\gamma^0 \\ &= \left(-i\gamma^0\gamma^2\psi\right)^T. \end{aligned} \tag{1921}$$

D. Summary of C, P and T:

Use $(-1)^\mu \equiv 1$ for $\mu = 0$ and $(-1)^\mu = -1$ for $\mu = 1, 2, 3$

	$\psi\bar{\psi}$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$
C	$+1$	$+1$	-1	$+1$	-1
CPT	$+1$	$+1$	-1	-1	$+1$

Also

P	$\partial_\mu \to (-1)^\mu$
T	$\partial_\mu \to -(-1)^\mu$
C	$\partial_\mu \to +1$
CPT	$\partial_\mu \to -1$

CPT-Theorem: One cannot build a Lorentz-invariant QFT with a Hamiltonian that violates CPT.

XVIII. APPENDIX-III: ADDITIONAL MATERIAL FOR CHAPTER-IV

Recall that the S-matrix was $e^{-iH\left(T_f-T_i\right)}$ where H is a second quantized Hamiltonian and $T_f - T_i \to \infty$,

$$S = e^{-iH\left(T_f-T_i\right)}. \tag{1922}$$

It is easy to show $S^\dagger S = 1 = SS^\dagger$. Note also we often define $S \equiv 1 + iT$. In the Schrodinger picture elements of the S-matrix is

$${}_S\!< b,\, T_f|e^{-iH\left(T_f-T_i\right)}|a,\, T_i >_S \tag{1923}$$

which just gives the amplitude to start with the state $|a>$at T_i and end up with the state $|b>$ at T_f. And, again, we take $T_f \to \infty$ and $T_i \to -\infty$.

Since Heisenberg picture is more convient in QFT. Let us see how the S-matrix elements look like in the Heisenberg picture.

$$|a>_H = e^{iHt}|a,\, t>_S, \tag{1924}$$

$$A_H\,(t) = e^{iHt}A_S e^{-iHt}. \tag{1925}$$

So

$$|a, T_i >_H = e^{iHT_i}|a, T_i >_S \ . \tag{1926}$$

So then

$$< b|S|a > = _H< b, T_f|a, T_i >_H \tag{1927}$$

which is the elements of the S-matrix in the Heisenberg picture. Then a generic S-matrix element reads (I will drop H)

$$< \vec{p}_1, \vec{p}_2, ..., \vec{p}_n; T_f|\vec{k}_1, \vec{k}_2, ..., \vec{k}_n; T_i > \tag{1928}$$

where $T_f \to \infty$ and $T_i \to -\infty$.

Note: For notational simplicity we just consider a single species of neutral scalar particle.

Figure-115

Note: In actuality we usually have two particles entering the process and many coming out. But at this stage we allow our jelues the freedom of having many particles interact.

Note: This is very important. The S-matrix elements is a generalization of the wave-function (which is the simplest probability amplitude). When particles are created, of course we will have such a complicated amplitude or a wave-function as above.

The question is how to compute this complicated object? We have found the remarkable LSZ reduction formula before. Let us recall it

$$\left(\prod_{i=1}^{m}\frac{i\sqrt{z}}{k_i^2-m^2}\right)\left(\prod_{j=1}^{n}\frac{i\sqrt{z}}{p_j^2-m^2}\right) < \vec{p}_1, \vec{p}_2, ..., \vec{p}_n|iT|\vec{k}_1, \vec{k}_2, ..., \vec{k}_m >$$

$$= \left(\prod_{i=1}^{m}\int d^4x_i\, e^{-ik_ix_i}\right)\left(\prod_{j=1}^{n}\int d^4y_j\, e^{-ip_jy_j}\right) < 0|T\left\{\phi\left(x_1\right)...\phi\left(x_m\right)\phi\left(y_1\right)...\phi\left(y_n\right)\right\}|0 >. \tag{1929}$$

So it is clear that everything boils down to the computation of N-point Green's function (or correlation amplitude)

$$G\left(x_1, ..., x_N\right) = < 0|T\left\{\phi\left(x_1\right)...\phi\left(x_n\right)\right\}|0 > . \tag{1930}$$

Note that here $\phi\left(x\right)$ are the full Heisenberg quantum fields. Recall that we were able to describe this correlation function in terms of free (Interacting picture) fields as

$$G\left(x_1, ..., x_N\right) = \frac{< 0|T\left\{\phi_I\left(x_1\right)...\phi_I\left(x_n\right)e^{-i\int d^4x\mathcal{H}_I}\right\}|0 >}{< 0|T\left\{e^{-i\int d^4x\mathcal{H}_I}\right\}|0 >} \tag{1931}$$

where

$$H_I\left(t\right) = e^{iH_0\left(t-t_0\right)}H_{\rm int}e^{-iH_0\left(t-t_0\right)}. \tag{1932}$$

XIX. APPENDIX-IV: SUPERCONDUCTORS

Maxwell's equations are

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}, \quad \vec{B} = \vec{\nabla}\times\vec{A}, \tag{1933}$$

$$\vec{J} = q\frac{1}{4m}\left[\psi^*\left(\vec{p}-\frac{q}{c}\vec{A}\right)\psi - \left(\left(\vec{p}-\frac{q}{c}\vec{A}\right)\psi\right)^*\psi\right]. \tag{1934}$$

Note: Electrons in a metal interact through the exchange of phonons (excitation quanta of the crystalline lattice) and form a weakly bound state cooper pair.

At low tempratures cooper pairs condense and are described by a sort of macroscopic wave function ψ. $|\psi(\vec{r})|^2$ describes the real density of particles not probability density. $q = -2|e|$ and m is the mass of the electron, and the wave function is

$$\psi = \sqrt{\rho}\, e^{i\theta} \tag{1935}$$

where

$$\rho(\vec{r}) = \psi^*\psi \neq 0. \tag{1936}$$

Then

$$\vec{J} = \frac{q\rho}{2m}\left(\hbar\vec{\nabla}\theta - \frac{q}{c}\vec{A}\right) \tag{1937}$$

where $\vec{\nabla}.\vec{J} = 0$ and $\nabla^2\theta = 0$ (in the $\vec{\nabla}.\vec{A} = 0$ gauge). Then θ is constant and

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \nabla^2\vec{A} = -\frac{4\pi}{c}j. \tag{1938}$$

So

$$\vec{J} = -\frac{q^2\rho}{2mc}\vec{A} \tag{1939}$$

and

$$\nabla^2\vec{A} = \frac{2\pi\rho q^2}{mc^2}\vec{A} = \lambda^2\vec{A} \tag{1940}$$

where λ is London penetration depth $\lambda \sim$?? (10^{-5})cm.

XX. APPENDIX-V: NOTES ON PARITY

A. $\tau - \theta$ Puzzle:

In 1947 Cecil F. Powell found the pi-meson in the cloud chambers. (Hideki Yukawa predicted the particle in 1935.)

In 1848 Louis Pasteur discovered optical isomerism: two forms of the same compound (chemically same) (called isomers) rotate the polarized light in two different directions: left and right. One isomer is the mirror image of the other.

Pasteur observed that living organisims were able to synthesize and use only one isomer not the other. But in reactions left and right-handed isomers were produced in equal amounts.

In 1924 O. Laporte classified the wave-functions of an atom as either even or odd.

He observed that when an atom makes a transition and a photon is emmitted the wave-function changes symmetry.

photon is assigned	-1 parity
even wave-function	$+1$ parity
odd-wave-function	-1 parity

in atomic transitions parity is conserved (as a multiplication rule).

In 1927 Wigner found that Laporte's result came from the mirror-symmetry of the electromagnetic force.

B. Lee-Yang:

Japan invaded Kweichow (province of China) and Tsung Dao Lee was a student there, he left to Kunmig where he met Chen Ning Yang (at National Southwest University). In 1946 both recieved fellowships to study in USA.

Yang went to Chicago to work with Fermi (following Fermi from Cuhahic). Lee went to Chicago (because he did not have intermediate degrees. Chicago was the only university which allowed an undegreed to work towards PhD degree), then Yang did his PhD with Edward Teller and Lee did his thesis with Fermi.

Fermi adjsted Yang: As a young man, work on practical problems; do not worry about things of fundamental ???????. Yang ignored his advice! They then went to IAS, Princenton.

In 1949 Powell discovered a new particle, he called the tau meson. It decayed into three pions. Theta mesons was discovered (who found it?). It decayed into two pions

$$\begin{aligned} \theta^+ &\to \pi^+ + \pi^0 \\ \tau^+ &\to \pi^+ + \pi^+ + \pi^- \end{aligned} \tag{1941}$$

via weak force. Their masses and life-time were identical same for their decay modes.

In 1953 R. H. Dalitz argued that since the pion has -1 parity, there was a problem of parity non-conservation if τ^+ and θ^+ were the same particle. He then said they could not be the same particle.

At Rochester in 1956 at a conference Lee and Yang made a proposal: parity doubling: certain kinds of elementary particles occur in two different forms.

Feynman was staying in the same room with Matis Block. Block suggested Feynman that parity could be violated.

Next day, after Yang's presentation, Feynman brought-up the question of parity non-conservation. Yang replied that Lee and himself thought about this and came to non-conservation.

Wigner said that perhaps parity was violated in weak interactions.

Lee and Yang studied the literature and found that no experimental check of parity conservation was made for weak interactions.

In October 1956 Phys. Rev. Lee-Yang wrote "Question of Parity Conservation in Weak Interactions". They proposed several experiments.

Co-60: β-decay

Figure-116

(But of course spins of the nuclei of the Cobalt-60 is oriented with a strong magnetic force).

Chien-Shinng Wu (came to Berkly in 1936). She was a professor at Columbia.

Tempretures $\sim \frac{1}{100}$ Kelvin were needed. 27 December 1956 they managed! Done the experiment many times.

January 9 at 2 a.m. in the morning.

At the same time Leon Lederman's group found similar results. They submitted their paper to Phys. Rev. 15 January 1957.