Quantum Electrodynamics (QED)

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Content of the lecture:

- The Dirac Equation
- Lorentz transformation of the DE.
- Bilinear Covariants
- Solutions of free DE.
- The photon sector
- The Gauge symmetry

- QED is the quantum theory of electron, positron & photon
- QED is one of the oldest, most sucessful theory in particle physics. It has inspired many other theories.
- One needs to understand the quantum theory of spin-1/2 particle (electron) & spin-1 boson (photon) to get a better understanding of QED.
- Relativistic Quantum Mechanics (RQM) $\equiv$ (NR) Quantum Mechanics + Special Theory of relativity

In NRQM: For any particle $\to$ Schrödinger eqn (SE)

$\downarrow$

RQM: spin-0 particle: Klein-Gordon eqn.

spin-1/2 particle: Dirac eqn.

spin-1 particle: Maxwell's eqns (massless case)

SE: $i\hbar \frac{\partial \psi}{\partial t} = H\psi = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V\right)\psi$.

- time & space are treated differently $\left(\frac{\partial}{\partial t} \text{ vs } \vec{\nabla}^2\right)$
- Not good for invariance under Lorentz transf.

- Energy-momentum relation:

NR case: $E = \frac{\vec{p}^2}{2m} + V \xrightarrow[\vec{p}\to -i\hbar\vec{\nabla}]{E\to i\hbar\frac{\partial}{\partial t}}$ SE for the wave function ψ.

Relativistic case: $\underbrace{E^2 - \vec{p}^2c^2}_{c^2 p_\mu p^\mu} = m^2c^4 \xrightarrow{p_\mu \to i\hbar\partial_\mu} -\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \vec{\nabla}^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$: KG eqn.

$(p_\mu p^\mu = m^2c^4)$

- Both time & space derivatives are 2nd order.

- There is a problem with KG eqn:

In SE; $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ where $\vec{J} = \frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$

$\rho = |\psi|^2 > 0$

In KG; $\rho \propto \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}$: Not a positive definite

- Dirac's idea: An equation being first order in time & space derivatives.

- Special case: Particle at rest $(\vec{p} = 0)$

$$\frac{E}{c} = \frac{i\hbar}{c}\frac{\partial}{\partial t}$$

$$p_\mu p^\mu - m^2c^2 = 0 \xrightarrow{\vec{p}=0} p_0 p^0 - (mc)^2 = 0 \quad \underbrace{(p^0 - mc)}_{=0}\underbrace{(p^0 + mc)}_{=0} = 0$$

Either is 1^{st} order in $\frac{\partial}{\partial t}$

- Free particle case $(\vec{p} \neq 0)$:

$$p_\mu p^\mu - m^2c^2 = (\beta^K p_K - mc)(\gamma^\lambda p_\lambda + mc) = 0 \quad , \quad K, \lambda, \mu = 0,1,2,3$$
(sum over all)

β^K, γ^λ : 8 unknown coefficients

One can show that

$$p_\mu p^\mu - m^2c^2 = \underbrace{(\gamma^\mu p_\mu - mc)}_{=0}\underbrace{(\gamma^\nu p_\nu + mc)}_{=0} = 0$$

↓

Dirac Eqn.

$\boxed{\text{HW1}}$: Show that the following relations

(a) β^K & γ^K should be equal, $\beta^K = \gamma^K$, $K = 0,1,2,3$

(b) The algebra γ^μ's satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

Here $g^{\mu\nu}$ is the element of the metric $\underset{\sim}{g} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

(c) Show also that the algebra can only be satisfied by matrices rather than simple numbers.

(d) To determine the dimension of γ^μ's, show first that

(α) All γ^μ matrices are traceless, $\mathrm{Tr}[\gamma^\mu]=0$, $\mu=0,1,2,3$

(β) If one diagonalizes γ matrices, eigenvalues are ± 1.

(e) From part (d), deduce that the lowest dimension possible to realize the algebra is indeed 4 (not 2).

(f) Show that one possible representation for γ^μ's is

$$\gamma^0=\begin{pmatrix} \mathbb{1}_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -\mathbb{1}_{2\times 2}\end{pmatrix}, \quad \gamma^i=\begin{pmatrix} 0_{2\times 2} & \sigma^i \\ -\sigma^i & 0_{2\times 2}\end{pmatrix}$$

Where σ^i $(i=1,2,3)$ are the Pauli matrices.

- Very important result:

$$(\gamma^\mu p_\mu - mc)\psi(x)=0$$

$p_\mu \to i\hbar\partial_\mu$ in x-space

$$\boxed{(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x)=0}$$

Since γ^μ's are 4×4 matrices $\psi(x)$ cannot be a simple function $\psi(x)$ must be a 4×1 column vector.

$\Downarrow$

Dirac spinor

- $$\psi(x)=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

$\psi_1, \psi_2 \to$ particle (Electron's) spinor

$\psi_3, \psi_4 \to$ Anti-particle's spinor (positron's)

- SE-like form: $i\hbar\dfrac{\partial\psi}{\partial t}=H_D\psi$ $\quad H_D \to$ Dirac Hamiltonian

$\boxed{\text{HW2}}$: Show that Dirac Hamiltonian can be written

$$H_D=\frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla}+mc^2\beta, \quad \vec{\alpha}\equiv\gamma^0\vec{\gamma}, \quad \beta\equiv\gamma^0$$

Lorentz transformation of spinor fields:

- Lorentz transformation:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \quad , \quad \underset{\sim}{\Lambda} : 4\times 4 \text{ LT matrix}$$

- Scalar field case ($\phi(x)$):

$$\phi(x) \longrightarrow \phi'(x') = \phi(x) \quad (\text{since it's a scalar})$$

$$x' = \Lambda x, \qquad x \to \Lambda^{-1} x$$

$$\longrightarrow \phi'(\Lambda\Lambda^{-1}x) = \phi(\Lambda^{-1}x)$$

$$= \phi'(x) = \phi(\Lambda^{-1}x)$$

- Vector field case ($A^{\mu}(x)$):

$$A^{\mu}(x) \longrightarrow \Lambda^{\mu}{}_{\nu} A^{\nu}(\Lambda^{-1}x)$$

- How about spinor case ($\psi(x)$)?

In general, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$ (infinitesimal transf)

Where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ (anti-symmetric)

(consider $\Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\sigma'} g^{\sigma\sigma'} = g^{\mu\nu}$)

One can parametrize $\omega^{\mu}{}_{\nu}$ as a linear combination of $M^{\alpha\beta}$ defined as

$$(M^{\alpha\beta})^{\mu\nu} \equiv g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}$$

: The generators of the Lorentz transf. (3 rots + 3 boosts)

$M^{\alpha\beta}$ → 6 anti-symmetric 4x4 matrices

Then: $\omega^{\mu}{}_{\nu} = \frac{1}{2} \Omega_{\alpha\beta} (M^{\alpha\beta})^{\mu}{}_{\nu}$

$\Omega_{\alpha\beta}$ → just six numbers then, the finite form of $\Lambda^\mu{}_\nu \; (= \delta^\mu{}_\nu + \omega^\mu{}_\nu)$ becomes

$$\Lambda = e^{\frac{1}{2}\Omega_{\alpha\beta} M^{\alpha\beta}}$$

Now, the spinor case ($\psi(x)$):

$\psi^a(x)$, $a = 1,2,3,4$ (the spinor index)

$$\psi^a(x) \longrightarrow \underbrace{S[\Lambda]^a{}_b}_{S[\Lambda] = e^{\frac{1}{2}\Omega_{\alpha\beta} S^{\alpha\beta}}} \psi^b(\Lambda^{-1}x)$$

Where $S^{\alpha\beta}$ is the Lorentz transf generator (like $M^{\alpha\beta}$) but this time defined in terms of γ^μ's, satisfying the same algebraic commutation like $M^{\alpha\beta}$,

$$[S^{\mu\nu}, S^{\alpha\beta}] = S^{\mu\beta} g^{\nu\alpha} - S^{\nu\beta} g^{\alpha\mu} + S^{\mu\alpha} g^{\nu\beta} - S^{\nu\alpha} g^{\mu\beta}$$

If $S^{\alpha\beta} \equiv \frac{1}{4}[\gamma^\alpha, \gamma^\beta]$, it satisfies the above eqn

- Note that $S[\Lambda]$ is <u>not</u> unitary ($S[\Lambda]^\dagger S[\Lambda] \neq 1$)

How to construct Action terms (bilinear covariants):

Notice the following:

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

thus,
$$(S^{\alpha\beta})^\dagger = -\gamma^0 S^{\alpha\beta} \gamma^0$$

then,
$$S[\Lambda]^\dagger = e^{\frac{1}{2}\Omega S^\dagger} = \gamma^0 S[\Lambda]^{-1} \gamma^0$$

Define the Dirac adjoint of ψ: $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$.

One can show that

- $\bar{\psi}\psi$ transforms as a scalar under LT.
$$\bar{\psi}(x)\psi(x) = \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x)$$
- $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz vector.
$$\bar{\psi}(x)\gamma^\mu\psi(x) = \Lambda^\mu{}_\nu\, \bar{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x)$$
- $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor
$$\bar{\psi}(x)\gamma^\mu\gamma^\nu\psi(x) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, \bar{\psi}(\Lambda^{-1}x)\gamma^\alpha\gamma^\beta\psi(\Lambda^{-1}x)$$

$\boxed{\text{HW3}}$: Show that the following identities

(a) $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

(b) $(S^{\alpha\beta})^\dagger = -\gamma^0 S^{\alpha\beta}\gamma^0$

(c) $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0$

(d) $\bar{\psi}\psi$ is a Lorentz scalar

(e) Euler-Lagrange eqn for the action $J = \int d^4x\, \mathcal{L}(\psi, \bar{\psi}, \partial_\mu\psi)$. with $\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$ produces the Dirac eqn.

γ^5 and Parity Symmetry:

Define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

- $\{\gamma^5, \gamma^\mu\} = 0$
- $\gamma^{5\dagger} = \gamma^5$
- $(\gamma^5)^2 = \mathbb{1}$.

One can define Lorentz invariant projection operators:

$$P_{\substack{L\\R}} \equiv \frac{1}{2}(1 \mp \gamma^5) \qquad \left(\cdot\ P_{\substack{L\\R}}^2 = P_{\substack{L\\R}}\ ,\ P_L + P_R = \mathbb{1},\ P_L P_R = P_R P_L = 0\right)$$

Chiral spinors: $\psi_{\substack{L\\R}} \equiv P_{\substack{L\\R}}\psi$ s.t $\psi = \psi_L + \psi_R$.

ψ_L → left-handed spinor

ψ_R → Right-handed spinor

Parity ($\mathcal{P}$): $\psi(\vec{x},t) \xrightarrow{\mathcal{P}} \mathcal{P}\psi(\vec{x},t) = \gamma^0\psi(-\vec{x},t)$

$$\psi_{\substack{L\\R}}(\vec{x},t) \xrightarrow{\mathcal{P}} \mathcal{P}\psi_{\substack{L\\R}}(\vec{x},t) = \psi_{\substack{R\\L}}(-\vec{x},t)$$

- Both $\psi(\vec{x},t)$ & $\mathcal{P}\psi(\vec{x},t)$ satisfy the Dirac eqn

New bilinear covariants:

$\mathcal{P}$: $\bar{\psi}\gamma^5\psi(\vec{x},t) = -\bar{\psi}\gamma^5\psi(-\vec{x},t)$ → Pseudoscalar

$$\mathcal{P}: \bar{\psi}\gamma^5\gamma^\mu\psi(\vec{x},t) = \begin{cases} -\bar{\psi}\gamma^5\gamma^0(-\vec{x},t), & \mu=0 \\ +\bar{\psi}\gamma^5\gamma^i(-\vec{x},t), & \mu=i \end{cases}$$ → Axial vector

Plane Wave Solutions of Dirac Eqn:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad \ldots (*)$$

- $\psi(x) = u(\vec{p})\, e^{-ip.x}$ (positive energy soln since $\psi \sim e^{-iEt}$)

$$(*) \ldots \begin{pmatrix} -m & p.\sigma \\ p.\bar{\sigma} & -m \end{pmatrix} u(\vec{p}) = 0 \quad , \qquad \begin{matrix} \sigma^\mu = (1, \sigma^i) \\ \bar{\sigma}^\mu = (1, -\sigma^i) \end{matrix}$$

soln: $\bar{u}(\vec{p}) = \begin{pmatrix} \sqrt{p.\sigma}\,\xi \\ \sqrt{p.\bar{\sigma}}\,\xi \end{pmatrix}$, ξ : 2x1 spinor with $\xi^\dagger \xi = 1$.

Further solution:

- use $\psi(x) = v(\vec{p})\, e^{+ip.x}$ (Negative energy soln since $\psi \sim e^{+iEt}$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \rightarrow (\gamma^\mu p_\mu + m) v(\vec{p}) = \begin{pmatrix} m & p.\sigma \\ p.\bar{\sigma} & m \end{pmatrix} v(\vec{p}) = 0$$

soln: $v(\vec{p}) = \begin{pmatrix} \sqrt{p.\sigma}\,\eta \\ -\sqrt{p.\bar{\sigma}}\,\eta \end{pmatrix}$, $\eta^\dagger\eta = 1$.

Completeness relations: (by using the above solns)

- $\displaystyle\sum_{s=1}^{2} u^s(\vec{p})\, \bar{u}^s(\vec{p}) = \not{p} + m$
- $\displaystyle\sum_{s=1}^{2} v^s(\vec{p})\, \bar{v}^{(s)}(\vec{p}) = \not{p} - m.$

Feynman slash notation: $\not{p} \equiv \gamma^\mu p_\mu$

The photon sector:

The Lagrangian of a vector field $A_\mu(x)$ (No source)
$\downarrow$ free field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: the field strength tensor

EoM: $\partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \dfrac{\partial \mathcal{L}}{\partial A_\nu} = 0 \Rightarrow \partial_\mu F^{\mu\nu} = 0$. (Non-Homogeneous maxwell eqns)

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \to \partial_\mu \tilde{F}^{\mu\nu} = 0 \xrightarrow{\text{OR}} \partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$$

Define: $A^\mu = (\varphi, \vec{A})$

$\vec{E} = -\vec{\nabla}\phi - \dfrac{\partial \vec{A}}{\partial t}$ $\longrightarrow$ $\vec{\nabla}\cdot\vec{B} = 0$ (No magnetic monopole) $\quad \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$ (Faraday's law)

$\vec{B} = \vec{\nabla}\times\vec{A}$

- $\partial_\mu F^{\mu\nu} = 0$ $\longrightarrow$ $\vec{\nabla}\cdot\vec{E} = 0$ (Gauss' law)
  $\longrightarrow$ $\vec{\nabla}\times\vec{B} - \dfrac{\partial \vec{E}}{\partial t}$ (Ampere's law)

The Gauge Symmetry:

- $A_0(x)$ is not dynamical in a sense that no kinetic term in $\mathcal{L}$.

  $A_0(x)$ can be determined from $\vec{A}(x)$.

Hence $A_\mu(x)$ with 4-components (4 DOF) $\longrightarrow$ 3 components (3 DoF).

But the photon has only 2 DoF.!!

- One can observe that the Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is invariant under

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \underbrace{\partial_\mu \Lambda(x)}_{\substack{\text{any function} \\ (\Lambda(x\to\infty)\to 0)}}$$

$$
\begin{aligned}
F_{\mu\nu}(x) \longrightarrow F'_{\mu\nu}(x) &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\
&= \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \cancel{\partial_\mu \partial_\nu \Lambda} - \cancel{\partial_\nu \partial_\mu \Lambda} \\
&= F_{\mu\nu}(x).
\end{aligned}
$$

$\therefore \exists$ a redundancy in the formulation

take $\partial_\mu F^{\mu\nu} = 0 \to$

$$
\begin{aligned}
&\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \\
&(\partial_\mu \partial^\mu) A^\nu - \partial_\mu \partial^\nu A^\mu = 0 \\
&(\partial_\mu \partial^\mu) g^{\alpha\nu} A_\alpha - \partial^\alpha \partial^\nu A_\alpha = 0 \\
&[g^{\alpha\nu}(\partial_\mu \partial^\mu) - \partial^\alpha \partial^\nu] \underbrace{A_\alpha}_{A_\alpha + \partial_\alpha \Lambda} = 0
\end{aligned}
$$

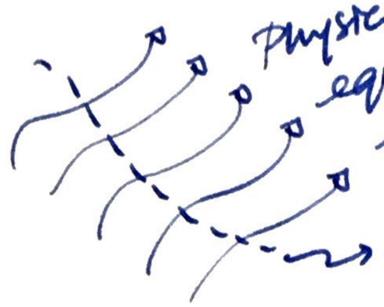
Then, $[g^{\alpha\nu}(\partial_\mu \partial^\mu) - \partial^\alpha \partial^\nu](\partial_\alpha \Lambda) = \cancel{(\partial_\mu \partial^\mu) \partial^\nu \Lambda} - \cancel{\partial^\alpha \partial^\nu \partial_\alpha \Lambda}$

$= 0$ for any Λ

$\therefore$ A_α & $A_\alpha + \partial_\alpha \Lambda$ cannot be distinguished

$\to$ A_α cannot be determined uniquely by the Maxwell's eqns.

Hence, A_α & $A_\alpha + \partial_\alpha \Lambda$ need to correspond the same physical state.



Physically equivalent states

Gauge fixing → Additional condition introduced to remove the redundancy

↓

Reduce DoF 3 → 2.

↓

Not unique, one choice is known as the Lorenz gauge, $\partial_\mu A^\mu = 0$.

EoM in Lorenz gauge:

$$\partial_\mu F^{\mu\nu} = 0 \rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu \underbrace{(\partial_\mu A^\mu)}_{=0 \text{ from Lorenz gauge}} = 0$$

$$\partial_\mu \partial^\mu A^\nu = 0, \quad \nu = 0,1,2,3$$

(No mixing among components of A^ν.)

What would be the corresponding Lagrangian?

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

→ Like a Lagrange-multiplier term

↓

Known as R_ξ-gauge.

$\boxed{\text{HW4}}$: Take the R_ξ-gauge Lagrangian of the photon field,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 .$$

(a) Show that the EoM (by using the Euler-Lagrange eqns) becomes

$$\left[\Box g^{\alpha\beta} - \left(1 - \frac{1}{\xi}\right) \partial^\alpha \partial^\beta \right] A_\alpha = 0 \qquad (\Box \equiv \partial_\mu \partial^\mu)$$

(b) Obtain the photon propagator in momentum space using $\partial_\alpha \to -i p_\alpha$ and show that it is

$$P_{\alpha\beta} = \frac{-i}{p^2} \left[g_{\alpha\beta} - (1-\xi) \frac{p_\alpha p_\beta}{p^2} \right]$$

Hint: the propagator in mom. space is naively defined as

$$P_{\alpha\beta} = i D^{-1}_{\alpha\beta} \quad \text{Where the EoM is } D_{\alpha\beta} A^\alpha = 0$$

Hence invert $D_{\alpha\beta}$ given in part (a).

Note that $\xi = 1$ is called the Feynman gauge.

The Full QED Lagrangian & the Feynman Rules:

$$\mathcal{L}_{free} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi$$ is <u>not invariant</u>

under the transformation

Gauge transf. $\begin{cases} \psi \to \psi' = e^{-i\underset{\uparrow}{e}\lambda(x)}\psi & \text{coupling constant} \\ A_\mu \to A'_\mu = A_\mu + \partial_\mu \lambda(x) \end{cases}$

$\therefore$ Free theories cannot be gauge invariant

How to make gauge invariant? $\Rightarrow$ Minimal coupling

$$\equiv$$

$$\forall\ \partial_\mu \to \underset{\text{the covariant derivative}}{D_\mu} \equiv \partial_\mu + ieA_\mu$$

Normally ψ & $\partial_\mu\psi$ do not transform in the same way.

That is, $\psi \to \psi' = e^{-ie\lambda(x)}\psi$.

$$\partial_\mu\psi \to \partial_\mu\psi' = e^{-ie\lambda(x)}\partial_\mu\psi \overbrace{- ie\underbrace{(\partial_\mu\lambda(x))}_{\neq 0} e^{-ie\lambda(x)}\psi}^{\text{unwanted term}}$$

$$\neq e^{-ie\lambda(x)}\partial_\mu\psi.$$

However, D_μ fixes the problem.

$$D_\mu\psi \to D'_\mu\psi' = (\partial_\mu + ie\underset{A_\mu + \partial_\mu\lambda(x)}{\underset{\shortparallel}{A'_\mu}})e^{-ie\lambda(x)}\psi$$

$$= \underbrace{\partial_\mu e^{-ie\lambda(x)}\psi}_{e^{-ie\lambda(x)}\partial_\mu\psi\ \cancel{-ie(\partial_\mu\lambda(x))e^{-ie\lambda}\psi}} + ieA_\mu e^{-ie\lambda(x)}\psi \cancel{+ie(\partial_\mu\lambda(x))e^{-ie\lambda(x)}\psi}. = e^{-ie\lambda}D_\mu\psi.$$

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi$ is gauge invariant.

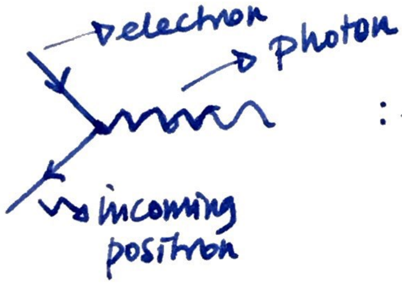
Note that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left[i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right]\psi$$

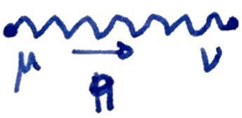
$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - \underbrace{e\,\overbrace{\bar{\psi}\gamma^\mu\psi}^{\equiv J^\mu} A_\mu}$$

The interaction term the form of which is dictated by the gauge invariance.

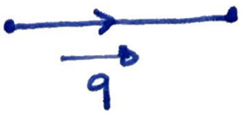
QED Feynman Rules:

- Vertex: 

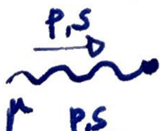
: $-ie\gamma^\mu$

- Photon Propagator: 

: $\dfrac{-ig_{\mu\nu}}{q^2}$ (in Feynman gauge)

- Fermion Propagator: 

: $\dfrac{i}{\not{q} - m} = \dfrac{i(\not{q} + m)}{q^2 - m^2}$

- External photon: 

: $\epsilon_\mu^{(s)}(p)$

: $\epsilon_\mu^{(s)*}(p)$

- External fermion: 

: $u^{(s)}(p)$ (incoming)


: $\bar{u}^{(s)}(p)$ (outgoing)

- External anti-ferm: 
: $\bar{v}^{(s)}(p)$ (incoming)


: $v^{(s)}(p)$ (outgoing)